1. Given $f(x) = \dfrac{x^2}{ln(x)}$, answer the following questions:

   (a) Find the domain of $f(x)$.

   (b) Find $f'(x)$.

   (c) Find all values of x for which the tangent line to the graph of the function is horizontal.

   (d) Find all relative maximum/minimum points. .

2. Given $f(x) = ln\left(\frac{1}{x^2}\right)$, answer the following questions:

   (a) Find the domain of $f(x)$.

   (b) Find $f'(x)$.

   (c) Find the intervals on which the function is increasing and those on which the function is decreasing.

   (d) Determine the concavity of the graph and identify all inflection points.

3. Given $f(x) = [2 - ln(x)]^2$, answer the following questions:

   (a) Find the domain of $f(x)$.

   (b) Find $f'(x)$.

   (c) Find the intervals on which the function is increasing and those on which the function is decreasing.

   (d) Identify any relative/absolute maximum and minimum points.

4. Given $f(x) = x^2 \cdot ln(x)$, answer the following questions:
   (a) Find the domain of $f(x)$.

   (b) Find $f'(x)$.

   (c) Find all values of x for which the slope of the tangent line to the graph is equal to 0.

Answer Key

1 **(a)** $(0,1)\cup(1,\infty)$ **(b)** $f'(x) = \dfrac{x[2ln(x)-1]}{[ln(x)]^2}$ **(c)** $x = e^{1/2} = \sqrt{e}$ **(d)** Relative minimum point at $(\sqrt{e}, 2e)$

2 **(a)** $(-\infty,0)\cup(0,\infty)$ **(b)** $f'(x) = -\dfrac{2}{x}$ **(c)** Increasing on $(-\infty,0)$; Decreasing on $(0,\infty)$ **(d)** $f''(x) = \dfrac{2}{x^2}$; Concave upward on $(-\infty,0)\cup(0,\infty)$; no inflection points.

3 **(a)** $(0,\infty)$ **(b)** $f'(x) = -\dfrac{2}{x}(2-ln(x))$ **(c)** Decreasing on $(0,e^2)$; Increasing on (e^2,∞) **(d)** Relative/absolute minimum point at $(e^2,0)$.

4 **(a)** $(0,\infty)$ **(b)** $f'(x) = x\cdot(1+2ln(x))$ **(c)** $x = \dfrac{1}{\sqrt{e}} = \dfrac{\sqrt{e}}{e}$

Detailed Solutions

1 (a) We first note that $ln(x)$ is only defined from $(0, \infty)$, so the domain cannot contain elements outside this. Second, $f(x)$ is undefined when the denominator is 0 which occurs when

$$ln(x) = 0 \longrightarrow x = e^0 = 1$$

therefore $x = 1$ is also excluded from the domain returning $(0, 1) \cup (1, \infty)$.

(b) We take the derivative using quotient rule

$$\frac{d}{dx}\left[\frac{x^2}{ln(x)}\right] = \frac{x^2 \frac{d}{dx}[ln(x)] - ln(x)\frac{d}{dx}[x^2]}{(ln(x))^2} = \frac{x^2 \frac{1}{x} - ln(x)2x}{(ln(x))^2} = \frac{x - 2x ln(x)}{(ln(x))^2}$$

which simplifies to $f'(x) = \frac{x[2ln(x) - 1]}{[ln(x)]^2}$.

(c) The tangent line is horizontal when $f'(x) = 0$, we can cancel the denominator knowing that the domain excludes $x = 1$

$$\frac{x[2ln(x) - 1]}{[ln(x)]^2} = 0 \longrightarrow x[2ln(x) - 1] = 0$$

$$x = 0 \text{ or } 2ln(x) - 1 = 0$$

However, $x = 0$ is excluded from our domain and $f'(0)$ is undefined, so we cannot use this value. Let's solve for x in the second equation we determined

$$2ln(x) - 1 = 0 \longrightarrow ln(x) = \frac{1}{2} \longrightarrow x = e^{\frac{1}{2}}$$

therefore the tangent line is horizontal at x=e$^{\frac{1}{2}} = \sqrt{e}$.

(d) We determine that $e^{\frac{1}{2}} \approx 1.649$ and

$$f(e^{\frac{1}{2}}) = \frac{(e^{\frac{1}{2}})^2}{ln(e^{\frac{1}{2}})} = \frac{e^{\frac{1}{2}\cdot 2}}{\frac{1}{2} \cdot ln(e)} = \frac{e^1}{\frac{1}{2}} = 2e$$

Then we use the first derivative test around this point: $f'(1.5) \approx -1.725$ and $f'(2) \approx 1.608$ therefore there is a relative minimum at $(\sqrt{e}, 2e)$.

