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DEPARTMENT OF MATHEMATICS & STATISTICS

MATH 1100: CALCULUS TECHNIQUES

CLASS WORKBOOK

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November, 2016

Contents

Part I

Derivatives

Lesson No. 1

Limits

Objectives

- Using correct notation, describe the limit of a function.
- Be able to determine if the limit of a function exists, and if so, determine the limiting value using a table, a graph, or a formula.
- Be able to determine one-sided limits.
- Be able to determine if the limit of a function exists, and if so, determine the limiting value using the relationship between one-sided and two sided limits.
- Determine continuity at a point on the graph of a function.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Limits and One-Sided Limits* in Lesson 1 of Chapter 1 (pages 73-77)
- Strang and Herman, *Calculus Volumn 1*[2]
  - *Section 2.2. The Limit of a Function*[3]
- Hoffman, *Contemporary Calculus I*[4]
  - *Section 1.3: Continuous Functions* (pages 98-103)[3]
    - Definition and Meaning of Continuous (page 98).
    - Why do we care whether a function is continuous? (page 101)
    - Which Functions Are Continuous? (page 102)

Key Terms:

- Limit of a function
- Limit Theorems
- "$x \to a$" or "x approaches a"
- "arbitrarily close"

[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2] Available free to download from https://openstax.org/details/books/calculus-volume-1 .
[3] Disregard any examples with trigonometry.
[4] Available free to download from https://www.opentextbookstore.com/details.php?id=11#tabs-3 .

Idea and Notation

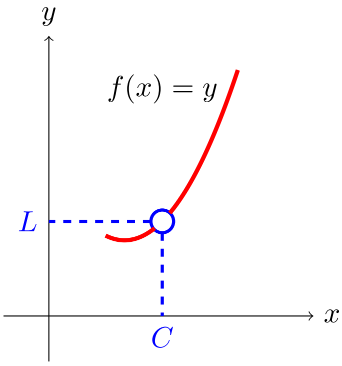

Figure 1.1

The concept of the **limit of a function** shows up in many areas in mathematics, and it is one of the fundamental concepts in calculus. The limit of a function describes the behavior of the function when the variable is near, but does not equal , a specified number (Figure 1.1). If the values of $f(x)$ gets closer and closer, as close as we want, to one number L as we take values of x very close to (but not equal to) a number c, then we *say* **"the limit of $f(x)$, as x approaches c, is L" and we *write* "$\lim_{x \to c} f(x) = L$" (The symbol "$\to$" means "approaches" or "gets very close to")**.

> $f(c)$ is a single number that describes the behavior (value) of f **AT** the point $x = c$.
>
> $\lim_{x \to c} f(x)$ is a single number that describes the behavior of f **NEAR, BUT NOT AT**, the point $x = c$.

Methods for Evaluating Limits

- **The Algebra Method:** The algebra method involves algebraically simplifying the function before trying to evaluate its limit. Often, this simplification just means factoring and dividing, but sometimes more complicated algebraic or even trigonometric steps are needed.

- **The Table and Graph Methods:** To evaluate a limit of a function $f(x)$ as x approaches c, the table method involves calculating the values of $f(x)$ for "enough" values of x very close to c so that we can "confidently" determine which value $f(x)$ is approaching. If $f(x)$ is well–behaved, we may not need to use very many values for x. However, this method is usually used with complicated functions, and then we need to evaluate $f(x)$ for lots of values of x. The graph method is closely related to the table method, but we create a graph of the function instead of a table of values, and then we use the graph to determine which value $f(x)$ is approaching.

- **Which Method Should You Use?** In general, the algebraic method is preferred because it is precise and does not depend on which values of x we chose or the accuracy of our graph or precision of our calculator. **If you can evaluate a limit algebraically, you should do so.** Sometimes, however, it will be very difficult to evaluate a limit algebraically, and the table or graph methods offer worthwhile alternatives. Even when you can algebraically evaluate the limit of a function, it is still a good idea to graph the function or evaluate it at a few points just to verify your algebraic answer.
  The table and graph methods have the same advantages and disadvantages. Both can be used on very complicated functions which are difficult to handle algebraically or whose algebraic properties you don't know. Often both methods can be easily programmed on a calculator or computer. However, these two methods are very time–consuming by hand and are prone to round off errors on computers. You need to know how to use these methods when you can't figure out how to use the algebraic method, but you need to use these two methods warily.

Example 1.1. Let $f(x) = \dfrac{2x^2 - x - 1}{x - 1}$. Using the table and the graph of $f(x)$ (Table 1.1, Table 1.2 and Figure 1.2), find $\lim_{x\to 1} f(x)$. Explain how you reach your answer.

x	$f(x)$
0.9	2.82
0.9998	2.9996
0.9999994	2.999988
0.9999999	2.9999998

Table 1.1

x	$f(x)$
1.1	3.2
1.003	3.006
1.0001	3.0002
1.000007	3.000014

Table 1.2

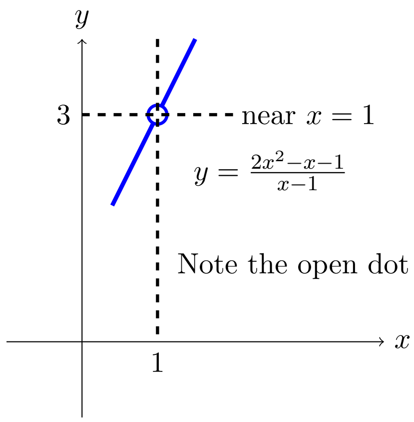

Figure 1.2

Finding Limits of Polynomial and Rational Functions by Substitution
By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x\to a} f(x) = f(a)$ is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Limits of Polynomial and Rational Functions

If $P(x)$ and $Q(x)$ are **polynomials** and a is any real number,
then $\lim_{x\to a} P(x) = P(a)$ and $\lim_{x\to a} \dfrac{P(x)}{Q(x)} = \dfrac{P(a)}{Q(a)}$ **if** $\boldsymbol{Q(a) \neq 0}$

In other words, we can calculate the limits of polynomials and rational functions by substituting *as long as the substitution does not result in a division by zero*.

Example 1.2. Evaluate the following limit: $\lim_{x\to 2} 5x^3 - x^2 + 3$.

Example 1.3. Evaluate the following limit: $\lim_{x\to 2} \dfrac{x^3 - 7x}{x^2 + 3x}$.

Example 1.4. Evaluate the following limit: $\lim_{x\to 2} 4x + 2$. *Hint: A linear function is a polynomial degree 1.*

For some functions, it is possible to calculate the limit as x approaches a simply by substituting $x = a$ into the function and then evaluating $\frac{f(a)}{g(a)}$, but sometimes this method does not work since $\frac{f(a)}{g(a)}$ has an indeterminate form $\left(ex.\ \frac{0}{0}, \frac{\infty}{\infty}, \frac{\infty}{0}\ etc. \right)$. These forms are common in calculus. In fact, the limit definition of the derivative which will be discussed in the next lesson is the limit of the indeterminate form of $\frac{0}{0}$.

Calculating a Limit When $\frac{f(a)}{g(a)}$ has the indeterminate form of $\frac{0}{0}$

To find $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ where $f(a) = 0$ and $g(a) = 0$, follow the steps below.

1. Find an expression that is equal to $\frac{f(x)}{g(x)}$ for all $x \neq a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
   (a) If $f(x)$ and $g(x)$ are polynomials in expanded form, we should factor each function and cancel out any common factors.
   (b) If $f(x)$ is polynomials in factored form and $g(x)$ is a monomial, we should first expand $f(x)$ and then simplify $\frac{f(x)}{g(x)}$.
   (c) If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   (d) If $\frac{f(x)}{g(x)}$ is a complex fraction, we begin by simplifying it.
2. Find the limit by substitution and/or limit rules (see page 6).

Example 1.5. Refer to Example 1.1. Algebraically, evaluate $\lim\limits_{x\to 1} f(x)$.

Example 1.6. Algebraically, evaluate $\lim\limits_{x\to -1} \frac{\sqrt{x+2}-1}{x+1}$.

Example 1.7. Given $f(x) = x^2 + 3$, algebraically evaluate $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Some Basic Limit Rules

Some Basic Limit Rules

For any real number a and any constant c where $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$

$$\lim_{x\to a} f(x) = f(a) \tag{1.1}$$

$$\lim_{x\to a} c = c \tag{1.2}$$

$$\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M \tag{1.3}$$

$$\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M \tag{1.4}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{m} \text{ for} M \neq 0 \tag{1.5}$$

Example 1.8. Use the graph of $f(x)$ in Figure 1.3 to evaluate the following limit:

(a) $\lim_{x\to 1}\{3 + f(x)\}$ (b) $\lim_{x\to 1} f(2+x)$ (c) $\lim_{x\to 0} f(3-x)$ (d) $\lim_{x\to 2}\{f(x+1) - f(x)\}$

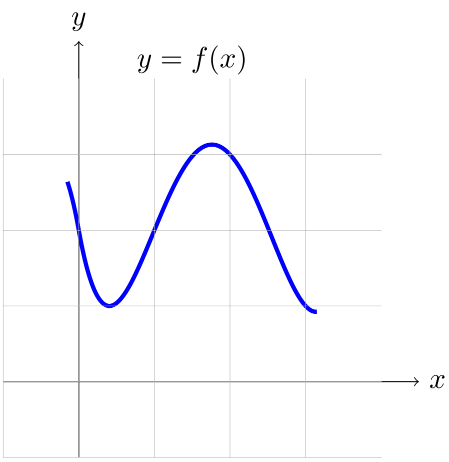

Figure 1.3

One-Sided Limits

Sometimes, what happens to us at a place depends on the direction we use to approach that place. If we approach Niagara Falls from the upstream side, then we will be 182 feet higher and have different worries than if we approach from the downstream side. Similarly, the values of a function near a point may depend on the direction we use to approach that point.

Definition of Left and Right Limits

The **left limit** as x approaches c of $f(x)$ is L if the values of $f(x)$ get as close to L as we want when x is very close to and *left of* c (or $x < c$)

$$\lim_{x \to c^-} f(x) = L \tag{1.6}$$

The **right limit** as x approaches c of $f(x)$ is L if the values of $f(x)$ get as close to L as we want when x is very close to and *right of* c (or $x > c$)

$$\lim_{x \to c^+} f(x) = L \tag{1.7}$$

Example 1.9. Given the function $f(x)$ below,

$$f(x) = \begin{cases} x+1 & \text{if} \quad x < 2 \\ x^2 - 4 & \text{if} \quad x \geq 2 \end{cases}$$

evaluate each of the following limits using Table 1.3 and the given graph of $f(x)$ (Figure 1.4). Explain how you reach your answers.

1. $\lim_{x \to 2^-} f(x)$
2. $\lim_{x \to 2^+} f(x)$

x	$f(x) = x+1$	x	$f(x) = x^2 - 4$
1.9	2.9	2.1	0.41
1.99	2.99	2.01	0.0401
1.999	2.999	2.001	0.004001
1.9999	2.9999	2.0001	0.00040001
1.99999	2.99999	2.00001	0.0000400001

Table 1.3

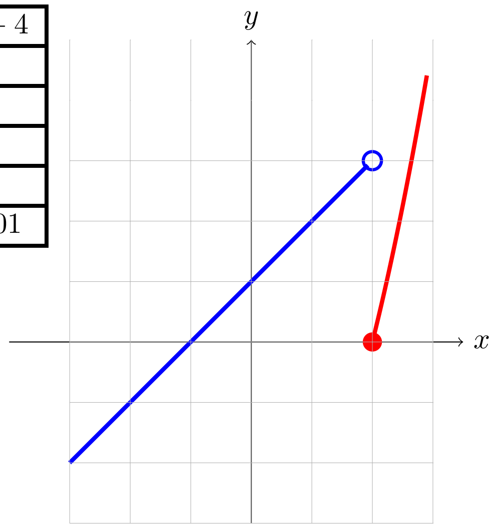

Figure 1.4

Example 1.10. Evaluate the following limit: $\lim_{x \to 0^+} \left(\frac{1}{x} + 2 \right)$.

The Existence of a Limit

Let us now consider the relationship between the limit of a function at a point versus the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in the box below:

One-Sided Limit Theorem:

$$\lim_{x\to c} f(x) = L \text{ if and only if } \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

Corollary:

$$\text{If } \lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x), \text{ then } \lim_{x\to c} f(x) \text{ does not exist.}$$

Example 1.11. Refer to the function $f(x)$ in Example 1.9, does $\lim_{x\to 2} f(x)$ exist? Explain your reasoning.

Example 1.12. Evaluate the following limit: $\lim_{x\to 2}\left(\frac{x^2+2x}{x^2-4}\right)$. If it does not exist, explain your reasoning.

Short Answers

1.1 3	**1.4** 10	**1.7** 2x	**1.10** ∞
1.2 39	**1.5** 3	**1.8** 5; 1; 1;-2	**1.11** No.
1.3 $-\frac{3}{5}$	**1.6** $\frac{1}{2}$	**1.9** 3;0	**1.12** No limit exists.

Lesson No. 2

Derivatives of Functions

Objectives

- Understand the difference between average velocity and instantaneous velocity.
- Understand the method for determining the **slope** of the graph of a function at a specific point using limits.
- Understand how the **derivative** of a function can be used to determine the slope of the graph of a function at any point.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*
  - *Precalculus Idea: Slope and Rate of Change* in Lesson 1 of Chapter 1 (page 73)
  - Lesson 2 of Chapter 1 (pages 80-92)
- Hoffman, *Contemporary Calculus I*
  - *Which functions have derivatives? differentiability implies continuity* in section 3.3 (pages 272-274)

Key Terms:

- slope of a secant line
- slope of a tangent line
- derivative of a function; differentiation
- slope of a curve using the derivative.

The **derivative of a function** is a topic that we will not stop discussing in this class. It will allow us to analyze the behavior of functions in a much more detailed manner than we were able to do using only the concepts developed in College Algebra. The importance of derivatives cannot be overstated. We will need to start out using an intuitive approach to determine a derivative, but we will see in subsequent sections that finding derivatives will be much easier and more efficient as we learn more sophisticated techniques of differentiation.

Slope and Rate of Change:The slope of a line measures how fast a *line* rises or falls as we move from left to right along the line. It measures the rate of change of the y-coordinate with respect to changes in the x-coordinate. If the line represents the distance traveled over time, for example, then its slope represents the **velocity**.

The geometric interpretation: The goal is determine the **slope of a curve** at a point. We have defined this slope to be the slope of the **tangent line** to the curve at the point of interest.

Figure 2.1 reminds you how to calculate slope (m) using two points on the line:

$$m = \frac{rise}{run} = \frac{\triangle y}{\triangle x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \quad (2.1)$$

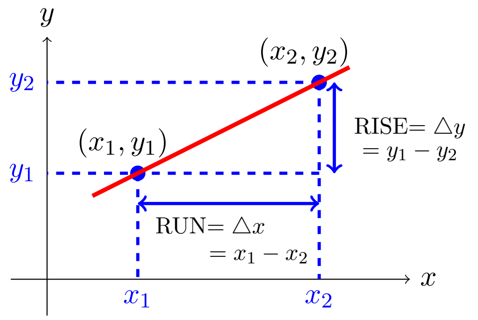

Figure 2.1: Calculate slope using two points on the line

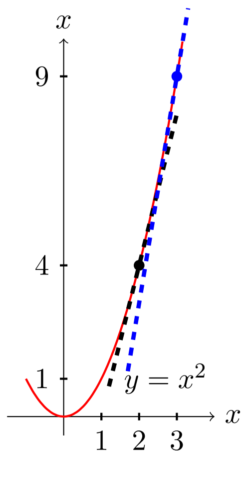

Figure 2.2: Calculate slope on a curve

But if we wants to know how fast a *curve* rises or falls, what should we do? The answer is to find the slope of the curve as in Figure 2.2 . There are two ways to find the slope of the curve:

1. Determine the **slope of a secant line** *passing through* two points on the curve. The slope tells us the **average velocity** between the points (i.e. how fast the curve rise (or fall) between the points).

2. Determine the **slope of a tangent line** which *touches* one point on the curve. The slope tells us the instantaneous velocity (i.e. how fast the curve rise (or fall) **at** the points).

Average Velocity (Rate of Change) vs. Instantaneous Velocity (Rate of Change)

Average Velocity $= \frac{\triangle position}{\triangle time}$ = Slope of the **secant line** through 2 points.

Instantaneous Velocity = Slope of the **tangent line** to the graph.

Approximate the slope of a Tangent Line from the slope of a secant line by using the idea that secant lines over tiny intervals approximate the tangent line.

Example 2.1. Approximate the slope of the tangent line at $(2, 4)$ from the slope of the secant line passing through $(2, 4)$ and $(3, 9)$.

Example 2.2. Approximate the slope of the tangent line at $(2, 4)$ from the slope of the secant line passing through $(2, 4)$ and $(2.5, 6.25)$.

Example 2.3. Compare the two secant lines in Example 2.1 and in Example 2.2. Which would be a better approximation of the tangent line to the curve at $(2, 4)$? Why?

As the interval got smaller and smaller, the secant line got closer to the tangent line and its slope got closer to the slope of the tangent line. We can continue picking points closer and closer to $(2, 4)$ on the graph of the function f, and then calculating the slopes of the lines through each of these points and the point $(2, 4)$.

Point to the left of (2,4)			Point to the right of (2,4)		
x	$y = x^2$	Slope	x	$y = x^2$	Slope
1.5	2.25	3.5	3	9	5
1.9	3.61	3.9	2.5	6.25	4.5
1.99	3.9601	3.99	2.01	4.0401	4.01

Table 2.1

Example 2.4. Using Table 2.1 above, guess the value of the slope of the tangent line at $x = 2$.

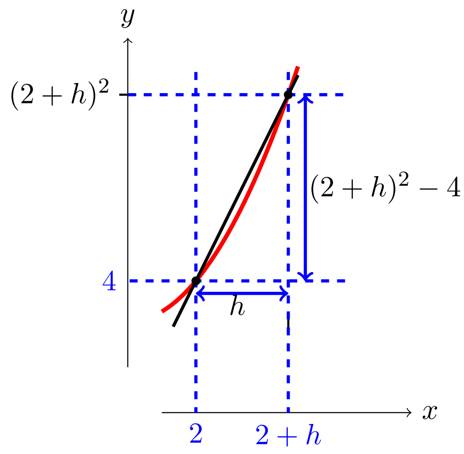

Figure 2.3: [1]

We can bypass much of the calculating by not picking the points one at a time: let's look at a general point near $(2, 4)$. Define $x = 2 + h$ so h is the increment from 2 to x (Figure 2.3). If h is small, then $x = 2 + h$ is close to 2 and the point $(2 + h, f(2 + h)) = (2 + h, (2 + h)^2)$ is close to $(2, 4)$.

<u>Secant lines and limits:</u> We now indicate, mathematically, how we will determine the slope of the tangent line to the curve at an arbitrary point. The end result will be a **function** that we will refer to as *"the derivative of $f(x)$"*, denoted by $f'(x)$ (*read aloud as "f prime of x*). The derivative function will take a value of x as an input and provide the slope of the graph of $f(x)$ at that point as the output.

Notation of Derivatives

The derivative of $y = f(x)$ with respect to x is written as
$f'(x)$ (read aloud as "*f prime of x*") or $\frac{dy}{dx}$ (read aloud as *"dee why dee ex"*).
In verb forms, we say we **find the derivative** of a function, or **take the derivative** of a function, or **differentiate** a function.

Formal Algebraic Definition of Derivatives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2.2}$$

Practical Definition of Derivatives

The derivative can be approximated by looking at an average rate of change, or the slope of a secant line, over a very tiny interval. The tinier the interval, the closer this is to the true instantaneous rate of change, slope of the tangent line, or slope of the curve.

[1] From Calaway, Hoffman, and Lippman, *Applied Calculus* ; page 85

Example 2.5. Given the function $f(x) = x^2 + 3$, answer the following questions:

1. Using the **limit** definition of the derivative in the equation 2.2 (page 12), determine the equation of the derivative, $f'(x) =$ ________________.
2. Find the slope of the graph of $f(x)$ at $x = 3$, $x = -1$, and $x = 0$.
3. How does the **sign of the derivative** relate to the **sign of the tangent line**?
4. What is the sign of the slope of a tangent line at any point on $(-\infty, 0)$? Is the function f increasing or decreasing on this interval?
5. What is the sign of the slope of a tangent line at any point on $(0, \infty)$? Is the function f increasing or decreasing on this interval?

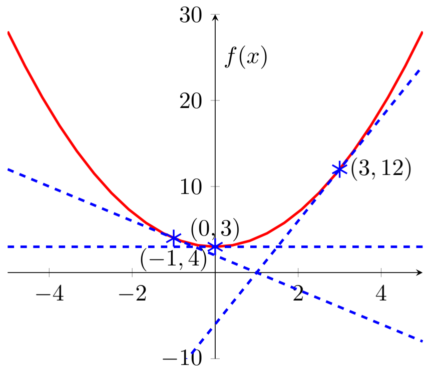

Figure 2.4: Graph of $f(x)$ with tangent lines at $x = -1$, $x = 0$ and $x = 3$

The Derivative and The Behavior of a Function

For a function f which is *differentiable*[2] on an interval I:

1. if $f'(x) > 0$ for all x in the interval I, then f is increasing on I.
2. if $f'(x) < 0$ for all x in the interval I, then f is decreasing on I.
3. if $f'(x) = 0$ for all x in the interval I, then f is constant on I.

The derivative of a function tells about the general shape of the function, and we can use that shape information to determine if an extreme point is a maximum or minimum or neither (See Figure 2.5[3]).

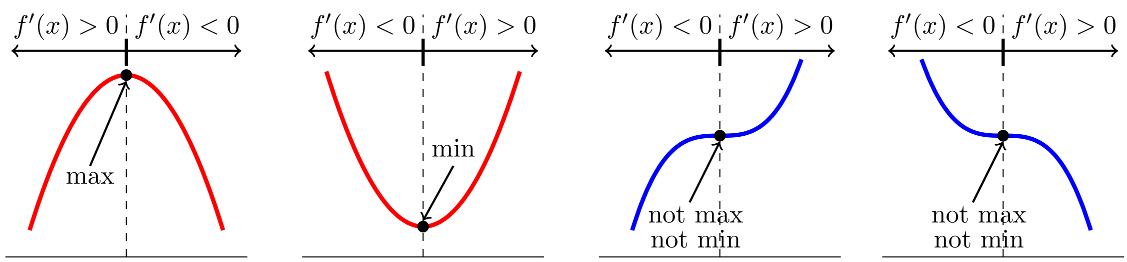

Figure 2.5

[3]See Lesson 3: Differentiability and Continuity
[3]From Hoffman, *Contemporary Calculus I* ; page 88

Interpretations of The Derivative

So far we have emphasized the derivative as the slope of the line tangent to a graph . That interpretation is very visual and useful when examining the graph of a function, and we will continue to use it. Derivatives, however,are used in a wide variety of fields and applications, and some of these fields use other interpretations. The following are a few interpretations of the derivative which are commonly used.

General

Rate of Change: $f'(x)$ is the **rate of change** of the function at x. If the units for x are years and the units for $f(x)$ are people, then the units for $\frac{df}{dx}$ are $\frac{\text{people}}{\text{year}}$, a rate of change in population.

Graphical

Slope: $f'(x)$ is the **slope of the line tangent to the graph of f at the point $(x, f(x))$**.

Physical

Velocity: If $f(x)$ is the position of an object at time x, then $f'(x)$ is the **velocity** of the object at time x. If the units for x are hours and $f(x)$ is distance measured in miles, then the units for $f'(x) = \frac{df}{dx}$ are $\frac{\text{miles}}{\text{hour}}$, miles per hour, which is a measure of velocity.

