1. The demand for some pairs of goods have a relationship, where the quantity demanded for one product depends somehow on the prices for both.

   Two goods are **complementary** if an increase in the price of either decreases the demand for both. Examples:

   - The demand for cars depends on both the price for cars and the price of gasoline.
   - The demand for hot dog buns depends on both the price for the buns and the price for the hot dogs.

   Two goods are **substitutes** if an increase in the price of one increases the demand for the other. Example:

   - The demand for Brand A depends on its price and also on the price of its main competitor Brand B. If the Brand B raises its price, consumers will switch brands – substitute – and demand for Brand A will increase.

   Suppose that the demand functions for two products are given as follows: $q_1 = 200 - 3p_1 - p_2$ and $q_2 = 150 - p_1 - 2p_2$ where p_1, p_2 are the prices (in dollars) and q_1, q_2 are the quantities for products 1 and 2.

   (a) What is the quantity demanded for each when the price for product 1 is \$20 per item and the price for product 2 is \$30 per item? **Include appropriate units**.

   (b) Find the four partial derivatives: $\dfrac{\partial q_1}{\partial p_1}, \dfrac{\partial q_1}{\partial p_2}, \dfrac{\partial q_2}{\partial p_1}$ and $\dfrac{\partial q_2}{\partial p_2}$.

   (c) Are these two products complementary goods or substitute goods? Justify your answer with brief numerical explanation $\left(\text{Hint: Consider the signs of } \dfrac{\partial q_1}{\partial p_2} \text{ and } \dfrac{\partial q_2}{\partial p_1}\right)$

2. Suppose that the demand functions for two products are given as follows: $q_1 = 200 - 3p_1 + p_2$ and $q_2 = 150 + p_1 - 2p_2$ where p_1, p_2, q_1, and q_2 are the prices (in dollars) and quantities for products 1 and 2.

   (a) What is the quantity demanded for each when the price for product 1 is \$20 per item and the price for product 2 is \$30 per item? **Include appropriate units**.

   (b) Find the four partial derivatives: $\frac{\partial q_1}{\partial p_1}$, $\frac{\partial q_1}{\partial p_2}$, $\frac{\partial q_2}{\partial p_1}$ and $\frac{\partial q_2}{\partial p_2}$.

   (c) Are these two products complementary goods or substitute goods? Justify your answer with brief numerical explanation $\left(\text{Hint: Consider the signs of } \frac{\partial q_1}{\partial p_2} \text{ and } \frac{\partial q_2}{\partial p_1}\right)$

3. Suppose the demand functions for two products are $q_1 = f(p_1, p_2)$ and $q_2 = g(p_1, p_2)$, where p_1, p_2, q_1 and q_2. are the prices (in dollars) and quantities for products 1 and 2. Consider the four partial derivatives $\frac{\partial q_1}{\partial p_1}$, $\frac{\partial q_1}{\partial p_2}$, $\frac{\partial q_2}{\partial p_1}$ and $\frac{\partial q_2}{\partial p_2}$. Tell the **sign** of each of these partial derivatives and **briefly explain** your reasoning IF

   (a) the products are complementary goods.

   (b) the products are substitute goods.

4. The volume, V, of a cylindrical can that has a height of h centimeters and a radius r centimeters for the circular top and bottom is given by the function $V(h, r) = \pi r^2 h$ cubic centimeters. The dimensions of a particular can used to ship a toxic chemical are $r = 31.8$ centimeters and $h = 37.3$ centimeters.

   (a) Using the concept of **partial derivatives**, determine the **rate** at which the volume of the can is changing with respect to a 1 centimeter increase in the height of the can with the radius being held constant. **Include the units** on the rate you compute. **Identify the partial derivative** that you are using. *Round your answer to 2 decimal places.* **Interpret the result** .

   (b) Using the concept of **partial derivatives**, determine the **rate** at which the volume of the can is changing with respect to a 1 centimeter increase in the radius of the can with the height being held constant? **Include the units** on the rate you compute. **Identify the partial derivative** that you are using. *Round your answer to 2 decimal places.* **Interpret the result** .

   (c) If you could increase only one of the dimensions of this can by 1 centimeter, which dimension would you choose to increase in order to have the **greatest impact** on the volume of the can. . Circle your answer.

      (i) Change the height.

      (ii) Change the radius.

      (iii) Change either the height <u>or</u> the radius as the impact on the volume will be the same.

5. Biologists model the shape of the bacillus bacteria as a cylinder with 2 spherical caps on each end. The cylindrical portion has a length denoted by L millimeters and the spherical caps have a radius denoted by r millimeters. The volume, V, of the shape is given by the function $V(L, r) = \pi r^2 L + \frac{4}{3}\pi r^3$

   (a) Determine the rate of change of the volume with respect to a 1 mm. increase in **length** with the radius being held constant.

   (b) Determine the rate of change of the volume with respect to a 1 mm. increase in **radius** with the length being held constant.

   (c) A certain bacteria is observed to have a length of 4 mm. and a radius of 1 mm. Which would have a greater impact on the volume, a 1 mm. increase in length or a 1 mm. increase in radius?

