STAT 5200 Handout #13a Random Effects (Ch. 11)

Intro. to Random Effects

- Until now, we could just throw terms in the model and let the software do all the work. From here on out, however, we need to pay closer attention to different kinds of factors and how to include them in the model. *Software can't think for us!*
- In previous models, the factors have set levels (A_i, $i = 1, \ldots, a$), and those are the only levels we care about. There, we make *inference on means* by testing differences between these factor levels (like the A_1 vs. A_3 comparison, for example). Such factors are called "fixed effects".
- But – what if we're actually using the levels of a factor as representative of all possible levels for that factor, and we're more interested in differences among all possible levels of the factor? Then we want to make *inference on variance*. Such factors are called "random effects".

Example: Handout #13

- Why these five machines? We don't just care about these five – they're representative of all possible machines, and we want to generalize to all machines
- Effects model:

$$Y_{ij} = \mu + A_i + \epsilon_{ij} \qquad i = 1, \ldots, 5$$
$$j = 1, \ldots, 5$$

strength ↗

effect of machine i

Fixed effects' w/ $\sum_i A_i = 0$

As before: ϵ_{ij} iid $N(0, \sigma^2)$

assumed indep

"A" is random effect → Also now: A_i iid $N(0, \sigma_A^2)$

Note: σ^2 & σ_A^2 are different parameters

- Interpretation of A_i's and their distribution:

Machine effects (A_i's) are from some population of effects, where $\underbrace{\text{ave. of dist. is } 0}_{\text{indirect constraint}}$ and variance in dist'n is σ_A^2.

(like $\sum_i A_i = 0$ in fixed effects)

Variance Components:

In such a design,

"by indep"

$$Var(Y_{ij}) = Var(\mu + A_i + \epsilon_{ij}) \overset{\downarrow}{=} Var(\mu) + Var(A_i) + Var(\epsilon_{ij})$$

$$= 0 + \sigma_A^2 + \sigma^2$$

and

$$Corr(Y_{ij}, Y_{\acute{\imath}\acute{\jmath}}) = \begin{cases} 0 & \text{if } i \neq \acute{\imath} \quad \text{(different machines)} \\ \frac{\sigma_A^2}{\sigma_A^2 + \sigma^2} & \text{if } i = \acute{\imath}\ ,\ j \neq \acute{\jmath} \quad \text{(different reps, same machine)} \\ 1 & \text{if } i = \acute{\imath}\ ,\ j = \acute{\jmath} \quad \text{(same rep, same machine)} \end{cases}$$

Inference on parameters of interest:

- Two main parameters here:
  1. σ_A^2 : variance component for factor A
  2. $\rho = \frac{\sigma_A^2}{\sigma_A^2+\sigma^2}$: intraclass correlation
- Inference here relies heavily on Expected Mean Squares (expected value of MS's within this model; derivation coming up in Ch. 12; for now, use GLM output)

Source	DF	MS	EMS
Machine	$5-1=$ 4	1040.84	$\sigma^2 + 5\sigma_A^2$ (p.3 HO#13)
Error	20	78.92	σ^2
Corrected Tot.	24 $= 25-1$		

- $H_0 : \sigma_A^2 = 0$ vs. H_a : $\sigma_A^2 > 0$ (not meaningful to have neg. variance)
  ↳ then $A_i = 0 \ \forall i$, so factor A has no effect
- Construct test statistic as ratio of MS's such that corresponding ratio of EMS's is 1 under H_0: when Ho true
  - Under H_0, $\frac{EMS_{Machine}}{EMS_E} = \frac{EMS_A}{EMS_E} = \frac{\sigma^2+5\sigma_A^2}{\sigma^2} = \frac{\sigma^2+5\cdot 0}{\sigma^2} = 1$ (when Ho true)
  - So use test statistic:

$$F = \frac{MS_{machine}}{MS_E} = 13.19 \sim F_{4,20}$$

sampling dist'n when Ho true

num ↑ ↑ denom df

⟹ conclude $\sigma_A^2 > 0$; machine-to-machine variation (among all machines) is signif.

- Get "point estimate" $\hat{\sigma}_A^2$ from $\sigma_A^2 = (EMS_{Machine} - EMS_E)/5$

$$\hat{\sigma}_A^2 = \frac{MS_{machine} - MS_E}{5} = 192.384$$

This ANOVA-based estimate is a "method of moments" estimate. It can sometimes be negative; we'll come back to this.

- Note that $\rho^2 = 0$ if and only if: $\sigma_A^2 = 0$

– so H_0: $\rho = 0$ is equivalent to: $H_0 : \sigma_A^2 = 0$

- Estimate correlation of tensile strength of reps from same machine using "point estimate" for $\rho = \sigma_A^2/(\sigma_A^2 + \sigma^2)$:

$$\hat{\rho} = \frac{\hat{\sigma}_A^2}{\hat{\sigma}_A^2 + \hat{\sigma}^2} = \frac{192.38}{192.38 + 78.92} \simeq .707$$

<u>Reconciling potential discrepancies:</u>

- In PROC MIXED, the COVTEST option gives estimates (and SE's, z-statistic, & p-values) for variance components.

$$z = \frac{Est.}{SE} \sim N(0,1) \qquad z = \frac{\hat{\sigma}_A^2}{SE[\hat{\sigma}_A^2]}$$

  - COVTEST allows for negative variance estimates (flexible, but <u>interpretation</u> may suffer)
  - These tests sometimes contradict the F-ratio tests (using ratio of MS's) – so what to do?