Acceleration: If $f(x)$ is the velocity of an object at time x, then $f'(x)$ is the **acceleration** of the object at time x. If the units for x are hours and $f(x)$ has the units $\frac{\text{miles}}{\text{hour}}$, then the units for the acceleration $f'(x) = \frac{df}{dx}$ are $\frac{\text{miles/hour}}{\text{hour}} = \frac{\text{miles}}{\text{hour}^2}$, miles per hour per hour.

Business

Marginal Cost, Marginal Revenue, and Marginal Profit: In business contexts, the word "*marginal*" usually means the derivative or rate of change of some quantity. We'll explore these terms in more depth later in the section. Basically, the marginal cost is approximately the additional cost of making one more object once we have already made x objects. If the units for x are bicycles and the units for $f(x)$ are dollars, then the units for $\frac{df}{dx}$ are $\frac{\text{dollars}}{\text{bicycle}}$, the cost per bicycle.

Example 2.6. Suppose the demand curve for widgets was given by $D(p) = \frac{1}{p}$, where D is the quantity of items widgets, in thousands at a price of p dollars.

1. Evaluate $f(3)$. Interpret the result in the context.

2. Determine $f'(x)$ using the limit definition in the equation 2.2 (page 12).

3. Evaluate $f'(3)$. Interpret the result in the context.

Short Answers to Examples

2.1 5

2.2 4.5

2.3 The secant line in Example 2.2.

2.4 2

2.5 $f'(x) = 2x$; (1) 6, -2, 0; (2) same sign; (3) negative;decreasing (4) positive;increasing.

2.6 $f(3) = \frac{1}{3}$;$f'(x) = \frac{1}{x^2}$;$f'(3) = \frac{1}{9}$

Lesson No. 3

Differentiability and Continuity

Objectives

- Be able to recognize when a function is not differentiable at a point.
- Be able to determine when a function is continuous at a point and/or over a specified interval.
- Be able to determine if a function has any points of discontinuity.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 2.1 Limit and Continuity*
    - Continuity
- Strang and Herman, *Calculus Volumn 1*[2,3]
  - *Section 2.4 Continuity*
    - Continuity at a Point
    - Types of Discontinuities
    - Continuity over an Interval
- Hoffman, *Contemporary Calculus I*[3,4]
  - *Section 1.3: Continuous Functions*)
    - Definition and Meaning of Continuous.
    - Graphic Meaning of Continuity
    - Why do we care whether a function is continuous?
    - Which Functions Are Continuous?

Key Terms:

- differentiable functions
- continuous functions
- points of discontinuity

[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2] Available free to download from https://openstax.org/details/books/calculus-volume-1 .
[3] Disregard any examples with trigonometry.
[4] Available free to download from https://www.opentextbookstore.com/details.php?id=11#tabs-3 .

Why do we care whether a function is continuous?

There are several reasons for us to examine continuous functions and their properties:

- Most of the applications in engineering, the sciences and business are continuous and are modeled by continuous functions or by pieces of continuous functions.
- Continuous functions have a number of useful properties which are not necessarily true if the function is not continuous. If a result is true of all continuous functions and we have a continuous function, then the result is true for our function.
- Differential calculus has been called the study of continuous change, and many of the results of calculus are guaranteed to be true only for continuous functions.

Continuity at a Point

Definition)

A function $f(x)$ is **continuous at a point** a if and only if the following **three conditions** are satisfied:

i $f(a)$ is defined

ii $\lim_{x\to a} f(x)$ exists

iii $\lim_{x\to a} f(x) = f(a)$

A function is **discontinuous at a point** a if it fails to be continuous at a.

Example 3.1. Given the graph of each function $f(x)$ in Figure 3.1 below, determine if $f(x)$ is continuous at $x = a$. Using the definition of *Continuity at a Point*, justify your answer.

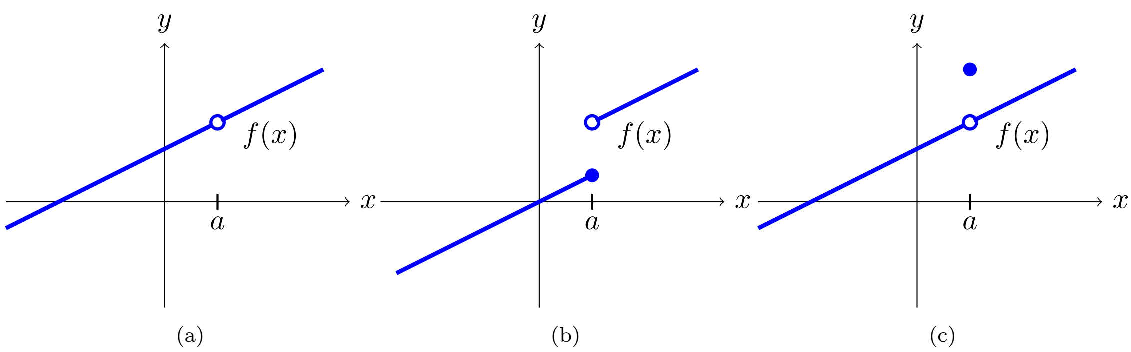

Figure 3.1

Example 3.2. Given the the function $f(x)$ below, answer the following questions:

$$f(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$$

1. Find $f(3)$.

2. Find $\lim_{x \to 3^-} f(x)$.

3. Find $\lim_{x \to 3^+} f(x)$.

4. Checking all **three conditions** in the definition of *Continuity at a Point*, determine whether the function $f(x)$ is continuous at $x = 3$. Justify your answer.

Example 3.3. Is the function $f(x) = \dfrac{1}{x^2}$ continuous at $x = 2$? Justify your answer by checking all **three conditions** for continuity.

Example 3.4. Is the function $f(x) = \dfrac{1}{x^2}$ continuous at $x = 0$? Justify your answer by checking all **three conditions** for continuity.

Differentiability

A function is called **differentiable** at a point if its derivative exists at that point.
Recall the limit definition of the derivative of the function $f(x)$ at $x = a$: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.
As we have seen in *Lesson* 1, there are certainly some case where a limit does not exist. If this is true for the limit described here, we say that $f(x)$ is nondifferentiable at $x = a$.

> Theorem:
>
> IF a function is **differentiable** at a point, THEN it is **continuous** at that point.

****IMPORTANT****It is important to clearly understand what is meant by this theorem and what is not meant:

- If the function is **differentiable** at a point, then the function is automatically **continuous** at that point.
- If the function is **continuous** at a point, then the function **may or may not** have a **derivative** at that point.
- If the function is **not continuous** at a point, then the function is **not differentiable** at that point.

Where can a slope not exist?

- *If there is a sharp corner (cusp) in the graph, the derivative will not exist at that point.* From the graph of the function $f(x) = x$ (see Figure 3.2), there are many "tangent lines" that could be drawn at x=a, so there is no "slope" of a unique tangent line at these points. On the left side of the graph, the slope of the line is $= 1$. On the right side of the graph, the slope is $+1$. There is no well-defined tangent line at the sharp corner at $x = 0$. So the function is **not differentiable** at that point.
- *If the tangent line is vertical, the derivative will not exist.* From the graph of the function $f(x) = x^{1/3}$ (see Figure 3.2), we can see that the tangent line to this curve at $x = 0$ is vertical with undefined slope, which is why the derivative does not exist at $x = 0$.

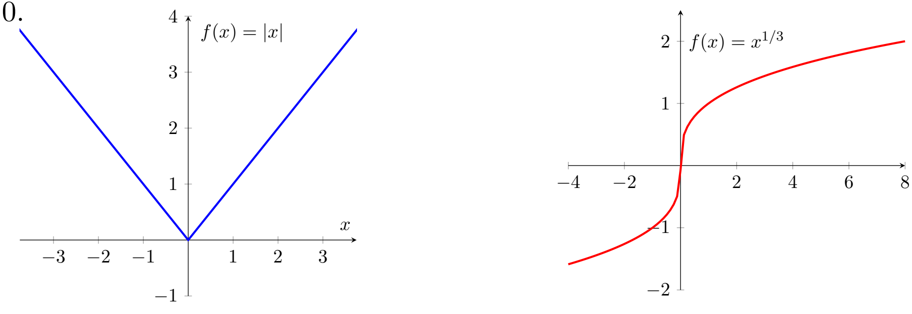

Figure 3.2

Example 3.5. Is the function $f(x) = x^2$ differentiable at $x = 2$? Is the function continuous at $x = 2$? Justify your answers.

Example 3.6. Is the function $f(x) = x+1$ differentiable at $x = 0$? Is the function continuous at $x = 0$? Justify your answers.

Short Answers to Examples

3.1 discont.;discont.;discont.

3.2 -5;-5;4;discont.

3.3 yes

3.4 yes

3.5 yes;yes

3.6 no;yes

Lesson No. 4

Basic Rules of Differentiation and Marginal Analysis

Objectives

- Be familiar with some of the basic rules for finding derivatives of functions
- Understand how the derivative of a function is used to determine the slope of the graph of the function at a point
- Understand the interpretation and use of the derivative in **marginal analysis** (business and economics).

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 2.3 Power and Sum Rules for Derivatives*
    - Disregard the derivatives for *Exponential Functions* and *Natural Logarithm*. We will discuss about these rules in the future lessons.
    - Skip *Example 7*
- Abramson, *College Algebra*[2]
  - Review rules of exponents; exponents and rational exponents from *sections 1.2-1.3* .

Key Terms and Concepts:

- Rules for finding derivatives of functions
- Notation for derivatives of functions
- Slope of a graph at a point
- Equation of a tangent line
- Marginal cost, Marginal profit and Marginal revenue

[1]Available free to download from `http://www.opentextbookstore.com/details.php?id=14` .
[2]Available free to download from `https://openstax.org/details/books/college-algebra` .

Using Basic Differentiation Rules

A formula for the derivative function (see equation 2.2 in lesson 2) is very powerful, but as you can see, calculating the derivative using the limit definition is very time consuming. In this section, we will use some simple rules[1] for finding derivatives of basic functions without needing the limit definition. In the following lessons, we will consider some additional rules for differentiation that will allow us to find derivatives for more complicated functions.

Recall that in section 2 the alternative notations for the derivative were introduced using $\frac{d}{dx}f(x)$ and $\frac{dy}{dx}$ (read as *"the derivative of y with respect to x".*). You will want to be familiar and comfortable with these various notations since they will be used when the rules are provided in this section.

Differentiation Rules: Basic Rules

In what follows, $f(x)$ and $g(x)$ are differentiable functions of x and k is a constant.

Constant Multiple Rule:

$$\frac{d}{dx}(kf(x)) = kf'(x) \tag{4.1}$$

Sum Rule:

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x) \tag{4.2}$$

Difference Rule:

$$\frac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x) \tag{4.3}$$

Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1} \tag{4.4}$$

Constant Rule:[2]

$$\frac{d}{dx}(k) = 0 \tag{4.5}$$

Examples 4.1-4.3 show that the sum, difference, and constant multiple rule combined with the power rule allow us to easily find the derivative of a function. However, converting the math expressions into exponents is needed before taking the derivative using the *Power Rule* in the equation 4.4.The review of some *Rules of Exponents*[3]that are useful when translating a math expression into exponents is shown on page 22.

[1] To make sense out of these rules, see *Building Blocks* (examples 1-3) in section 2.3 from Calaway, Hoffman, and Lippman, *Applied Calculus*.

[2] The derivative of a constant is zero because $k = kx^0$.

[3] See sections 1.2-1.3 from Abramson, *College Algebra* for full review on *Exponents and Rational Exponents*. For the online version, visit `https://openstax.org/details/books/college-algebra`

Review: Some Rules of Exponents

The Negative Rule of Exponents: For any real number a and natural numbers n, the negative rule of exponents states that

$$\frac{1}{a^n} = a^{-n} \tag{4.6}$$

Rational Exponents: Rational exponents are another way to express principal nth roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is

$$\sqrt[n]{a^m} = a^{\frac{m}{n}} \tag{4.7}$$

The Quotient Rule of Exponents: For any real number a and natural numbers m and n, such that $m > n$, the quotient rule of exponents states that

$$\frac{a^m}{a^n} = a^{m-n} \tag{4.8}$$

The Product Rule of Exponents: For any real number a and natural numbers m and n, the product rule of exponents states that

$$a^m \cdot a^n = a^{m+n} \tag{4.9}$$

The Power Rule of Exponents: For any real number a and natural numbers m and n, the power rule of exponents states that

$$(a^m)^n = a^{m \cdot n} \tag{4.10}$$

Example 4.1. Given $y = \dfrac{2}{x^2}$, find $\dfrac{dy}{dx}$.

Example 4.2. Given $f(x) = \dfrac{3x^5 - 4x^4 - 6x^3}{x^4}$, find $f'(x)$.

Example 4.3. Given $f(x) = 2\sqrt{x} + 3x$, find $f'(x)$. Then, evaluate $f'(1)$

Finding the Equation of a Tangent Line

In section 2, we first discussed about the derivative as the slope of the line tangent to a graph. We will revisit this concept again with using the basic differentiation rules instead of using limits.

Example 4.4. Given $f(x) = \dfrac{3}{x^2} + x$,

(1) Using the differentiation rule(s), find $f'(x)$.

(2) Using the derivative of f, find the slope of the line tangent to the graph of f at the point $(-1, 2)$.

(3) Find the equation of the line tangent to the graph of f. Use the point-slope form: $y = mx + b$.

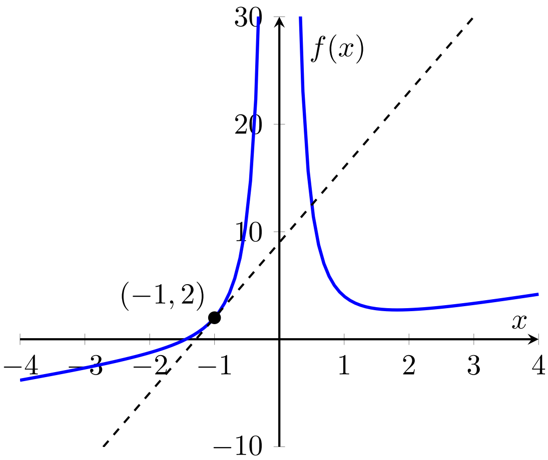

Figure 4.1: Graphing can verify the line found in part (3) is indeed tangent to the graph of f.

Example 4.5. Given $f(x) = 2x^3 + 6x^2 - 144x + 18$, find $f'(x)$ **using the differentiation rules.** Then determine **all points** for which the slope of the tangent line is zero (horizontal line). Use a graphing calculator to verify your answers by graphing the function f and determine if the lines tangent to these points are actually zeros.[4]

[4]**Desmos** online graphing calculator or **Geogebra** online graphing calculator are strongly recommended. They are available online for free from https://www.desmos.com/calculator and https://www.geogebra.org/graphing?lang=en. You may also download their apps on your smartphone for free.

[4]To graph the function in Desmos graphing calculator, type $f(x) = 2x^3 + 6x^2 - 144x + 18$ and then click the wrench icon on the top right to adjust the x-axis and y-axis settings. Change the default settings to $-15 \le x \le 15$ and $-4000 \le y \le 4000$. You may find the following tutorial videos helpful for basic graphing in Desmos: https://youtu.be/7oVOs9TX57s and https://youtu.be/En_PkyA-4_4

Marginal Analysis

In economics, production processes are often analyzed by using cost (C), revenue (R), and profit (P) functions. Each function uses the production level, x, as the independent variable. A question of interest is finding the approximate additional cost, revenue, or profit that results when the production level is increased by one unit above the current level. From the previous result, we can **approximate** (or **predict**) the change in these functions by using their derivatives . For example, $C'(a)$ is used to approximate the change in total cost when the production level is increased from $x = a$ to $x = a + 1$: $C'(a) \approx C(a + 1) - C(a)$. In this context, $C'(a)$ is called the "marginal cost at production level a", or the "**marginal cost** of producing a units". It is interpreted as the **additional cost** of the *"next unit"* of production. Similar interpretations are given to $R'(a)$ and $P'(a)$. In general, the marginal cost, revenue, and profit functions are denoted by $C'(a)$, $R'(a)$, and $P'(a)$.

Example 4.6. Suppose a manufacturer has determined that the monthly profit associated with producing x gallons of pesticide may be modeled by the function $P(x) = -.01x^2 + 6x - 100$ dollars, $x \geq 0$.

(1) Find the marginal profit function.

(2) Find the marginal profit when production is $x = 100$ gallons and interpret the result. Compare this **approximate** change in profit to the **actual** change when increasing production from 100 to 101 gallons.

(3) Using the concept of **marginal analysis**, by how much would you **predict** the profit will change if the production is increased from $x = 400$ gallons to $x = 401$? Compare this prediction to the actual change in profit when increasing production from 400 to 401 gallons.‘

(4) Determine the production level for which the marginal profit is zero and interpret the result.

(5) Note that the profit function is a quadratic function. Use the concept of a vertex of a parabola to determine the production level for which the profit is the maximum. Compare this result to part (4) and then interpret graphically.

Short Answers to Examples

4.1 $\dfrac{dy}{dx} = -\dfrac{4}{x^3}$

4.2 $f'(x) = 3 + \dfrac{6}{x^2}$

4.3 $f'(x) = \dfrac{1}{\sqrt{x}} + 3$; 3

4.4 (1) $f'(x) = \dfrac{-6}{x^3} + 1$ (2) $m = 7$ (3) $y = 7x + 9$

4.5 (-6,666) and (4,-334)

4.6 (1) $P'(x) = -.02x^2 + 6$ (2) \$4 per gallon vs.\$3.99 (3) -\$4 per gallon vs.-\$3.99 (4) $x = 300$ gallons

Lesson No. 5

Chain Rule and General Power Rule

Objectives

- State the chain rule for the composition of two functions.
- Be able to differentiate a composite function using the *Chain Rule* together with *Power Rule*, so called *General Power Rule*.
- Apply the *General Power Rule* with the *elasticity of demand* in Economics.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 2.5 Chain Rule*
    - Skip the following examples: 4, 7 and 8.[2]
  - Section 2.10 Other Applications
    - Elasticity
- Strang and Herman, *Calculus Volumn 1*[3,4]
  - *Section 3.6 The Chain Rule*
    - The Chain and Power Rules Combined
      - Example 3.48 and Example 3.50
    - The Chain Rule Using Leibniz's Notation
      - Example 3.58
- Abramson, *College Algebra*[5]
  - *Review* section 3.4 Composition of Functions

Key Terms and Concepts:

- Composition of functions
- Chain Rule
- General Power Rule
- Elasticity of Demand

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[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .

[2] Disregard any examples with exponential functions and logarithmic functions. We will discuss the General Power Rule together with these functions in the future lessons.

[3] Available free to download from https://openstax.org/details/books/calculus-volume-1 .

[4] Disregard any examples with trigonometry.

[5] Available free to download from https://openstax.org/details/books/college-algebra .

We have learned how to apply the basic differentiation rules on basic functions in section 4. In this section, we will discuss how to differentiate *composition of functions*[1] by using the *Chain Rule* together with the *Power Rule*. These two rules together are so called the *General Power Rule*. We will then apply the rule with the *elasticity of demand* in Economics.

The Chain Rule

Example 5.1. Given $g(x) = (4x^3 + 15x)^2$, find $g'(x)$ using the basic differentiation rules in section 4.

Solution:

This is not a simple polynomial, so we can't use the basic rules yet. It is a product, so we could first "multiply it out" and then use the basic rule(s) to find the answer.

$g(x) = (4x^3 + 15x)^2 = (4x^3 + 15x)(4x^3 + 15x) = 16x^6 + 120x^4 + 225x^2$

$g'(x) =$ ____

Suppose we want to find the derivative of $h(x) = (4x^3 + 15x)^{20}$. Similar to Example 5.1, we could write it as a product with 20 factors and use the product rule, or we could use the *Chain Rule* together with the *Power Rule*, but how?

The *Chain Rule* is the most common place for students to make mistakes. Part of the reason is that the notation takes a little getting used to. And part of the reason is that students often forget to use it when they should. Fortunately, with some practice, the Chain Rule is also easy to use. When should you use the Chain Rule? Almost every time you take a derivative. You will need the Chain Rule hundreds of times in this course. It is a powerful tool that leads to important applications in a variety of field.

The Chain Rule

Let $f(x)$ and $g(x)$ be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the *composite function*

$$h(x) = (f \circ g)(x) = f(g(x)) \tag{5.1}$$

is given by

$$h'(x) = f'(g(x)) \cdot g'(x) \tag{5.2}$$

In other words, the derivative of a composition of a function (denoted as $(f \circ g)'(x)$ is the derivative of the outside function f (with respect to the original inside function) *times* the derivative of the inside function g.

[1]You may review this topic from section 3.4 on Abramson, *College Algebra* which is available free to download from https://openstax.org/details/books/college-algebra

Steps to Apply The Chain Rule

1. To differentiate $h(x) = f(g(x))$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'(g(x))$.
3. Find $g'(x)$.
4. Write $h'(x) = f'(g(x)) \cdot g'(x)$

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

Example 5.2. Again, let's consider the function $h(x) = (4x^3 + 15x)^{20}$. Following the *Steps to Apply The Chain Rule*, find the derivative of $h(x)$.

1. Identifying $f(x)$ and $g(x)$:

   Outside Function: $f(x) =$ ______________________________

   Inside Function: $g(x) =$ ______________________________

2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'(g(x))$.

3. Find $g'(x)$.

4. Write $h'(x) = f'(g(x)) \cdot g'(x)$

Example 5.3. Given $h(x) = \sqrt{x^2 + 2}$, find $h'(x)$ by following the four Steps to Apply The Chain Rule. Then, determine all values of x for which the slope of the tangent line is zero (horizontal line).

As you can see from Example 5.2 and 5.3, the *Chain Rule* provides an easier way to differentiate a more complicated function like the composition in the examples. The *Chain Rule* is a little complicated, but it saves us the much more complicated algebra of multiplying something like ones in these examples. It will also handle compositions where it would not be possible to "multiply it out."

The Chain and Power Rules Combined[1]

We can now apply the chain rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h(x) = [g(x)]^n$, we need to use the chain rule combined with the power rule. To do so, we can think of $h(x) = [g(x)]^n$ as $f(g(x))$ where $f(x) = x^n$. Then $f'(x) = n \cdot x^{n-1}$. Thus, $f'(g(x)) = n \cdot [g(x)]^{n-1}$. This leads us to the derivative of a *power function* using the *chain rule*, so called *the General Power Rule*.

The General Power Rule

For all values of x for which the derivative is defined, if

$$h(x) = [g(x)]^n \tag{5.3}$$

Then

$$h'(g(x)) = n \cdot [g(x)]^{n-1} g'(x) \tag{5.4}$$

Example 5.4. Again, let's consider the function $h(x) = (4x^3+15x)^{20}$. Using the General Power Rule in equation 5.4, find the derivative of $h(x)$. Which method do you prefer? Using the Chain Rule in Example 5.2 or using the General Power Rule in this example?

Example 5.5. Given $f(x) = x^2$ and $g(x) = \dfrac{1}{x-1}$, let $h(x) = (f \circ g)(x)$. Find $h(x)$ and $h'(x)$. Then, find the slope of the tangent line to the graph of h at $x = 0$.

[1] In the future lessons, we will further examine how to combine the chain rule with the other rules we have not learned. In particular, we can use it with the *product rule* or with the *quotient rule* which will be discussed in lesson 11

Example 5.6. Given $f(x) = x^2$ and $g(x) = \dfrac{1}{x-1}$, let $w(x) = (g \circ f)(x)$. Find $w(x)$ and $w'(x)$. Then, find the equation of the tangent line to the graph of w at point $(0, -1)$. What is the slope of the tangent line?

Elasticity of Demand

A word on notation is often more confusing than the concepts themselves. In applications, we often try to give names to the variables that are intuitively related to the quantities of interest as opposed to always using x and y.

