1. Find the slope of the tangent line to the graph of the equation $x^2+y^2=16$ at the point $(3,\sqrt{7})$.
   Question Help: Watch this tutorial video from https://youtu.be/kroPpRJI0QM

2. Given the equation $26x^3+8x^{26}y+y^5=35$, answer the following questions:
   Question Help: Watch this tutorial video from https://youtu.be/3J0m53qkb0k

   (a) Find $\dfrac{dy}{dx}$.

   (b) Find the equation of the **tangent line** to the curve at $(1,1)$.

3. Suppose the x and y are functions of another variable t, and x and y are related by the equation $y^2=4+xy$. Use *implicit differentiation* with respect to t to determine $\dfrac{dy}{dt}$ in terms of x, y, and $\dfrac{dx}{dt}$.

4. Given the equation $\frac{x^2}{16} + \frac{y^2}{25} = 1$, answer the following questions:
Question Help: Watch this tutorial video from https://youtu.be/kroPpRJI0QM

 (a) Use implicit differentiation to find $\frac{dy}{dx}$ without first solving for y.

 (b) Find $\left.\frac{dy}{dx}\right|_{(3,3.31)}$.

5. Let A be the area of a circle with radius r. If $\frac{dr}{dt} = 5$, find $\frac{dA}{dt}$ when $r = 1$.
Question Help: Watch this tutorial video from https://youtu.be/rHtpggJEfaw.

6. Find the **slope** to two decimal places of the tangent line to the curve (a lemniscate) $2(x^2+y^2)^2 = 25(x^2 - y^2)$ at the point $(-3, -1)$. Graph the curve to see a familiar shape!
Question Help: Watch this tutorial video from https://youtu.be/J8t0aG8-aB0.

7. Given the equation $y = \dfrac{x^3 + 8x + 8}{x^2 + 9}$, find $\dfrac{dy}{dx}$.

8. A company selling widgets has found that the number of items sold, x, depends upon the price, p at which they're sold, according the equation $x = \dfrac{40000}{\sqrt{2p+1}}$. Due to inflation and increasing health benefit costs, the company has been increasing the price by \$4 per month. Find the **rate** at which revenue is changing when the company is selling widgets at \$150 each. Include the units and indicate whether the revenue is **increasing or decreasing** at this price of \$150 each.
Question Help: Watch this tutorial video from https://youtu.be/SxxOlblHgIQ.

9. A factory's daily production cost, C (in thousands of dollars), and the daily production quantity, x (in thousands of units), are related by the equation $C^2 - 4x = 34$. The factory has begun to **increase** its production level by **0.1 thousands of units per day** (note that this is the rate of change of the daily production level). As such, the daily production costs will be increasing each day. This means that both x and C are functions of time t, measured in days.

   (a) Using implicit differentiation, find the **related rates equation** which relates the rate of change of the daily production costs to the rate of change of daily production.

   (b) At a certain point in time, the daily production quantity is at a level of $\boldsymbol{x = 0.5}$ thousands of units with a corresponding daily production cost of $C = 6$ thousand dollars. Determine the **time rate of change** of daily production costs at this time. Include the units and indicate whether the daily production costs are increasing or decreasing at this time.

   (c) At a later point in time, the daily production quantity is a level of $x = 3.75$ thousands of units with a corresponding daily production cost of $C = 7$ thousand dollars. Determine the **time rate of change** of daily production costs at this time. Include the units and indicate whether the daily production costs are increasing or decreasing at this time.

   (d) Is the daily production cost rising at a slower rate or faster rate as the production level increases?

10. The level of a pollutant in the air of a county, $\boldsymbol{L}$, depends partially on the **daily amount of coal, $\boldsymbol{x}$** (measured in pounds), that is burned at a nearby power plant. Health officials predict the pollutant level using the functional relationship: $\boldsymbol{L = 0.01x^{3/2} + 100}$ parts per million (ppm);$x \geq 0$. Due to alternative sources of energy that are becoming available, analysts have determined that at the point in time when $\boldsymbol{x = 10,000}$ pounds of coal are being burned in a day, the daily amount being burned is **decreasing at a rate of 20 pounds per day**. (NOTE that both variables $\boldsymbol{L}$ and $\boldsymbol{x}$ are functions of (or depend on) **time $\boldsymbol{t}$** measured in days. As such, they may be denoted by $L(t)$ and $x(t)$.

