1. Given the function $f(x) = \dfrac{1}{\sqrt{4x^2 + 3x + 7}}$, find $f'(x)$.

   Question Help:

   i First, rewrite the radical form in a rational exponent form using the following rules:

   - *The Rule of Rational Exponents*: Rational exponents are another way to express principal nth roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is $\sqrt[n]{a^m} = a^{\frac{m}{n}}$
   - *The Negative Rule of Exponents*: For any real number a and natural numbers n, the negative rule of exponents states that $\dfrac{1}{a^n} = a^{-n}$

   ii Watch this tutorial video from https://youtu.be/15uZr7eggkc to review *The Chain Rule*.

2. Given the function $g(x) = (9x^3+6)^5(7x^5+8)^4$, answer the following questions <u>without using the graph of g</u>:

   Question Help:

   - Watch this tutorial video from https://youtu.be/W6J6KN_OVWO.
   - Review Lesson 8: Optimization.

   (a) Find $g'(x)$.

   (b) Find the **slope of the tangent line** to the graph of the function at $x = 0$.

(c) Determine if the interval $(-1, 0)$ increasing or decreasing. *Explain your reasoning using Calculus.*

(d) Determine if the interval $(0, 1)$ increasing or decreasing. *Explain your reasoning using Calculus.*

(e) Is the point at $x = 0$ on the graph of g a relative (local) maximum, a relative (local) minimum, or neither? *Explain your reasoning using Calculus.*

3. Given the function $f(x) = \dfrac{-x+3}{x-3}$, answer the following questions:

(a) Find $f'(x)$.

(b) Find $f''(x)$.

4. Given the function $f(x) = \dfrac{x+5}{x-3}$, answer the following questions:

(a) Find $f'(x)$.

(b) Find $f''(x)$.

5. Given the function $f(x) = \left(\frac{x+1}{x+2}\right)^9$, find the first derivative of $f(x)$.

   Question Help: Watch this tutorial video from https://youtu.be/IHENYo3ogv8.

6. Given the function $h(x) = \frac{(2x+4)^3}{5x+8}$, find $h'(x)$.

7. An offshore oil well is leaking oil which is forming a circular slick on the water's surface that is 0.005 meters thick. The radius of a circular oil spill after $\boldsymbol{t}$ minutes is given by $r = \sqrt{16t}$ meters per minute. The volume, $\boldsymbol{V}$, of the slick can be determined by the equation $V = 0.005\pi r^2$.

   (a) Using the **Chain Rule** with the *Leibniz* notation form, determine the **related rates equation** that gives the relationship between the rate of change of the volume of the oil spill, $\frac{dV}{dt}$ and the rate of change of the radius of the oil spill, $\frac{dr}{dt}$.

   (b) Find the **radius** of oil spill after 36 minutes.*Include the appropriate units.*

   (c) Determine the equation of $\frac{dr}{dt}$.

   (d) Find the **rate of change** of the slick (oil spill) radius after 32 minutes. *Include the units on the rate of change. Give an exact value for your answer (that is, in a fraction form).*

   (e) Find the *instantaneous rate* at which the volume of the slick is changing and at which the radius is growing after 36 minutes. Indicate if the volume is increasing or decreasing at this time. *Include the appropriate UNITS on the rate you compute. Give an exact value for your answer. Leave π in the answer.*

8. The volume, $\boldsymbol{V}$, of a spherical cancer tumor is given by $V = \frac{\pi}{6} \cdot x^3$, where $\boldsymbol{x}$ is the diameter of the tumor, in millimeters. Past experience with tumors of this type has indicated that the volume of the tumors grows at a rate of 20 cubic millimeters per day.

   (a) Determine the **related rates** equation which relates the rate of change of the volume per day to the rate of change of the diameter per day of the tumor.

   (b) At **what rate** is the diameter of the tumor increasing at the point in time when the diameter is $\boldsymbol{x} = \mathbf{10}$ mm.? *Include the units on the rate of change. Round the answer to 2 decimal places.*

   (c) If the volume continues to grow at the same rate over time as given above, will the diameter continue to grow at the same rate as you determined in part (b)? YES or NO, then convince yourself mathematically why or why not.

