1. The function $f(x) = 2x^3 - 33x^2 + 168x - 5$ has two critical numbers. Find these critical numbers. Show your work.

   Question Help: https://youtu.be/T1iF26hqdpI

2. Given the function $g(x) = 4x^3 + 6x^2 - 72x$, answer the following questions.

   Question Help: https://youtu.be/vX-jvpNIHek

   (a) Find the first derivative $g'(x)$.

   (b) Find the second derivative $g''(x)$.

   (c) Evaluate $g''(-3)$.

   (d) Based on the sign of this number, does this mean the graph of $g(x)$ is concave up or concave down at $x = -3$.

   (e) Based on the concavity of g(x) at $x = -3$, does this mean that there is a local minimum or local maximum at $x = -3$?.

3. Consider the function $f(x) = 5 - 5x^2$, $-3 \leq x \leq 1$. Fill in the blanks below. Show your work. Use the **First Derivative Test** in your work.

   Question Help: https://youtu.be/N37AqpdN3Wc

   (a) The absolute maximum value is ____________________ and this occurs at $x =$ ____________________.

   (b) The absolute minimum value is ____________________ and this occurs at $x =$ ____________________.

4. The function $f(x) = -2x^3 + 33x^2 - 144x + 9$ has one local minimum and one local maximum. Fill in the blanks below. Show your work. Use the **Second Derivative Test** in your work.

   Question Help: https://youtu.be/kv_W7259jlY

   (a) The function has a local minimum at $x =$ ________________ with function value ________________.

   (b) The function has a local maximum at $x =$ ________________ with function value ________________.

5. The function $f(x) = 6x + \dfrac{8}{x}$ has one local minimum and one local maximum. Fill in the blanks below. Show your work. Use **either** the *First Derivative Test* **or** the *Second Derivative Test* in your work. Round your final answers to 4 decimal places.

   Question Help: https://youtu.be/FP0WMdJH2jE

   (a) The function has a local maximum at $x =$ ________________ with function value ____________________.

   (b) The function has a local minimum at $x =$ ________________ with function value ____________________.

6. Find the global maximum and global minimum values of $f(x) = x + \frac{9}{x}$ on $[0.2, 12]$.

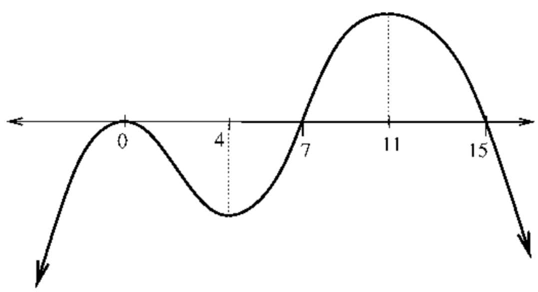
7. Suppose the function $f(x)$ has a domain of all real numbers except $x = -7$. The first derivative of $f(x)$ is shown below.

$$f'(x) = \frac{-5(x-1)}{(x+7)^7}$$

   (a) Find all intervals on which $f(x)$ is increasing.
   (b) Find all intervals on which $f(x)$ is decreasing.
   (c) Find the $x-$coordinates of all local extrema on the graph of $f(x)$.

8. Use the graph of $f'(x)$ to answer questions about the function $f(x)$.

This is graph of $f'(x)$.



(a) Find all critical values of $f(x)$.

(b) Find all intervals on which $f(x)$ is increasing and all intervals on which $f(x)$ is decreasing.

(c) Find the $x-$coordinates of all local extrema on the graph of $f(x)$.

9. For each of the following functions, identify the following by using the *First Derivative Test*, the *Second Derivative Test* and/or the *Concavity Test*. Use the given graph of each function to help determine global extrema.