   (a) Determine the **related rates equation** that gives the relationship between the rate of change of the level of the pollutant, $\boldsymbol{\frac{dL}{dt}}$, and the rate of change of the daily amount of coal being burned, $\boldsymbol{\frac{dx}{dt}}$.

   (b) Determine the **rate** at which the level of the pollutant is changing at the point in time when the daily amount being burned is $x = 10,000$ pounds as given above. Indicate if the level of the pollutant is **increasing or decreasing** at this point in time. Include the **units** on your answer.

   (c) At a later point in time, analysts have determined that the daily amount of coal being consumed continues to decrease at the same rate of **20 pounds per day** as previously recorded. According to the model above, which of the following would be TRUE:

      (i) The level of the pollutant is changing at a **faster rate** than the previous time.
      (ii) The level of the pollutant is changing at a **slower rate** than the previous time.
      (iii) The level of the pollutant is changing at the **same rate** than the previous time.

Answer Key

1 $\frac{dy}{dx} = -\frac{x}{y}$; slope$= -\frac{3\sqrt{7}}{7}$

2 **(a)** $\frac{dy}{dx} = \frac{-78x^2 - 208x^{25}y}{8x^{26} + 5y^4}$ **(b)** $y = -22x + 23$.

3 $\frac{dy}{dx} = \frac{y}{2y - x} \cdot \frac{dx}{dt}$

4 **(a)** $\frac{dy}{dx} = -\frac{25x}{16y}$ **(b)** $-\frac{75}{52.96}$

5 31.42

6 -0.69

7 $\frac{dy}{dx} = \frac{x^4 + 19x^2 - 16x + 72}{(x^2 + 9)^2}$

8 4,626.44 dollars per month; increasing

9 **(a)** $2C\frac{dC}{dt} - 4\frac{dx}{dt} = 0$ **(b)** $\frac{dC}{dt} = 0.0\bar{3} \Longrightarrow$daily production costs are increasing at a rate of $0.0\bar{3}$ thousands of dollars per day. **(c)** $\frac{dC}{dt} \approx 0.029 \Longrightarrow$daily production costs are increasing at a rate of 0.029 thousands of dollars per day. **(d)** Costs are rising at a slower rate.

10 **(a)** $\frac{dL}{dt} = 0.015\sqrt{x} \cdot \frac{dx}{dt}$ **(b)** $\frac{dL}{dt} = -30$ ppm per day; decreasing. **(c)** Slower rate.

Detailed Solutions

1 To find the **slope** of a tangent line we need an equation for $\frac{dy}{dx}$. We begin by differentiating both sides of the given equation with respect to x. If the term contains y we apply the chain rule

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}[16]$$

We first evaluate the left side

$$\frac{d}{dx}\left[x^2+y^2\right]=2x+\frac{d}{dx}\left[y^2\right]=2x+\frac{dy}{dx}\cdot\frac{d}{dy}\left[y^2\right]=2x+2y\cdot\frac{dy}{dx}$$

and the right side is simply $\frac{d}{dx}16=0$. Together we have

$$2x+2y\cdot\frac{dy}{dx}=0\rightarrow\frac{dy}{dx}=-\frac{x}{y}$$

We can find the slope at the point $(3,\sqrt{7})$ by plugging in those values

$$\left.\frac{dy}{dx}\right|_{(3,\sqrt{7})}=-\frac{3}{\sqrt{7}}=-\frac{3\sqrt{7}}{7}$$

2 (a) To find $\frac{dy}{dx}$ we take the derivative of both sides with respect to x

$$\frac{d}{dx}\left[26x^3+8x^{26}y+y^5\right]=\frac{d}{dx}[35]$$

The left side evaluates as

$$\begin{aligned}\frac{d}{dx}\left[26x^3+8x^{26}y+y^5\right]&=78x^2+\frac{d}{dx}\left[8x^{26}y\right]+\frac{d}{dx}\left[y^5\right]=78x^2+208x^{25}\cdot y+8x^{26}\cdot\frac{dy}{dx}\frac{d}{dy}[y]+\frac{dy}{dx}\frac{d}{dy}\left[y^5\right]\\&=78x^2+208x^{25}\cdot y+8x^{26}\cdot\frac{dy}{dx}+5y^4\frac{dy}{dx}=78x^2+208x^{25}\cdot y+\frac{dy}{dx}\left(8x^{26}+5y^4\right)\end{aligned}$$

where the product rule is color coded. The right side simply evaluates to $\frac{d}{dx}35=0$. Together we have

$$\begin{aligned}78x^2+208x^{25}y+\frac{dy}{dx}\left(8x^{26}+5y^4\right)&=0\\\frac{dy}{dx}&=\frac{-78x^2+-208x^{25}y}{8x^{26}+5y^4}\end{aligned}$$

