1. The monthly sales, S (in dollars), of a new product at time t months from the present is predicted using the model $S(t) = 5000e^{0.5\sqrt{t+1}}$ dollars.

   (a) Determine the derivative $S'(t)$ [you may leave the square root term in the denominator]. Then find the rate of change of monthly sales at time $t = 3$ months and time $t = 8$ months from the present. Round your answers to the nearest dollar. Indicate if the monthly sales is increasing or decreasing at these times. Include the units on the rates of change. .

   (b) **DIRECT APPROACH**. Use the results in part (a) to determine the **relative and percentage rates of change** at $t = 3$ and $t = 8$. Round the percentage rates to one decimal place. Interpret the percentage rates of change in the context of this problem.

(c) **LOGARITHMIC DIFFERENTIATION**. Find $ln[S(t)]$ and simplify the result. Note: $ln(a \cdot b) = ln(a) + ln(b)$

(d) Find $\frac{d}{dt}ln[S(t)]$ using the result in part (c) [you may leave the square root term in the denominator]. Use this result to determine the relative and percentage rates of change at $t = 3$ and $t = 8$. For this particular function $S(t)$, which approach appears to be preferred?

2. You have invested \$50,000 in an account that pays 3.5% annual interest **compounded continuously**.

   (a) Identify the function $A(t)$ that determines the amount of money in the account at time t years from the present.

   (b) Find $ln[A(t)]$. Simplify the result.

(c) Find $\frac{d}{dt}ln[A(t)]$ using the result in part (b).

(d) Use the result from part (c) to find the **percentage rates of change** at time $t = 1$ year and $t = 5$ years. Based on the result in part (c), make a general statement about the percentage rate of change of an account for which the interest is compounded continuously.

Answer Key

1 (a) $S'(t) = \frac{1250e^{0.5\sqrt{t+1}}}{\sqrt{t+1}}$; $S'(3) \approx 1699$ \$/month; increasing; $S'(8)$ =1867 \$/month; increasing; **(b)** $\frac{S'(3)}{S(3)} = 0.125$ or 12.5% per month; monthly sales is expected to grow by 12.5% per month at time $t = 3$ months; $\frac{S'(8)}{S(8)} = 0.08\bar{3}$ or $8.\bar{3}$% per month; monthly sales is expected to grow by $8.\bar{3}$% per month at time $t = 8$ months **(c)** $ln[S(t)] = ln(50000)+0.5\sqrt{t+1}$ **(d)** $\frac{d}{dt}ln[S(t)] = \frac{1}{4\sqrt{t+1}}$

2 (a) $A(t) = 50000e^{0.035t}; t \geq 0$ **(b)** $ln[A(t)] = ln(50000)+0.35t$ **(c)** $\frac{d}{dt}ln[A(t)] = 0.035$ **(d)** At $t = 1$: 3.5% per year; at $t = 5$: 3.5% per year. The % rate of increase is **constant** for all years at 3.5%.

Detailed Solutions

1 (a) Let's evaluate the derivative using chain rule

$$S'(t) = \frac{d}{dx}\left[5000e^{0.5\sqrt{t+1}}\right] = 5000e^{0.5\sqrt{t+1}} \cdot \frac{d}{dx}\left[0.5\sqrt{t+1}\right] = 5000e^{0.5\sqrt{t+1}} \cdot \left(0.5(0.5)\frac{1}{\sqrt{t+1}}\right)$$

which simplifies further to $S'(t) = \frac{1250e^{0.5\sqrt{t+1}}}{\sqrt{t+1}}$. We evaluate this at the given times, recalling that t is measured in months: $S'(3) \approx \$1699$ per month and $S'(8) \approx \$1867$ per month. Since both answers are positive sales are increasing at both times.

(b) We calculate the relative rate of change as $\frac{S'(t)}{S(t)}$. Let's evaluate $S(3) \approx \$13591$ and $S(8) \approx \$22408$ to find $\frac{S'(3)}{S(3)} = \frac{1699}{13591} \approx 0.125$ or 12.5% per month and $\frac{S'(8)}{S(8)} = \frac{1867}{22408} \approx 0.083$ or 8.3% per month. We can interpret these results as saying that after 3 months, monthly sales are expected to grow by 12.5% per month, and after 8 months monthly sales are expected to grow by 8.3% per month.

(c) We perform the following algebraic manipulations using logarithm rules

$$ln[S(t)] = ln[5000 \cdot e^{0.5\sqrt{t+1}}] = ln(5000) + ln(e^{0.5\sqrt{t+1}}) = ln(5000) + 0.5\sqrt{t+1}$$

(d) We now take the derivative of our answer from part (d)

$$\frac{d}{dt}ln[S(t)] = \frac{d}{dt}\left[ln(5000) + 0.5\sqrt{t+1}\right] = 0.5(0.5)\frac{1}{\sqrt{t+1}} = \frac{1}{4\sqrt{t+1}}$$

Importantly, this is the same thing as the relative rate of change. Using the chain rule

$$\frac{d}{dt}ln[S(t)] = \frac{1}{S(t)}\frac{d}{dt}[S(t)] = \frac{S'(t)}{S(t)}$$

and we can evaluate $\frac{1}{4\sqrt{3+1}} = 0.125$ and $\frac{1}{4\sqrt{8+1}} = 0.08\bar{3}$ which are the same answers we obtained previously, but with no approximations!

2 (a) We fill the continuously compounding interest formula $A(t) = A_0e^{rt}$ with known information to obtain $A(t) = 50000e^{0.035t}$. Note that this is only valid for $t \geq 0$ since no money is invested before this time.

(b) We use logarithm rules to evaluate

$$ln[A(t)] = ln[50000e^{0.035t}] = ln(50000) + ln(e^{0.035t}) = ln(50000) + 0.035t$$

(c) We evaluate the derivative of our answer from part (b)

$$\frac{d}{dt}ln[A(t)] = \frac{d}{dt}\left[ln(50000) + 0.035t\right] = 0.035$$

(d) The result from part (c) is the percentage rate of change, which is constant for all time (see 1 part (d) for justification). Specifically at $t = 1$ year the account is growing at 3.5% per year, and at $t = 5$ years the account is growing at 3.5% per year.