1. Find the 1st derivative of the function $f(x) = x(x^2 - 5)^2$.
   (a) Find $f'(x)$.

   (b) Find all values of x for which the tangent line to the graph of the function is horizontal. (Start with the **Product Rule**.)

2. Given the function $f(x) = \dfrac{x^2}{x-2}$.
   (a) Find $f'(x)$.

   (b) Find the coordinates of all relative maximum and relative minimum points.

3. Given the function $f(x) = \dfrac{x}{x^2+1}$.

   (a) Find $f'(x)$.

   (b) Find the intervals on which the function is increasing and the intervals on which the function is decreasing.

4. Given the function $f(x) = \left(\dfrac{x}{x+1}\right)^3$.

   (a) Find $f'(x)$.

   (b) Find the intervals on which the function is increasing and the intervals on which the function is decreasing.

5. Given the function $f(x) = \dfrac{2x}{(x+1)^2}$.
   (a) Find $f'(x)$.

   (b) Find the intervals on which the function is increasing and the intervals on which the function is decreasing.

6. Given the function $f(x) = \dfrac{(x+2)^3}{x}$.
   (a) Find $f'(x)$.

   (b) Find the coordinates of all relative maximum and relative minimum points.

7. A patient is given an injection of a drug that acts as a sedative. The **amount**, $\boldsymbol{A}$ (in milligrams), of the drug in the patient's bloodstream at time t hours after the injection is to be predicted using the following function: $\boldsymbol{A(t) = \frac{50t}{t^2+9}}$ **mg.**; $\boldsymbol{t \geq 0}$.

   (a) *Fill in the following*: According to this model, at time $\boldsymbol{t = 1}$ **hour** after the injection, the amount of the drug in the patient's bloodstream will be ______________________________

   (*increasing/ decreasing*) at a rate of __________ (*amount*) ____________________(*units*).

   (b) Due to the nature of the drug, it is desired that the maximum amount of the drug in the bloodstream at any time does not exceed 10 mg. According to the model above, will the amount ever reach or exceed a value of 10 mg.? SHOW ALL WORK NECESSARY TO JUSTIFY YOUR ANSWER.

Answer Key

1 **(a)** $f'(x) = (x^2 - 5)(5x^2 - 5)$ or $5(x^2 - 5)(x^2 - 1)$ **(b)** $x = \pm\sqrt{5},\ x = \pm 1$

2 **(a)** $f'(x) = \dfrac{x^2 - 4x}{(x-2)^2}$ **(b)** Relative max at $(0, 0)$; relative min at $(4, 8)$

3 **(a)** $f'(x) = \dfrac{1 - x^2}{(x^2+1)^2}$ **(b)** decreasing on $(-\infty, -1) \cup (1, \infty)$; increasing on $(-1, 1)$

4 **(a)** $f'(x) = \dfrac{3x^2}{(x+1)^4}$ **(b)** increasing on $(-\infty, -1) \cup (-1, \infty)$

5 **(a)** $f'(x) = \dfrac{-2x + 2}{(x+1)^3}$ or $\dfrac{-2x^2 + 2}{(x+1)^4}$ **(b)** decreasing on $(-\infty, -1) \cup (1, \infty)$; increasing on $(-1, 1)$

6 **(a)** $f'(x) = \dfrac{(x+2)^2(2x-2)}{x^2}$ or $\dfrac{2(x+2)^2(x-1)}{x^2}$ **(b)** Relative minimum at $(1, 27)$

7 **(a)** $A(1) = 5$ mg; increasing; 4 mg. per hour. **(b)** No since the relative and absolute maximum value of the amount at $t = 3$ hours is $8.\bar{3}$ mg.

Detailed Solutions

1 (a) We need to use both the product and chain rule to evaluate this derivative

$$\frac{d}{dx}\left(x(x^2-5)^2\right) = \left(\frac{d}{dx}x\right)(x^2-5)^2 + x\left(\frac{d}{dx}(x^2-5)^2\right) \text{ product rule}$$
$$= 1\cdot(x^2-5)^2 + x\left(2(x^2-5)^{2-1}\right)\left(\frac{d}{dx}(x^2-5)\right) \text{ chain rule, power rule}$$
$$= (x^2-5)^2 + 2x(x^2-5)2x \text{ power rule}$$
$$= (x^2-5)(x^2-5+4x^2) = (x^2-5)(5x^2-5) = 5(x^2-5)(x^2-1)$$

(b) If the tangent line is horizontal, then its slope, or the derivative, is 0. Let us find where the slope is zero by finding x that satisfy the condition

$$5(x^2-5)(x^2-1) = 0$$

which is satisfied by $x = \pm\sqrt{5},\ x = \pm 1$.