   (i) intervals over which the graph of the function is increasing/decreasing;

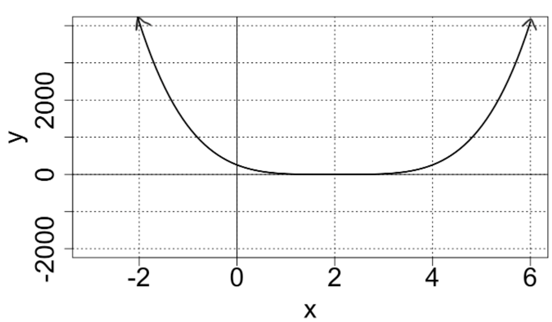
   (ii) local maximum/minimum points;

   (iii) global maximum/minimum points;

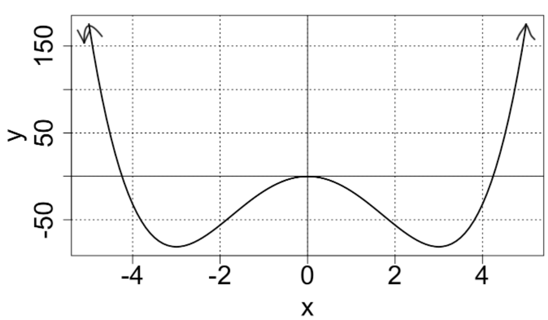
   (iv) intervals over which the graph of the function is concave up/down.

   (v) inflection points

   (a) $f(x) = (2x - 4)^4$



   (b) $f(x) = x^4 - 18x^2$.



Answer Key

1 $x = 4$; $x = 7$

2 **(a)** $g'(x) = 12x^2 + 12x - 72$ **(b)** $g''(x) = 24x + 12$ **(c)** -60 **(d)** down **(e)** local maximum

3 **(a)** 5; 0 **(b)** -40; -3

4 **(a)** 3; -180 **(b)** 8; -55

5 **(a)** 1.1547; 13.8564 **(b)** -1.1547; -13.8564

6 Global max of 45.2 at $x = 0.2$; global min of 6 at $x = 3$

7 **(a)** $(-7, 1)$ **(b)** $(-\infty, -7) \cup (1, \infty)$ **(c)** local max at $x = 1$; no local min.

8 **(a)** $x = 0$; $x = 7$; and $x = 15$ **(b)** Increasing: $(7, 15)$; Decreasing: $(-\infty, 0) \cup (0, 7) \cup (15, \infty)$ **(c)** local max at $x = 15$; local min at $x = 7$.

9 **part (a):** (i) Decreasing on $(-\infty, 2)$; Increasing on $(2, \infty)$ (ii)-(iii) local (and global) minimum point at (2,0) (iv) Concave up for all values of x (v) No inflection point.

part (b): (i) Decreasing on $(-\infty, -3) \cup (0, 3)$; Increasing on $(-3, 0) \cup (3, \infty)$ (ii)-(iii) local (and global) minimum point at $(-3, -81)$ and $(3, -81)$; local max point at $(0, 0)$ (iv) Concave up for $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$; Concave down on $(-\sqrt{3}, \sqrt{3})$.(v) Inflection points at $(-\sqrt{3}, -45)$ and $(\sqrt{3}, -45)$.

Detailed Solutions

1 Critical numbers occur when the derivative is zero. We take the derivative of $f(x)$ using the power rule as follows

$$\begin{aligned} f(x) &= 2x^3 - 33x^2 + 168x - 5 \\ &= 2x^3 - 33x^2 + 168x^1 - 5x^0 \\ f'(x) &= 3 \cdot 2x^{3-1} - 2 \cdot 33x^{2-1} + 1 \cdot 168x^{1-1} - 0 \cdot 5x^{0-1} \\ &= 6x^2 - 66x + 168 \end{aligned}$$

We can find the zeros/roots of $f'(x)$ using the quadratic formula or factoring $f'(x) = 6(x-4)(x-7)$ to determine the critical numbers x=4 and x=7.

2 **(a)** To find the first derivative we use the power rule to obtain $g'(x) = 12x^2 + 12x - 72$.
(b) Use the power rule once more to find the second derivative $g''(x) = 24x + 12$.
(c) $g''(-3) = 24 \cdot (-3) + 12 = -60$.
(d) Since the second derivative is negative at $x = -3$ it is concave down at that point.
(e) First note that $x = -3$ is a critical number since $g'(-3) = 0$. Since our function is concave down at this critical number it corresponds to a local maximum.