Answer Key

1 $f'(x) = -\frac{1}{2}(4x^2 + 3x + 7)^{-\frac{3}{2}} \cdot (8x + 3)$

2 **(a)** $g'(x) = (9x^3 + 6)^5 \cdot 4 \cdot (7x^5 + 8)^3 \cdot 35x^4 + (7x^5 + 8)^4 \cdot 5 \cdot (9x^3 + 6)^4 \cdot 27x^2$ **(b)** $g'(0) = 0$ **(c)** Increasing **(d)** Increasing **(e)** Neither

3 **(a)** 0 **(b)** 0

4 **(a)** $f'(x) = -\frac{8}{(x-3)^2}$ **(b)** $f''(x) = \frac{16}{(x-3)^3}$

5 $f'(x) = 9\left(\frac{x+1}{x+2}\right)^8 \cdot \frac{1}{(x+2)^2}$

6 $h'(x) = \frac{6 \cdot (2x+4)^2(5x+8) - 5 \cdot (2x+4)^3}{(5x+8)^2}$

7 **(a)** $\frac{dV}{dt} = 0.01\pi r \frac{dr}{dt}$ **(b)** 24 meters **(c)** $\frac{dr}{dt} = \frac{2}{\sqrt{t}}$ **(d)** $\frac{1}{2\sqrt{2}}$ meters per minute **(e)** 0.08π cubic meters per minute; $\frac{1}{3}$ meters per minute

8 **(a)** $\frac{dV}{dt} = \frac{\pi}{2}x^2 \cdot \frac{dx}{dt}$ **(b)** 0.13 mm.per day **(c)** No; check $\frac{dx}{dt}$; for example, when $x = 15$ mm.

Detailed Solutions

1 We begin by rewriting $f(x)$ in radical form

$$f(x) = \frac{1}{\sqrt{4x^2+3x+7}} = (4x^2+3x+7)^{-1/2}$$

and then evaluate the derivative with chain rule

$$\begin{aligned} &\frac{d}{dx}(4x^2+3x+7)^{-1/2} \\ &= -\frac{1}{2}(4x^2+3x+7)^{-3/2} \cdot \frac{d}{dx}(4x^2+3x+7) \\ &= -\frac{1}{2}(4x^2+3x+7)^{-3/2} \cdot (8x+3) \end{aligned}$$

We can conclude $f'(x) = -\frac{1}{2}(4x^2+3x+7)^{-3/2} \cdot (8x+3)$

2 (a) We begin evaluating $g'(x)$ with the product rule

$$g'(x) = \frac{d}{dx}\left[(9x^3+6)^5(7x^5+8)^4\right] = (9x^3+6)^5\frac{d}{dx}\left[(7x^5+8)^4\right] + (7x^5+8)^4\frac{d}{dx}\left[(9x^3+6)^5\right]$$

Then using the chain rule

$$\begin{aligned} &(9x^3+6)^5\frac{d}{dx}\left[(7x^5+8)^4\right] + (7x^5+8)^4\frac{d}{dx}\left[(9x^3+6)^5\right] \\ &= (9x^3+6)^5 \cdot 4(7x^5+8)^3 \cdot \frac{d}{dx}\left[7x^5+8\right] + (7x^5+8)^4 \cdot 5(9x^3+6)^4 \cdot \frac{d}{dx}\left[9x^3+6\right] \\ &= (9x^3+6)^5 \cdot 4(7x^5+8)^3 \cdot 35x^4 + (7x^5+8)^4 \cdot 5(9x^3+6)^4 \cdot 27x^2 \end{aligned}$$

to conclude $g'(x) = (9x^3+6)^5 \cdot 4(7x^5+8)^3 \cdot 35x^4 + (7x^5+8)^4 \cdot 5(9x^3+6)^4 \cdot 27x^2$.