Elasticity of Demand is a measure of how demand reacts to price changes. The calculation of *Elasticity of Demand* requires a demand function which is expressed the demand of a product or service as a function of its price and other factors (e.g. substitutes and complementary goods etc.)

In example 2 and example 3 from section 2.10 by Calaway, Hoffman, and Lippman, *Applied Calculus*, the notation q representing the demand quantity is used as the *dependent variable*. The notation p representing the price is used as the *independent variable*. As such, the demand function is written as $q(p)$. We would then express the derivative as $q'(p)$ or[1] $\dfrac{dq}{dp}$. Of course, the rules for finding derivatives are the same and not dependent on the choice of variables.

The calculation and the interpretation of the *Elasticity of Demand* are explained in great details in section 2.10 from Calaway, Hoffman, and Lippman, *Applied Calculus*[2].

Example 5.7. See Question 6 (last question) from Homework Assignment 2: Lesson 5.

Short Answers to Examples

5.1 $g'(x) = 96x^5 + 480x^3 + 450x$

5.2 $h'(x) = 20(4x^3 + 15x)^{19} \cdot (12x^2 + 15)$

5.3 $h'(x) = \dfrac{x}{\sqrt{x^2+2}}$; $x = 0$

5.4 $h'(x) = 20(4x^3 + 15x)^{19} \cdot (12x^2 + 15)$

5.5 $h(x) = \dfrac{1}{(x-1)^2}$; $h'(x) = \dfrac{-2}{(x-1)^3}$; $m = 2$

5.6 $w(x) = \dfrac{1}{x^2-1}$; $w'(x) = \dfrac{-2x}{(x^2-1)^2}$; $y = -1$; $m = 0$

5.7 0.4091; Inelastic ; Raise Prices

[1] The *Leibniz* notation form: $\dfrac{dp}{dq}$ (read aloud as *"dee que dee pee"*) is used in the *Elasticity of Demand* formula in Calaway, Hoffman, and Lippman, *Applied Calculus*

[2] Available free to download from `http://www.opentextbookstore.com/details.php?id=14` .

Lesson No. 6

Derivatives as Rates of Change

Objectives

- Understand the interpretation of the derivative of a function as the **rate of change** (*instantaneous*) of the function
- Understand the concept of instantaneous rate of change of a function.
- Understand how the rate of change of a function can be used in applied analysis.
- Understand the concept of "time rate of change" of a function (ex. profit)

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 2.2 The Derivative Definition of the Derivative*
    - The *Dropping Totmato* example and the *Growth Bacteria* example. Address the following concepts:
      - Average velocity vs. Instantaneous velocity
      - Secant line vs. tangent line

Key Terms and Concepts:

- Average rate of change
- Instantaneous velocity
- Application to position, velocity and acceleration

[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .

Average Rate of Change vs. Instantaneous Rate of Change

Recall that the slope of the tangent line to the graph of a function at a point is the **rate of change** of the function at that point. In general (graphs of lines being an exception), the slope of a graph will be changing as we move along the graph. We also know that the slope of the graph at a point x=a is given by $f'(a)$. The result is that we interpret the value of $f'(a)$ as the rate of change of the function at $x = a$.

Read the *Dropping Totmato* example and the *Growth Bacteria* example in section 2.2 from Calaway, Hoffman, and Lippman, *Applied Calculus*.

Example 6.1. Suppose a county has just begun development for large-scale oil extraction. The population, P, of the county (measured in thousands of people) at a point in time t years from the present is modeled by the function $P(t) = \sqrt{4t^2 + 5}$, $0 \le t \le 10$

(A) Find the **predicted rate of change** of the population at time $t = 1$ years from the present.

(B) By how many people would you expect the population to change over the year from $t = 1$ to $t = 2$? Note: $P'(1) \approx P(1 + 1) - P(1)$ or equivalently $P(2) \approx P(1) + P'(1)$. Compare this approach to the exact value of $P(2)$.

Result: We interpret $P'(t)$ as the expected change in the population if time is increased by 1 year.

Example 6.2. After toxic levels of radon gas are detected in a suburban area, the population, P, is to be predicted using the function $P(t) = 5000 + \dfrac{2000}{(t+1)}$ people, $t \ge 0$, where t is the number of years from the present ($t = 0$).

(A) What is the **average** rate of change of the population over the time interval $t = 1$ to $t = 4$?

(B) What is the **average** rate of change of the population over the time interval $t = 1$ to $t = 1.5$?

(C) What is the **instantaneous** rate of change of the population at time $t = 1$?

(D) When is the population decreasing at a *faster* rate: At $t = 3$ years or $t = 5$ years?

Example 6.3. A ball is launched vertically upward from a building that is 100 meters tall with an initial velocity of 150 meters per second. The position/height, s, of the ball with respect to the ground at time t seconds is given by the function $s(t) = -16t^2 + 150t + 100$ meters, $t \geq 0$.

(A) What is the velocity of the ball at time $t = 0$? ?

(B) What is the velocity of the ball at time $t = 4$?

(C) What is the velocity of the ball equal to zero? (Rounded.)

(D) Is the distance from the ball to the ground *increasing* or *decreasing* at time $t = 0$ and $t = 4$? Why are the values for the velocities negative?

(E) When is the ball dropping at a faster rate: At what time ($t = 0$ or $t = 4$)? Why is there a difference in the velocities?

(F) With what velocity will the ball hit the ground?

The *Leibniz* Notation Form and Intuition of Chain Rule

Recall the Chain Rule in lesson 5: $h'(x) = f'(g(x)) \cdot g'(x)$ where $h(x) = (f \circ g)(x) = f(g(x))$. Now, consider a variable y that is a function of an independent variable $u : y = f(u)$. Also, the variable u is itself a function of another independent variable $x : u = g(x)$. Using the *Leibniz* notation form, $h'(x)$ can be written as $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Recall that the derivative of a function may be interpreted as the rate of change of the function, that is, it indicates the rate at which the value of the function is changing as the independent variable changes. Now consider the "flow of information": An input, x, to the function g determines an output u; this value of u then becomes an input to the function f, which results in an output y. As such, we see that y is ultimately a function of x (think of compositions). What is now of interest is to find the **rate of change** of y with respect to x , which we recognize as the concept of the derivative of a function. The result is given by: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

Example 6.4. The daily demand for diesel fuel (measured in 50-gallon barrels) in a city is expected to increase over time during the summer months. The demand, x, at a time t days from the present may be modeled by the function $x(t) = .5t^2 + 900$ barrels; $0 \leq t \leq 90$. At the local refinery, the profit, P, (measured in thousands of dollars) when the demand for diesel fuel is x barrels is given by $P(x) = \sqrt{(x + 1400)}$ thousands of dollars;$x \geq 1$.

(A) Find the **marginal profit** function, $\dfrac{dP}{dx}$. Include the units.

(B) Write the profit function P as a function of time, t using the composition of functions $P(x)$ and $x(t)$.

(C) Find the "**time rate of change of profit**", $\dfrac{dP}{dt}$ by taking the derivative of the function $P(t)$ with respect to t. Include the appropiate units.

(D) Again, find the "**time rate of change of profit**", $\dfrac{dP}{dt}$ by using the *Leibniz* notation form as described above this example, $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$. Include the appropiate units. Compare the result to one in the previous part.

(E) Find the rate at which the refinery's profit will be changing at time $t = 20$ days.

Short Answers to Examples

6.1 (A) $P'(1) = \frac{4}{3}$ thousands of people per year. (B) Expect an increase of approximately 1333 people over the next year.

6.2 (A) -200 people per year (B) -400 people per year (C) -500 people per year (D) $t = 3$ years; the rate of decrease is *"slowing down"* as time goes on.

6.3 (A) 150 meters per second (B) 22 meters per second (C) $t \approx 4.6875$ seconds (D) *Decreasing*; the distance from the ball to the ground keeps *decreasing.*(E) At $t = 4$; the ball is accelerating due to gravity (F) -170 meters per second

6.4 (A) $\dfrac{dP}{dx} = \dfrac{1}{2\sqrt{x + 1400}}$ thousands of dollars per barrel. (B) $P(t) = (P \circ x)(t) = P(x(t)) = \sqrt{0.5t^2 + 2300}$ (C) $\dfrac{dP}{dx} = \dfrac{1}{(2\sqrt{(0.5t^2 + 2300)}}$ (D) same as part (C) (E) 0.2 thousands of dollars per day

Lesson No. 7

Second Derivative and Concavity

Objectives

- Be able to compute the 2^{nd} derivative of a function using the basic rules
- Be able to interpret second derivatives in some real-world applications.
- Be able to interpret the second derivative in terms of the change in the (first) derivative or the change the slope of a curve.
- Be able to identify the sign of the (first) derivative the sign of the second derivative from the graph or the table of a function.
- Be able to determine concavity and find the inflection points from the graph of a function.
- Be able to determine concavity and find the inflection points given a function.
- Be able to determine the point of diminishing returns as an inflection point.
- Be able to interpret and apply the point of diminishing returns in the business context.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 2.6 Second Derivative and Concavity*
- Hoffman, *Contemporary Calculus I*[2]
  - *Section 3.4: Second Derivative and the shape of* f)
    - Skip f'' *and Extreme Values of* f.

Key Terms and Concepts:

- 1^{st} derivative vs. 2^{nd} derivative
- Increasing and decreasing functions
- Concave up vs. Concave down
- Inflection points
- Velocity vs. Acceleration
- Point of Diminishing Return

[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2] Available free to download from https://www.opentextbookstore.com/details.php?id=11#tabs-3.

Second Derivatives

Next, we consider the concept of the 2^{nd} derivative of a function. In short, for a given function $f(x)$, the 2^{nd} derivative, denoted by $f''(x)$, is found by determining the derivative of $f'(x)$.

Example 7.1. Given $f(x) = \frac{1}{x} + x^2$, find $f''(x)$.

Example 7.2. Given $f(x) = \frac{2}{x-3}$, find $f''(x)$.

The (*first*) derivative of a function f is a function that gives information about the slope of f. **The (*first*) derivative tells us if the original function is increasing or decreasing**.

Not all increasing (or decreasing) functions behave in the same manner. The second derivative allows us to examine in more detail how a function increases or decreases in an interval. Specifically, **the second derivative, tells us how fast the original function is increasing (or decreasing)**. See Example 7.3.

Example 7.3. Let $s(t) = 6t^3 - 81t^2 + 360t$ be the equation of the height (in feet) of a particle at time t seconds.

(a) Find a function for the **velocity**: $v(t) =$_______________.

(b) Find a function for the **acceleration** of the particle: $a(t) =$_______________.

Concavity

The second derivative is important when we analyze graphs of functions and the behavior of functions. It gives us a mathematical way to tell how the graph of a function is curved. **The second derivative tells us if the original function is concave up or down.**

Graphically, a function is **concave up** if its graph is curved with the **opening upward** (a in the figure). Similarly, a function is **concave down** if its graph **opens downward**. Notice that a function can be concave up regardless of whether it is increasing or decreasing.

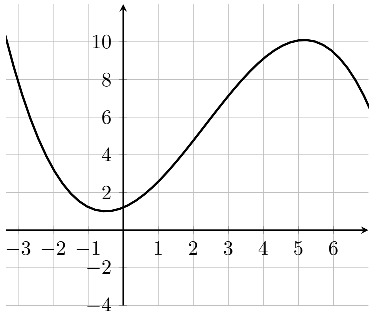


Figure 7.1

Example 7.4. Given the graph of a function in figure 7.1, determine the concavity on each interval below.
$(-\infty, 2)$:__________ *(concave up/concave down).*

$(2, \infty)$:__________ *(concave up/concave down).*

The Second Derivative Condition for Concavity

(a) If $f''(x) > 0$ on an interval I, then $f'(x)$ is increasing on I and f is concave up on I.

- $f'(x)$ is *increasing* on I means f is increasing (or decreasing) **at an increasing rate**.

(b) If $f''(x) < 0$ on an interval I, then $f'(x)$ is decreasing on I and f is concave down on I.

- $f'(x)$ is decreasing on I means f is increasing (or decreasing) **at an decreasing rate**.

(c) If $f''(x) = 0$, then $f(x)$ may be concave up or concave down or neither at a.

Example 7.5. For each of four tables in table 7.1, identify if the table is increasing/decreasing and concave up/down

x	$f(x)$
1	3
2	12
3	27
4	48
5	75
6	108

(a)

x	$g(x)$
1	108
2	75
3	48
4	27
5	12
6	3

(b)

x	$h(x)$
1	92
2	125
3	152
4	173
5	188
6	197

(c)

x	$k(x)$
1	197
2	188
3	173
4	152
5	125
6	92

(d)

Table 7.1

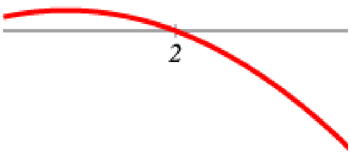


Figure 7.2

Example 7.6. Determine the signs (*positive, negative,* or *zero*) of $y = f(x)$ (shown in figure 7.2) and $f'(x)$ and $f''(x)$ when $x = 2$.

(a) The sign of $f(2)$ is _____.

(b) The sign of $f'(2)$ is _____.

(c) The sign of $f''(2)$ is _____.

Example 7.7. Given the following statements, answer the questions below.

Let $x = a$ be on an interval I.

1. $h''(a) > 0$ ($h''(a)$ is positive.)
2. $h''(a) < 0$ ($h''(a)$ is negative.)
3. $h'(a) > 0$ ($h'(a)$ is positive.)
4. $h'(a) < 0$ ($h'(a)$ is negative.)
5. $h'(a)$ is increasing.
6. $h'(a)$ is decreasing.
7. The slope of a line tangent to the graph of h is decreasing on an interval I.
8. The slope of a line tangent to the graph of h is increasing on an interval I .
9. The slope of a line tangent to the graph of h at $x = a$ is positive.
10. The slope of a line tangent to the graph of h at $x = a$ is negative.

(a) Which of the statements given above are equivalent to the statement that $h(x)$ is **increasing** at $x = a$? Choose all that apply.

(b) Which of the statements given above are equivalent to the statement that $h(x)$ is **decreasing** at $x = a$? Choose all that apply.

(c) Which of the statements given above are equivalent to the statement that $h(x)$ is **concave up** on an interval I? Choose all that apply.

(d) Which of the statements given above are equivalent to the statement that $h(x)$ is **concave down** on an interval I? Choose all that apply.

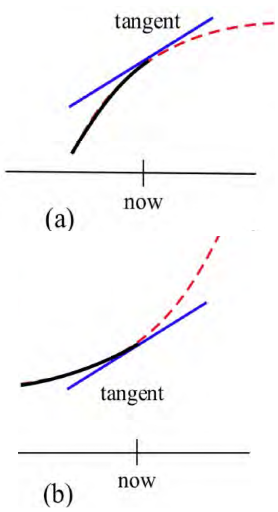


Figure 7.3

Example 7.8. An Epidemic: Suppose an epidemic has started, and you, as a member of congress, must decide whether the current methods are effectively fighting the spread of the disease or whether more drastic measures and more money are needed. In the figure 7.3, $f(x)$ is the number of people who have the disease at time x, and two different situations are shown. In both (a) and (b), the number of people with the disease, $f(now)$, and the rate at which new people are getting sick, $f'(now)$, are the same. The difference in the two situations is the concavity of f, and that difference in concavity might have a big effect on your decision.

In (a), f is concave __________(*up or down*) at "now", the slopes are __________________ (*increasing or decreasing*), and it looks as if it's tailing off. We can say "f is increasing at a __________(*increasing rate or decreasing rate*)". Does it appear that the current methods are starting to bring the epidemic under control? __________(*Yes or No*)

In (b), f is concave __________(*up or down*) at "now", the slopes are __________________ (*increasing or decreasing*). We can say "f is increasing at a __________(*increasing rate or decreasing rate*)". Does it appear that the current methods are starting to bring the epidemic under control? __________(*Yes or No*)

Inflection Points

Definition

An **inflection point** is a point on the graph at which the function is *continuous* and the concavity of the graph changes from concave up to down or from concave down to up.

Example 7.9. Let $f(x) = x^3$, $g(x) = x^4$ and $h(x) = x^{1/3}$ For which of these functions is the point (0,0) an inflection point? Use figure 7.5 and fill in table 7.2 to help you answer the question.

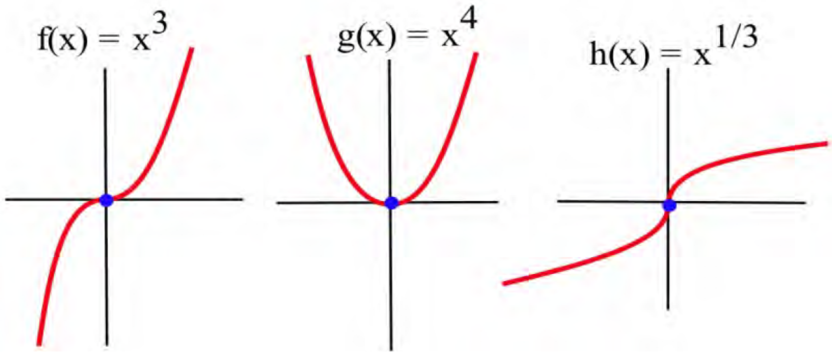


Figure 7.4

	$f(x) = x^3$	$f(x) = x^4$	$f(x) = x^{1/3}$
Concave up/down on $(-\infty, 0)$			
Concave up/down on $(0, \infty)$			
Change in concavity (yes/no)			
Inflection point (yes/no)			

Table 7.2

Example 7.10. Which of the labeled points in the graph below are inflection points?

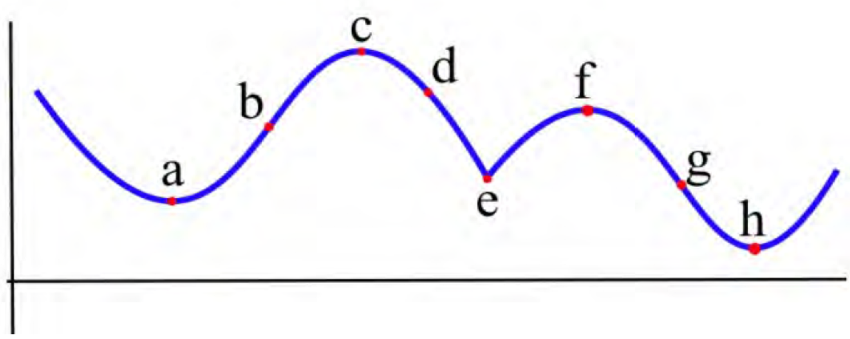


Figure 7.5

Concavity Test: Determine Concavity and Inflection Values

1. Find the values $x = c$ where $f''(c) = 0$ or $f"(c)$ does not exist.
2. Place these values of a number line and use the second derivative to generate a sign chart.
3. The point $(c, f(c))$ is an inflection point if $f''(x)$ changes sign at $x = c$ and if $x = c$ is in the domain of $f(x)$.

Example 7.11. Consider $f(x) = x^4 - 12x^3 + 30x^2 + 5x - 7$. Give the intervals where $f(x)$ is concave up or down and find the x-values of any inflection points. *Check your answer with the graph of f on a graphing calculator.*[1]

Example 7.12. Consider $f(x) = \dfrac{1}{x-1}$.Give the intervals where $f(x)$ is concave up or down and find the x-values of any inflection points. *Check your answer with the graph of f on a graphing calculator.*[1]

Result: The concavity of a graph may change at a value of x for which the function is not defined (not in the domain). As such, this is not considered an inflection point.

[1] Access Desmos Graphing Calcuator from https://www.desmos.com/calculator. Access Geogebra Graphing Calculator from https://www.geogebra.org/graphing. You may find the following tutorial videos helpful for basic graphing in Desmos: https://youtu.be/7oVOs9TX57s and https://youtu.be/En_PkyA-4_4

Point of Diminishing Return

Law of Diminishing Returns[1]

In economics the **law of diminishing returns** says that anytime you increase one factor of production (e.g. employees, machinery, fertilizer, etc) while keeping all other factors of production constant, eventually you will hit a **point of diminishing returns**,where the incremental per-unit returns begin to drop.

Example 7.13. A company estimates that it will have the revenue R(x) from sales after spending x on advertising their product, as given by

$$D(x) = \frac{1}{10000}(300x^2 - x^3), \quad 0 \leq x \leq 200$$

where the revenue $R(x)$ and the advertising cost x are both measured in million of Euros. Given the graph of $R(x)$ shown in figure 7.6 , answer the following questions.

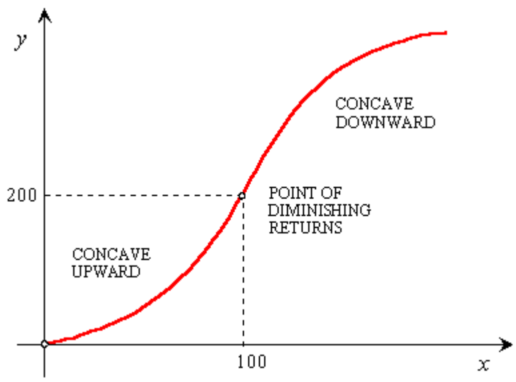


Figure 7.6

(a) Using the *Concavity Test*, show that the **point of diminishing returns** is the inflection point of R.

[1]https://math.mit.edu/seminars/esme/misc/2017-11-21$_{Gerhardt_{B}usiness - Lab.pdf}$

(b) **Fill in the blanks:** In the interval $(0, 100)$, each additional Euro invested in advertising increases sale __________ (*more or less*) than the previous Euro invested in advertising. The is because the slope of the curve is __________ (*increasing or decreasing*) on $(0, 100)$.

(c) **Fill in the blanks:** In the interval $(100, 200)$, the slope of the curve is __________ (*increasing or decreasing*) and each additional Euro invested in advertising increases sale __________ (*more or less*) than the previous Euro invested in advertising.

(d) If the company currently put 100 millions of Euros in advertising, would you recommend the company to invest more on the advertising? Why or why not?

Short Answers to Examples

7.1 $f''(x) = \dfrac{2}{x^3} + 2$

7.2 $f''(x) = \dfrac{4}{(x-3)^3}$

7.3 (a) $v(t) = 18t^2 - 162t + 360$

(b) $a(t) = 36t - 162$

7.4 concave up on $(-\infty, 2)$; concave down on $(2, \infty)$

7.5 (a) increasing; concave up (b) decreasing; concave up (c) increasing; concave down (d) decreasing; concave down

7.6 (a) zero

(b) negative

(c) negative

7.7 (a) 3,9 (b) 4,10 (c) 1,5,8 (d) 2,6,7

7.8 (a) down;decreasing;decreasing rate;yes. (b) up;increasing;increasing rate;no.

7.9 yes;no;yes

7.10 b and g

7.11 concave up on $(-\infty, 1)$; concave down on $(1, 5)$; concave up on $(5, \infty)$

7.12 concave down on $(-\infty, 1)$; concave up on $(1, \infty)$; no inflection point.

7.13 (a) concave up on $(-\infty, 100)$; concave down on $(100, \infty)$; Change in concavity at $x = 100 \Longrightarrow$ inflection point at $x = 100$. (b) more;increasing (c) decreasing;less (d) no.

Lesson No. 8

Optimization

Objectives

- Define absolute extrema and local extrema.
- Use first derivative test and second derivative test to find local extrema.
- Locate absolute extrema over a closed interval.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 2.7 Optimization*

Key Terms and Concepts:

- Partition Numbers vs. Critical Numbers
- Sign Chart
- First Derivative Test
- Second Derivative Test
- Local Maximum and Local Minimum
- Absolute Maximum and Absolute Minimum

[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .

Without calculus, we only know how to find the optimum points in a few specific examples (for example, we know how to find the vertex of a parabola). But what if we need to optimize an unfamiliar function?
The best way we have without calculus is to examine the graph of the function, perhaps using technology. But our view depends on the viewing window we choose – we might miss something important. In addition, we'll probably only get an approximation this way. (In some cases, that will be good enough.)
Calculus provides ways of drastically narrowing the number of points we need to examine to find the exact locations of maximums and minimums, while at the same time ensuring that we haven't missed anything important.
Before we examine how calculus can help us find maximums and minimums, we need to be familiar with the vocabulary and the concepts we will develop and use to describe **behavior of a function**.

Increasing and Decreasing Functions in an interval

Simply put, if there is some interval of values of x for which the value of a function is getting larger as we move from left to right, we say that the function is increasing in that interval. A decreasing function is defined similarly, with the value of the function becoming smaller vs. larger.

For a function f which is *differentiable*[1] on an interval I:

1. if $f'(x) > 0$ for all x in the interval I, then f is **increasing** on I.
2. if $f'(x) < 0$ for all x in the interval I, then f is **decreasing** on I.
3. if $f'(x) = 0$ for all x in the interval I, then f is **constant** on I.

The derivative of a function tells about the general shape of the function, and we can use that shape information to determine if an extreme point is a maximum or minimum or neither (See Figure 8.1).

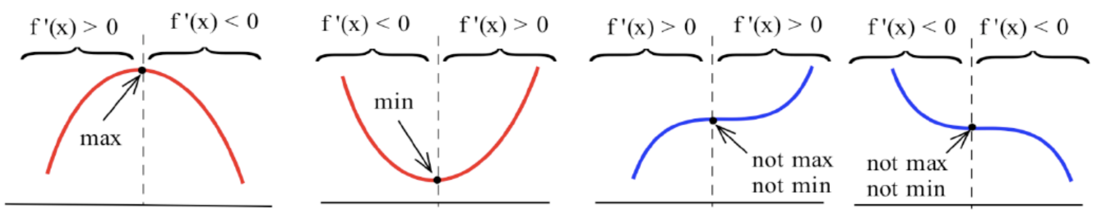
[2]


Figure 8.1

[1]See Lesson 3: Differentiability and Continuity
[2]From Hoffman, *Contemporary Calculus I* ; page 88

Extreme Points

Local Extreme Points

- A **local maximum** point is a point on the graph of the function at which the function changes from an increasing function to a decreasing function. A point is a local maximum if it is higher than all the **nearby points**. Formally, we say
  - f has a **local maximum** at c if $f(c) \geq f(x)$ for all x near c.
- A **local minimum** point is a point on the graph of the function at which the function changes from a decreasing function to an increasing function. A point is a local minimum if it is lower than all the **nearby points**. Formally, we say
  - f has a **local minimum** at c if $f(c) \leq f(x)$ for all x near c.

Notes:

- These points are also referred to as "relative" maximum and minimum points.
- f has a **local extremum** at c if $f(c)$ is a local maximum or minimum. The plural of local extremum is **local extrema**.
- The plurals of these are **maxima** and **minima**. We often simply say **"max"** or **"min"**.
- The process of finding *maxima* or *minima* is called **optimization**.

Global Extreme Points

- A **global maximum** point is the point at which the largest value of the function occurs.Formally, we say
  - f has a **global maximum** at c if $f(c) \geq f(x)$ for all x in the domain of f.
- A **global minimum** point is the point at which the smallest value of the function occurs. Formally, we say
  - f has a **global minimum** at c if $f(c) \leq f(x)$ for all x in the domain of f.

Notes:

- These points are also referred to as "absolute" maximum and minimum points.
- Every global extreme is also a local extreme but there are local extremes that are NOT global extremes.

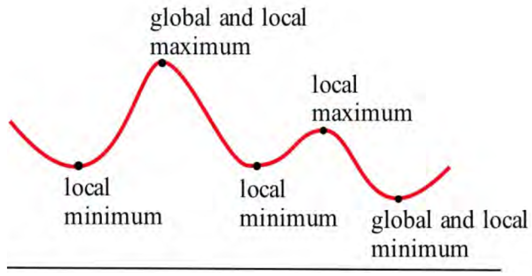


Figure 8.2

Finding Maxima and Minima of a Function

A Partion Number, Critical Number and A Critical Point

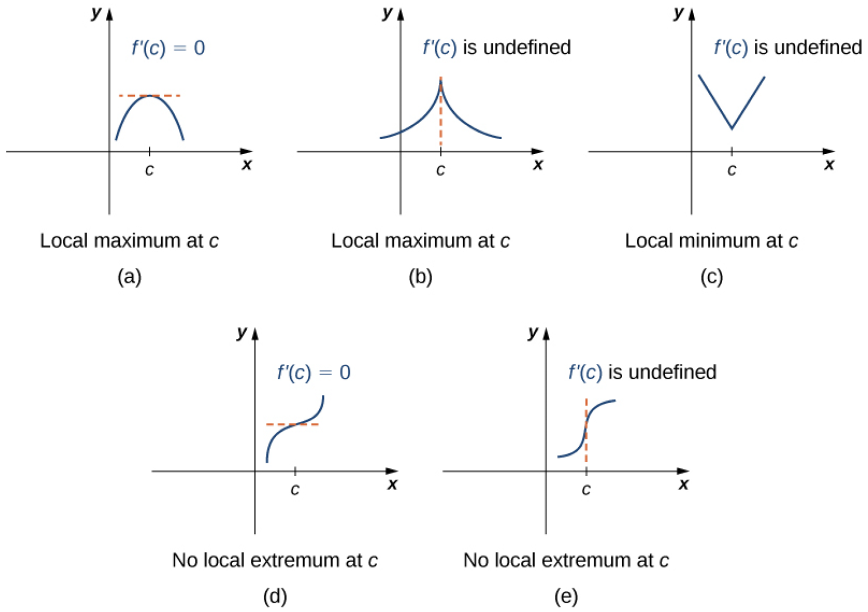
- A **partition number** of f' is a number p for which either $f'(p) = 0$ or $f'(p)$ is undefined.
- A **critical number** of a function f is a number c for which either $f'(c) = 0$ or $f'(c)$ is undefined AND c is in the domain of f.
- A **critical point** of a function f is a point $(c, f(c))$ where c is a critical number of f.
  - See the point marked by an arrow in figure 8.1, where $f'(x) = 0$ and $f'(x)$ is undefined.

Notes:

- Critical numbers are also partition numbers, but partition numbers may not be critical numbers.
- A local max or min of f can only occur at a critical point.
- The critical numbers only give the **possible** locations of extremes, and some critical numbers are not the locations of extremes.
- If the function has only one critical point and it's a local max (or min), then it must be the global max (or min).

Figure 8.3

[1]From Strang and Herman, *Calculus Volumn 1*; section 4.3 Maxima and Minima; Figure 4.15.



1

Figure 8.4: Different possibilities for critical points and corresponding local extrema.

The First Derivative Test for Extremes:

Suppose that c is a **critical number** of a continuous function f. For each critical number c, examine the sign of f' to the left and to the right of c. What happens to the sign as you move from left to right?

(a) If $f'(x)$ changes from **positive to negative** at c, then f has a **local max** at $(c, f(c))$.

(b) If $f'(x)$ changes from **negative to positive** at c, then f has a **local min** at $(c, f(c))$.

(c) If $f'(x)$ **does not change sign** at c, then $(c, f(c))$ is **neither** a local max nor a local min.

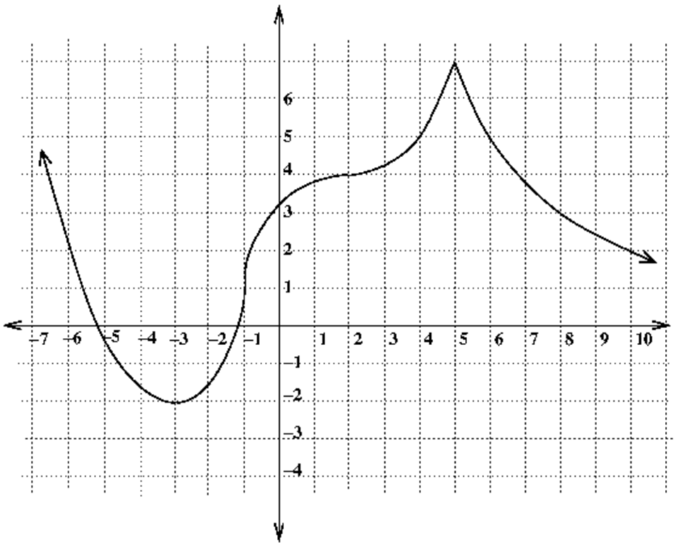
Steps for finding where f is Increasing/Decreasing or any Local Extrema

1. Find the **domain** of f.
2. Find all **partition numbers** of f'.
3. Find all **critical numbers** of f.
4. Make a **sign chart** to track where $f' > 0$ or $f' < 0$ by following these steps:
   (i) Plot all partition and critical numbers on a number line.
   (ii) Choose values to test regions on the number line around the critical/partition numbers.
   (iii) Plug test values into f' and record the sign $(\pm)$
5. Determine any **local extrema** using the *First Derivative Test for Extremes.*

Locating Global Extrema Over An Open Interval

If you are trying to find a global max or min on an open interval (or the whole real line), and there is more than one critical point, then you need to look at the graph to decide whether there is a global max or min. Be sure that all your critical points show in your graph, and that you go a little beyond – that will tell you what you want to know.

Example 8.1. Use the graph of the function $f(x)$ displayed below to answer the following questions.



(a) Find the critical values where $f'(x)$ does not exist.

(b) Find the critical values where $f'(x) = 0$.

(c) Find the $x-$coordinate(s) of the relative maxima for $f(x)$.

(d) Find the $x-$coordinate(s) of the relative minima for $f(x)$.

Example 8.2. Given the function $f(x) = \dfrac{x^2+3}{x-1}$ and its first derivative is shown below:

$$f'(x) = \frac{(x-3)(x+1)}{(x-1)^2}$$

Answer the following questions.

(a) Find the domain of f.

(b) Find the partition number(s) of f'.

(c) Find the critical number(s) of f.

(d) Make a sign chart for the function and use the chart to find the following:

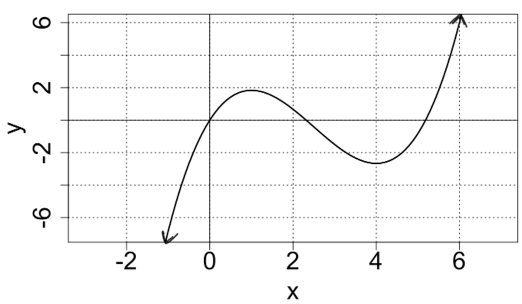
(i) all intervals on which $f(x)$ is increasing.

(ii) all intervals on which $f(x)$ is decreasing.

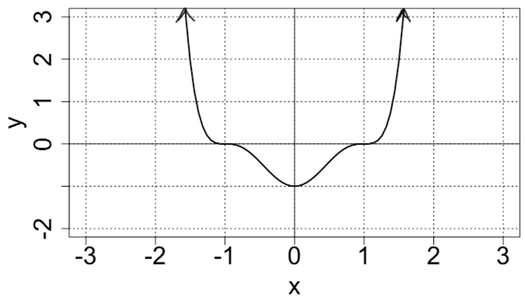
(iii) the $x-$coordinate(s) of all relative extrema on the graph of $f(x)$.

Example 8.3. For each of the following functions, determine all **local extrema** using the *First Derivative Test.* Follow the *Five Steps for finding where f is increasing/decreasing or any local extrema.* Then, use the given graph of each function to help determine any **global extrema**.

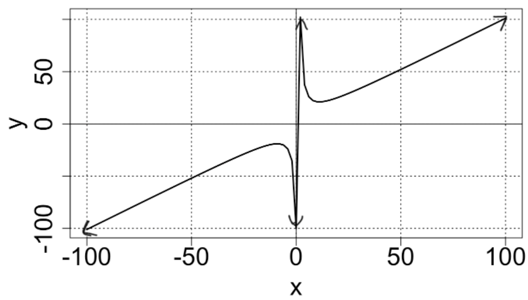
(a) $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x.$



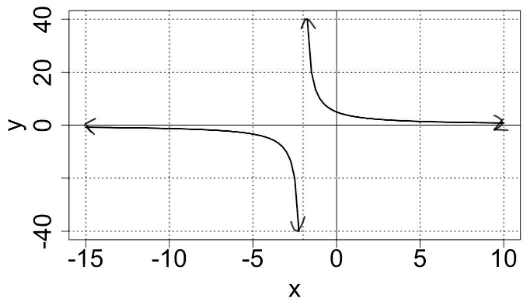
(b) $f(x) = (x^2 - 1)^3$.



(c) $f(x) = x + \dfrac{100}{x-1}$.



(d) $f(x) = \dfrac{10}{x+2}$.



Steps For Locating Global Extrema Over A Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.
3. Compare all values found in (1) and (2). Since the global extrema must occur at endpoints or critical points, the largest of these values is the global maximum of f. The smallest of these values is the global minimum of f.

Example 8.4. Given $f(x) = x^3 - 3x^2 - 9x + 5$ for $-2 \leq x \leq 6$, determine all **local extrema** using the *First Derivative Test.* Follow the *Five Steps for finding where f is increasing/decreasing or any local extrema.* Then, use the steps described above to locate any **global extrema**.

The concavity of a function can also help us determine whether a critical point is a maximum or minimum or neither. For example, if a point is at the bottom of a concave up function, then the point is a minimum (see figure 8.3).

The Second Derivative Test for Extremes:

1. Find the **domain** of f.
2. Find all **critical numbers** of f.
3. For those critical points where $f'(c) = 0$, find $f''(c)$.
4. Locate any **local extrema** by determining the sign of $f''(c)$ as follows:
   (a) If $f''(c) < 0$ then f is **concave down** and has a **local maximum** at $x = c$.
   (b) If $f''(c) > 0$ then f is **concave up** and has a **local minimum** at $x = c$.
   (c) If $f''(c) = 0$ then f may have a local maximum, a minimum or neither at $x = c$.

Example 8.5. Given $f(x) = x + \dfrac{100}{x-1}$, determine all **local extrema** by using the *Second Derivative Test for Extremes* as described above. You may use the *critical numbers* of f that were already found in part c of Example 8.3. Compare your results to the results from using *First Derivative Test for Extremes* in the example.

Short Answers to Examples

8.1 (a) $x = -1, 5$ (b) $x = -3, 2$ (c) $x = 5$ (d) $x = -3$

8.2 (a) $(-\infty, 1) \cup (1, \infty)$ (b) $x = -1; x = 1; x = 3$ (c) $x = -1; x = 3$ (d)(i) $(-\infty, -1) \cup (3, \infty)$ (d)(ii) $(-1, 1) \cup (1, 3)$ (d)(iii) local max at $x = -1$; local min at $x = 3$

8.3 (a) local max at $x = 1$;local min at $x = 4$;no global extrema. (b) only local (and global) minimum at $x = 0$. (c) local max at $x = -9$; local min at $x = 11$; no global extrema. (d) no extrema.

8.4 Local max of 10 at $x = -1$; local (and global) min of -22 at $x = 3$; global max of 59 when $x = 6$.

8.5 local max at $x = -9$; local min at $x = 11$; same results.

Lesson No. 9

Applied Optimization

Objectives

- Set up an objective equation and an constraint equation from real-world problems.
- Use calculus to find optimal values from the objective question and constraint equation in the context of real-world problems.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 2.9 Applied Optimization*
- Strang and Herman, *Calculus Volumn 1*[2,3]
  - *Section 4.7 Applied Optimization Problems*

Key Terms and Concepts:

- Objective Equation
- Constraint Equation

[1]Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2]Available free to download from https://openstax.org/details/books/calculus-volume-1 .
[3]Disregard any examples with trigonometry.

Like many concepts and techniques that you have been exposed to in math classes, derivatives also have importance in applied problems. We will consider examples of a certain type of application known as ”constrained optimization” problems. The common theme in these problems is that we will determine a function that we hope to maximize or minimize subject to a constraint that does not allow us to make the value of the function arbitrarily large or small.

Objective Function

The process of finding maxima or minima is called optimization. The function we are optimizing is called the **objective function**. The objective function can be recognized by its proximity to “est” words (greatest, least, highest, farthest, most, . . .)

Constraint Equation

In many cases, there are two (or more) variables in the problem. If there is an equation that relates the variables we can solve for one of them in terms of the others, and write the objective function as a function of just one variable. Equations that relate the variables in this way are called **constraint equations**. Include the appropriate units.

Solving Optimization Problems: Problem-Solving Strategy

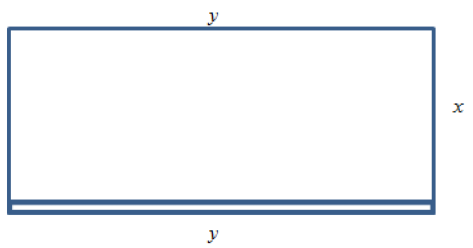
1. Introduce all **variables**. If applicable, draw a figure and label all variables.
2. Determine which **quantity is to be maximized or minimized**, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula **(objective equation)** for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations **(constraint equations)** relating the independent variables in the formula from step 3.
5. Solve the constraint equation for one variable and substitute into the objective function. Now you have **the objective equation of one variable**.
6. Identify the **domain** of consideration for the function in step 4 based on the physical problem to be solved.
7. Use calculus to find the maximum or minimum values. (Use either the First Derivative Test or the Second Derivative test to find **global extrema on the domain**.)
8. Look back at the question to make sure you answered what was asked. **Translate your number** answer back into English.

Example 9.1. Find two non-negative numbers whose sum is 44 and whose product is a minimum. Also, **find the minimum value** of the product of these numbers.

Follow the strategy for solving optimization problems. Identify the objective equation and the constraint equation.

Example 9.2. The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of \$7 per running foot. The fourth side will be built of cement blocks, at a cost of \$14 per running foot.

(a) **Find the dimensions** of the least costly such enclosure.
Follow the strategy for solving optimization problems. Identify the objective equation and the constraint equation.



(b) **Find the minimum cost** of the enclosure using the dimensions found in part (a).

Example 9.3. A concert promoter has found that if she sells tickets for \$50 each, she can sell 1200 tickets, but for each \$5 she raises the price, 50 less people attend. Answer the following questions.

(a) The problem provides information about the demand relationship between price $(\boldsymbol{p})$ and quantity $(\boldsymbol{q})$ - as price increases, demand decreases. Use the given information, complete the table below.

Price, p	50	55		
Quantity, q	1200	1150		

(b) Using the numbers provided in the table in part (a), find a formula for the relationship between price $(\boldsymbol{p})$ and quantity $(\boldsymbol{q})$. *Hint: Based on the numbers in the table, you might recognize this as a linear relationship. Recall that the point-slope formula for a linear relationship is given by* $y - y_1 = m(x - x_1)$

(c) Find a function that gives the revenue with the price (p) as the *independent variable*.

(d) What **price** should she sell the tickets at to maximize her revenue?
Follow the strategy for solving optimization problems. Identify the objective equation and the constraint equation.

(e) At the price found in the previous part, **how many tickets** will be sold?

(f) What is her **maximum revenue** from selling the tickets?

Profit has critical points when Marginal Revenue and Marginal Cost are equal.

Example 9.4. A company sells $\boldsymbol{q}$ ribbon winders per year at \$ **p** per ribbon winder. The demand function for ribbon winders is given by: $\mathbf{p = 300 - 0.02q}$. The ribbon winders cost \$30 apiece to manufacture, plus there are fixed costs of \$9000 per year. **Find the quantity** where profit is maximized.

(a) Find a function that gives the **revenue** with the quantity (q) as the *independent variable*.

(b) Find a function that gives the **cost** with the quantity (q) as the *independent variable*.

(c) Find a function that gives the **profit** with the quantity (q) as the *independent variable*.

(d) Find the **quantity** that maximizes the profit.
Follow the strategy for solving optimization problems. Identify the objective equation and the constraint equation.

(e) Find a function that gives the **marginal revenue**.

(f) Find a function that gives the **marginal cost**.

(g) Confirm that the marginal revenue and marginal cost are equal at your answer to part (d).

> Average Cost has critical points when Average Cost and Marginal Cost are equal.

Example 9.5. The total cost in dollars for Alicia to make $\boldsymbol{q}$ oven mitts is given by $\boldsymbol{C(q) = 0.01q^2 + 1.5q + 64}$.

(a) Based on the given cost function, what is the **fixed cost**?

(b) Find a function that gives the **marginal cost**.

(c) Find a function that gives the **average cost**.

(d) Find the quantity that minimizes the **average cost**. *Follow the strategy for solving optimization problems. Identify the objective equation and the constraint equation.*

(e) Confirm that the average cost and marginal cost are equal at your answer to part (d).

Short Answers to Examples

9.1 smaller number: 0 ; bigger number:44 ; minimum value of the product: 0

9.2 **(a)** $y = 20$ feet; $x = 30$ feet **(b)** \$840

9.3 **(a)** $(60, 1100)$; $(65, 1050)$ **(b)** $q = 1700 - 10p$ **(c)** $R(p) = 1700p - 10p^2$
(d) p_{max} =\$85 per ticket **(e)** q_{max} =850 ticket **(f)** R_{max} =\$72,250

9.4 **(a)** $R(q) = 300q - 0.02q^2$ **(b)** $C(q) = 9000 + 30q$ **(c)** $P(q) = -0.02q^2 + 270q - 9000$
(d) q_{max} =6750 ribbon winders **(e)** $MR(q) = 300 - 0.04q$ **(f)** $MC = 30$
(g) $MR(6750) = MC(6750) = 30$

9.5 **(a)** \$64 **(b)** $MC(q) = 0.02q + 1.5$ **(c)** $AC(q) = 0.01q + 1.5 + \frac{64}{q}$ **(d)** $q_{min} = 80$ oven mitts **(e)** $AC(80) = MC(80) = 3.1$

Lesson No. 10

Product and Quotient Rules

Objectives

- Understand when and how to use the Product Rule for finding the derivative of a product functions.
- Understand when and how to use the Quotient Rule for finding the derivative of a quotient of a functions.
- Be able to find derivatives for a polynomial or rational functions that require the use of more than one rule for differentiation.
- Be able to use the Quotient Rule for solving applications.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1],[2]
  - *Section 2.4 Product and Quotient Rules*
- Strang and Herman, *Calculus Volumn 1*[3,4]
  - *Section 3.3 Differentiation Rules*
    - The Product Rule
    - The Quotient Rule
    - Combining Differentiation Rules

Key Terms and Concepts:

- Product Rule
- Quotient Rule
- Solving optimization problems using the Product rule, the Quotient rule and/or combination of more than one rule.
- Determining behavior of the graph of a function using the Product rule, the Quotient rule and/or combination of more than one rule.

[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2] Disregard any examples involving logarithmic functions and/or exponential functions.
[3] Available free to download from https://openstax.org/details/books/calculus-volume-1 .

Using the techniques of differentiation that we currently have, there are many types of functions for which we cannot determine the derivative. We will expand our capabilities for finding derivatives by introducing two more techniques, the Product Rule and the Quotient Rule.

Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions.

Using the *Leibniz* notation, the Product rule can be written as follows:

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x). \tag{10.1}$$

Using the *prime* notation, the Product rule can be written as follows:

If $h(x) = f(x)g(x)$, then

$$h'(x) = f'(x)g(x) + g'(x)f(x). \tag{10.2}$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Notes:

- What this rule does NOT say; that is, the derivative of a product of functions is NOT the product of the derivatives.
- The product rule can extend to a product of several functions; the pattern continues – take the derivative of each factor in turn, multiplied by all the other factors left alone, and add them up.

Example 10.1. Given $f(x) = x^2 \cdot (x^2 + 1)^4$, find $f'(x)$ and express the answer in factored form. Then, identify any relative maximum and relative minimum points. A sign char will help.