2 (a) Let us use the quotient rule to evaluate this derivative and simplify

$$\frac{d}{dx}\left[\frac{x^2}{x-2}\right] = \frac{(x-2)\left(\frac{d}{dx}x^2\right) - x^2\left(\frac{d}{dx}(x-2)\right)}{(x-2)^2} = \frac{(x-2)2x - x^2(1)}{(x-2)^2} = \frac{2x^2-4x-x^2}{(x-2)^2} = \frac{x^2-4x}{(x-2)^2}$$

so we conclude $f'(x) = \dfrac{x^2-4x}{(x-2)^2}$

(b) We can find critical numbers by looking for where $f'(x) = 0$

$$\frac{x^2-4x}{(x-2)^2} = 0$$
$$x^2-4x = 0$$
$$x(x-4) = 0$$
$$x = 0 \text{ or } x = 4$$

We then evaluate $f(0) = 0$ and $f(4) = 8$. Finally we determine which are maxima and minima with the first derivative test $f'(-1) = 0.556$, $f'(1) = -3$, and $f'(5) = 0.556$ to conclude there is a local maximum at (0,0) and a local minimum at (4,8).

3 (a) Let us use the quotient rule to evaluate this derivative and simplify

$$\frac{d}{dx}\left[\frac{x}{x^2+1}\right] = \frac{(x^2+1)\left(\frac{d}{dx}x\right) - x\left(\frac{d}{dx}(x^2+1)\right)}{(x^2+1)^2} = \frac{(x^2+1)1 - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

so we conclude $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$

(b) We first find where the derivative can change sign by looking for critical numbers

$$\begin{aligned}\frac{1-x^2}{(x^2+1)^2} &= 0\\ 1-x^2 &= 0\\ x^2 &= 1\\ x &= \pm 1\end{aligned}$$

and test numbers between these points $f'(-2) = -0.12$, $f'(0) = 1$, $f'(2) = -0.12$ to conclude that our function is decreasing on $(-\infty, -1) \cup (1, \infty)$ and increasing on $(-1, 1)$.

4 (a) We evaluate the derivative by first using the chain rule, then the quotient rule

$$\frac{d}{dx}\left(\frac{x}{x+1}\right)^3 = 3\left(\frac{x}{x+1}\right)^2 \cdot \frac{d}{dx}\left(\frac{x}{x+1}\right) = 3\left(\frac{x}{x+1}\right)^2 \cdot \frac{(x+1)\left(\frac{d}{dx}x\right) - x\left(\frac{d}{dx}(x+1)\right)}{(x+1)^2}$$
$$= 3\left(\frac{x}{x+1}\right)^2 \cdot \frac{(x+1)(1) - x(1)}{(x+1)^2} = 3\left(\frac{x}{x+1}\right)^2 \cdot \frac{1}{(x+1)^2} = \frac{3x^2}{(x+1)^4}$$

so we conclude $\frac{3x^2}{(x+1)^4}$

(b) We first find where the derivative can change sign by looking for critical numbers

$$\begin{aligned}\frac{3x^2}{(x+1)^4} &= 0\\ 3x^2 &= 0\\ x &= 0.\end{aligned}$$

and note a vertical asymptote at $x = -1$. Now we test numbers between these points $f'(-2) = 12$, $f'(-\frac{1}{2}) = 12$, and $f'(1) = 0.1875$. Because the sign of the derivative is positive everywhere the function is increasing on $(-\infty, -1) \cup (-1, \infty)$ which is everywhere besides $x = -1$ where a vertical asymptote occurs.

