1. A moving company wants to build a storage facility that will have a square base with each side of length $\boldsymbol{x}$ meters. The height of the shed is to be $\boldsymbol{h}$ meters. The company's goal is to **minimize** the materials cost for the facility, which gives an **objective equation**: $\boldsymbol{C = 25x^2 + 120xh}$. The requirement the company has is that the facility needs to have a volume, $\boldsymbol{V}$, of 5000 cubic meters. This gives the **constraint equation:** $\boldsymbol{V = x^2 \cdot h = 5000}$ cubic meters. Determine the dimensions $\boldsymbol{x}$ and $\boldsymbol{h}$ that will result in the minimum materials cost. Round your answers to **1 decimal place**.

2. Find two positive numbers whose product is 300 such that the sum $x + 3y$ is a minimum.

3. ABC Daycare wants to build a fence to enclose a rectangular playground. The area of the playground is 910 square feet. The fence along three of the sides costs \$5 per foot and the fence along the fourth side, which will be made of brick, costs \$15 per foot. Find the length of the brick fence that will minimize the cost of enclosing the playground. (Round your answer to one decimal place.)

4. A baseball team plays in a stadium that holds 58,000 spectators. With ticket prices at \$10, the average attendance had been 49,000. When ticket prices were lowered to \$8, the average attendance rose to 51,000. Using this information, answer the following questions.

   Question Help: https://youtu.be/-TzmLSQZFoI

   (a) Find the demand function (price p as a function of attendance x), assuming it to be linear. *Hint: Recall that the point-slope formula for a linear relationship is given by* $y - y_1 = m(x - x_1)$ *and* $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

   (b) How many tickets need to be sold to maximize revenue?

   (c) How should ticket prices be set to maximize revenue? (Round your answer to the nearestcent.)

   (d) What is the maximum revenue that can be expected?

5. Suppose a company's revenue function is given by $R(q) = -q^3 + 280q^2$ and its cost function is given by $C(q) = 500 + 11q$, where q is hundreds of units sold/produced, while $R(q)$ and $C(q)$ are in total dollars of revenue and cost, respectively.

   Question Help: https://youtu.be/zR1CK4-Jg0E

   (a) Find the profit function.
   (b) Find the marginal profit function
   (c) How many items (in hundreds) need to be sold to maximize profits? *Use 3 decimal places.*
   (d) What is the maximum profit that can be expected?
   (e) Find the marginal revenue function.
   (f) Find the marginal cost function.
   (g) Confirm that the marginal revenue is equal to the marginal cost when the profit being maximized.

6. Given the cost function $C(x) = 128\sqrt{x} + \frac{x^2}{216000}$ where x represents the production level in units, answer the following questions.

   Question Help: https://youtu.be/9rg3HPxPsCo

   (a) Find the cost at the production level of 1450 units.
   (b) Find the average cost at the production level of 1450 units.
   (c) Find the marginal cost at the production level of 1450 units.
   (d) What is the production level that minimize the average cost?
   (e) What is the minimal average cost that can be expected?
   (f) Confirm that the average cost is equal to the marginal cost when the average cost being minimized.

Answer Key

1 $x \approx 22.9$ meters; $h \approx 9.5$ meters

2 smaller number: 10 ; bigger number:30

3 21.3 feet

4 **(a)** $p = -0.001x + 59$ **(b)** 29500 tickets **(c)** \$29.5 per ticket **(d)** \$870,250

5 **(a)** $P(q) = -q^3 + 280q^2 - 11q - 500$ **(b)** $MP(q) = -3q^2 + 560q - 11$ **(c)** 186.647 hundred units **(d)** \$ 3,249,594.92 **(e)** $MR(q) = -3q^2 + 560q$ **(f)** $MC(q) = 11$ **(g)** $MR(186.647) \approx 11$ and $MC(186.647) = 11$ *Note:* $MR(186.647) \approx 11.012$ *which is slightly off from* 11 *due to the result of rounding-off error.*

6 **(a)** $C(1450)$ =\$4883.83 **(b)** $AC(1450)$ =\$3.37 **(c)** $MC(1450)$ =\$1.69 **(d)** $x_{min} = 57600$ units **(e)** $AC_{min} = AC(57600)$ =\$0.8 **(f)** $AC(57600) = MC(57600) = 0.8$

Detailed Solutions

1 Let us first express our objective equation entirely in terms of x so we can find extrema using derivatives. We can do this by rearranging the constraint equation $V = x^2 \cdot h = 5000$ to obtain $h = \frac{5000}{x^2}$. We then plug this result into our objective equation

$$C = 25x^2 + 120xh = 25x^2 + 120x \cdot \frac{5000}{x^2} = 25x^2 + \frac{600,000}{x}$$