(b) To find the slope of the tangent line at $x = 0$ we evaluate $g'(0)$

$$g'(0) = (9(0)^3+6)^5 \cdot 4(7(0)^5+8)^3 \cdot 35(0)^4 + (7(0)^5+8)^4 \cdot 5(9(0)^3+6)^4 \cdot 27(0)^2 = 0 + 0 = 0$$

therefore $g'(0) = 0$.

(c) To determine if a function is increasing or

3 (a) We begin by rewriting

$$f(x) = \frac{-x+3}{x-3} = (-x+3)(x-3)^{-1}$$

and start with the product rule

$$f'(x) = \frac{d}{dx}\left[(-x+3)(x-3)^{-1}\right] = (x-3)^{-1}\frac{d}{dx}\left[(-x+3)\right] + (-x+3)\frac{d}{dx}\left[(x-3)^{-1}\right]$$

and then use the chain and product rule

$$= (x-3)^{-1}(-1) + (-x+3)(-1)(x-3)^{-2}$$

and simplifying

$$\frac{-1}{x-3} + \frac{x-3}{(x-3)^2} = \frac{-(x-3)}{(x-3)^2} + \frac{x-3}{(x-3)^2} = \frac{-(x-3)+(x-3)}{(x-3)^2} = 0$$

to conclude that $f'(x) = 0$. Note that neither $f(x)$ nor $f'(x)$ are defined at $x = 3$ since the denominator is zero at that value.

(b) Since $f'(x) = 0$ its derivative is $f''(x) = 0$.

4 (a) We start by rewriting

$$f(x) = \frac{x+5}{x-3} = (x+5)(x-3)^{-1}$$

and start with the product rule

$$f'(x) = \frac{d}{dx}\left[(x+5)(x-3)^{-1}\right] = (x-3)^{-1}\frac{d}{dx}\left[(x+5)\right] + (x+5)\frac{d}{dx}\left[(x-3)^{-1}\right]$$

and then use the chain and product rule

$$= (x-3)^{-1}(1) + (x+5)(-1)(x-3)^{-2}$$

and simplifying

$$= \frac{1}{x-3} + \frac{x+5}{(x-3)^2} = \frac{(x-3)}{(x-3)^2} - \frac{x+5}{(x-3)^2} = \frac{(x-3)-(x+5)}{(x-3)^2} = \frac{-8}{(x-3)^2}$$

therefore $f'(x) = \dfrac{-8}{(x-3)^2}$.

(b) We start by rewriting

$$f'(x) = \frac{-8}{(x-3)^2} = -8(x-3)^{-2}$$

and then use the chain rule

$$f''(x) = \frac{d}{dx}\left[-8(x-3)^{-2}\right] = -8 \cdot -2(x-3)^{-3} \cdot \frac{d}{dx}\left[x-3\right] = 16(x-3)^{-3} \cdot (1)$$

to conclude that $f''(x) = \dfrac{16}{(x-3)^3}$.

5 We start by expressing

$$f(x) = \left(\frac{x+1}{x+2}\right)^9 = \left((x+1)(x+2)^{-1}\right)^9$$

and use the chain rule

$$f'(x) = \frac{d}{dx}\left((x+1)(x+2)^{-1}\right)^9 = 9\left((x+1)(x+2)^{-1}\right)^8 \cdot \frac{d}{dx}\left((x+1)(x+2)^{-1}\right)$$

then evaluate the term being differentiated

$$\frac{d}{dx}\left((x+1)(x+2)^{-1}\right) = (x+2)^{-1}\frac{d}{dx}(x+1) + (x+1)\frac{d}{dx}\left((x+2)^{-1}\right)$$

$$= (x+2)^{-1}(1) + (x+1)(-1)(x+2)^{-2} = \frac{1}{x+2} - \frac{x+1}{(x+2)^2} = \frac{(x+2)-(x+1)}{(x+2)^2} = \frac{1}{(x+2)^2}$$

We then multiply this by the term obtained from the chain rule $f'(x) = 9\left(\frac{x+1}{x+2}\right)^8 \cdot \frac{1}{(x+2)^2}$.