(b) To find the equation of the **tangent line**, $y=mx+b$, to the curve at (1,1) we first find the slope via

$$m=\left.\frac{dy}{dx}\right|_{(1,1)}=\frac{-78-208}{8+5}=-22$$

so our line must be of the form $y=-22x+b$. We find b using the known point $(1,1)$

$$1=-22(1)+b\rightarrow b=23$$

so the tangent line is given by y = -22x + 23.

3 Our goal is to have an equation for $\frac{dy}{dt}$, so let's differentiate both sides of our equation with respect to t

$$\frac{d}{dt}\left[y^2\right] = \frac{d}{dt}\left[4 + xy\right] \rightarrow \frac{dy}{dt}\frac{d}{dy}\left[y^2\right] = 0 + \frac{d}{dt}\left[xy\right]$$
$$= 2y\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$$
$$= \frac{dy}{dt}(2y - x) = y\frac{dx}{dt}$$
$$\frac{dy}{dt} = \frac{y}{2y - x} \cdot \frac{dx}{dt}$$

4 (a) To find $\frac{dy}{dx}$ we take the derivative of both sides with respect to x

$$\frac{d}{dx}\left[\frac{x^2}{16} + \frac{y^2}{25}\right] = \frac{d}{dx}\left[1\right]$$

The left side evaluates as

$$\frac{d}{dx}\left[\frac{x^2}{16} + \frac{y^2}{25}\right] = \frac{x}{8} + \frac{dy}{dx}\frac{d}{dy}\left[\frac{y^2}{25}\right] = \frac{x}{8} + \frac{2y}{25}\frac{dy}{dx}$$

The right side simply evaluates to $\frac{d}{dx}1 = 0$. Together we have

$$\frac{x}{8} + \frac{2y}{25}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{25x}{16y}$$

(b) We evaluate the slope at the point given

$$\left.\frac{dy}{dx}\right|_{(3,3.31)} = -\frac{25(3)}{16(3.31)} = -\frac{75}{52.96}$$

5 The area of a circle with radius r is $A = \pi r^2$. Let us find $\frac{dA}{dt}$ with implicit differentiation

$$\frac{dA}{dt} = \frac{d}{dt}\left[\pi r^2\right] = \frac{dr}{dt}\frac{d}{dr}\left[\pi r^2\right] = 2\pi r\frac{dr}{dt}$$

and we are given $\frac{dr}{dt} = 5$, so we know can evaluate $\frac{dA}{dt}$ when $r = 1$

$$\frac{dA}{dt} = 2\pi(1)(5) = 10\pi \approx 31.42$$

6 We need to differentiate both sides with respect to x

$$\frac{d}{dx}\left[2(x^2+y^2)^2\right] = \frac{d}{dx}\left[25(x^2-y^2)\right]$$
$$4(x^2+y^2)\cdot\frac{d}{dx}\left[x^2+y^2\right] = 25(2x-2y\frac{dy}{dx})$$
$$2(x^2+y^2)(2x+2y\frac{dy}{dx}) = x-25y\frac{dy}{dx}$$
$$\frac{4}{25}x(x^2+y^2)+\frac{4}{25}y(x^2+y^2)\frac{dy}{dx}+y\frac{dy}{dx} = x$$
$$y\frac{dy}{dx}(1+\frac{4}{25}(x^2+y^2)) = x(1-\frac{4}{25}(x^2+y^2))$$
$$\frac{dy}{dx} = \frac{x(1-\frac{4}{25}(x^2+y^2))}{y((1+\frac{4}{25}(x^2+y^2))}$$

We can find the slope at the point $(-3,-1)$ by plugging in those values for x and y returning -0.69.