Quotien Rule

Let $f(x)$ and $g(x)$ be differentiable functions.

Using the *Leibniz* notation, the Quotient rule can be written as follows:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{[g(x)]^2}. \tag{10.3}$$

Using the *prime* notation, the Quotient rule can be written as follows:

If $h(x) = \dfrac{f(x)}{g(x)}$, then

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}. \tag{10.4}$$

The numerator of the result resembles the product rule, but there is a minus instead of a plus; the minus sign goes with the g'. The denominator is simply the square of the original denominator – no derivatives there.

Example 10.2. Given $f(x) = \dfrac{2x}{x^2+1}$, find $f'(x)$ and determine all values of x for which the tangent line to the graph is horizontal. Then, identify the intervals for which the function is increasing and the intervals for which the function is decreasing.

Combining Differentiation Rules

Example 10.3. Given $f(x) = \left(\dfrac{2}{x+1}\right)^2$, find $f'(x)$ and identify any relative (local) maximum and relative (local) minimum points. A sign chart will help.

Example 10.4. Given $f(x) = \dfrac{x}{(x^2+1)^3}$, find $f'(x)$ and identify the intervals for which the function is increasing and the intervals for which the function is decreasing.

Quotient Rule Applications

A note on mathematical modeling: A mathematical function may be used to model a process in which one variable demonstrates a relationship with another variable. In many applications, time is the independent variable, and some other quantity, the dependent variable, is viewed as a "function" of time. Through observation of the process, it may be possible to identify a mathematical function that can be used to predict the value of the dependent variable from the value of the independent variable. That is, we use the function to "model" the process. In the following two examples, note that the function we use is chosen to fit the "data" one would observe in the process. In this case, we are observing how the amount of a drug in a patient's bloodstream changes over time.

Example 10.5. A Bolus Injection ("Shot") of a Drug(Note: The drug will seep into the bloodstream and then it will be metabolized by the body.) A patient is given an injection of a drug into a shoulder muscle and the following function is used to model the amount, $\boldsymbol{A}$ (in milligrams), of the drug in the patient's bloodstream at time $\boldsymbol{t}$ hours after the injection:

$$A(t) = \frac{10t}{1 + 0.25t^2}\ mg.;\ t \geq 0$$

(a) Determine the **time** at which the amount of the drug will be at its largest value.

(b) Determine the maximum amount of the drug that will be in the bloodstream.

(c) Determine the **rate** at which the amount of the drug in the bloodstream is changing at time $t = 4$ hours. Indicate if the amount is increasing or decreasing at this time, and include the units on the rate of change.

Example 10.6. A continuous intravenous injection of a drug (Note: The drug drips directly into a vein and at the same time the drug is being metabolized by the body.) A patient is given an intravenous injection of a drug and the following function is used to model the amount, $\boldsymbol{A}$ (in milligrams), of the drug in the patient's bloodstream at time $\boldsymbol{t}$ hours after the injection::

$$A(t) = \frac{50t}{t+1}\ mg.;\ t \geq 0$$

(a) Compare the rates of change of the drug at times $t = 1$ hour and $t = 4$ hours. What do you notice?

(b) According to this function, will the amount of the drug in the bloodstream ever reach a maximum value? Use the derivative of the function to explain why or why not.

Short Answers to Examples

10.1 $f'(x) = 2x(x^2+1)^3(5x^2+1)$; relative (local) minimum at $(0,0)$

10.2 $f'(x) = \dfrac{-2x^2+2}{(x^2+1)^2}$; $x = -1,\ 1$; decreasing on $(-\infty,-1)\cup(1,\infty)$; increasing on $(-1,1)$.

10.3 $f'(x) = \dfrac{-8}{(x+1)^3}$; No relative extreme points; Increasing on $(-\infty,1)$, decreasing on $(-1,\infty)$, but $x=-1$ is not in the domain of the function and therefor does not correspond to a relative (local) maximum point.

10.4 $f'(x) = \dfrac{-5x^2+1}{(x^2+1)^4}$; increasing on: $\left(-\frac{\sqrt{5}}{5},\frac{\sqrt{5}}{5}\right)$; decreasing on: $\left(-\infty,-\frac{\sqrt{5}}{5}\right)\cup\left(\frac{\sqrt{5}}{5},\infty\right)$

10.5 **(a)** $A'(t) = \dfrac{-2.5t^2+10}{(1+0.25t^2)^2}$; Relative (and Absolute) max at $t=2$ hours **(b)** $A(2) = 10$mg.
(c) $A'(4) = -1.2$ mg. per hour; decreasing at a rate 1.2 mg. per hour.

10.6 **(a)** $A'(t) = \dfrac{50}{(t+1)^2}$; $A'(1) = 12.5$ mg. per hour ; $A'(4) = 2$ mg. per hour; The amount is increasing at both times, but it is increasing more rapidly at time $t=1$ hour. **(b)** No. Note that $A'(t) > 0$ for all values of t, so $A(t)$ is increasing on $(0,\infty)$. However, $\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\dfrac{50t}{t+1} = 50$ mg. As such, the amount is NOT increasing without bound (horizontal asymptote) consistent with this observation is $\lim_{t\to\infty}\dfrac{50}{(t+1)^2} = 0$. The rate of graph is diminishing over time (the graph is becoming flatter).

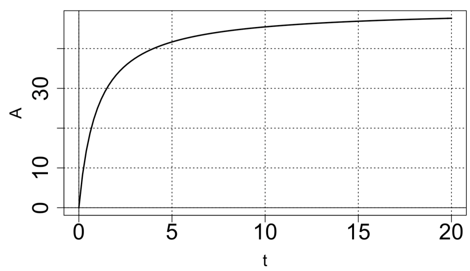


Figure 10.1: The graph of $A(t)$

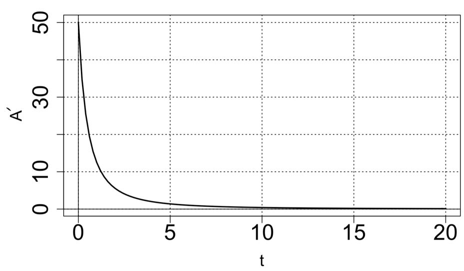


Figure 10.2: The graph of $A'(t)$

Lesson No. 11

More on Chain Rule and Related Rates

Objectives

- Apply the **Chain Rule** and the product/quotient rules correctly in combination when both are necessary.
- Be able to apply the concept of the *Chain Rule* to **related rates** application.

Suggested Reading:

- Strang and Herman, *Calculus Volumn 1*[1,2]
  - Section 3.6 The Chain Rule
- Calaway, Hoffman, and Lippman, *Applied Calculus*[3]
  - Section 2.11 Implicit Differentiation and Related Rates
    - Only Examples 3-4.

Key Terms and Concepts:

- The Chain and Power Rules combined
- The Chain and Product Rules combined
- The Chain and Quotient Rules combined
- Related Rates

[1]Available free to download from https://openstax.org/details/books/calculus-volume-1 .
[2]Disregard any examples with trigonometry.
[3]Available free to download from http://www.opentextbookstore.com/details.php?id=14 .

The Chain Rule was introduced in lesson 5 using the prime notation. You have learned how to use the Chain Rule with the Power Rule. The combination of the two rules is so called the General Power Rule. In lesson 6, the *Leibniz* notation was introduced to help with the intuition of the Chain Rule. In this lesson, we will learn how to use the **Chain Rule** in a more complex way—-with the **Product Rule** and with the **Quotient Rule**.

Review: The Chain Rule

Let $f(x)$ and $g(x)$ be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the *composite function*

$$h(x) = (f \circ g)(x) = f(g(x)) \tag{11.1}$$

is given by

$$h'(x) = f'(g(x)) \cdot g'(x) \tag{11.2}$$

In other words, the derivative of a composition of a function (denoted as $(f \circ g)'(x)$ is the derivative of the outside function f (with respect to the original inside function) *times* the derivative of the inside function g.

For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = f(u)$. Using the *Leibniz* notation form, the $h'(x)$ can be written as follows:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{11.3}$$

Review: The Chain and Power Rules Combined

Example 11.1. Given $h(x) = (2x + 1)^5$,

(a) Find the derivative of $h(x)$ using the **Chain Rule** with the prime notation form in equation 11.2.

(b) Find the derivative of $h(x)$ using the **Chain Rule** with the *Leibniz* notation form in equation 11.3.

The Chain and Product Rules Combined

Example 11.2. Given $h(x) = (2x+1)^5(3x-2)^7$, find the derivative of $h(x)$ using the **Chain Rule** with the prime notation form in equation 11.2.

The Chain and Quotient Rules Combined

Example 11.3. Given $h(x) = \dfrac{x}{(2x+3)^3}$, find the derivative of $h(x)$ using the **Chain Rule** with the prime notation form in equation 11.2.

Example 11.4. Given $h(x) = \left(\dfrac{x}{3x+2}\right)^5$, find the derivative of $h(x)$ using the **Chain Rule** with the *Leibniz* notation form in equation 11.3.

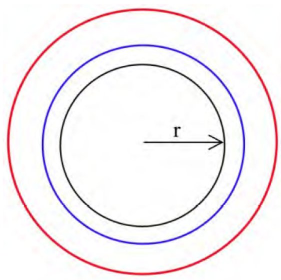
Application Using The Chain Rule: Related Rates

Strategy for Finding Related Rates

When working with a related rates problem,

1. Identify the quantities that are changing, and assign them variables.
2. Find an equation that relates those quantities.
3. Differentiate both sides of that equation with respect to time.
4. Plug in any known values for the variables or rates of change.
5. Solve for the desired rate.

Example 11.5. Suppose the border of a town is roughly circular, and the radius of that circle has been increasing at a rate of 0.1 miles each year.



(a) Determine the equation that gives the relationship between the **town area** (in mile^2) , $\boldsymbol{A}$ and the town radius (in mile), $\boldsymbol{r}$. *Include the appropriate unit.*

(b) Determine the equation that gives the relationship between the **town radius** , $\boldsymbol{r}$ and the time (in year), $\boldsymbol{t}$. *Include the appropriate unit.*

(c) Using the **Chain Rule** with the <u>*Leibniz* notation form</u> in equation 11.3, determine the **related rates equation** that gives the relationship between the rate of change of the circumference of the town border, $\frac{dC}{dt}$ and the rate of change of the radius of the town border, $\frac{dr}{dt}$. *Include the appropriate unit.*

(d) Find **how fast the area** of the town has been increasing when the radius is 5 miles. *Include the appropriate unit.*

Short Answers to Examples

11.1 **(a)** $h'(x) = 10(2x+1)^4$ **(b)** $\frac{dy}{dx} = 10(2x+1)^4$

11.2 $h'(x) = (2x+1)^4(3x-2)^6(72x+1)$

11.3 $h'(x) = \frac{3-4x}{(2x+3)^4}$

11.4 $\frac{dy}{dx} = \frac{10x^4}{(3x+2)^6}$

11.5 **(a)** $A(r) = \pi r^2$ mile^2 **(b)** $r(t) = 0.1t$ mile per year **(c)** $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ mile^2 **(d)** Around 3.14 mile^2 per year

Lesson No. 12

Implicit Differentiation

Objectives

- Understand the method of *implicit differentiation* and when it is appropriate to use it to determine $\frac{dy}{dx}$.
- Use *implicit differentiation* to determine the equation of a tangent line.
- Be able to apply the concept and method of *implicit differentiation* to real-word problems.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 2.11 Implicit Differentiation and Related Rates*
    - Skip *Related Rates* which is the topic covered in lesson 11.
- Strang and Herman, *Calculus Volumn 1*[2,3]
  - *Section 3.8 Implicit Differentiation*

Key Terms and Concepts:

- Explicit vs. implicit relationships
- Implicit differentiation technique

[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2] Available free to download from https://openstax.org/details/books/calculus-volume-1 .
[3] Disregard any examples with trigonometry.

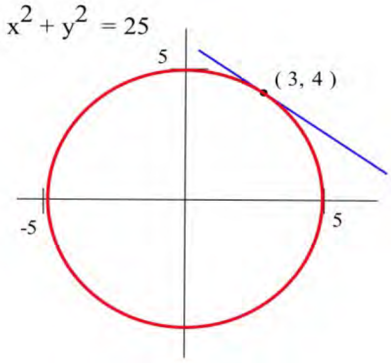
To this point, we have dealt with functions in which the value of y is described **"explicitly"** as a function $x : y = f(x)$. We say that x and y have a functional relationship. In some applications, the relationship between x and y is described by an equation (ex. $x^2 + y^2 = 5$). In this case, we say that y is defined **"implicitly"** in terms of x, since an assignment of a specific value for x will determine a value or values for y. Our interest will still be in determining slopes of tangent lines to curves and in the rate of change of y with respect to x, that is, the derivative $\frac{dy}{dx}$. However, the technique for determining $\frac{dy}{dx}$ will need to be modified from the previous approach when the relationship was explicitly stated.

Problem-Solving Strategy: Implicit Differentiation

To perform implicit differentiation on an equation that defines a function y implicitly in terms of a variable x, use the following steps:

1. Take the derivative of both sides of the equation. Keep in mind that ***y* is a function of *x*** even if we cannot *explicitly* solve for y.
2. Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and all terms that do not contain $\frac{dy}{dx}$ are on the right.
3. Factor out $\frac{dy}{dx}$ on the left.
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

Example 12.1. Find the slope of the tangent line to the graph of the equation $x^2 + y^2 = 25$ at the point $(3, 4)$.



Example 12.2. If x and y are related by the equation $xy^2 = 10$, answer the following questions:

(a) Find $\frac{dy}{dx}$.

(b) Evaluate: $\left.\frac{dy}{dx}\right|_{(2,-\sqrt{5})}$ and interpret the result.

Example 12.3. If x and y are related by the equation $x \cdot (2x+1)^3 = 27$, answer the following questions:

(a) Find $\frac{dy}{dx}$.

(b) Find the slope of the tangent line to the graph of the equation at the point $(1, 1)$.

Example 12.4. A company has determined the demand curve for their product is

$$q = \sqrt{5000 - p^2}$$

where p is the price in dollars, and q is the quantity in millions. The quantities changing are p and q, and we assume they are both functions of time, t, in weeks. We already have an equation relating the quantities.

If weather conditions are driving the price up \$2 a week, find **the rate at which demand is changing when the price is \$40**. Include the appropriate units. Interpret the result.

Short Answers to Examples

12.1 $\dfrac{dy}{dx} = -\dfrac{x}{y}$; slope$= -\dfrac{3\sqrt{7}}{7}$

12.2 **(a)** $\dfrac{dy}{dx} = -\dfrac{y}{2x}$ **(b)** $\dfrac{5}{4}$; slope of the tangent line at this point.

12.3 **(a)** $\dfrac{dy}{dx} = \dfrac{-2y-1}{6x}$ **(b)** Slope $= -\frac{1}{2}$

12.4 -1.37 million items per week ; Demand is <u>falling</u> by 1.37 million items per week.

Part II

Derivatives of Exponential and Logarithmic Functions

Lesson No. 13

Exponential Functions

Objectives

- Be able to use the **basic** derivative rule of the Natural Exponential Function in conjunction with the other rules (ex. product rule, quotient rule).
- Be able to use the **general** derivative rule of the Natural Exponential Function in conjunction with the other rules.
- Understand the interpretation and use of the derivative in the model for exponential growth or decay.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 2.3 Power and Sum Rules for Derivatives*
    - Derivative Rule for Exponential Functions
    - Example 7
  - *Section 2.4 Chain Rule*
    - Example 4 and Example 8.
- Strang and Herman, *Calculus Volumn 1*[2,3]
  - *Section 3.9 Derivative of the Exponential Function*

Key Terms and Concepts:

- Exponential functions, base e.
- Differentiation of exponential functions
- Model for Exponential Growth or Decay

[1]Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2]Available free to download from https://openstax.org/details/books/calculus-volume-1 .
[3]Disregard any examples with trigonometry.

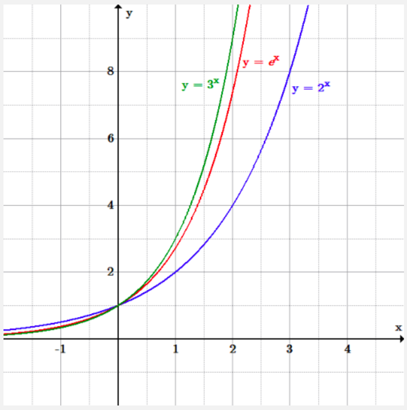
Chapter 1 Section 7 from Calaway, Hoffman, and Lippman, *Applied Calculus* provides a quick review of exponential functions which you should read and focus on exponential functions with the base being the mathematical constant denoted by *Euler's Number*: e, where $e \approx 2.71828\ldots$. Since calculus studies continuous change, we will almost always use the e-based form of exponential equations in this course.

Review: The Natural Exponential Function

The exponential function

$$f(x) = e^x$$

with base e is so prevalent in the sciences that it is often referred to as *the* exponential function or the natural exponential function.
$e \approx 2.718281828459045.....$
The number is irrational.



Basic Derivative Rule of the Natural Exponential Function

Derivative of the Natural Exponential Function: Basic Rule

Let $f(x) = e^x$ be the natural exponential function. Then

$$f'(x) = e^x \tag{13.1}$$

We can use the rule in equation 13.1, in conjunction with the rules we have already been given, to find derivatives of functions that include the term ke^x, where k is a constant.

Example 13.1. Given $g(x) = x^2 \cdot e^x$, answer the following questions:

(a) Find $g'(x)$.

(b) Find all values of x for which the slope of the tangent line to the graph of $g(x)$ equals 0.

(c) Find the intervals in which $g(x)$ is increasing and the intervals in which $g(x)$ is decreasing.

Example 13.2. Given $g(x) = \dfrac{2x+1}{e^x}$, answer the following questions:

(a) Find $g'(x)$.

(b) Find all values of x for which the slope of the tangent line to the graph of $g(x)$ equals 0.

(c) Find the intervals in which $g(x)$ is increasing and the intervals in which $g(x)$ is decreasing.

Example 13.3. Find the slope of the tangent line to the graph of $g(x) = \dfrac{e^x}{x + e^x}$ at the point $(0, 1)$.

Example 13.4. Given $g(x) = e^x(1 - e^x)^2$, answer the following questions:

(a) Find $g'(x)$.

(b) Find all values of x for which the slope of the tangent line to the graph of $g(x)$ equals 0.

General Derivative Rule of the Natural Exponential Function

Section 5 uses the Chain Rule to provide a more general rule for differentiating exponential functions, a rule that is needed when the exponent on the base e is not simply x, but rather a function of x.

Derivative of the Natural Exponential Function: General Rule

If $f(x) = e^{g(x)}$, then

$$f'(x) = e^{g(x)} \cdot g'(x) \tag{13.2}$$

Example 13.5. Given $g(x) = x \cdot e^{x^2}$, answer the following questions:

(a) Find $g'(x)$.

(b) Find all values of x for which the slope of the tangent line to the graph of $g(x)$ equals 0.

(c) Find all relative maximum and relative minimum points.

Example 13.6. Given $g(x) = \dfrac{e^{2x+1}}{x^2}$, answer the following questions:

(a) Find $g'(x)$.

(b) Find all values of x for which the slope of the tangent line to the graph of $g(x)$ equals 0.

(c) Find the intervals in which $g(x)$ is increasing and the intervals in which $g(x)$ is decreasing.

Applying the Natural Exponential Function

Review: Continuous Compound Interest Formula

$$A = Pe^{rt}$$

$A =$ amount after t years
$P =$ principal
$r =$ annual interest rate (expressed as a decimal)
$t =$ number of years

Example 13.7. Suppose you invest \$10,000 in an account that pays *continuously compounded interest* at an annual rate of 5%.

(a) Determine the appropriate function $A(t)$.

(b) How much money will be in the account at time $t = 5$ years from the present?

(c) Find $A'(5)$. Interpret the result.

> **Review: Model for Exponential Growth or Decay**
>
> For $k > 0$, **Exponential Growth:** $A(t) = A_0 e^{kt}$,
> **Exponential Decay:** $A(t) = A_0 e^{-kt}$
> where,
> $A(t)$ = the amount at time t
> $A_0 = A(0)$, the initial amount (the amount at $t = 0$)
> k =relative rate of growth or decay.
> t = time
> ***Continuously Compound Interest*** *is an example for Exponential Growth.*

Example 13.8. A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$.

(a) The **relative rate of growth** or the **growth factor** is the ratio of the rate of change of the population, $A'(t)$, to the population , $A'(t)$, to the population, $A(t)$. Show that the **relative rate of growth** of mosquitoes population is constant.

(b) What is the **rate of change** of mosquitoes population after 4 days?. Interpret the result.

(c) What is the **growth factor** of mosquitoes population after 4 days? Interpret the result.

Short Answers to Examples

13.1 **(a)** $g'(x) = xe^x(x+2)$ **(b)** $x = -2, 0$ **(c)** Increasing on $(-\infty, -2) \cup (0, \infty)$; decreasing on $(-2, 0)$

13.2 **(a)** $g'(x) = \dfrac{-2x+1}{e^x}$ **(b)** $x = \frac{1}{2}$ **(c)** Increasing on $\left(-\infty, \frac{1}{2}\right)$; decreasing on $\left(\frac{1}{2}, \infty\right)$

13 slope $= -1$

13.4 **(a)** $x = 0$ **(b)** $x = ln\left(\frac{1}{3}\right)$

13.5 **(a)** $g'(x) = e^{x^2}(2x^2+1)$ **(b)** None **(c)** None

13.6 **(a)** $g'(x) = \dfrac{2e^{2x+1} \cdot (x-1)}{x^3}$ **(b)** $x = 1$ **(c)** Increasing on $(-\infty, 0) \cup (1, \infty)$; decreasing on $(0, 1)$

13.7 **(a)** $A(t) = 10,000e^{0.05t}$ **(b)** $A(5) \approx 12840.25$ dollars. **(c)** $A'(t) = 500e^{0.05t}$;$A'(5) \approx 642.01$; At the 5^{th} year, your investment is increasing at the rate of (around) 642.01 dollars per year.

13.8 **(a)** $A'(t) = 300e^{0.3t} \Longrightarrow \dfrac{A'(t)}{A(t)} = \dfrac{300e^{0.3t}}{A(t)} = \dfrac{300e^{0.3t}}{1000e^{0.3t}} = 0.3$ **(b)** After 4 days, the mosquitoes population is increasing at the rate of $A'(4) \approx 996$ mosquitoes per day. **(c)** For each day, the mosquitoes population has multiplied itself by 103% (by a factor of 3).

Lesson No. 14

Natural Logarithm Functions

Objectives

- Understand how and when to use the **inverse properties** for natural exponential and natural log functions.
- Be able to **simplify expressions** involving natural exponential and natural log functions.
- Be able to **solve equations** involving natural exponential and natural log functions.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 1.8 Logarithmic Functions*
- Abramson, *College Algebra*[2]
  - *Section 6.3 Logarithmic Functions*
    - Using Natural Logarithms
  - *Section 6.6 Exponential and Logarithmic Equations*
    - Solving Exponential Equations Using Logarithms

Key Terms and Concepts:

- Natural log functions (base e)
- Inverse Properties
- Solving exponential and log equations.

[1]Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2]Available free to download from https://openstax.org/details/books/college-algebra .

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**.
This lesson provides a brief review of the natural log function $f(x) = ln(x)$ and its two important inverse properties. We use these properties often when solving equations involving exponential and log terms.