Now that everything is in terms one variable we can find the minimum by looking for critical numbers

$$C'(x) = 50x - \frac{600,000}{x^2}$$
$$50x - \frac{600,000}{x^2} = 0$$
$$50x = \frac{600,000}{x^2}$$
$$x^3 = 12,000$$
$$x = (12,000)^{1/3} \approx 22.9 \text{ meters}.$$

We can confirm this critical point is a minimum using the second derivative test. We then find the height in meters by plugging $x = 22.9$ into our constraint equation $h = \frac{5000}{22.9^2} \approx 9.5$ meters.

2 We designate our two numbers x and y and formulate the equation $x \cdot y = 300$ to match the problem description. Since we are minimizing $x + 3y$ we should get everything in terms of x (or y) in order to find extrema. We rearrange our constraint equation to read $y = \frac{300}{x}$ and substitute

$$x + 3y = x + 3 \cdot \frac{300}{x} = x + \frac{900}{x}.$$

We then look for extrema with the first derivative $\frac{d}{dx}(x + \frac{900}{x}) = 1 - \frac{900}{x^2}$

$$1 - \frac{900}{x^2} = 0$$
$$x^2 = 900$$
$$x = 30 \text{ or } -30.$$

We then see which one of these is a minimum using the second derivative test with $\frac{d}{dx}(1 - \frac{900}{x^2}) = \frac{1800}{x^3}$ to find $\frac{1800}{30^3} = 0.0667$ and $\frac{1800}{(-30)^3} = -0.0667$. Since the sign of the second derivative is positive for the critical number $x = 30$ we know that we have a minimum at $x = 30$. Knowing that the product of our numbers is 300 we can conclude the other number must be $y = 10$.

3 Since the area we are considering is rectangular we can designate our two side lengths x and y such that the area is $A = x \cdot y = 910$. Let us **choose** x as the length in feet of the sides that are made of brick on one side and fence on the other. This means y is the length in feet of the sides that are both made of fence. We can then develop a cost formula knowing the cost per foot of material $C = 5y + 5y + 5x + 15x$ which has four terms, one for each side of the playground. This simplifies to $C = 10y + 20x$. We can express this cost in terms of one variable by rearranging the constraint equation for the area $y = \frac{910}{x}$ and obtaining

$$C = 10y + 20x = 10 \cdot \frac{910}{x} + 20x = \frac{9100}{x} + 20x$$

We then take the derivative with respect to x

$$\frac{d}{dx}\left(\frac{9100}{x} + 20x\right) = -\frac{9100}{x^2} + 20$$

and search for critical numbers

$$\begin{aligned} -\frac{9100}{x^2} + 20 &= 0 \\ 455 &= x^2 \\ x \approx 21.3 \text{ feet or } x &\approx -21.3 \text{ feet.} \end{aligned}$$

We can confirm that $x = 21.3$ is a minimum while $x = -21.3$ is a maximum using the second derivative test, or understand that negative lengths are nonsensical in this context and determine $x = 21.3$ feet as one of the two side lengths. This is the length of the side constructed from brick, which is all the question asks us to find.

4 (a) We want to develop a linear relationship expressing price p as a function of attendance x, or $p = mx + b$. We first find the slope m

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 10}{51,000 - 49,000} = -0.001$$

the difference between prices belongs in the numerator corresponding to the function outputs (or y-values), and attendance in the denominator corresponding to the inputs (or x-values). It is also important that the subtractions are ordered in this manner so the denominator is positive. Now we find the intercept b using known values $p = 8$ and $x = 51,000$

$$\begin{aligned} p &= -0.001x + b \\ 8 = -0.001(51,000) + b &= -51 + b \\ b &= 59 \end{aligned}$$

so our equation is $p = -0.001x + 59$.

(b) Revenue can be expressed $R = x \cdot p$ or the number of tickets sold times the price of a single ticket. We can express this entirely in terms of x, the number of tickets sold

$$R = x \cdot p = x \cdot (-0.001x + 59) = -0.001x^2 + 59x$$

We find how many tickets need to be sold to maximize revenue using the first derivative test for critical numbers

$$\frac{d}{dx}\left(-0.001x^2 + 59x\right) = -0.002x + 59$$
$$-0.002x + 59 = 0$$
$$x = 29,500$$

so we need to sell 29,500 tickets to maximize revenue.