7 To find $\frac{dy}{dx}$ we need only evaluate the derivative using quotient rule and simplify

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^3+8x+8}{x^2+9}\right] = \frac{(x^2+9)\frac{d}{dx}(x^3+8x+8)-(x^3+8x+8)\frac{d}{dx}(x^2+9)}{(x^2+9)^2}$$
$$= \frac{(x^2+9)(3x^2+8)-(x^3+8x+8)2x}{(x^2+9)^2} = \frac{3x^4+35x^2+72-2x^4-16x^2-16x}{(x^2+9)^2} = \frac{x^4+19^2-16x+72}{(x^2+9)^2}$$

8 We want to the find the rate of change of revenue R with respect to time when $p=\$150$. We also know that $\frac{dp}{dt}=\$4$ per month from the problem statement. We eventually want to express the change of revenue in terms of known information: p and $\frac{dp}{dt}$ The revenue is the price of a product p multiplied by the number of products sold x, or $R=px$. We evaluate the derivative with respect to time

$$\frac{dR}{dt} = \frac{d}{dt}[px] = p\frac{dx}{dt}+x\frac{dp}{dt} = p\frac{d}{dt}\left[\frac{40000}{\sqrt{2p+1}}\right]+x\frac{dp}{dt} = p\cdot\frac{-40000}{(2p+1)^{3/2}}\cdot\frac{dp}{dt}+\frac{40000}{\sqrt{2p+1}}\cdot\frac{dp}{dt} \text{ dollars per month}$$

We determine the units knowing that revenue is measured in dollars and t in months. Now we plug in the known information to obtain

$$(150)\cdot\frac{-40000}{(2(150)+1)^{3/2}}\cdot(4)+\frac{40000}{\sqrt{2(150)+1}}\cdot(4) \approx \$4626.44 \text{ per month}$$

Therefore the revenue is increasing at this price.

9 (a) To relate the rates of change for cost C and daily production x we implicitly differentiate $C^2-4x=34$

$$\frac{d}{dt}\left[C^2-4x\right] = \frac{d}{dt}[34] \longrightarrow 2C\frac{dC}{dt}-4\frac{dx}{dt}=0$$

(b) From the problem statement, we know that $\frac{dx}{dt}=0.1$ thousands of units per day, and we substitute this along with $C=6$ into the equation from part (a)

$$2(6)\frac{dC}{dt}-4(0.1)=0 \longrightarrow \frac{dC}{dt}=\frac{0.4}{12}=0.0\bar{3} \text{ thousand dollars per day}$$

Cost is increasing at this time. Note that $x = 0.5$ is relevant to maintain consistency with the initial equation given but not for finding the value.

(c) We again use $\frac{dx}{dt} = 0.1$ thousands of units per day and $C = 7$ thousand dollars in our equation from part (a)

$$2(7)\frac{dC}{dt} - 4(0.1) = 0 \longrightarrow \frac{dC}{dt} = \frac{0.4}{14} = 0.029 \text{ thousand dollars per day}$$

Cost is increasing at this time.

(d) As the production level x increases (like from $x = 0.5$ to $x = 3.75$) the production cost rises at a slower rate ($\frac{dC}{dt} = 0.0\bar{3}$ to $\frac{dC}{dt} \approx 0.029$).

10 (a) We implicitly differentiate

$$\frac{d}{dt}[L] = \frac{d}{dt}\left[0.01x^{3/2} + 100\right] \longrightarrow \frac{dL}{dt} = 0.01(3/2)x^{1/2} \longrightarrow \frac{dL}{dt} = 0.015\sqrt{x}\frac{dx}{dt}$$

(b) The rate at which the pollutant level L is changing when $x = 10,000$ pounds and the amount of coal being burned x is decreasing at $\frac{dx}{dt} = -20$ pounds per day can be found using the relation from part (a)

$$\frac{dL}{dt} = 0.015(10000)^{1/2}(-20) = -30 \text{ ppm per day}$$

The pollutant level is decreasing at this time.

(c) If the amount of coal being consumed is decreasing, then at sometime T after the measurement taken in part (b) the amount of coal burned will be less than $x = 10,000$. We also know that the amount of coal being consumed is still decreasing at 20 pounds per day. Consider $\frac{dL}{dt} = 0.015x^{1/2}(-20)$ at this later time when $x < 10000$, so $\frac{dL}{dt}$ will be smaller than previously. This implies that the level of the pollutant is changing at a **slower rate** than the previous time.