1. Given $g(x) = e^x(1 + 2e^x)^2$, answer the following questions:
   (a) Find $g'(x)$.

   (b) Find all values of x for which the tangent line to the graph of the function is horizontal.

2. Given the function $g(x) = \dfrac{e^x}{x+1}$.
   (a) Find $g'(x)$.

   (b) Find all values of x for which the slope of the tangent line to the graph of $g(x)$ equals 0.

   (c) Find the intervals in which $g(x)$ is increasing and the intervals in which $g(x)$ is decreasing.
   Hint: Remember to check to determine where the derivative is **undefined** when creating the sign chart for g'(x).

3. Given the function $g(x) = \dfrac{x}{e^x}$.
   (a) Find $g'(x)$.

   (b) Find all values of x for which the slope of the tangent line to the graph of $g(x)$ equals 0.

   (c) Identify any relative/absolute maximum and relative/absolute minimum points.

4. Given the function $g(x) = x^3 e^x$.
   (a) Find $g'(x)$.

   (b) Find all values of x for which the slope of the tangent line to the graph of $g(x)$ equals 0.

   (c) Find the intervals in which $g(x)$ is increasing and the intervals in which $g(x)$ is decreasing.

5. Given the function $g(x) = \dfrac{e^{\frac{x^2}{2}}}{x}$.

   (a) Find $g'(x)$ and *simplify*.

   (b) Find all values of x for which the tangent line to the graph of the function is horizontal.

   (c) Find the intervals in which $g(x)$ is increasing and the intervals in which $g(x)$ is decreasing. **Hint:** Remember to check to determine where the derivative is **undefined** when creating the sign chart for g'(x).

6. Given the function $g(x) = \dfrac{x+1}{e^{2x}}$.

   (a) Find $g'(x)$ in *factored/simplified* form.

   (b) Find the intervals in which $g(x)$ is increasing and the intervals in which $g(x)$ is decreasing. **Hint:** Remember to check to determine where the derivative is **undefined** when creating the sign chart for g'(x).

   (c) Identify any relative/absolute maximum and relative/absolute minimum points.

7. Given the function $g(x) = (e^{4x} + 1)^2$.
   (a) Find $g'(x)$ in *factored/simplified* form.

   (b) Find the intervals in which $g(x)$ is increasing and the intervals in which $g(x)$ is decreasing. **Hint:** Remember to check to determine where the derivative is **undefined** when creating the sign chart for g'(x).

   (c) Identify any relative/absolute maximum and relative/absolute minimum points.

8. Given the function $g(x) = (1 - x)e^{2x}$.
   (a) Find $g'(x)$ in *factored/simplified* form.

   (b) Find the intervals in which $g(x)$ is increasing and the intervals in which $g(x)$ is decreasing.

   (c) Find $g''(x)$.

   (d) Find the intervals in which $g(x)$ is concave upwards and the intervals in which $g(x)$ is concave downwards; identify any inflection points.

9. The concentration of a drug in a patient's bloodstream t hours after injection is modeled by the function $C(t) = 100e^{-0.05t}$ mg./liter; $t \geq 0$. Find the rate of change of the concentration at times t=1,5, and 24 hours after the injection. Include the appropriate units. Interpret the results.

10. The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1964 is modeled by the function

$$N(t) = 5.3e^{0.093t^2 - 0.87t} \qquad 0 \leq t \leq 4$$

where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.

(a) Evaluate $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.

(b) Evaluate $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in New York City.

11. The **relative rate of change** of a differentiable function $y = f(x)$ is given by $\dfrac{100 \cdot f'(x)}{f(x)}\%$.
One model for population growth is a *Gompertz* growth function, given by

$$P(x) = ae^{-b \cdot e^{-cx}}$$

where a, b, and c are constants.

(a) Find the **relative rate of change formula** for the generic *Gompertz* function.

(b) Use part (a) to find the **relative rate of change** of a population in $x = 20$ months when $a = 204$, $b = 0.0198$, and $c = 0.15$.

