# STAT 5200 Handout #10a

## Contrasts in Two-Way Factorial Design (Ch. 9)

Main effects with contrasts: (See HO # 9)

↳ interaction driven by Alcohol = 1

- Best to use effects parameterization to define contrasts here:

Base, $j=1,2$

$$Y_{ijk} = \mu + A_i + B_j + AB_{ij} + \epsilon_{ijk} \qquad k = 1 \dots 4$$

Alcohol $i=1,2,3$ ... interaction

- After accounting for effect of Base, does Alcohol have any effect?

Type III SS do this - test terms (like A) after accounting for (or while controlling for) all other terms in the model

- $H_0$: $A_1 = A_2 = A_3 = 0$

↳ by $\sum_i A_i = 0$ constraint

  - This is actually the intersection of multiple nulls:

2 DF. $H_0^{(1)}: A_1 = A_2$ and $H_0^{(2)}: A_1 = A_3$ -or- $A_1 = A_2$ and $A_2 = A_3$ -or- $A_2 = A_3$ and $A_1 = A_3$

  - Think of these as multiple contrasts:

$H_0^{(1)}: \psi^{(1)} = 0$ where $\psi^{(1)} = A_1 - A_2$

and

$H_0^{(2)}: \psi^{(2)} = 0$ $\quad \psi^{(2)} = A_1 - A_3$

(see pp. 5-6 of Ho #9)

- We can similarly test effect of Base, after accounting for effect of Alcohol:

Contrasts to test interactions: (See HO # 10; $T$ = Temp, $S$ = Sucrose)

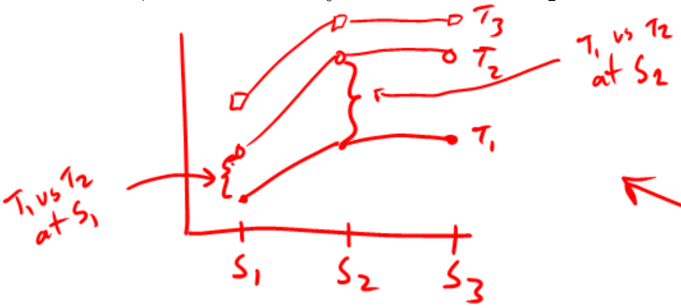
signif. T×S interaction is "driven" by Temp=20 / Sucr=20 combination

- Best to use effects parameterization to define contrasts here:

$$Y_{ijk} = \mu + T_i + S_j + TS_{ij} + \epsilon_{ijk}$$

$i=1..3$ $\quad j=1..3$ $\quad k=1..3$

- In testing interaction, look carefully at interaction plot:




- $H_0$: there is no interaction involving T & S

same

$H_0$: the effect of T does not depend on S (and vice-versa)

same

$H_0$: the $\underbrace{\text{T-level comparisons}}_{T_1 \text{ vs } T_2 \text{ and } T_1 \text{ vs } T_3}$ are the same for each $\underbrace{\text{S level}}_{S_1, S_2 \,\&\, S_3}$

no - profiles/lines aren't roughly parallel

same

- Consider this as the intersection of multiple hypotheses:

4 d.f.

$$\left\{\begin{array}{lll}
H_0^{(1)} & : & TS_{11} - TS_{21} = TS_{12} - TS_{22} \quad (T_1 - T_2 \text{ is same at S levels } 1 \,\&\, 2) \\
H_0^{(2)} & : & TS_{11} - TS_{31} = TS_{12} - TS_{32} \quad (T_1 - T_3 \text{ is same at S levels } 1 \,\&\, 2) \\
H_0^{(3)} & : & TS_{11} - TS_{21} = TS_{13} - TS_{23} \quad (T_1 - T_2 \text{ is same at S levels } 1 \,\&\, 3) \\
H_0^{(4)} & : & TS_{11} - TS_{31} = TS_{13} - TS_{33} \quad (T_1 - T_3 \text{ is same at S levels } 1 \,\&\, 3)
\end{array}\right.$$


- What about the "vice-versa"?
  - Note that $H_0^{(1)}$ can be rearranged as $TS_{11} - TS_{12} = TS_{21} - TS_{22}$ ($S_1 - S_2$ is the same at $T$ levels 1 and 2)
  - Similarly:
    - $H_0^{(2)}$ is $S_1 - S_2$ at T levels 1 & 3
    - $H_0^{(3)}$ is $S_1 - S_3$ at T levels 1 & 2
    - $H_0^{(4)}$ is $S_1 - S_3$ at T levels 1 & 3