5 (a) Let us use the quotient rule to evaluate this derivative and simplify

$$\frac{d}{dx}\left[\frac{2x}{(x+1)^2}\right] = \frac{(x+1)^2\left(\frac{d}{dx}2x\right) - 2x\left(\frac{d}{dx}(x+1)^2\right)}{(x+1)^4} = \frac{(x+1)^2 2 - 2x(2(x+1))}{(x+1)^4}$$
$$= \frac{2x^2+4x+2-4x^2-4x}{(x+1)^4} = \frac{-2x^2+2}{(x+1)^4}$$

so we conclude $f'(x) = \frac{-2x^2+2}{(x+1)^4}$ which can also be expressed

$$\frac{-2x^2+2}{(x+1)^4} = -2\frac{1-x^2}{(x+1)^4} = -2\frac{(1-x)(1+x)}{(x+1)^4} = -2\frac{1-x}{(x+1)^3} = \frac{-2x+2}{(x+1)^3}$$

(b) We first find where the derivative can change sign by looking for critical numbers

$$\begin{aligned} \frac{-2x^2+2}{(x+1)^4} &= 0 \\ -2x^2+2 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

and note a vertical asymptote at $x = -1$, implying that we only have a critical number at $x = 1$. Now we test numbers between these values
$f'(-2) = -6$, $f'(0) = 2$, and $f'(2) = -0.074$ to conclude $f(x)$ is decreasing on $(-\infty, -1) \cup (1, \infty)$ and increasing on $(-1, 1)$.

6 (a) Let us use the quotient rule to evaluate the derivative and simplify

$$\begin{aligned} \frac{d}{dx}\left[\frac{(x+2)^3}{x}\right] = \frac{x\left(\frac{d}{dx}(x+2)^3\right) - (x+2)^3\left(\frac{d}{dx}x\right)}{x^2} &= \frac{x3(x+2)^2 - (x+2)^3(1)}{x^2} \\ = \frac{(x+2)^2(3x-x-2)}{x^2} &= \frac{(x+2)^2(2x-2)}{x^2} \end{aligned}$$

so we conclude $f'(x) = \dfrac{(x+2)^2(2x-2)}{x^2}$

(b) We can find critical numbers by looking for where $f'(x) = 0$

$$\begin{aligned} \frac{(x+2)^2(2x-2)}{x^2} &= 0 \\ (x+2)^2(2x-2) &= 0 \\ x &= -2 \text{ or } x = 1 \end{aligned}$$

and note a vertical asymptote at $x = 0$. Then evaluate $f(-2) = 0$ and $f(1) = 27$. Finally we determine which are maxima and minima with the first derivative test by evaluating
$f'(-3) = -0.889$, $f'(-1) = -4$, $f'(\frac{1}{2}) = -25$, and $f'(2) = 8$. We can then conclude there is a local minimum at $(1, 27)$ and no local maximum.

7 (a) We use the quotient rule to evaluate the derivative and simplify

$$\begin{aligned} A'(t) = \frac{d}{dt}\left(\frac{50t}{t^2+9}\right) = \frac{(t^2+9)\left(\frac{d}{dt}50t\right) - 50t\left(\frac{d}{dt}(t^2+9)\right)}{(t^2+9)^2} &= \frac{(t^2+9)50 - 50t(2t)}{(t^2+9)^2} \\ = 50\frac{t^2+9-2t^2}{(t^2+9)^2} &= 50\frac{9-t^2}{(t^2+9)^2} \end{aligned}$$

Note that the units of $A'(t)$ are mg per hour since t is measured in hours. At $t = 1$ hour after injection the amount of drug in the patient's bloodstream will be $A(1) = 5$ mg and increasing at a rate of $A'(1) = 4$ mg per hour.

(b) Let us first locate critical numbers via

$$50\frac{9-t^2}{(t^2+9)^2} = 0$$
$$9 - t^2 = 0$$
$$t = \pm 3$$

and we exclude $t = -3$ since we are only interested in $t \geq 0$. Then use the first derivative test with values surrounding the critical numbers $A'(0) = 5.556$ mg and $A'(4) = 0.56$ mg to determine there is a relative maximum at $t = 3$ of $A(3) = 8.333$. Because this is a relative maximum $A(t)$ will have no greater output for $t > 3$. We then evaluate $A(0) = 0$ and know that $A(t) < 8.333$ for all t in the interval $[0, 3)$. Therefore, there is a absolute maximum at $t = 3$ hours of 8.333 mg, so we never exceed 10 mg.