(c) Briefly interpret what the result of part (b) means.

Answer Key

1 **(a)** $g'(x) = e^x(1 + 2e^x)(6e^x + 1)$ **(b)** None

2 **(a)** $g'(x) = \dfrac{xe^x}{(x+1)^2}$ **(b)** $x = 0$ **(c)** $g'(x)$ is undefined at $x = -1$; $g(x)$ is decreasing on $(-\infty, -1) \cup (-1, 0)$; increasing on $(0, \infty)$

3 **(a)** $g'(x) = \dfrac{1-x}{e^x}$ **(b)** $x = 1$ **(c)** $\left(1, \dfrac{1}{e}\right)$ is a relative/absolute maximum point; no relative/absolute minimum point.

4 **(a)** $g'(x) = e^x x^2(x + 3)$ **(b)** $x = -3, 0$ **(c)** $g(x)$ is decreasing on $(-\infty, -3)$; increasing on $(-3, \infty)$

5 **(a)** $g'(x) = \dfrac{e^{\frac{x^2}{2}}(x^2 - 1)}{x^2}$ **(b)** $x = \pm 1$ **(c)** $g'(x)$ is undefined at $x = 0$; $g(x)$ is increasing on $(-\infty, -1) \cup (1, \infty)$; decreasing on $(-1, 0) \cup (0, 1)$

6 **(a)** $g'(x) = \dfrac{-2x - 1}{e^{2x}}$ **(b)** $g(x)$ is increasing on $\left(-\infty, -\dfrac{1}{2}\right)$; decreasing on $\left(-\dfrac{1}{2}, \infty\right)$ **(c)** Relative/absolute maximum point at $\left(-\dfrac{1}{2}, \dfrac{1}{2}e\right)$

7 **(a)** $g'(x) = 8e^{4x}(e^{4x} + 1)$ **(b)** $g'(x) \neq 0$ and moreover $g'(x) > 0$ for all x so $g(x)$ is increasing for all x. **(c)** None.

8 **(a)** $g'(x) = e^{2x}(1 - 2x)$ **(b)** $g'(x) = 0$ for $x = \dfrac{1}{2}$; $g(x)$ is increasing on $\left(-\infty, \dfrac{1}{2}\right)$; decreasing on $\left(\dfrac{1}{2}, \infty\right)$. **(c)** $g''(x) = -4xe^{2x}$ **(d)** Inflection point occurring at $x = 0$; $g(x)$ is concave upwards on $(-\infty, 0)$; concave downwards on $(0, \infty)$

9 $C'(1) \approx -4.756$ (mg.per liter)per hour; $C'(5) \approx -3.894$ (mg.per liter)per hour; $C'(24) \approx -1.506$ (mg.per liter)per hour.

10 **(a)** $N(0) = 5.3$; $N(4) \approx 0.723$; At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1964 there were approximately 723 cases of the disease in the United States. **(b)** $N'(0) = -4.611$; $N'(3) \approx -0.2808$; At the beginning of 1960 the number of cases of the disease was **decreasing** at rate of 4.611 thousand per year; at the beginning of 1963, the number of cases of the disease was **decreasing** at a rate of 0.2808 thousand per year.

11 **(a)** The relative rate of change$= \dfrac{100 \cdot f'(x)}{f(x)} = 100 \cdot (bc) \cdot e^{-cx}$ **(b)** $P'(x) = a \cdot b \cdot c \cdot e^{-be^{-cx} - cx}$; $P'(20) \approx 0.01479$ % **(c)** In 20 months, the population is increasing at the **percentage rate** around 0.015% per month

Detailed Solutions

1 (a) We evaluate the derivative starting with the product rule

$$\begin{aligned}\frac{d}{dx}\left[e^x(1+2e^x)^2\right] &= \frac{d}{dx}\left[e^x\right]\cdot(1+2e^x)^2 + e^x\cdot\frac{d}{dx}\left[(1+2e^x)^2\right]\\ &= e^x\cdot(1+2e^x)^2 + e^x\cdot 2(1+2e^x)\frac{d}{dx}\left[1+2e^x\right]\\ &= e^x\cdot(1+2e^x)^2 + e^x\cdot 2(1+2e^x)\,(2e^x)\\ &= e^x(1+2e^x)(1+2e^x+4e^x)\end{aligned}$$

which simplifies to $g'(x) = e^x(1+2e^x)(6e^x+1)$.