3 We first determine the critical numbers within the interval by finding where $f'(x) = 0$. By the power rule $f'(x) = -10x$ which is zero when $x = 0$. Note that 0 is within the provided interval $-3 \leq x \leq 1$. We then observe the output of $f(x)$ for the critical number and endpoints of the interval, corresponding to the ordered pairs (-3, -40), (0, 5), (1, 0). The absolute maximum is 5 and occurs at $x = 0$. The absolute minimum is -40 and occurs at $x = -3$.

4 We first find the critical numbers by evaluating where the first derivative $f'(x) = -6x^2+66x-144$ is zero. Factoring yields $f'(x) = -6(x-3)(x-8)$ so we know that critical numbers occur at $x = 3$ and $x = 8$. We then use the second derivative test to determine if these points are local maxima or minima. A second application of the power rule gives us $f''(x) = -12x + 66$ and we then find $f''(3) = 30$ and $f''(8) = -30$. This means that we have a local minimum at $x = 3$ with function value $f(3) = -180$ and a local maximum at $x = 8$ with function value $f(8) = -55$. We know which is which by the sign of the second derivative at the critical numbers.

5 We first use the power rule on $f(x) = 6x + \dfrac{8}{x} = 6x^1 + 8x^{-1}$

$$f'(x) = 1 \cdot 6x^{1-1} + -1 \cdot 8x^{-1-1} = 6x^0 - 8x^{-2} = 6 - \frac{8}{x^2}$$

We then find the critical numbers

$$6 - \frac{8}{x^2} = 0 \longrightarrow 6 = \frac{8}{x^2} \longrightarrow x^2 = \frac{8}{6} \longrightarrow x = \pm\sqrt{\frac{4}{3}} \longrightarrow x \approx 1.1547 \text{ or } x \approx -1.1547$$

Finally we plug our critical points into the second derivative $f''(x) = \dfrac{16}{x^3}$ to find $f''(1.1547) = 10.3923$, $f''(-1.1547) = -10.3923$ and conclude from the signs of the outputs that there is a local maximum at $x = 1.1547$ with function value $f(1.1547) = 13.8564$ and a local minimum at $x = -1.1547$ with function value $f(-1.1547) = -13.8564$.

6 We begin by finding the critical numbers from the derivative $f'(x) = 1 - \dfrac{9}{x^2}$

$$\begin{aligned} 1 - \frac{9}{x^2} &= 0 \\ x^2 &= 9 \\ x = 3 \text{ or } x &= -3 \end{aligned}$$

plugging critical numbers into the second derivative $f''(x) = \dfrac{18}{x^3}$ returns $f''(3) = \dfrac{2}{3}$ and $f''(-3) = -\dfrac{2}{3}$. This means that $x = 3$ is a local minimum, and $x = -3$ is a local maximum. However, -3 does not lie within the interval of interest, so we do not consider it further. Examining the function outputs for the endpoints and critical number $x = 3$ we have $f(0.2) = 45.2$, $f(3) = 6$, $f(12) = 12.75$. We can then conclude that there is a global maximum of 45.2 at $x = 0.2$ and a global minimum of 6 at $x = 3$.

7 We first find local extrema

$$\begin{aligned} \frac{-5(x-1)}{(x+7)^7} &= 0 \\ -5(x-1) &= 0 \\ x &= 1 \end{aligned}$$

Since $x = -7$ is excluded from our domain, we can safely multiply both sides by $(x + 7)^7$ in the second step and conclude there is a local extrema at $x = 1$. There is also an asymptote at $x = -7$. The sign of the derivative (which indicates whether a function is increasing or decreasing) can flip across local extrema and asymptotes, so we pick some sample values between and outside to determine the function's behavior: $f'(-8) = -45$, $f'(0) \approx 0.000006$, $f'(2) \approx -0.000001$. From this data we conclude

(a) $f(x)$ is increasing on $(-7, 1)$.

(b) $f(x)$ is decreasing on $(-\infty, -7) \cup (1, \infty)$.