Definition of The Natural Logarithms

The logarithm with **base *e*** is called the ***natural* logarithm** and is denoted by $\ln(x)$. That is,

$$\ln(x) = \log_e(x)$$

Thus,

$$y = \ln(x) \quad \text{if and only if} \quad x = e^y$$

$e \approx 2.718281828459045...$

Recall that the natural log function $f(x) = ln(x)$ is the inverse of the exponential function $g(x) = e^x$. As such, we have two important inverse properties:

Inverse Properties of Logarithms

Since the functions $y = e^x$ and $y = ln(x)$ are inverse functions,

$$ln(e^x) = x \quad \text{for all} \quad x \tag{14.1}$$

and

$$e^{ln(x)} = x \quad \text{for} \quad x > 0 \tag{14.2}$$

Most values of $ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $ln(1) = 0$. For other natural logarithms, we can use the ln key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms in equation 14.1. In addition, we use these properties often when solving equations involving exponential and log terms.

Example 14.1. Simplify the expression, making use of laws of exponents: $e^{4ln(x)}$

Example 14.2. Simplify the expression, making use of laws of exponents: $e^{ln(x)-ln(3)}$

Example 14.3. Solve for x: $ln(x^2) = 10$

Example 14.4. Solve for x: $4ln(3x) = 9$

Example 14.5. Solve for x: $ln\left(\frac{3}{x}\right) = 2$

Example 14.6. Solve for x: $2ln(x + 1) = 5$

Example 14.7. Given $f(x) = 2x - 4e^x$, find the coordinates of any relative maximum or relative minimum points.

Example 14.8. Find the point on the graph of the function $f(x) = e^{2x}$ where the slope of graph is equal to 4.

Short Answers to Examples

14.1 x^4

14.2 $\frac{x}{3}$.

14.3 $x = \pm\sqrt{e^{10}}$ or $x = \pm e^5$.

14.4 $x = \frac{1}{3}e^{9/4}$.

14.5 $x = \frac{3}{e^2}$.

14.6 $x = e^{5/2} - 1$.

14.7 $(ln(0.5), 2ln(0.5) - 2)$.

14.8 $\left(\frac{ln(2)}{2}, 2\right)$.

Lesson No. 15

Derivatives of Natural Logarithm Functions

Objectives

- Understand the techniques for differentiating log functions.
- Be able to analyze functions involving log functions.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 2.4 Product and Quotient Rules*
    - Example 3
  - *Section 2.4 Chain Rule*
    - Example 7.
- Strang and Herman, *Calculus Volumn 1*[2,3]
  - *Section 3.9 Derivatives of Exponential and Logarithmic Functions*
    - Derivative of the Logarithmic Function
- Review previous lessons:
  - Lesson 7: Second Derivative and Concavity
    - Concavity Test: Determine Concavity and Inflection Values
  - Lesson 8: Optimization
    - Finding Maxima and Minima of a Function

Key Terms and Concepts:

- Differentiation of natural log functions (base e).

[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2] Available free to download from https://openstax.org/details/books/calculus-volume-1 .
[3] Disregard any examples with trigonometry.

The Derivative of the Natural Logarithmic Function[1]

If $x > 0$ and $y = ln(x)$, then

$$\frac{dy}{dx} = \frac{1}{x} \tag{15.1}$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g'(x) > 0$, the derivative of $h(x) = ln(g(x))$ is given by

$$h'(x) = \frac{1}{g(x)} g'(x) \tag{15.2}$$

Example 15.1. Given $f(x) = \dfrac{x}{ln(x)}$, answer the following questions:

(a) Find $f'(x)$.

(b) Find all values of x for which the slope of the tangent line to the graph is equal to 0.

(c) Find all relative maximum/minimum points.

[1]Theorem 3.15 from Strang and Herman, *Calculus Volumn 1*

Example 15.2. Given $f(x) = ln\left(\frac{1}{x}\right)$, answer the following questions:

(a) Find $f'(x)$.

(b) Find the intervals on which the function is increasing and those on which the function is decreasing.

(c) Determine the concavity of the graph and identify all inflection points.

Example 15.3. Given $f(x) = x \cdot ln(x^2)$, answer the following questions:

(a) Find $f'(x)$.

(b) Find the intervals on which the function is increasing and those on which the function is decreasing.

(c) Determine the concavity of the graph and identify all inflection points.

Example 15.4. Given $f(x) = [1 + ln(x)]^2$, answer the following questions:

(a) Find $f'(x)$.

(b) Find the intervals on which the function is increasing and those on which the function is decreasing.

(c) Determine the concavity of the graph and identify all inflection points.

Example 15.5. Given $f(x) = ln\left(\frac{x}{x+1}\right)$, answer the following questions:

(a) Find the domain of $f(x)$.
Hint: Reveiw *Finding the Domain of a Logarithmic Function* on *6.4 Graphs of Logarithmic Functions* from Abramson, *College Algebra* [1].

(b) Find $f'(x)$.

(c) Find the intervals on which the function is increasing and those on which the function is decreasing.

(d) Find all relative maximum/minimum points.

[1] Available free to download from https://openstax.org/details/books/college-algebra .

Short Answers to Examples

15.1 **(a)** $f'(x) = \dfrac{ln(x) - 1}{[ln(x)]^2}$ **(b)** $x = e$ **(c)** Relative minimum point at (e, e)

15.2 **(a)** $f'(x) = -\dfrac{1}{x}$ **(b)** Decreasing on $(0, \infty)$ **(c)** $f''(x) = \dfrac{1}{x^2}$; Concave up on $(0, \infty)$; no inflection points.

15.3 **(a)** $f'(x) = 2 + ln(x^2)$ **(b)** Increasing on $\left(-\infty, -\dfrac{1}{e}\right) \cup \left(\dfrac{1}{e}, \infty\right)$ **(c)** $f''(x) = \dfrac{2}{x}$; Concave down on $(-\infty, 0)$; Concave up on $(0, \infty)$; no inflection points.

15.4 **(a)** $f'(x) = \dfrac{2}{x}(1+ln(x))$ **(b)** Increasing on $\left(\dfrac{1}{e}, \infty\right)$; Decreasing on $\left(0, \dfrac{1}{e}\right)$ **(c)** $f''(x) = -\dfrac{2}{x^2} \cdot ln(x)$; Concave up on $(0, 1)$; Concave down on $(1, \infty)$; inflection point at $(1, 1)$.

15.5 **(a)** $(-\infty, -1) \cup (0, \infty)$ **(b)** $f'(x) = \dfrac{1}{x(x+1)}$ **(c)** Increasing on $(\infty, -1) \cup (0, \infty)$

Lesson No. 16

Natural Logarithm Functions: Properties and Differentiation

Objectives

- Be familiar with the properties of log functions and how to use them in solving equations.
- Be able to use log properties in differentiating complicated functions that involve log functions.

Suggested Reading:

- Abramson, *College Algebra*[1]
  - *Section 6.3 Logarithmic Functions*
  - *Section 6.5 Logarithmic Properties*

Key Terms and Concepts:

- Natural log functions (base e)
- Properties of log functions.
- Differentiation using log properties.

[1] Available free to download from https://openstax.org/details/books/college-algebra .

The following logarithmic properties are useful in solving equations:

Review: Logarithmic Properties

$$\ln(e) = 1 \tag{16.1}$$

$$\ln(1) = 0 \tag{16.2}$$

$$\ln(e^x) = x \quad \text{for any real number} x \tag{16.3}$$

$$e^{\ln(x)} = x \quad \text{for any} \quad x > 0 \tag{16.4}$$

$$\text{Product Rule: } \ln(M \cdot N) = ln(M) + ln(N) \tag{16.5}$$

$$\text{Quotient Rule: } \ln\left(\frac{M}{N}\right) = \ln(M) - \ln(N) \tag{16.6}$$

$$\text{Power Rule: } \ln(M^r) = r \cdot \ln(M) \tag{16.7}$$

Example 16.1. Given $ln(x)+ln(x^2)-ln(x^4)$, simplify the expression and write your answer in terms of $ln(x)$.

Example 16.2. Which of the following is algebraically equivalent to the expression: $\boldsymbol{ln(8x^3)}$?
HINT: $8 = 2^3$
(a) $3 \cdot ln(8x)$ **(b)** $[3 \cdot ln(2)] \cdot [3 \cdot ln(x)]$ **(c)** $3 \cdot ln(2x)$ **(d)** $3 \cdot [ln(8) + ln(x)]$

Example 16.3. Solve for x: $ln(\sqrt{x}) - 2 \cdot ln(3) = 0$

Example 16.4. Solve for x: $ln(x^2) - ln(2x) + 1 = 0$

Example 16.5. Solve for x: $ln(x + 4) - ln(x - 2) + 1 = ln(x)$

Example 16.6. Solve for x: $ln(x^3) - 4 \cdot ln(x) = 1$

Example 16.7. Solve for x: $ln(x + 1) - ln(x) = 1$

Example 16.8. Find the derivative of the function $f(x) = ln\left(\frac{x}{x+1}\right)$. NOTE: Use properties of logs to re-write the function before differentiating. Simply your answer.

Example 16.9. Find the derivative of the function $f(x) = ln\left[(x+1)(x^2-1)\right]$. NOTE: Use properties of logs to re-write the function before differentiating. Simply your answer.

Short Answers to Examples

16.1 $-ln(x)$

16.2 c

16.3 $x = 81$

16.4 $x = \frac{2}{e}$

16.5 $x = 4$

16.6 $x = \frac{1}{e}$

16.7 $x = \frac{1}{e-1}$

16.8 $f'(x) = \frac{1}{x(x+1)}$ or $\frac{1}{x^2+x}$

16.9 $f'(x) = \frac{3x-1}{(x+1)(x-1)}$ or $\frac{3x-1}{x^2-1}$

Lesson No. 17

Exponential Growth and Decay Models

Objectives

- Be able to identify a model as being an exponential growth or exponential decay model.
- Understand the concept of **half-life** in the context of an exponential decay model.
- Understand the unique qualities of the **rate of change** of exponential growth and decay models.

Suggested Reading:

- Abramson, *College Algebra*[1]
  - *Section 6.7 Exponential and Logarithmic Models*
    - Model exponential growth and decay.

Key Terms and Concepts:

- Differential equations.
- Growth constant and decay constant.
- Models for exponential growth and decay.
- Half-life in an exponential decay process.

[1] Available free to download from https://openstax.org/details/books/college-algebra .

Exponential Growth and Decay Models

The exponential growth (or decay) occurs when a quantity grows(or decreases) at a rate proportional to its size. The mathematical model for **exponential growth** or **decay** is given by

$$f(t) = Ce^{kt} \tag{17.1}$$

If $k > 0$, the function models the amount, or size, of a growing entity. C is the original amount, or size, of the growing entity at time $t = 0$, f is the amount at time t, and k is a constant representing the **growth rate**.

If $k < 0$, the function models the amount, or size, of a density entity. C is the original amount, or size, of the decay entity at time $t = 0$, f is the amount at time t, and k is a constant representing the **decay rate**.

An exponential function of the form $f(t) = Ce^{k}t$ has the interesting property that $f'(t) = k \cdot f(t)$. If the independent variable t represents time, the interpretation is that the rate at which these functions are changing at any point in time is proportional to the size of the function at that time. We will look at various applications of functions that fit this form.

Exponential Growth Models

Example 17.1. The number of a certain type of mountain beetle that kills pine trees has been increasing in forests across western Canada. Sampling at various locations in a particular area has indicated that the number, N, of beetles is growing at a rate that is proportional to the number of beetles that are present (we say that the number of beetles is growing "*exponentially*" or growing at an "*exponential rate*"). Given this observation, the number of beetles in this area at time t months from the present ($t = 0$) may be modeled by the function

$$\boldsymbol{N(t) = N_0 e^{k} t \qquad ; t \geq 0}$$

(a) Suppose that initially, 2500 beetles were assumed to be in this area of the forest. Four months later the number had grown to 4000. Determine the function $N(t)$ that would be appropriate.

(b) Use a differential equation to determine the **rate of growth** of the number of beetles at time $t = 4$ months.

(c) For this area, it is believed that the forest cannot recover from the damage done by the beetles if the number of beetles reaches 10,000. According to the model determined in part (a), at what time will the number of beetles reach 10,000?

Exponential Decay Models

Example 17.2. After an injection of a drug into a patient's bloodstream, the amount, A, of the drug tends to decrease at a rate that is proportional to the amount of the drug still present in the blood. If $A(t) =$ the amount (in milligrams) of the drug in the blood at time t hours after the injection, then $A'(t) = -\lambda \cdot A(t)$, where λ is a positive constant referred to as the **decay constant**, or in the context of this application, the "elimination rate" of the drug. For a particular drug, suppose $\lambda = 0.45$.

(a) Identify the appropriate form of the function A(t).

(b) How long will it take for the amount of the drug remaining in the bloodstream to be reduced to 40% of the initial amount injected?

(c) At what point in time is the amount of the drug decreasing at its fastest rate?

(d) DEFINITION: The **half-life** of a drug is the amount of **time** that it takes a drug to be reduced to an amount that is one-half of its' initial (or current) amount. Determine the **half-life** of the drug.

NOTE:

(1) The half-life of the drug does not depend on the initial amount of the drug. For example, 10 mg. of the drug will be reduced to 5 mg. in about 1.54 hours, and 100 mg. will be reduced to 50 mg. in the same amount of time.

(2) The amount of the drug will continue to be reduced by a factor of one-half over each time interval equal to the half-life. In this case, the amount will be reduced to one-half of the current amount approximately every 1.54 hours.

(e) If 80 mg. of the drug are injected, according to this model what is the rate of decrease of the drug at time $t = 3$ hours after injection?

Short Answers to Examples

17.1 **(a)** $N(t) = 2500e^{0.1175t}$ **(b)** $N'(4) \approx 470$ beetles per month **(c)** $t \approx 11.8$ months

17.2 **(a)** $A(t) = A_0e^{-0.45t}$; $t \geq 0$ **(b)** $t \approx 2.04$ hours **(c)** $t = 0$ **(d)** $t \approx 1.54$ hours **(e)** $A'(3) \approx -9.33$ mg. per hour

Lesson No. 18

Continuously Compounded Interest

Objectives

- Understand the use of exponential growth functions when interest is compounded continuously.
- Understand how the amount of money in an grows when interest is compounded continuously.
- Understand the concept of **present value** and how it is used in analysis and decision-making.

Suggested Reading:

- Abramson, *College Algebra*[1]
  - *Section 6.1 Exponential Functions*
    - Use compound interest formulas.
    - Evaluate exponential functions with base e.

Key Terms and Concepts:

- Compounding of interest
- Present value

[1] Available free to download from https://openstax.org/details/books/college-algebra .

Consider accounts where money is placed by an investor/bank customer. Typically, interest on an account is compounded periodically over the course of a year. For example, the compounding may be done annually (once per year), quarterly (4 times per year), monthly (12 times per year), etc. In some cases, the compounding of interest occurs continuously over time (or we may assume this type of compounding). In this case, the amount of money, A, at time t is given by the model $A(t) = Pe^{rt}$, where P is the initial amount invested, and r is the annual interest rate.

Continuous Compound Interest Formula

$$A = Pe^{rt}$$

$A =$ amount after t years
$P =$ principal
$r =$ annual interest rate (expressed as a decimal)
$t =$ number of years
$e \approx 2.718281828459045.....$

From lesson 17, recall that the **exponential growth** model which is given by $f(t) = Ce^{kt}$ has the interesting property that $f'(t) = k \cdot f(t)$ which tells us that the rate at which these functions are changing at any point in time is proportional to the size of the function at that time. The application of **continuously compound interest** also fits this form. Specifically, $A'(t) = r \cdot A(t)$; that is, money grows at a rate proportional to the size of the account.

Example 18.1. A person has purchased a house as an investment for $100,000. Six years later, a developer offers to buy the house for $205,000. At what rate of annual interest compounded continuously did this investment earn over this time period?

Example 18.2. Mary invests $60,000 in an account that pays 5.5% annual interest compounded continuously.

(a) At what time will the amount in her account be increasing at a rate of $4000 per year?

(b) Tom invests \$70,000 in an account that pays 5% annual interest compounded continuously. Which account is growing at a faster rate at time $t = 5$ years? At time $t = 8$ years?

(c) At what point in time are the amounts in the two accounts increasing at the SAME RATE?

Present Value

Consider the equation $A = Pe^{rt}$ and solve for P: $A = Pe^{-rt}$. We can use this equation as follows: You are to receive an amount of money, A, at some point in time t years in the future. Assume that throughout this time period, money can be invested at an annual interest rate, r, compounded continuously. As an option, what amount would you accept (and invest) now in order to have the amount A after t years? This amount, P, is referred to as "the present value of the amount A to be received in t years".

Example 18.3. You have won a contest and 2 options are provided for your award: (1) You will receive \$25,000 4 years from now, or (2) You will receive \$20,000 now. A bank will guarantee an annual interest rate of 3.5% compounded continuously over the next 4 years. Which option should you choose in order to have the most money after 4 years?

Short Answers to Examples

18.1 $r \approx 0.1196$ or 11.96%

18.2 **(a)** $t \approx 3.50$ years **(b)** At time $t = 5$ years **(c)** $t \approx 11.77$ years

18.3 Present value is $P \approx \$21734$; choose 1st option. You would need \$21734 now to have \$25000 in 4 years at 3.5% annual rate.

Lesson No. 19

Applications of Natural Log Functions

Objectives

- Understand the concepts of relative/percentage rate of change vs. the rate of change of a function given by $f'(x)$.
- Understand when and how logarithmic differentiation is used to determine relative/percentage rates of change.
- Be convinced that logarithmic differentiation is definitely preferred in some cases (and know which cases).

Suggested Reading:

- Abramson, *College Algebra*[1]
  - *Section 6.5 Logarithmic Properties*
- Strang and Herman, *Calculus Volumn 1*[2,3]
  - *Section 3.9 Derivatives of Exponential and Logarithmic functions*
    - Logarithmic Differentiation

Key Terms and Concepts:

- Relative rates of change
- Percentage rate of change
- Logarithmic differentiation

[1]Available free to download from https://openstax.org/details/books/college-algebra .
[2]Available free to download from https://openstax.org/details/books/calculus-volume-1 .
[3]Disregard any examples with trigonometry.

The rate of change of a function $f(t)$ has been defined as being given by the derivative $f'(t)$. While the rate of change does provide useful information about the function, it is sometimes more informative to consider the rate of change relative to the size of the function. We refer to this as the **relative rate of change** and determine it by finding the ratio $\frac{f'(t)}{f(t)}$. When this ratio is express as a percentage (multiply by 100), it is referred to as the **percentage rate of change** of $f(t)$.

Finding Relative Rate of Change (Direct Approach)

1. Find $f'(t)$.
2. Form the ratio $\frac{f'(t)}{f(t)}$.

Since it can be shown that $\frac{d}{dt}ln[f(t)] = \frac{f'(t)}{f(t)}$, the **logarithmic differentiation** approach as described below can be an alternative approach to find the relative rate of change. For *some functions*, this approach may be more efficient than a direct approach.

Finding Relative Rate of Change (Logarithmic Differentiation Approach)

1. Given a function $f(t)$, find $ln[f(t)]$. Use properties of logarithms to expand $ln[f(t)]$ as much as possible.
2. Differentiate $ln[f(t)]$.

Review: Logarithmic Properties

$$\ln(e) = 1 \tag{19.1}$$

$$\ln(1) = 0 \tag{19.2}$$

$$\ln(e^x) = x \quad \text{for any real number} x \tag{19.3}$$

$$e^{\ln(x)} = x \quad \text{for any} \quad x > 0 \tag{19.4}$$

$$\text{Product Rule: } \ln(M \cdot N) = ln(M) + ln(N) \tag{19.5}$$

$$\text{Quotient Rule: } \ln\left(\frac{M}{N}\right) = \ln(M) - \ln(N) \tag{19.6}$$

$$\text{Power Rule: } \ln(M^r) = r \cdot \ln(M) \tag{19.7}$$

Example 19.1. The value, V, of a stock portfolio at time t years in the future is to be modeled by the function $V(t) = 200,000e^{0.4\sqrt{t}}$ dollars (not an exponential growth model). Determine the **percentage** rate of change of the value of the portfolio at time t =3,6, and 10 years and interpret the results.

(a) Using the **Direct approach**, find the **relative rate of change** of V. Then, interpret the results.

(b) Using the **Logarithmic approach**, find the **relative rate of change** of V.

(c) Compare the results to $V'(3), V'(6).V'(10)$.

Short Answers to Examples

19.1 (a) $\dfrac{V'(t)}{V(t)} = \dfrac{0.2}{\sqrt{t}}$; Value is expect to increase by 11.5%/years; 8.2%/year; 6.3%/year at these times. **(b)** $ln(V(t)) = ln(200,000)+0.4\sqrt{t}$; $\dfrac{d}{dt}ln[f(t)] = \dfrac{0.2}{\sqrt{t}}$ **(c)** $V'(t) = \dfrac{40,000e^{0.4\sqrt{t}}}{\sqrt{t}}$; $V'(3) \approx$ \$46,173 per year; $V'(6) \approx$ \$43,502 per year; $V'(10) \approx$ 44,813 per year. Value is increasing at each time.

Part III

Integration

Lesson No. 20

Antidifferentiation

Objectives

- Understand the concept of an **antiderivative**.
- Be able to use basic rules of integration (antidifferentiation) for finding antiderivatives (indefinite integrals).

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 3.3 Antiderivatives of Formulas*
    - *Examples 1-4*
- Strang and Herman, *Calculus Volumn 1*[2]
  - *Section 4.10 Antiderivatives*[3]

Key Terms and Concepts:

- Antiderivative
- Indefinite integrals
- Integrand
- Rules for integration/antidifferentiation
- Initial Value

[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2] Available free to download from https://openstax.org/details/books/calculus-volume-1 .
[3] Disregard any examples with trigonometry.

We have spent a lot of time discussing the derivatives of various types of functions, derivatives that were determined by a process called differentiation. We now consider the following question: If we know that a function $f(x)$ is a **derivative** of a function $F(x)$, that is $F'(x) = f(x)$, how can we determine $F(x)$ [we will call $F(x)$ an **antiderivative** of $f(x)$.] ?

A simple example would be $f(x) = 2x$. In this case, we can see that $F(x) = x^2$ would be an antiderivative of $f(x)$, because clearly $F'(x) = 2x$. But we also note that $F(x) = x^2 + 4$ would be an antiderivative of $f(x)$. The collection of all functions of the form $x^2 + C$, where C is any real number, is known as the **family of antiderivatives** of $2x$.

From this observation, we can conclude that $f(x)$ does not have a single antiderivative, but all antiderivatives of $f(x)$ will have the form $F(x) + C$, where C is a constant. The process of determining antiderivatives is referred to as **antidifferentiation**. For cases in which $f(x)$ is a more complicated function, we will make use of rules for finding antiderivatives just as we used rules for finding derivatives.

Antiderivatives

An antiderivative of a function $f(x)$ is any function $F(x)$ where $F'(x) = f(x)$.

The antiderivative of a function $f(x)$ is a whole **family** of functions written

$$F(x) + C$$

where $F'(x) = f(x)$ and C respresents any constant. The antiderivative is also called the **indefinite integral**.

Verb forms: We **antidifferentiate, or integrate**, or **find the indefinite integral** of a function. This process is called **antidifferentiation** or **integration**.

Notation for the antiderivative:

The antiderivative of f is written

$$\int f(x)dx$$

The $\int$ symbol is still called an **integral sign**; the dx on the end still must be included; you can still think of $\int$ and dx as left and right parentheses. The dx tells us that the function of interest is a function of the *independent* variable x. The function f is still called the **integrand**.

Example 20.1. Find **an** antiderivative of x.

Example 20.2. Find **the** antiderivative of x.

Example 20.3. Each of the following statements is of the form $\displaystyle\int f(x)dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$

(a) $\displaystyle\int (x + e^x)dx = \frac{x^2}{2} + e^x + C.$

(b) $\int (xe^x)dx = xe^x - e^x + C.$

Antidifferentiation is going backwards through the derivative process. So the easiest antiderivative rules are simply backwards versions of the easiest derivative rules. The corresponding rules for antiderivatives are listed in the table below – each of the antiderivative rules is simply rewriting the derivative rule. All of these antiderivatives can be verified by differentiating.