6. The plans for a truck service center will use aluminum material to build a rectangular structure with a square roof and walls on 3 sides (it is open at one end). The length, in **feet**, of the edge of the square roof is denoted by $\boldsymbol{x}$, and the height, in feet, of the walls is denoted by $\boldsymbol{h}$. The total cost, $\boldsymbol{C}$, of materials is given by the function $\boldsymbol{C(x,h) = 100x^2 + 120xh}$ dollars.

   (a) Determine the **rate of change** of cost with respect to the length of the edges of the roof, $\frac{\partial C}{\partial x}$, and the **rate of change** of cost with respect to the height, $\frac{\partial C}{\partial h}$. Include the **units** associated with each rate.

   (b) Currently, the plan for the dimensions calls for x=50 feet and h=10 feet. **Using Partial Derivatives**, find the rate at which the cost is expected to be changing (marginal cost) if we decide to increase the height by 1 foot and keep the length of the edges of the roof the same. Then complete the following:

   At the current dimensions, the cost is expected to ________________ (increase/decrease) by approximately ________________ (provide both the **amount** and the **units** on the amount).

Answer Key

1 **(a)** $q_1 = 200 - 3(20) - 30 = 110$ units; $q_2 = 70$ units **(b)** $\dfrac{\partial q_1}{\partial p_1} = -3$; $\dfrac{\partial q_1}{\partial p_2} = -1$, $\dfrac{\partial q_2}{\partial p_1} = -1$ and $\dfrac{\partial q_2}{\partial p_2} = -2$. **(c)** Since the signs of $\dfrac{\partial q_1}{\partial p_2}$ and $\dfrac{\partial q_2}{\partial p_1}$ are both negative, an increase in either price decreases both demands. The two products are complementary goods.

2 **(a)** $q_1 = 170$ units; $q_2 = 110$ units **(b)** $\dfrac{\partial q_1}{\partial p_1} = -3$; $\dfrac{\partial q_1}{\partial p_2} = 1$, $\dfrac{\partial q_2}{\partial p_1} = 1$ and $\dfrac{\partial q_2}{\partial p_2} = -2$. **(c)** The two products are substitute goods.

3 **(a)** $\dfrac{\partial q_1}{\partial p_1}$ $(-)$; $\dfrac{\partial q_1}{\partial p_2}$ $(-)$; $\dfrac{\partial q_2}{\partial p_1}$ $(-)$ and $\dfrac{\partial q_2}{\partial p_2}$ $(-)$. **(b)** $\dfrac{\partial q_1}{\partial p_1}$ $(-)$; $\dfrac{\partial q_1}{\partial p_2}$ $(+)$; $\dfrac{\partial q_2}{\partial p_1}$ $(+)$ and $\dfrac{\partial q_2}{\partial p_2}$ $(-)$.

4 **(a)** $\dfrac{\partial V}{\partial h} = \pi r^2$; $\left.\dfrac{\partial V}{\partial h}\right|_{r=12.5,h=14.7} \approx 490.87$ cubic inches per 1 inch increase in height. If the can is made slightly (1 inch) taller with the same radius, its volume will increase about 490.87 cubic inches. **(b)** $\dfrac{\partial V}{\partial r} = 2\pi hr$; $\left.\dfrac{\partial V}{\partial r}\right|_{r=12.5,h=14.7} \approx 1154.54$ cubic inches per 1 inch increase in radius. **(c)** 1154.54 vs/ 4490.87 $\Longrightarrow$ ii.

5 **(a)** $\dfrac{\partial V}{\partial L} = \pi r^2$ cubic mm. per mm. length. **(b)** $\dfrac{\partial V}{\partial r} = 2\pi r L + 4\pi r^2$ cubic mm. per mm. radius. **(c)** $\left.\dfrac{\partial V}{\partial L}\right|_{L=4,r=1} = \pi(1)^2 \approx 3.14$ cubic mm. per mm. length; $\left.\dfrac{\partial V}{\partial r}\right|_{L=4,r=1} = 2\pi(1)(4) + 4\pi(1)^2 \approx 37.7$ cubic mm. per mm. radius; 1 mm. increase in radius has more impact on the volume.

6 **(a)** $\frac{\partial C}{\partial x} = 200x + 120h$ dollars per 1 foot increase in the edge of the square roof; $\frac{\partial C}{\partial h} = 120x$ dollars per 1 foot increase in height. **(b)** Increase by $\left.\frac{\partial C}{\partial h}\right|_{x=50,h=10} = 6000$ dollars per 1 foot increase in height.