2 (a) Since $ln(x)$ is only defined over the interval $(0, \infty)$, the argument (or input) of $ln\left(\frac{1}{x^2}\right)$ **cannot** take on any values ≤ 0. In other words $\frac{1}{x^2}$ cannot be allowed to output values ≤ 0. We know that $\frac{1}{x^2}$ is always positive, but it can output 0 when $x = 0$, so that value is the only value excluded from our domain $(-\infty, 0) \cup (0, \infty)$.

(b) We first express $ln\left(\frac{1}{x^2}\right) = ln\left(x^{-2}\right)$ then take the derivative starting with chain rule

$$f'(x) = \frac{d}{dx} ln\left(x^{-2}\right) = \frac{1}{x^{-2}} \cdot \frac{d}{dx} x^{-2} = x^2 \cdot (-2x^{-3}) = \frac{-2x^2}{x^3} = -\frac{2}{x}$$

(c) We first find critical points by looking for zeros of the derivative

$$-\frac{2}{x} = 0$$

which is not satisfied by any x in our domain. Therefore the sign of the derivative can only change across the undefined point at $x = 0$, so we test values around 0: $f'(-1) = 2$ and $f'(1) = -2$ to conclude that $f(x)$ increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

(d) To determine inflection points we express $-\frac{2}{x} = -2x^{-1}$ and determine

$$f''(x) = \frac{d}{dx}\left[-2x^{-1}\right] = 2x^{-2} = \frac{2}{x^2}$$

which does not output zero for any value of x. Therefore there are no inflection points. The concavity can only change across $x = 0$, so we test values: $f''(-1) = 2$ and $f''(1) = 2$ to conclude the function is concave up over the whole domain $(-\infty, 0) \cup (0, \infty)$

3 (a) Since $ln(x)$ is only defined over the interval $(0, \infty)$, the function $f(x) = [2 - ln(x)]^2$ is likewise defined over $(0, \infty)$.

(b) We take the derivative starting with the chain rule

$$f'(x) = \frac{d}{dx}\left[(2 - ln(x))^2\right] = 2(2 - ln(x))\frac{d}{dx}\left[2 - ln(x)\right] = 2(2 - ln(x))(\frac{1}{x}) = -\frac{2}{x}(2 - ln(x))$$

(c) We first look for critical values where $f'(x) = 0$, recalling that $x \neq 0$

$$-\frac{2}{x}(2 - ln(x)) = 0 \longrightarrow 2 - ln(x) = 0 \longrightarrow ln(x) = 2 \longrightarrow x = e^2 \approx 7.389$$

There are no asymptotes so we only need test values around this point to determine where the derivative changes sign: $f'(7) \approx -0.015$ and $f'(8) \approx 0.020$ to conclude that $f(x)$ decreasing on $(0, e^2)$ and increasing on (e^2, ∞)

(d) Using the data from part (c) we can conclude that there is a relative minimum at $x = e^2$. We can further determine

$$f(e^2) = [2 - ln(e^2)]^2 = [2 - 2ln(e)]^2 = 0^2 = 0$$

Since there are no other critical points or vertical asymptotes we can conclude that there is an absolute minimum at $(e^2, 0)$

4 (a) Since $ln(x)$ is only defined over the interval $(0, \infty)$, the function $f(x) = x^2 \cdot ln(x)$ is likewise defined over $(0, \infty)$.

(b) We take the derivative beginning with product rule

$$\frac{d}{dx}\left[x^2 ln(x)\right] = x^2\frac{d}{dx}\left[ln(x)\right] + ln(x)\frac{d}{dx}\left[x^2\right] = x^2\frac{1}{x} + ln(x)2x = x + 2x \cdot ln(x)$$

which simplifies to $f'(x) = x \cdot (1 + 2ln(x))$.

(c) We need to find where $f'(x) = 0$

$$x \cdot (1 + 2ln(x)) = 0$$
$$x = 0 \text{ or } 1 + 2ln(x) = 0$$

but $x = 0$ is excluded from our domain. We solve the second equation for x

$$1 + 2ln(x) = 0 \longrightarrow ln(x) = -\frac{1}{2} \longrightarrow x = e^{-\frac{1}{2}}$$

which we can also express as $x = \frac{1}{\sqrt{e}} = \frac{\sqrt{e}}{e}$