In what follows, $f(x)$ and $g(x)$ are differentiable functions of x, k, n and C is a constant.

	Derivative Rules	Indefinite Integral
Power Rule	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C \ ; \ n \neq -1$
Constant Rule	$\frac{d}{dx}(k) = 0$	$\int kdx = \int kx^0 dx = kx + C$
Exponential Rule	$\frac{d}{dx}(e^{kx}) = ke^{kx}$	$\int e^{kx} dx = \frac{1}{k} e^{kx} + C$
Natural Logarithmic Rule	$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln \lvert x \rvert + C \ ; \ x \neq 0$
Constant Multiple Rule	$\frac{d}{dx}(kf(x)) = kf'(x)$	$\int kf(x)dx = k \int f(x)dx + C$
Sum (or Difference) Rule	$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$	$\int f(x) \pm g(x)dx$ $= \int f(x)dx \pm \int g(x)dx + C$

Note: For the *Natural Logarithmic Rule*, the antiderivative of $\frac{1}{x}$ is actually NOT simply $\ln(x)$, is is $\ln \lvert x \rvert$. Why? The domain of $\frac{1}{x}$ is $(-\infty, 0) \cup (0, \infty)$ which is bigger than the domain of $\ln(x)$ which is $(0, \infty)$. To match the domain of the antiderivative of $\frac{1}{x}$ with the domain of $\frac{1}{x}$, you must careful to include those absolute values so that you do not have to worry about whether our x's are positive or negative. Otherwise, you could end up with domain problems.

Example 20.4. Evaluate $\int (-3)dx$.

Example 20.5. Evaluate $\int \left(\frac{1}{x^3}\right) dx$.

Example 20.6. Evaluate $\int \left(e^{-4x}\right) dx$.

Example 20.7. Evaluate $\int \left(\sqrt{x} + e^{2x}\right) dx$.

Example 20.8. Evaluate $\int \left(\frac{2}{x} - 3e^{\frac{1}{2}x} + 5\right) dx$.

Example 20.9. Determine the function $P(t)$ given the following: $P'(t) = 2t - e^{-2t}$ and $P(0) = 2$ (this is referred to as the **initial value** of the function.)

Example 20.10. A manufacturer of powdered dishwasher soap estimates that the **marginal cost** at a production level of x tons of soap each day is given by $\boldsymbol{C'(x) = 0.25x + 120}$ dollars per ton. The fixed costs of production (when no soap is produced; $x = 0$) are \$750 each day. Determine the total daily cost function, $C(x)$, which provides the total cost of producing x tons of soap each day.

Short Answers to Examples

20.1 You can choose any function you like as long as its derivative is x, so you can pick ; for example, $F(x) = \frac{x^2}{2} - 5$, $F(x) = \frac{x^2}{2} + 2.3, etc.$

20.2 Now you need to write the entire family of functions whose derivatives are x. You can use the notation: $\int x\ dx = \frac{x^2}{2} + C$

20.3 **(a)** $\frac{d}{dx}\left(\frac{x^2}{2} + e^x + C\right) = x + e^x$ **(b)** $\frac{d}{dx}\left(xe^x - e^x + C\right) = xe^x$

20.4 $F(x) = -3x + C$

20.5 $F(x) = -\frac{1}{2x^2} + C$

20.6 $F(x) = -\frac{1}{4}e^{-4x} + C$

20.7 $F(x) = \frac{2}{3}x^{3/2} + \frac{1}{2}e^{2x} + C$

20.8 $F(x) = 2\ln|x| - 6e^{\frac{1}{2}x} + 5x + C$

20.9 $P(t) = t^2 + \frac{1}{2}e^{-2t} + \frac{3}{2}$

20.10 $C(x) = 0.125x^2 + 120x + 750$ dollars

Lesson No. 21

Definite Integrals and Net Change of a Function

Objectives

- Understand the procedure for computing the value of a **definite integral**.
- Be able to use definite integrals to determine the **net change** in the value of a function.
- Be able to use and interpret the net change in the value of a function in application problems.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 3.2 The Fundamental Theorem and Antidifferentiation*
    - *The Fundamental Theorem of Calculus*
- Strang and Herman, *Calculus Volumn 1*[2,3]
  - *Section 5.3 The Fundamental Theorem of Calculus*
    - *Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem*
  - *Section 5.4 Integration Formulas and the Net Change Theorem*
    - *The Net Change Theorem*
    - *Applying the Net Change Theorem*

Key Terms and Concepts:

- Definite integrals
- Limits of integration
- Net change in the value of a function

[1]Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2]Available free to download from https://openstax.org/details/books/calculus-volume-1 .
[3]Disregard any examples with trigonometry.

The relationship between differentiation and integration was discovered and explored by both *Sir Isaac Newton and Gottfried Wilhelm Leibniz* (among others) during the late 1600s and early 1700s, and it is codified in what we now call the **Fundamental Theorem of Calculus**. Its very name indicates how central this theorem is to the entire development of calculus. Calculus is one of the most significant intellectual structures in the history of human thought, and the Fundamental Theorem of Calculus is a most important brick in that beautiful structure. The theorem provides scientists with the necessary tools to explain many phenomena. Using calculus, astronomers could finally determine distances in space and map planetary orbits. Everyday financial problems such as calculating marginal costs or predicting total profit could now be handled with simplicity and accuracy. Engineers could calculate the bending strength of materials or the three-dimensional motion of objects. Our view of the world was forever changed with calculus.

The Fundamental Theorem of Calculus is comprised of two parts. The *Fundamental Theorem of Calculus, Part 1*, establishes the relationship between differentiation and integration. This lesson however does not cover the first part of the Fundamental Theorem of Calculus[1]. In this lesson, we examine the **Fundamental Theorem of Calculus, Part 2** which is perhaps the most important theorem in calculus. It gives us powerful and useful techniques for evaluating **definite integrals**.

The Definite Integral and Its Notation

Suppose we are given a function $f(x)$ that is continuous over an interval $[a, b]$, and the function $F(x)$ is any antiderivative of $f(x)$ (that is, $F'(x) = f(x)$. The definite integral of f from a to b is written

$$\int_a^b f(x)dx \tag{21.1}$$

The numbers a and b are called the **limits of integration**; a is the **lower limit**, and b is the **upper limit**.

The Fundamental Theorem of Calculus, Part 2 (also known as the evaluation theorem)

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is any antiderivative of $f(x)$ (that is, $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a) \tag{21.2}$$

We often see the notation $F(x)\Big|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated **limits of integration** a and b to indicate that we should evaluate the function $F(x)$ at the **upper limit** b, and subtract the value of the function $F(x)$ evaluated at the **lower limit** a.

The theorem states that if we can find an **antiderivative** for the integrand, then we can evaluate the **definite integral** by evaluating the antiderivative at the endpoints of the interval and subtracting.

[1] The explanation of the first part of the theorem can be found in section 5.3 from Strang and Herman, *Calculus Volumn 1*

Example 21.1. Let $f(x) = x^2$.

(a) Find **the** antiderivative of $f(x)$. That is, evaluate **the indefinite integral**: $\int f(x)dx$.

(b) Evaluate **the definite integral**: $\int_1^3 f(x)dx$.

Look Back: Notice the similarity and the difference among *Antiderivative, Indefinite Integral and Definite Integral.* The final answer of the definite integral is a number while the result of the indefinite integral is a function.

Net Change Theorem

Using the definition of a definite integral from the *Fundamental Theorem of Calculus, Part 2*, we can alter the notation in Equation 23.4 slightly and re-write the definition as follows (recall that $F'(x) = f(x)$):

$$\int_a^b F'(x)dx = F(b) - F(a) \tag{21.3}$$

By adding $F(a)$ from both sides of the equation above yields the equivalent equation, which one we use depends on the application.

$$F(b) = F(a) + \int_a^b F'(x)dx \tag{21.4}$$

The net change theorem considers the integral of a *rate of change*, that is $F'(x)$ or $f(x)$ above. The formula can be expressed in two ways. The first in equation 21.3 is more familiar; it is simply the definite integral defined in equation 23.4. The second formula in equation 21.4 says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral.

Example 21.2. The value, V, of an investment is expected to grow over time. The expectation is that the rate of increase in value can be approximated by $V'(t) = 1000e^{0.08t} + 2000$ dollars per year. Determine the net increase in the value of the investment over the first 5 years, that is, over the time interval from $t = 0$ to $t = 5$. Round your answer to the nearest dollar.

Example 21.3. During spring, the bird population, P, at a refuge is changing at a rate given by $P'(t) = .8e^{0.04t} - 0.2e^{0.0625t}$ hundreds of birds per day. Let the function $P(t)$ represent the total number of birds at the refuge (in hundreds) at time t days from the first day of spring.

(a) Find the net change in the population size from time $t = 10$ days to $t = 20$ days. Is the population increasing or decreasing over this time interval? Use one decimal place in your answer.

(b) Find the net change in the population size from time $t = 65$ days to $t = 75$ days. Is the population increasing or decreasing over this time interval? Use one decimal place.

Short Answers to Examples

21.1 **(a)** $\frac{1}{3}x^3$ **(b)** $8\frac{2}{3}$

21.2 $V(t) = 12500e^{0.08t} + 2000t$; $V(t)\Big|_0^5 \approx 16,148$; We are expecting the value of the investment to increase by approximately \$16,148 over the first 5 years.

21.3 **(a)** $P(t) = 20e^{0.04t} - 3.2e^{0.0625t}$; $P(t)\Big|_{10}^{20} \approx 9.5$; We expect the population to increase by approximately$\approx$ 9.5 hundreds or 950 birds over the interval $t = 10$ to $t = 20$ days. **(b)** $P(t) = 20e^{0.04t} - 3.2e^{0.0625t}$; $P(t)\Big|_{65}^{75} \approx -29.0$; We expect the population to decrease (net change is negative or ≈ -29.0 hundreds of birds) by approximately hundreds 2900 birds over the interval $t = 65$ to $t = 75$ days.

Lesson No. 22

Integration by Substitution

Objectives

- Be able to use the method of substitution to determine antiderivatives for complicated functions.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 3.4 Substitution*
- Strang and Herman, *Calculus Volumn 1*[2,3]
  - *Section 5.5 Substitution*

Key Terms and Concepts:

- u-substitution

[1] Available free to download from `http://www.opentextbookstore.com/details.php?id=14` .
[2] Available free to download from `https://openstax.org/details/books/calculus-volume-1` .
[3] Disregard any examples with trigonometry.

While the rules presented in section 22 are helping in determining antiderivatives of functions, we are limited in the types of functions which we can apply these rules. For example, consider the following: $\int \left[(x^2-2)^4 \cdot 2x\right] dx$. Unless we expand the integrand (unappealing), the rules from section 22 will not provide a means to evaluate this integral. We solve this problem by introducing another technique of integration which will allow us, through a substitution to transform the integrand into a form for which the rules from section 22 will apply. This technique is often referred to as "**u-substitution**". The method is one way of algebraically manipulating an integrand so that the rules apply. Specifically, this method helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

At first, the approach to the substitution procedure may not appear very obvious. However, it is primarily a visual task——that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see? We are looking for an integrand of the form $f[g(x)]g'(x)$.

For example, in the integral $\int \left[(x^2-2)^4 \cdot 2x\right] dx$, we have $f(x) = x^4$, $g(x) = x^2 - 2$, and $g'(x) = 2x$. Then, $f[g(x)]g'(x) = (x^2-2)^4 \cdot 2x$. With substituition, we will substitute $u = g(x)$. This means $\dfrac{du}{dx} = g'(x)$, so $du = g'(x)dx$. Making this substitutions, $\int \left[(x^2-2)^4 \cdot 2x\right] dx = \int u^4 du$, or $\int f[g(x)]g'(x)dx = \int f(u)du$.

Theorem: Substitution with Indefinite Integrals[1]

Let $u = g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned} \int f[g(x)]g'(x)dx &= \int f(u)du \\ &= F(u) + C \\ &= F(g(x)) + C. \end{aligned} \tag{22.1}$$

Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x)$ such that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$ into the integral.
3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.
4. Evaluate the integral in terms of u. If you still cannot evaluate it, go back to step 1 and try a different choice of u.
5. Substitute back x's for u's everywhere in your answer.

[1] Strang and Herman, *Calculus Volumn 1*

Example 22.1. Evaluate $\int \left[(x^2 - 2)^4 \cdot 2x\right] dx$.

Example 22.2. Evaluate $\int \left(3x^2 \cdot e^{x^3+2}\right) dx$.

Example 22.3. Evaluate $\int \left(\frac{e^x}{2 + 3e^x}\right) dx$.

Example 22.4. Evaluate $\int \left(8x\sqrt{2x^2 + 4}\right) dx$.

Example 22.5. Evaluate $\int \left(\frac{\ln(4x)}{x}\right) dx$.

Example 22.6. Evaluate $\int \left(\frac{x}{(x^2+3)^2}\right) dx$.

Short Answers to Examples

22.1 $F(x) = \frac{1}{5}(x^2-2)^5 + C$

22.2 $F(x) = e^{x^3+2} + C$

22.3 $F(x) = \frac{1}{3}\ln(2+3e^x) + C$

22.4 $F(x) = \frac{4}{3}(2x^2+4)^{\frac{3}{2}} + C$

22.5 $F(x) = \frac{1}{2}[\ln(4x)]^2 + C$

22.6 $F(x) = -\frac{1}{2(x^2+3)} + C$

Lesson No. 23

Definite Integrals and Area

Objectives

- Understand the use of definite integrals to determine the area of a region bounded by the x-axis and the graph of a function.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 3.1 The Definite Integral*
    - Approximating with Rectangles
    - Definition of the Definite Integral
    - *Signed Area*
- Strang and Herman, *Calculus Volumn 1*[2,3]
  - *Section 5.2 The Definite Integral*
    - Area and the Definite Integral

Key Terms and Concepts:

- Area under the graph of a function; area as a limit of sum of areas of rectangles.
- Total Area vs. Net Signed Area

[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2] Available free to download from https://openstax.org/details/books/calculus-volume-1 .
[3] Disregard any examples with trigonometry.

Recall that the process of differentiation is used to provide a function, called the derivative, which is used to find the slope of a tangent line to the graph of the function. It turns that the process of integration also provides information about the function being integrated.

One reason areas are so useful is that they can represent quantities other than simple geometric shapes. If the units for each side of the rectangle are meters, then the area will have the units *meters times meters* $= m^2$. But if the units of the base of a rectangle are hours and the units of the height are *miles/hour*, then the units of the area of the rectangle are *hours times miles/hour* $=$ *miles*, a measure of distance. Similarly, if the base units are *centimeters* and the height units are *grams*, then the area units are *gram-centimeters*, a measure of work.

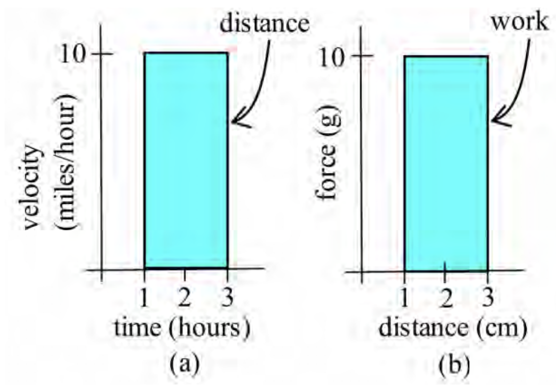


Figure 23.1

For functions representing other **rates** such as the production of a factory (bicycles per day), or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week), the area will still represent the **total** amount of something.

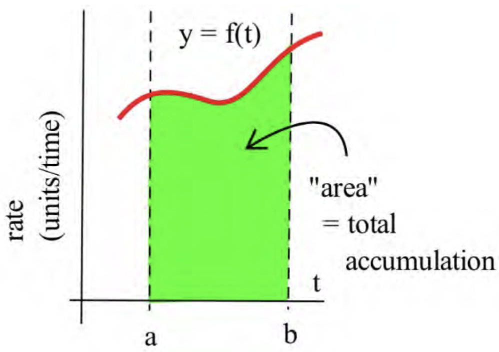


Figure 23.2

Suppose we want to calculate the area between the graph of a positive function f and the x–axis on the interval $[a, b]$ (graphed on figure 23.3). The **Riemann Sum method** is to build several rectangles with bases on the interval $[a, b]$ and sides that reach up to the graph of f. Then the areas of the rectangles can be calculated and added together to get a number called a Riemann Sum of f on $[a, b]$. The area of the region formed by the rectangles is an **approximation** of the area we want.

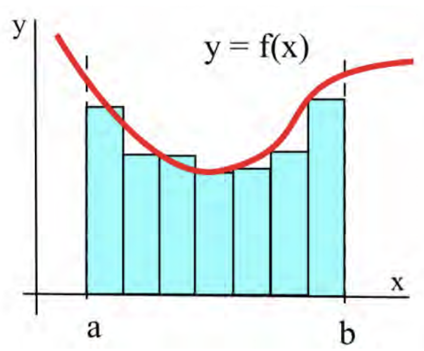


Figure 23.3

These sums of areas of rectangles are called **Riemann sums**. You may see a shorthand notation used when people talk about sums. We won't use it much in this class, but you should know what it means.

Riemann Sum

A **Riemann sum** for a function $f(x)$ over an interval $[a, b]$ is a sum of areas of rectangles that approximates the area under the curve. Start by dividing the interval [a, b] into n subintervals; each subinterval will be the base of one rectangle. We usually make all the rectangles the same width $\triangle x$. The height of each rectangle comes from the function evaluated at some point in its sub interval. Then the **Riemann sum** is:

$$f(x_1)\triangle x + f(x_2)\triangle x + f(x_3)\triangle x + ... + f(x_n)\triangle x \tag{23.1}$$

Sigma Notation: The upper-case Greek letter Sigma $\sum$ is used to stand for Sum. Sigma notation is a way to compactly represent a sum of many similar terms. Using the Sigma notation, the Riemann sum can be written

$$\sum_{i=1}^{n} f(x_i)\triangle x \tag{23.2}$$

This is read aloud as " the sum as $i = 1$ to n of f of x sub i Delta x." The "i" is a counter, like you might have seen in a programming class.

Formal Definition of Definite Integral and Riemann Sum

If $f(x)$ is continuous and over an interval $[a, b]$, and the function $F(x)$ is any antiderivative of $f(x)$ (that is, $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i)\triangle x \tag{23.3}$$

Practical Definition: The definite integral can be approximated with a *Riemann sum* (dividing the area into rectangles where the height of each rectangle comes from the function, computing the area of each rectangle, and adding them up). The more rectangles you use, the narrower the rectangles are, the better your approximation will be.

Since finding definite integrals using limits of *Riemann sums* is cumbersome and is beyond the scope of this course, we will use the techniques of integration we learned in the previous sections that are developed from the **Fundamental Theorem of Calculus, Part 2** or also known as the **Evaluation Theorem** .

Review: Calculating Definite Integral using The Evaluation Theorem

If $f(x)$ is continuous and over an interval $[a, b]$, and the function $F(x)$ is any antiderivative of $f(x)$ (that is, $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a) \tag{23.4}$$

We often see the notation $F(x)\big|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated **limits of integration** a and b to indicate that we should evaluate the function $F(x)$ at the **upper limit** b, and subtract the value of the function $F(x)$ evaluated at the **lower limit** a.

Before introducing the formal definition of the definite integral as **Net Signed Area** and as

Total Area, let's consider the following application of velocity function to get some basic intuition of the difference between these two types of area.

One application of the definite integral is finding displacement when given a velocity function. If $v(t)$ represents the velocity of an object as a function of time, then the area under the curve tells us how far the object is from its original position. In the context of displacement, **net signed area** allows us to take *direction* into account. If a car travels straight *north* at a speed of $v(t) = 60$ mph for 2 hours, it is 120 miles north of its starting position. using integral notation, we have

$$\int_0^2 60dt = 60t\Big|_0^2 = (60 \cdot 2) - (60 \cdot 0) = 120 \tag{23.5}$$

If the car then turns around and travels *south* at a speed of $v(t) = -40$ mph for 3 hours, it is 120 miles *south* of its the turning-around point. Notice that the velocity now has a negative sign since the car is heading to the opposite direction (i.e. heading south instead of heading north). Again, using integral notation, we have

$$\int_2^5 -40dt = -40t\Big|_2^5 = (-40)(5) - (-40)(2) = -200 + 80 = -120 \tag{23.6}$$

As a result, the car will be back at the starting position. In this case the displacement is zero or the **net signed area** is zero as shown in equation 23.7 and in figure 23.4

$$\int_0^2 60dt + \int_2^5 -40dt = 120 - 120 = 0 \tag{23.7}$$

Suppose we want to know how far the car travels overall, regardless of direction. In this case, we want to know the area between the curve and the x-axis, regardless of whether that area is above or below the axis. This is called the **total area**.

Graphically, it is easiest to think of calculating **total area** by adding the areas above the axis and the areas below the axis (rather than subtracting the areas below the axis, as we did with net signed area). To accomplish this mathematically, we use the absolute value function. Thus, the total distance traveled by the car is 240 miles. Using integral notation, we have

$$\int_0^2 |60|dt + \int_2^5 |-40|dt = 120 + 120 = 240 \tag{23.8}$$

Bringing these ideas together formally, we state the following definitions.

Net Signed Area vs. Total Area

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the $x-$axis that lies *above* the axis and let A_2 represent the are between $f(x)$ and the $x-$axis that lies *below* the axis. Then, the **net signed area** between $f(x)$ and the $x-$axis is given by

$$\int_a^b f(x)dx = A_1 - A_2$$

The **total area** between $f(x)$ and the $x-$axis is given by

$$\int_a^b |f(x)|dx = A_1 + A_2$$

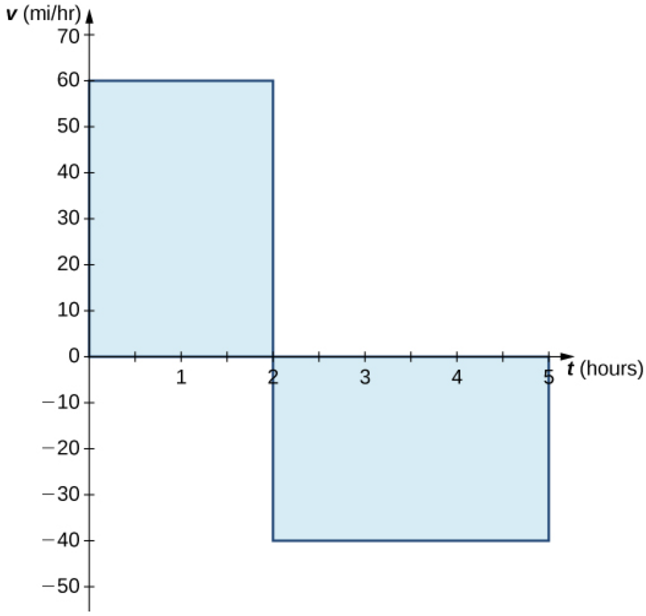


Figure 23.4: The area above the axis and the area below the axis are equal, so the **net signed area** is zero but the **total area** is 240.

Review: Absolute Value Function

For $f(x) \geq 0$ (the graph of f lies *above* the $x-$axis),

$$|f(x)| = f(x) \tag{23.9}$$

For $f(x) < 0$ (the graph of f lies *below* the $x-$axis),

$$|f(x)| = -f(x) \tag{23.10}$$

Rule: Some Useful Properties of the Definite Integral

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \tag{23.11}$$

for constant c. The integral of the product of a constant and a function is equal to the constant multiplied bythe integral of the function.