- So we have multiple contrasts:

$$\begin{aligned}
\psi^{(1)} &= TS_{11} - TS_{21} - TS_{12} + TS_{22} \\
\psi^{(2)} &= TS_{11} - TS_{31} - TS_{12} + TS_{32} \\
\psi^{(3)} &= TS_{11} - TS_{21} - TS_{13} + TS_{23} \\
\psi^{(4)} &= TS_{11} - TS_{31} - TS_{13} + TS_{33}
\end{aligned}$$

- Note that these contrasts are zero if and only if lines in interaction plot are parallel
  (i.e., iff only if no interaction)
  - So testing $H_0$: "no interaction involving T & S" is equivalent to testing:
    $H_0^{(1)}$: $\psi^{(1)} = 0$ and $H_0^{(2)}$: $\psi^{(2)} = 0$ and $H_0^{(3)}$: $\psi^{(3)} = 0$ and $H_0^{(4)}$: $\psi^{(4)} = 0$
  - Then can do these as post-hoc tests to characterize non-parallel nature of plot (or to check specific hypotheses) (p.7 of HO #10)
- Contrasts involving interactions (or other custom comparisons) are notoriously awkward. Without some thought and organization, CONTRAST statement in SAS will fail. (no output!) (contrast not "estimable")
  Think of interaction subscripts being ordered as (row, column). Consider $\psi^{(1)}$ above:

"Row" term is first in CLASS statement

T level ↑ ↓ S level

$$\begin{array}{cc|ccc|}
 & & & S & \\
 & & 1 & 2 & 3 \\
 & 1 & 1 & -1 & 0 \\
T & 2 & -1 & 1 & 0 \\
 & 3 & 0 & 0 & 0
\end{array}$$

← coeffs. in $\psi^{(1)}$, and in first block of p.7 of HO#10; could also write as

$\boxed{1 \;\; -1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0}$

<u>Contrasts for custom comparisons</u>

- Example: In Handout #10 (T=Temp, S=Sucrose), compare Temp20/Sucrose60 ($T_1$ / $S_3$) with Temp30/Sucrose20 ($T_2$ / $S_1$)
- General steps to build a meaningful CONTRAST statement:

  even though only looks at means of 2 of 9 (3x3) factor level comb., the larger sample size of full structure of design will give greater power to detect a difference.

  1. Write out $H_0$ in terms of $\mu_{ij}$'s (from means model)

     $H_0$: $\mu_{13} = \mu_{21}$

  2. Substitute in effects model (to "talk" to SAS)

     $$\begin{aligned}
     \underset{\text{means}}{\mu_{13}} &= \underset{\text{effects}}{\mu + T_1 + S_3 + TS_{13}} \\
     \mu_{21} &= \mu + T_2 + S_1 + TS_{21}
     \end{aligned}$$

     $H_0$: $(\mu + T_1 + S_3 + TS_{13}) = (\mu + T_2 + S_1 + TS_{21})$

  3. Simplify to $H_0$: $\psi = 0$ (in terms of effect model parameters)

     $H_0$: $\underbrace{T_1 - T_2 - S_1 + S_3 + TS_{13} - TS_{21}}_{\psi} = 0$

4. Use coefficients of $\psi$ to define CONTRAST coefficients (in order)

T 1 -1 0 S -1 0 1

$$T*S \quad \begin{array}{ccc} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \quad \left.\begin{array}{l} \text{row} = i \text{ corresp. to } T_i \\ \text{col} = j \text{ corresp. to } S_j \end{array}\right.$$

(row term is first in CLASS statement)

$T_1$ $S_1, S_2, S_3$

- Custom comparison example: Does average of Temp20 (over all Sucrose levels) differ significantly from Temp30/Sucrose20? (See top plot on p. 3 of Handout #10)

$T_2$ $S_1$

$\mu_{ij}$ = mean at T level $i$ & S level $j$ ; $\underbrace{\mu_{ij}}_{\text{means model}} = \underbrace{\mu + T_i + S_j + TS_{ij}}_{\text{effects model}}$

$$H_0: \frac{\mu_{11} + \mu_{12} + \mu_{13}}{3} = \mu_{21}$$

subs. in effects param.

$$H_0: (\mu + T_1 + S_1 + TS_{11}) + (\mu + T_1 + S_2 + TS_{12}) + (\mu + T_1 + S_3 + TS_{13})$$
$$= 3(\mu + T_2 + S_1 + TS_{21})$$

simplify

$$H_0: \underbrace{3T_1 - 3T_2 - 2S_1 + S_2 + S_3 + TS_{11} + TS_{12} + TS_{13} - 3TS_{21}}_{\psi} = 0$$

SAS: T 3 -3 0 S -2 1 1 T*S $\begin{array}{ccc} 1 & 1 & 1 \\ -3 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ ;

- Custom comparison example: Does average of Temp40 over Sucrose20 and Sucrose40 differ significantly from average of Temp30 over Sucrose40 and Sucrose60? (See bottom plot on p. 3 of Handout #10)