Detailed Solutions

1 (a) The problem statement asks us to find q_1 and q_2 when $p_1 = \$20$ and $p_2 = \$30$

$$q_1 = 200 - 3(20) - (30) = 110 \text{ units}$$
$$q_2 = 150 - (20) - 2(30) = 70 \text{ units}$$

(b) Let's evaluate the four partial derivatives directly, recalling that p_2 is treated as a constant when evaluating $\frac{\partial}{\partial p_1}$ and vice versa

$$\frac{\partial q_1}{\partial p_1} = \frac{\partial}{\partial p_1}[200 - 3p_1 - p_2] = -3$$
$$\frac{\partial q_1}{\partial p_2} = \frac{\partial}{\partial p_2}[200 - 3p_1 - p_2] = -1$$
$$\frac{\partial q_2}{\partial p_1} = \frac{\partial}{\partial p_1}[150 - p_1 - 2p_2] = -1$$
$$\frac{\partial q_2}{\partial p_2} = \frac{\partial}{\partial p_1}[150 - p_1 - 2p_2] = -2$$

(c) These products are complementary goods as when the price of one increases the demand for both decrease. We conclude this because the partial derivatives with respect to the prices are all negative, so whenever a the price of either product increases incrementally the demand is decreasing.

2 (a) The problem statement asks us to find q_1 and q_2 when $p_1 = \$20$ and $p_2 = \$30$

$$q_1 = 200 - 3(20) + (30) = 170 \text{ units}$$
$$q_2 = 150 + (20) - 2(30) = 110 \text{ units}$$

(b) Let's evaluate the four partial derivatives directly, recalling that p_2 is treated as a constant when evaluating $\frac{\partial}{\partial p_1}$ and vice versa

$$\frac{\partial q_1}{\partial p_1} = \frac{\partial}{\partial p_1}[200 - 3p_1 + p_2] = -3$$
$$\frac{\partial q_1}{\partial p_2} = \frac{\partial}{\partial p_2}[200 - 3p_1 + p_2] = 1$$
$$\frac{\partial q_2}{\partial p_1} = \frac{\partial}{\partial p_1}[150 + p_1 - 2p_2] = 1$$
$$\frac{\partial q_2}{\partial p_2} = \frac{\partial}{\partial p_1}[150 + p_1 - 2p_2] = -2$$

(c) While the signs of $\frac{\partial q_1}{\partial p_1}$ and $\frac{\partial q_2}{\partial p_2}$ are negative, $\frac{\partial q_1}{\partial p_2}$ and $\frac{\partial q_2}{\partial p_1}$ are positive indicating that an incremental increase in the price of one product implies that the demand of the other is increasing. Therefore the products are substitute goods.

3 (a) If the products are complementary goods then the signs of every partial derivative are negative. Whenever the price of either product increases incrementally the demand is decreasing for both products.

(b) If the products are substitute goods then the signs of $\frac{\partial q_1}{\partial p_1}$ and $\frac{\partial q_2}{\partial p_2}$ are negative, $\frac{\partial q_1}{\partial p_2}$ and $\frac{\partial q_2}{\partial p_1}$ are positive. This indicates that an incremental increase in the price of one product implies that the demand of the other is increasing.

4 (a) We need to evaluate the partial derivative of V with respect to h when $h = 37.3 + 1 = 38.3$ cm and $r = 31.8$ cm

$$\frac{\partial V}{\partial h} = \frac{\partial}{\partial h}\pi r^2 h = \pi r^2 \longrightarrow \left.\frac{\partial V}{\partial h}\right|_{h=38.3,r=31.8}$$

5 (a) We evaluate the partial derivative with respect to length

$$\frac{\partial V}{\partial L} = \frac{\partial}{\partial L}\left[\pi r^2 L + \frac{4}{3}\pi r^3\right] = \pi r^2 \text{ cubic mm per mm length}$$

(b) We evaluate the partial derivative with respect to radius

$$\frac{\partial V}{\partial 4} = \frac{\partial}{\partial 4}\left[\pi r^2 L + \frac{4}{3}\pi r^3\right] = 2\pi r L + 4\pi r^2 \text{ cubic mm per mm radius}$$

(c) Let's plug the values $L = 4$ mm and $r = 1$ mm into our partial derivatives

$$\left.\frac{\partial V}{\partial L}\right|_{L=4,r=1} = \pi(1)^2 = \pi \text{ cubic mm per mm radius}$$

$$\left.\frac{\partial V}{\partial r}\right|_{L=4,r=1} = 2\pi(1)(4) + 4\pi(1)^2 = 12\pi \text{ cubic mm per mm radius}$$

and we can see that a change in radius would have a greater impact on the volume for the given bacteria.

6 (a) We evaluate the partial derivatives directly

$$\frac{\partial C}{\partial x} = \frac{\partial}{\partial x}\left[100x^2 + 120xh\right] = 200x + 120h \text{ dollars per 1 foot increase in the edge of the roof}$$

$$\frac{\partial C}{\partial h} = \frac{\partial}{\partial h}\left[100x^2 + 120xh\right] = 120x \text{ dollars per 1 foot increase in the edge of the height}$$

(b) Since we are considering the edges fixed, we need to evaluate $\frac{\partial C}{\partial h}$ for $x = 50$ and $h = 10$ to find that the cost is expected to increase by

$$\left.\frac{\partial C}{\partial h}\right|_{x=50,h=10} = 120(50) = 6000 \text{ dollars per 1 foot increase in height}$$