6 Let's use the quotient rule to evaluate

$$\frac{d}{dx}\left[\frac{(2x+4)^3}{5x+8}\right] = \frac{(5x+8)\cdot\frac{d}{dx}(2x+4)^3 - (2x+4)^3\frac{d}{dx}(5x+8)}{(5x+8)^2}$$

$$= \frac{(5x+8)\cdot 3(2x+4)^2(2) - (2x+4)^3\cdot(5)}{(5x+8)^2}$$

$$= \frac{6(5x+8)(2x+4)^2 - 5(2x+4)^3}{(5x+8)^2}$$

so we conclude $h'(x) = \frac{6(2x+4)^2(5x+8) - 5(2x+4)^3}{(5x+8)^2}$

7 (a) Since the volume is a function of the radius $V(r)$ and the radius is a function of time $r(t)$ we can use the **chain rule** on $\frac{dV}{dt}$

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} = \frac{d}{dr}\left(0.005\pi r^2\right)\cdot\frac{dr}{dt} = 0.01\pi r\cdot\frac{dr}{dt}$$

so the relationship between the rates of change for volume and radius can be stated $\frac{dV}{dt} = 0.01\pi r\frac{dr}{dt}$

(b) The radius of the oil spill after 36 minutes is $r(36)$ since the variable t is measured in minutes

$$r(36) = \sqrt{16\cdot 36} = \sqrt{16}\cdot\sqrt{36} = 4\cdot 6 = 24 \text{ meters.}$$

(c) We can evaluate $\frac{dr}{dt}$ and simplify it to obtain

$$\frac{dr}{dt} = \frac{d}{dt}\sqrt{16t} = \frac{d}{dt}(16t)^{1/2} = \frac{1}{2}(16t)^{-1/2}\cdot 16 = \frac{8}{\sqrt{16t}} = \frac{8}{4\sqrt{t}} = \frac{2}{\sqrt{t}}$$

(d) The rate of change of the radius after 32 minutes is $\frac{dr}{dt}|_{t=32}$

$$\frac{dr}{dt}(32) = \frac{2}{\sqrt{32}} = \frac{2}{\sqrt{2\cdot 16}} = \frac{2}{4\sqrt{2}} = \frac{1}{2\sqrt{2}} \text{ meters per minute.}$$

(e) We want to evaluate $\frac{dV}{dt}|_{t=36}$ and $\frac{dr}{dt}|_{t=36}$. Let's first find the rate of change of volume, which can be expressed entirely in terms of t

$$\frac{dV}{dt} = 0.01\pi r\frac{dr}{dt} = 0.01\pi\sqrt{16t}\frac{2}{\sqrt{t}} = 0.08\pi \text{ cubic meters per minute}$$

so the volume is changing at 0.08π cubic meters per minute since the rate of change is constant for all time. Let's now find the rate of change of radius

$$\frac{dr}{dt}|_{t=36} = \frac{2}{\sqrt{36}} = \frac{2}{6} = \frac{1}{3} \text{ cubic meters per minute}$$

so the radius is changing at $\frac{1}{3}$ meters per minute at 36 minutes.

8 (a) We can find the desired relationship using the chain rule

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} = \frac{d}{dx}\left(\frac{\pi}{6} \cdot x^3\right) \cdot \frac{dx}{dt} = \frac{\pi}{2}x^2\frac{dx}{dt}$$

(b) We want to find the rate of change of the diameter, so we rearrange the equation from part (a) as $\frac{dx}{dt} = \frac{dV}{dt} \cdot \frac{2}{\pi x^2}$. We also know that $\frac{dV}{dt} = 20$ cubic mm per day from the problem statement. When the diameter is $x = 10$ mm the diameter is increasing at the rate

$$\frac{dx}{dt}\Big|_{x=10} = \frac{dV}{dt} \cdot \frac{2}{10^2\pi} = 20 \cdot \frac{1}{50\pi} = \frac{2}{5\pi} \approx 0.13 \text{ mm per day}$$

(c) The diameter will NOT increase at the same rate since $\frac{dx}{dt}$ depends on x. In other words the rate of change of the diameter can change depending on the diameter.