- Example – Handout #13: Three tests of H_0: $\sigma_A^2 = 0$
  - Cov. Table test (REML): → restricted maximum likelihood (iterative)
    - restricts $\hat{\sigma}_A^2 \geq 0$ p-value = .0957 (p.4)
    - uses H_A: $\sigma_A^2 > 0$
  - F-ratio test (Type 3):
    - sidesteps estimate of σ_A^2 p-value < 0.0001 (p.3 & 5) ← trust here
  - Cov. Table test (Type 3):
    - allows $\hat{\sigma}_A^2 < 0$ p-value = .1915 (p.5)
    - uses H_A: $\sigma_A^2 \neq 0$

  So is there a Machine effect?

- Cov. Table tests rely heavily on two things:
  1. normality of error terms (strict!)
  2. <u>large</u> sample size

  Lacking both of these, we don't trust Cov. Table tests (especially Type 3 result, which uses method of moments estimation – poorly behaved for inference compared to maximum likelihood estimates)

  So in Ho #13 example, we have okay (but not strict) normality of ε's & modest sample size
  ⇒ we'd trust F-ratio test

Side notes:

- ρ is the proportion of variability in tensile strength that is explained by variability among <u>all machines</u>

  - similar to (but not same as) R^2

- pp. 267-271 of text discusses "interval estimates" of variance components (confidence intervals for σ_A^2 and ρ)

$$\begin{tikzcd} & M \arrow[dl, dash] \arrow[dr, dash] & \\ (A) \arrow[dr, dash] & & (B) \arrow[dl, dash] \\ & (AB) \arrow[d, dash] & \\ & (E) & \end{tikzcd}$$

use ( ) to note random effects in Hasse Diagram

Two-factor Random Effects Model

- In a one-way random effects model, tests don't change from fixed effects – but for more than one factor, treating a factor as fixed vs. random can drastically alter test statistics (and conclusions).

- Example: Handout #14

$$\underbrace{Y_{ijk}}_{\text{strength}} = \mu + \underbrace{A_i}_{\substack{\text{operator} \\ i=1,\dots,10}} + \overbrace{B_j}^{\substack{\text{Machine} \\ j=1\dots10}} + AB_{ij} + \underbrace{\epsilon_{ijk}}_{\substack{\text{random error} \\ k=1\dots4}}$$

Error distribution:	ϵ_{ijk}	iid	$N(0, \sigma^2)$	} check with residuals (plots)
Random effects:	A_i	iid	$N(0, \sigma_A^2)$	} check with random effects estimates (pp. 6-7 of HO #14) – we'll do this last
	B_j	iid	$N(0, \sigma_B^2)$	
	AB_{ij}	iid	$N(0, \sigma_{AB}^2)$	

Var. Comp.	Source	DF	MS	EMS
σ_A^2	A	9	987.424	$\sigma^2 + 4\sigma_{AB}^2 + 40\sigma_A^2$
σ_B^2	B	9	300.645	$\sigma^2 + 4\sigma_{AB}^2 + 40\sigma_B^2$
σ_{AB}^2	AB	81	20.772	$\sigma^2 + 4\sigma_{AB}^2$
σ^2	Error	300	30.791	σ^2

} HO#14 p.4

Test H_0: $\sigma_A^2 = 0$ ("no A effect") using F-statistic (ratio of MS's) where ratio (of EMS's) $= 1$ under H_0

$\frac{EMS_A}{EMS_{AB}} = 1$ when $H_0: \sigma_A^2 = 0$ is true, so use $F_A = \frac{MS_A}{MS_{AB}} \sim \underbrace{F_{9,81}}_{\text{sampling dist'n}}$

The denominator of F is called the factor's "error term".

so MS_{AB} is the "error term" for testing factor A

A factor's error term is the next eligible random term below it in the Hasse diagram.

What would be different in test of "no A effect" if all factors were fixed?

We'd have $F_A = \frac{MS_A}{MS_E} \sim F_{9,300}$, which could lead to very different p-value & conclusion

Misc. Notes on Random Effects Models

- Both PROC GLM and PROC MIXED (with METHOD=TYPE3) give EMS's and construction of appropriate F-tests

- Can also use EMS's to estimate variance components

if X is a random variable, then $E[X]$ is the first moment of X. ($E[X^2]$ is second moment)

  - Example from Handout #14: $E[MS_A] = \ldots$

$$\sigma_A^2 = \frac{EMS_A - EMS_{AB}}{40}$$

So "method of moments" estimate of σ_A^2 is:

plug in obs. values for expected values

$$\hat{\sigma}_A^2 = \frac{MS_A - MS_{AB}}{40} \simeq 24.2$$

  - Another:

$$\sigma_{AB}^2 = \frac{EMS_{AB} - EMS_E}{4}$$

So

$$\hat{\sigma}_{AB}^2 = \frac{MS_{AB} - MS_E}{4} \simeq -2.5$$

But variance can't be negative!

(So be careful w/ method of moments estimates - don't have best properties)

- Can get variance estimates using restricted maximum likelihood (REML; constrain all $\sigma_*^2 \geq 0$) → restrict $\hat{\sigma}_*^2 \geq 0$
  - Get estimate $\hat{\sigma}_*^2$ and SE of estimate
  - Test H_0: $\sigma_*^2 = 0$ using $z = \hat{\sigma}_*^2/(SE[\hat{\sigma}_*^2]) \sim N(0,1)$
  - BUT – validity here assumes normality and large sample size (actually strictly requires) → So lacking these, we rely on F-ratio tests

- If $\hat{\sigma}_*^2 \leq 0$, in practice can usually just drop term from model (see p.5 & HO #14)
  - same as resetting / assuming $\sigma_*^2 \equiv 0$
  - $*$ - A, B, AB, ...

→ or just use COVTEST w/ REML

- Can check normality assumption of random effects
  - See pp. 6-7 of HO #14