(b) The tangent line is horizontal where the derivative is zero, so we look for values satisfying $g'(x) = 0$. Since $e^x > 0$ for all values of x, none of the three terms in our expression of $g'(x)$ can output 0, so there is nowhere in our graph with a horizontal tangent line.

2 (a) We evaluate the derivative starting with the quotient rule

$$g'(x) = \frac{d}{dx}\left[\frac{e^x}{x+1}\right] = \frac{(x+1)\frac{d}{dx}[e^x] - e^x\frac{d}{dx}[x+1]}{(x+1)^2} = \frac{(x+1)e^x - e^x(1)}{(x+1)^2} = \frac{xe^x}{(x+1)^2}$$

(b) The tangent line is horizontal where the derivative is zero, so we look for values satisfying $g'(x) = 0$

$$\frac{xe^x}{(x+1)^2} = 0 \rightarrow xe^x = 0$$

which is satisfied by $x = 0$.

(c) There is a critical point at $x = 0$ and the derivative has an asymptote at $x = -1$, so we test values between and around these: $g(-2) = -0.271$, $g(-0.5) = -1.213$, and $g(1) = 0.680$. We can conclude from the sign of the derivative that $g(x)$ is decreasing on $(-\infty,-1)\cup(-1,0)$ and increasing on $(0,\infty)$.

3 (a) We begin evluating the derivative with the quotient rule

$$\begin{aligned}g'(x) = \frac{d}{dx}\left[\frac{x}{e^x}\right] &= \frac{(e^x)\frac{d}{dx}[x] - x\frac{d}{dx}[e^x]}{(e^x)^2}\\ = \frac{(e^x)(1) - x(e^x)}{e^{2x}} &= \frac{(1-x)e^x}{e^{2x}} = \frac{1-x}{e^x}\end{aligned}$$

(b) We look for values satisfying $g'(x) = 0$. Note that we can safely cancel the denominator since $e^x > 0$ for all values of x

$$\frac{1-x}{e^x} = 0 \rightarrow 1-x = 0$$

which is satisfied by $x = 1$.

(c) Let's use the first derivative test around $x = 1$: $g'(0) = 1$ and $g'(2) = -0.135$ so there is a relative maximum at $x = 1$ where $g(1) = \frac{1}{e}$. Because there are no other critical points or asymptotes, we conclude there is an absolute maximum at $(1, \frac{1}{e})$ and no minima.

4 (a) Let's start evaluating the derivative with the product rule

$$\frac{d}{dx}\left[x^3 e^x\right] = 3x^2 e^x + x^3 e^x = e^x(x^3 + 3x^2)$$

which simplifies further to $g'(x) = e^x x^2 (x+3)$.

(b) Looking for zeros of the derivative: $e^x x^2(x+3) = 0$ is satisfied by $x = 0$ and $x = -3$.

(c) Lets test values around and between $x = 0$ and $x = -3$: $g(-4) = -0.293$, $g(-1) = 0.736$, and $g(1) = 10.873$ from which we can determine that $g(x)$ is decreasing on $(-\infty, -3)$ and increasing on $(-3, \infty)$.