(c) Since $f(x)$ is increasing to the right of $x = 1$ and decreasing to its left, we know that there is a local maximum at $x = 1$.

8 **(a)** Given a graph of $f'(x)$, the critical values correspond to x-intercepts where $f'(x) = 0$. We can see these occurring at $x = 0$, $x = 7$, and $x = 15$.

(b) $f(x)$ is increasing over $(7, 15)$ as it is the only interval where $f'(x) > 0$. The function is decreasing over $(-\infty, 0) \cup (0, 7) \cup (15, \infty)$ as the derivative is negative throughout this interval. Note that we exclude x=0 from the decreasing interval as it is an extrema.

(c) We can observe a local maximum at $x = 15$ since the derivative, or slope, is positive to the left of $x = 15$ and negative to the right. There is also a local minimum at $x = 7$ as the derivative is positive to the right of this value and negative to the left. (The point $x = 0$ is neither a maximum or minimum since not all critical points are extrema: the derivative is negative on both the left and right of $x = 0$.)

9 part (a):

(i) From the graph, we can see that f(x) is decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$. This is corroborated by the local/global minimum found in (ii)-(iii).

(ii) Taking the first derivative of $f(x)$ returns $f'(x) = 4(2x-4)^3 \cdot 2 = 8(2x-4)^3$. We then look for critical values

$$\begin{aligned} 8(2x-4)^3 &= 0 \\ 2x-4 &= 0 \\ x &= 2 \end{aligned}$$

so there is a critical point at $x = 2$. Using the graph we can see this corresponds to a local minimum at the point (2,0).

(iii) Again using the graph, we observe the local minimum at (2,0) is also a global minimum.

(iv) It appears that the graph is concave up everywhere, but let's confirm this. Taking the second derivative returns $f''(x) = 24(2x-4)^2 \cdot 2 = 48(2x-4)^2$. Since $2x-4$ is squared the second derivative will always be positive and the function is concave up over $(-\infty, \infty)$.

(v) Inflection points occur when the sign of a function's concavity changes. Since the function is concave up everywhere, there are no inflection points.

part (b):

(i) From the graph we can observe that $f(x)$ is decreasing on $(-\infty, -3) \cup (0, 3)$ and increasing on $(-3, 0) \cup (3, \infty)$. This is corroborated by the local extrema determined in (ii)-(iii).

(ii) We again take the first derivative $f'(x) = 4x^3 - 36x = 4x(x^2-9)$ and find critical values via $4x(x^2-9) = 0$. This equation is satisfied by the critical values $x = 0$, $x = -3$, and $x = 3$. These correspond to the points (0,0), (-3, -81), (3, -81). Using (i) and the graph we can determine which points are local maxima and which are local minima. There is a local maximum at (0,0), and local minima at the points (-3, -81) and (3, -81).

(iii) From the graph we can observe that (0,0) is **not** a global maximum as the function takes on larger values away from this point. The points (-3,-81) and (3, -81) are global minima since -81 is the smallest value the function takes, and it does so at multiple points.

(iv) From the graph we can observe that the concavity behavior is more complex than in **part (a)**. We need to find the roots of the second derivative $f''(x) = 12x^2 - 36 = 12(x^2-3)$ to determine where the concavity changes sign: $12(x^2-3) = 0$ is satisfied by $x = \sqrt{3}$ and $x = -\sqrt{3}$. Our function can only change concavity at these points, so we need to find the sign of the second derivative for points between and outside these values: $f''(-3) = 72$, $f''(0) = -36$, and $f''(3) = 72$. We can then conclude that $f(x)$ is concave up for $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$ and concave down on $(-\sqrt{3}, \sqrt{3})$.

(v) The x-values for the inflection points were determined in (iv) where the sign of the concavity changed: $x = \sqrt{3}$ and $x = -\sqrt{3}$. Since $f(-\sqrt{3}) = -45$ and $f(\sqrt{3}) = -45$ inflection points occur at $(-\sqrt{3}, -45)$ and $(\sqrt{3}, -45)$.