(c) To find how ticket prices should be set to maximize revenue we first rearrange our linear function to find x in terms of p, or $x = -1000p + 59,000$ we then plug this into our revenue equation

$$R = x \cdot p = (-1000p + 59,000) \cdot p = -1000p^2 + 59,000p$$

We then use the first derivative test to find critical numbers

$$\frac{d}{dp}\left(-1000p^2 + 59,000p\right) = -2000p + 59,000$$
$$-2000p + 59,000 = 0$$
$$x = 29.5$$

so we need to sell tickets at \$29.50 to maximize revenue.

(d) The maximum revenue is the product of the previous two answers

$$R_{max} = x_{max} \cdot p_{max} = 29,500 \cdot \$29.5 = \$870,250$$

5 (a) Profit $P(q)$ is the difference of revenue $R(q)$ and cost $C(q)$, so we formulate

$$P(q) = R(q) - C(q) = -q^3 + 280q^2 - (500 + 11q) = -q^3 + 280q^2 - 11q - 500$$

(b) The marginal profit $MP(q)$ is the derivative of $P(q)$

$$\frac{d}{dq}\left(-q^3 + 280q^2 - 11q - 500\right) = -3q^2 + 560q - 11$$

(c) To find how many items need to be sold to maximize profits we search for critical numbers using the first derivative and the quadratic formula

$$-3q^2 + 560q - 11 = 0$$
$$x = \frac{-560 \pm \sqrt{(560)^2 - 4(-3)(-11)}}{2(-3)}$$
$$x \approx 0.0196 \text{ or } x \approx 186.647$$

To find which of these maximizes profit we use the second derivative test

$$\frac{d}{dq}\left(-3q^2 + 560q - 11\right) = -6q + 560$$

to find $-6(0.0196) + 560 = 559.8824$, $-6(186.647) + 560 = -559.882$ so we conclude that there is a maximum at $q = 186.647$ corresponding to 186.647 hundred units since q measures items in hundreds.

(d) The maximum profit that can be expected is

$$P(186.647) = -(186.647)^3 + 280(186.647)^2 - 11(186.647) - 500 = \$3,249,594.92$$

since maximum profit occurs at $q = 186.647$.

(e) To find the marginal revenue function $MR(q)$ we take the derivative of the revenue function $R(q)$

$$\frac{d}{dq}(-q^3 + 280q^2) = -3q^2 + 560q.$$

(f) To find the marginal cost function $MC(q)$ we take the derivative of the cost function $C(q)$

$$\frac{d}{dq}(500 + 11q) = 11.$$

(g) The profit is maximized when $q = 186.647$. We can then find $MR(186.647) = -3(186.647)^2+560(186.647) \approx 11.012$ and $MC(186.647) = 11$. These answers would be exactly equal but are not since we rounded our answer in part (c). This is a natural consequence of the sum rule for derivatives.

6 (a) The cost at the production level of 1450 units is $C(1450) = 128\sqrt{1450}+\dfrac{1450^2}{216000} = \$4883.83$.

(b) The average cost $AC(x)$ is the total cost divided by the number of units, which at 1450 is $AC(1450) = \dfrac{\$4883.83}{1450} = \$3.37$.

(c) We find the marginal cost $MC(x)$ by taking the derivative of the cost

$$MC(x) = \frac{d}{dx}\left(128\sqrt{x} + \frac{x^2}{216000}\right) = \frac{64}{\sqrt{x}} + \frac{x}{108000}$$

The marginal cost at the production level of 1450 units is then $MC(1450) = \$1.69$.

(d) To minimize the average cost we first develop the average cost function

$$AC(x) = \frac{C(x)}{x} = \frac{128}{\sqrt{x}} + \frac{x}{216000}$$

and then take the first derivative

$$AC'(x) = -\frac{64}{\sqrt{x^3}} + \frac{1}{216000}$$

which allows us to look for critical numbers

$$-\frac{64}{\sqrt{x^3}} + \frac{1}{216000} = 0$$

$$\frac{x^{3/2}}{64} = 216000$$

$$x = (64 \cdot 216000)^{2/3} = 57600$$

so we have one critical number of products sold which we can safely assume minimizes the average cost at $x_{min} = 57,600$.

(e) The minimal average cost expected is $AC(x_{min}) = AC(57,600) = \frac{128}{\sqrt{57,600}} + \frac{57,600}{216000} = \$0.8$.

(f) We know the minimal average cost is $AC(57,600) = \$0.8$ and evaluate the marginal cost $MC(57,600) = \frac{64}{\sqrt{57,600}} + \frac{57,600}{108000} = \$0.8$ which confirms $AC(57,600) = MC(57,600) = \$0.8$. This is a direct and natural consequence of the quotient rule.