5 (a) We begin taking the derivative with the quotient rule

$$\frac{d}{dx}\left[\frac{e^{\frac{x^2}{2}}}{x}\right] = \frac{x\frac{d}{dx}[e^{\frac{x^2}{2}}] - e^{\frac{x^2}{2}}\frac{d}{dx}[x]}{x^2}$$

$$= \frac{xe^{\frac{x^2}{2}}\frac{d}{dx}[\frac{x^2}{2}] - e^{\frac{x^2}{2}}(1)}{x^2} = \frac{xe^{\frac{x^2}{2}} \cdot (x) - e^{\frac{x^2}{2}}}{x^2}$$

which simplifies to $g'(x) = \dfrac{e^{\frac{x^2}{2}}(x^2-1)}{x^2}$.

(b) We look for zeros of the derivative, first noting that the derivative is undefined at $x = 0$

$$\frac{e^{\frac{x^2}{2}}(x^2-1)}{x^2} = 0 \to e^{\frac{x^2}{2}}(x^2-1) = 0 \to x^2 - 1 = 0$$

which is satisfied by $x = -1$ and $x = 1$.

(c) We check values between and around our critical points at $x = \pm 1$ and the asymptote at $x = 0$: $g(-2) = 5.542$, $g(-0.5) = -3.399$, $g(0.5) = -3.399$, and $g(2) = 5.542$. From these we can conclude that $g(x)$ is increasing on $(-\infty, -1) \cup (1, \infty)$; decreasing on $(-1, 0) \cup (0, 1)$.

6 (a) We use the quotient rule to evaluate the derivative

$$\frac{d}{dx}\left[\frac{x+1}{e^{2x}}\right] = \frac{e^{2x}\frac{d}{dx}[x+1] - (x+1)\frac{d}{dx}[e^{2x}]}{(e^{2x})^2}$$

$$= \frac{e^{2x}(1) - (x+1)2e^{2x}}{(e^{2x})^2} = \frac{e^{2x}(1-2x-2)}{(e^{2x})^2}$$

which simplifies to $g'(x) = \dfrac{-2x-1}{e^{2x}}$

(b) To find which intervals it is increasing/decreasing over we first look for critical points

$$\frac{-2x-1}{e^{2x}} = 0 \to -2x - 1 = 0$$

so there is a critical point at $x = -\frac{1}{2}$ and no places where $g'(x)$ is undefined. We then examine the outputs of the derivative for surrounding values: $g'(-1) = 7.389$ and $g'(0) = -1$ verifying that $g(x)$ is increasing on $\left(-\infty, -\frac{1}{2}\right)$ and decreasing on $\left(-\frac{1}{2}, \infty\right)$.

(c) We can determine that there is a relative maximum at $x = -\frac{1}{2}$ from the data in part (b). We also find $g(-\frac{1}{2}) = \frac{1}{2}e$. Because this is the only critical value for $g(x)$ and our function is defined everywhere, there is an absolute maximum at $\left(-\frac{1}{2}, \frac{1}{2}e\right)$.

7 (a) We evalaute the derivative as follows

$$\frac{d}{dx}(e^{4x}+1)^2 = 2(e^{4x}+1)\cdot\frac{d}{dx}(e^{4x}+1) = 2(e^{4x}+1)(4e^{4x})$$

which simplifies to $g'(x) = 8e^{4x}(e^{4x}+1)$.

(b) We first want to find zeros for our derivative $g'(x) = 8e^{4x}(e^{4x}+1)$, but since $e^x > 0$ for all x there are no points for which the tangent line is horizontal. We can then test $g'(0) = 16$ to conclude $g(x)$ is increasing everywhere. We could have tested any point to confirm this.

(c) Since there are no critical values there are no maxima or minima.

8 (a) We start using the product rule

$$\frac{d}{dx}[(1-x)e^{2x}] = (-1)e^{2x} + (1-x)2e^{2x} = e^{2x}(-1+2-2x)$$

which simplifies to $g'(x) = e^{2x}(1-2x)$.

(b) To determine increasing/decreasing intervals we first want to find critical values using

$$g'(x) = e^{2x}(1-2x) = 0$$

which is by $x = \frac{1}{2}$. Let us check surrounding values: $g'(0) = 1$ and $g'(1) = -e$ to conclude that $g(x)$ is increasing on $\left(-\infty, \frac{1}{2}\right)$ and decreasing on $\left(\frac{1}{2}, \infty\right)$.

(c) We again begin with the product rule for derivatives

$$\frac{d}{dx}\left[e^{2x}(1-2x)\right] = e^{2x}(-2) + 2e^{2x}(1-2x) = e^{2x}(-2+2-4x)$$

which simplifies to $g''(x) = -4xe^{2x}$.

(d) Let us first determine the inflection points by finding zeros of the second derivative

$$g''(x) = -4xe^{2x} = 0$$

which is satisfied by $x = 0$. Then test values around this $g''(-1) = 0.541$ and $g''(1) = -29.556$ to conclude that $g(x)$ is concave upwards on $(-\infty, 0)$ and concave downwards on $(0, \infty)$.

9 To find the rate of change $C'(t)$ we take the derivative

$$\frac{d}{dt}100e^{-0.05t} = -5e^{-0.05t} \text{ mg./liter per hours.}$$

so the rate of change at the given times is $C'(1) \approx 4.756$, $C'(5) \approx 3.894$, and $C'(24) \approx 1.506$ all of which have units mg. per liter per hours since t is measuring hours. The results indicate that the drug is rapidly decreasing when initially introduced to the bloodstream and decreases more slowly after longer times when less of the drug is in the bloodstream.

10 (a) We can evaluate $N(0) = 5.3e^{0.093(0)^2-0.87(0)} = 5.3e^0 = 5.3$ and $N(4) = 5.3e^{0.093(4)^2-0.87(4)} \approx 0.723$. In 1960 the cases were at a high of 5.3 thousand, and by the beginning 1964 reduced to 0.723 thousand, or 723 hundred cases. These dates are determined from the fact that $t = 0$ represents the beginning of 1960 and t is measured in years.

(b) We first evaluate the derivative $N'(t)$

$$\frac{d}{dx}\left[5.3e^{0.093t^2-0.87t}\right] = 5.3e^{0.093t^2-0.87t} \cdot \frac{d}{dx}\left[0.093t^2 - 0.87t\right] = 5.3e^{0.093t^2-0.87t}(0.186t - 0.87)$$

then we evaluate $N'(0) = -4.611$ and $N'(3) \approx -0.2808$. At the beginning of 1964 the cases were decreasing at a rate of -4.611 thousand cases per year, and at the beginning of 1963 they were decreasing at a rate of -0.2808 thousand cases per year, or -280.8 hundred cases per year.

11 (a) We need to find the relative rate of change using $P(x)$ instead of $f(x)$, or $\dfrac{100 \cdot P'(x)}{P(x)}\%$. We first evaluate the derivative $P'(x)$

$$\frac{d}{dx}\left[ae^{-b \cdot e^{-cx}}\right] = ae^{-b \cdot e^{-cx}} \frac{d}{dx}\left[-b \cdot e^{-cx}\right] = a \cdot e^{-b \cdot e^{-cx}}\left(bc \cdot e^{-cx}\right) = (abc)e^{-b \cdot e^{-cx}} e^{-cx}$$

and then substitute this into the formula

$$\frac{100 \cdot P'(x)}{P(x)} = \frac{100(abc)e^{-b \cdot e^{-cx}} e^{-cx}}{(a)e^{-b \cdot e^{-cx}}} = 100 \cdot (bc) \cdot e^{-cx}$$

(b) To find the relative rate of change for the given parameters ($x = 20$ months when $a = 204$, $b = 0.0198$, and $c = 0.15$) we substitute them into the formula we determined in (a)

$$P'(20) = 100 \cdot (0.0198)(0.15) \cdot e^{-(0.15)(20)} \approx 0.01479\%$$

(c) We can interpret this result as saying in 20 months, the population is increasing at the relative rate (or percent rate) 0.015% per month since t is measured in months.