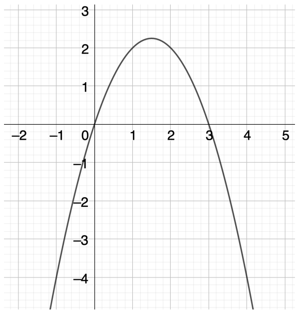
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{23.12}$$

Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

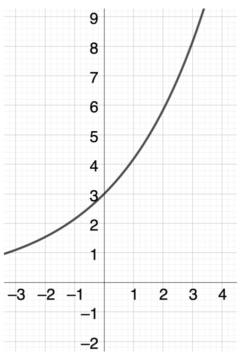
For more rules, see Strang and Herman, *Calculus Volumn 1*[1].

[1] Available free to download from https://openstax.org/details/books/calculus-volume-1 .

Example 23.1. Determine and shade the (total) area bounded by the $x-$axis and the graph of the function $f(x) = -x^2 + 3x$ on the interval from $x = 0$ to $x = 3$. The graph of f is given below.



Example 23.2. Determine and shade the (total) area bounded by the $x-$axis and the graph of the function $f(x) = 3e^{\frac{1}{3}x}$ on the interval from $x = 0$ to $x = 3$. Give the exact form of the area, then a 2-decimal place approximation. The graph of f is given below.



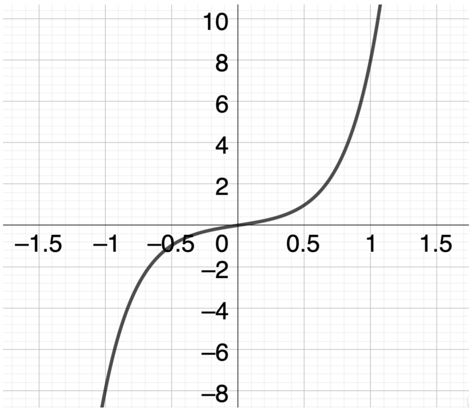
Example 23.3. Given the function $f(x) = x(x^2 + 1)^3$ and its graph given below, answer the following questions:

(a) Determine and shade the **(total) area** bounded by the $x-$axis and the graph of f on the interval from $x = -1$ to $x = 0$.

(b) Determine and shade the **(total) area** bounded by the $x-$axis and the graph of f on the interval from $x = 0$ to $x = 1$.

(c) Determine and shade the **(total) area** bounded by the $x-$axis and the graph of f on the interval from $x = -1$ to $x = 1$.

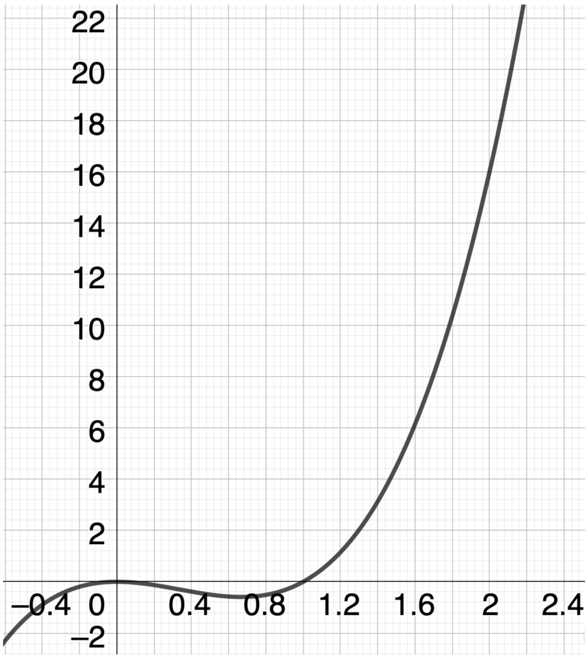
(d) Determine the **net signed area** bounded by the $x-$axis and the graph of f on the interval from $x = 0$ to $x = 1$.



Example 23.4. Given the function $f(x) = 4x^3 - 4x^2$ and its graph given below, answer the following questions:

(a) Determine and shade the **(total) area** bounded by the $x-$axis and the graph of f on the interval from $x = 0$ to $x = 2$.

(b) Determine the **net signed area** bounded by the $x-$axis and the graph of f on the interval from $x = 0$ to $x = 2$.



Short Answers to Examples

23.1 $\int_0^3 |f(x)|\,dx = \int_0^3 f(x)\,dx = -\frac{1}{3}x^3 + \frac{3}{2}x^2\Big|_0^3 = \frac{9}{2} = 4.5$. Note that $|f(x)| = f(x)$ since $f(x) \geq 0$ (the graph of f is above the $x-$axis) on the interval $[0, 3]$.

23.2 $\int_0^3 |f(x)|\,dx = \int_0^3 f(x)\,dx = 9e^{\frac{1}{3}x}\Big|_0^3 = 9e - 9 \approx 15.46$

23.3 **(a)** $\int_{-1}^0 |f(x)|\,dx = \int_{-1}^0 -f(x)\,dx = -\int_{-1}^0 f(x)\,dx = -\frac{1}{8}(x^2+1)\Big|_{-1}^0 = \frac{15}{8} = 1.875$. Note that $|f(x)| = -f(x)$ since $f(x) < 0$ (the graph of f is below the $x-$axis) on the interval $[-1, 0)$.

(b) $\int_0^1 |f(x)|\,dx = \int_0^1 f(x)\,dx = \frac{1}{8}(x^2+1)\Big|_0^1 = \frac{15}{8} = 1.875$. Note that $|f(x)| = f(x)$ since $f(x) \geq 0$ (the graph of f is above the $x-$axis) on the interval $[0, 1]$.

(c) $\int_{-1}^1 |f(x)|\,dx = \int_{-1}^0 |f(x)|\,dx + \int_0^1 |f(x)|\,dx = 1.875 + 1.875 = 3.75$

(d) $\int_{-1}^1 f(x)\,dx = \int_{-1}^0 f(x)\,dx + \int_0^1 f(x)\,dx = -1.875 + 1.875 = 0$

23.4 **(a)** $\int_0^2 |f(x)|\,dx = \int_0^1 |f(x)|\,dx + \int_1^2 |f(x)|\,dx = \int_0^1 -f(x)\,dx + \int_1^2 f(x)\,dx = \left[-(x^4 - \frac{4}{3}x^3)\right]_0^1 + \left[x^4 - \frac{4}{3}x^3\right]_1^2 = \frac{1}{3} + \frac{17}{3} = 6$

(b) $\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \left[x^4 - \frac{4}{3}x^3\right]_0^1 + \left[x^4 - \frac{4}{3}x^3\right]_1^2 = -\frac{1}{3} + \frac{17}{3} = \frac{16}{3}$

Lesson No. 24

Average Value of A Function

Objectives

- Be able to apply the concept and techniques of definite integration to find *Average Value of A Function.*
- Be able to find the *Average Value of A Function* in different applications.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 3.6 Area, Volume, and Average Value*
    * Average Value
- Strang and Herman, *Calculus Volumn 1*[2,3]
  - *Section 5.2 The Definite Integral*
    * Average Value of a Function

Key Terms and Concepts:

- Average value of a function

[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .
[2] Available free to download from https://openstax.org/details/books/calculus-volume-1 .
[3] Disregard any examples with trigonometry.

Suppose we are using an exponential growth model to predict the population, P, of a county at time t years in the future: $P(t) = P_0 e^{kt}$; $t \geq 0$. Clearly, at any point in time, the population will be different from any previous or future point in time. One question that may be asked is what the **average value** of the population would be over some particular time interval, say $t = a$ to $t = b$.

The concept of the average of a set of measurements is understood, that is the sum of n numbers $(a_1, a_2, ..., a_n)$ is their sum divided by n. However, there are an infinite number of points in time, and hence an infinite number of values for the population size. So we cannot sum up all of the population sizes and divide by the number of observations since there is an infinite number of each. The key is to remember our previous approach using definite integrals to determine the area under the graph of a function, in which the sum of an infinite number of terms approached the value of a definite integral. As such, we have the following method for determining the average value:

Average Value of A Function

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the **average value of the function** $f(x)$ (or $\boldsymbol{f_{ave}}$) on $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{24.1}$$

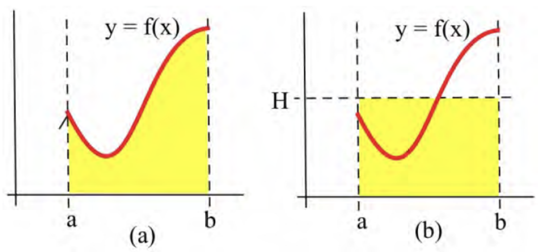


Figure 24.1

The average value of a positive f has a nice geometric interpretation. Imagine that the area under f (Figure 24.1 (a)) is a liquid that can "leak" through the graph to form a rectangle with the same area (Figure 24.1 (b)).

If the height of the rectangle is H, then the area of the rectangle is $H \cdot (b-a)$. We know the area of the rectangle is the same as the area under f so $H \cdot (b-a) = \int_b^a f(x)\,dx$. Then $H = \frac{1}{b-a}\int_b^a f(x)\,dx$, the average value of f on $[a, b]$. The average value of a positive function f is the height H of the rectangle whose area is the same as the area under f.

Example 24.1. During a 9 hour work day, the production rate at time t hours after the start of the shift was given by the function $r(t) = 5 + \sqrt{t}$ cars per hour. Find the average hourly production rate.

Look Back: A note about the units – remember that the definite integral has units cars per hour × hours = cars. But the $\frac{1}{b-a}$ in front has units 1/hours – the units of the average value are cars per hour, just what we expect an average rate to be. **In general, the average value of a function will have the same units as the integrand.**

Example 24.2. After an aggressive campaign to lure high-tech industries, a county with a current population of 10,000 people is expected to see an increase in population over the next 20 years. The county planning commission is using an exponential growth model to predict the population size at time t years from the present: $P(t) = 10000e^{0.08t}$; $0 \leq t \leq 20$ (Note: An annual growth rate of 8% compounded continuously.)

(a) Find $P(0)$ and $P(5)$. Interpret the results.

(b) Determine the predicted average population of the county over the next 5 years. Round to the nearest whole number.

(c) Determine the predicted average population of the county over the interval from $t = 10$ years to $t = 15$ years.

Look Back: Exponential Growth means that the population will grow at a faster rate as time goes on: $P'(t) = k \cdot P(t)$. These are both 5-year intervals but the average value is greater for the 2^{nd} interval as the population continues to increase. DO NOT confuse **rate** and **average**.

Extra Note:

Function averages, involving means and more complicated averages, are used to "*smooth*" data so that underlying patterns are more obvious and to remove high frequency "*noise*" from signals. In these situations, the original function f is replaced by some "average of f." If f is rather jagged time data, then the ten year average of f is the integral $g(x) = \frac{1}{10}\int_{x-5}^{x+5} f(t)\,dt$, an average of f over 5 units on each side of x. For example, the figure 24.2 shows the graphs of a Monthly Average (rather "*noise*" data) of surface temperature data, an Annual Average (still rather "*jagged*"), and a Five Year Average (a much smoother function). Typically the average function reveals the pattern much more clearly than the original data. This use of a "*moving average*" value of "*noisy*" data (weather information, stock prices) is a very common.

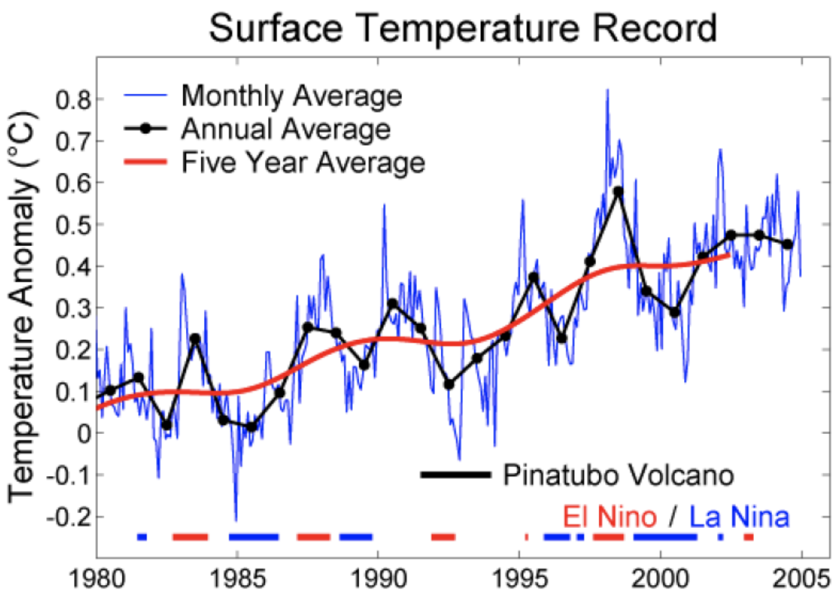


Figure 24.2

Short Answers to Examples

24.1 $f_{ave} = \frac{1}{2} \cdot \left[50000e^{0.08x}\right]_0^2 = 7$ cars per hour.

24.2 **(a)** The current population size is $P(0) = 10000$ people; the population size after 5 years (at time $t = 5$) is $P(5) \approx 14,918$ people

(b) $f_{ave} = 25000e^{0.08t}\Big|_0^5 \approx 12,296$ people.

(c) $f_{ave} = 25000e^{0.08t}\Big|_{10}^{15} \approx 27,364$ people.

Lesson No. 25

Future Value and Present Value

Objectives

- Be able to apply the concept and techniques of definite integration to compute and interpret the *future value of an income stream.*
- Be able to apply the concept and techniques of definite integration to compute and interpret the *present value of an income stream.*

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 3.7 Applications to Business*
    - Continuous Income Stream

Key Terms and Concepts:

- Continuous Income Stream
- Future value of an income stream
- Present value of an income stream

[1]Available free to download from http://www.opentextbookstore.com/details.php?id=14 .

In lesson 18, you reviewed about continuously compound interest that you learned in college algebra. In this simple situation, you initially made a single deposit into an interest-bearing account and let it sit undisturbed, earning interest, for some period of time.

Recall Compounded Continuously

$$A = Pe^{rt} \tag{25.1}$$

A = amount after t years
P = principal
r = annual interest rate (expressed as a decimal)
t = number of years

If you are using the formula in equation 25.1 to find what an account will be worth in the future, $t > 0$ and $A(t)$ is called the **future value**.

$$\text{Future Value} = P \cdot e^{rt} \tag{25.2}$$

By solving the same equation 25.1 for P, you will find what you need to deposit today to have a certain value P sometime in the future and $A(t)$ is called the **present value**.

$$\text{Present Value} = A \cdot e^{-rt} \tag{25.3}$$

The assumption of the calculation of the future value in equation 25.2 is that there is no future deposits or withdrawals once the initial deposit is made. Since this assumption is quite unrealistic, we will consider a more realistic situation where deposits are "flowing continuously" into an account that earns interest. We will refer to this framework as **continuous income stream**

Continuous Income Stream

Suppose money can earn interest at an annual interest rate of r, compounded continuously. Let $K(t)$ be the rate of continuous income function (in dollars per year) that applies between year 0 and year N. The **total income** (TI) for the first N years is

$$TI = \int_0^N K(t)\, dt \tag{25.4}$$

Practical Definition: We will refer to a **continuous income stream** as a sequence of future deposits is made into the account after the initial one and over a long period of time. If the deposits are made regularly enough and the time between deposits is relatively short compared to the overall lifetime of the account, we can think of the money as "flowing" continuously into the account rather than in a large number of discrete chunks[1].

If you want to know the **current worth** of a continuous income stream over time; this is referred to as the **present value of a continuous stream of income**. One formulation of the question is: What amount (present value) would you be willing to invest now (the present) in return for a continuous income stream (with a rate of $K(t)$ dollars per year) over a certain number of years (N) with interest on the income being compounded continuously

[1] Source: https://www.math.ubc.ca/~malabika/teaching/ubc/spring11/math105/value.pdf

at a guaranteed rate over the entire time interval? Another formulation would be: What amount (present value) would you accept right now in exchange for a continuous income stream over a specified number of years?

Calculus allows us to handle situations where "deposits" are flowing continuously into an account that earns interest. As long as we can model the flow of income with a function, we can use a definite integral to calculate the present and future value of a continuous income stream. The idea is – each little bit of income in the future needs to be multiplied by the exponential function to bring it back to the **present**, and then we will add them all up (a definite integral).

Present Value of Income Stream

Suppose money can earn interest at an annual interest rate of r, compounded continuously. Let $K(t)$ be the rate of continuous income function (in dollars per year) that applies between year 0 and year N. The **present value of the continuous income stream** is:

$$PV = \int_0^N K(t)e^{-rt}\,dt \tag{25.5}$$

where $t = 0$ to $t = N$ is the time interval.

Example 25.1. A company that is attempting to downsize its workforce is offering an early retirement option that provides \$20,000 per year for the next 10 years. Another option gives an employee a lump sum payment of \$165,000. Annual interest rates for invested money are assumed to remain at 4.5% compounded continuously for the 10-year time frame.

(a) If an employee would place the lump sum payment in a savings account for the next 10 years, which option would be most beneficial?

(b) Some employees have argued that the constant payment of \$20,000 per year does not take into account inflation over the period. The company changes the offer and will provide an income stream of $20,000e^{.01t}$ dollars per year. Would this change the employee's decision?

Now consider the following scenario (hypothetical): You are depositing money in a retirement account steadily (continuously) throughout the year (this is called a *continuous income stream*) so that in total, $K(t)$ dollars are deposited per year. Assume the account pays an annual interest rate of r compounded continuously. The information we would like to have is the amount that will be in the account after N years. This amount is referred to as the **future value of an income stream**. Similar to the future value with a single one-time deposit in equation 25.2, the **future value of an income stream** can be calculated by replacing the principle (P) in the equation 25.2 with the equation of the present value (PV) (eq.25.5):

$$\text{Equation 25.2}: \text{ Future Value} = P \cdot e^{rt} \implies FV = PV \cdot e^{rN}$$
$$= e^{rN} \cdot \int_0^N K(t)e^{-rt}\,dt = \int_0^N K(t)e^{-rt} \cdot e^{rN}\,dt$$
$$= \int_0^N K(t)e^{rN-rt}\,dt = \boldsymbol{\int_0^N K(t)e^{r(N-t)}\,dt}$$

Future Value of Income Stream

Suppose money can earn interest at an annual interest rate of r, compounded continuously. Let $K(t)$ be the rate of continuous income function (in dollars per year) that applies between year 0 and year N. The **future value of the continuous income stream** after N years is:

$$FV = \int_0^N K(t)e^{r(N-t)}\,dt \tag{25.6}$$

where $t = 0$ to $t = N$ is the time interval.

Example 25.2. Suppose that at age 50, you decide you want to retire at age 60 (10 years) and you set up a supplemental retirement account. You plan to deposit money into the account steadily throughout the year, depositing a total of \$24000 per year. The account is guaranteed to pay an annual interest rate of 4% compounded continuously.

(a) Determine the future value of your account at the end of 10 years. Compare this value with the actual amount deposited over 10 years.

Look Back: The difference between the actual amount deposited and the future value of the account is due to interest, which is being computed on a continuous basis. If we could deposit the entire $240,000 now, the FV would be $A(10) = 24,000e^{0.04(10)} \approx$ $358,038 (more interest; why?).

(b) You were hoping that the supplemental account would be worth $350,000 before your retire. If you are going to wait until the account reaches a value of $350000, after how many years will you be able to retire? (One decimal place)

Look Back: An additional 11.5-10=1.5 years of work increases the FV by 350,000-295,095=$54,905 (only (1.5)(24,000)=$36,000 deposited from the addition 1.5 years of work).

(c) Suppose you do not want to wait past age 60 to retire. At what rate per year do you need to contribute to the account in order to have $350000 after 10 years?

Look Back: Similar analysis can be used to determine the effect on FV if the interest rate is to be modified to achieve and amount at a given number of years in the future for a fixed amount deposited each year.

Example 25.3. A small-business owner is projecting that her annual profit stream over the next 5 years will be constant at \$60,000 per year. Suppose that the annual interest rate for invested money is guaranteed at 4% compounded continuously for the next 5 years. Answer the following questions:

(a) Determine the **present** value of the income stream.

(b) Suppose that an investor has offered to buy her business for \$300,000 payable immediately. Based on the **present** value in part (a), what should the owner decide to do if her desire is to maximize her value after 5 years? Should she sell the business? Why?

(c) Determine the **future** value of the income stream.

(d) If she takes $300,000 and deposit the money in her bank account, how much money would she have in the account in the next 5 year? Comparing this amount to the **future** value in part (c), should she sell the business? Why?

Look Back:

(1) Both the present value and the future value of the income stream suggest that she should sell her business and take $300,000 now.

(2) If she accept $271,904 (the amount of the present value) for the income stream now and put it in her bank account, the amount in her account in the next 5 year will be $A(5) = 271,904e^{0.04(5)}\approx$ $332,104 which is equal to the amount of the **future value !** So the **present value** is telling us what we should accept now to have the same future value as the income stream guarantees us over the same time frame.

Short Answers to Examples

25.1 **(a)** $PV = 20,000\left[\frac{1}{-0.045}e^{-0.045t}\right]_0^{10}$ ≈$161,054; take the lump sum payment since you are being offered more than the income stream is currently worth.

(b) $PV = 20,000\left[\frac{1}{-0.035}e^{-0.035t}\right]_0^{10}$ ≈$168,750. With this adjustment, the income stream is now worth more than the lump sum being offered. So, take the 2nd option

25.2 **(a)** $FV = -600,000\left[e^{0.04(10-t)}\right]_0^{10}$ ≈$295,095

(b) $FV = 350,000 \Longrightarrow \left[-600,000e^{0.04(N-t)}\right]_0^N = 350,000 \Longrightarrow N = \dfrac{\ln\left(\frac{19}{12}\right)}{0.04} \approx 11.5$ years.

(c) $FV = 350,000 \Longrightarrow K(t)\left[\frac{1}{-0.04}e^{0.04(10-t)}\right]_0^{10} = 350,000 \Longrightarrow K(t) \approx$ $28,465 per year deopsit.

25.3 **(a)** $PV = 60,000\left[\frac{1}{-0.04}e^{-0.04t}\right]_0^5 \approx$ $271,904.

(b) The owner is being offered more (right now) than the income stream is worth (right now). So, she should see the business and take the $300,000.

(c) $FV = 60,000\left[\frac{1}{-0.04}e^{0.04(5-t)}\right]_0^5$ ≈$332,104.

(d) $A(5) = 300,000e^{0.04(5)} \approx$ $366,421 which is more than the future value. Therefore, she should sell her business.

Lesson No. 26

Functions of Two Variables and Partial Derivatives

Objectives

- Be able to determine and interpret **partial derivatives** of a function of two variables in different applications.

Suggested Reading:

- Calaway, Hoffman, and Lippman, *Applied Calculus*[1]
  - *Section 4.1 Functions of Two Variables*
    - Functions of Two Variables
    - Formulas and Tables
    - Graphs
    - Functions of Two Real-Life Variables

Key Terms and Concepts:

- Rate of change of a function of more than one variable

[1] Available free to download from http://www.opentextbookstore.com/details.php?id=14 .

To this point, all of the functions we have analyzed have been functions of one variable such as a population function for a community which we have denoted by $P(t)$. Clearly, there are many "factors" that come into play that have an effect on a community's population over time; for example, there are economic factors and social factors that may play a role in whether the population is predicted to be increasing or decreasing over time. Real life is rarely as simple as one input – one output. Many relationships depend on lots of variables. Here are some more examples:

- If we consider the value of an account which compounds interest continuously, we have seen that the value depends on the initial amount we invest, P, the interest rate, r, and time, t, over which the initial investment grows. So technically, the model $A(t) = Pe^{rt}$ is an oversimplification since the only independent variable is time. It may be more informative in terms of analysis to represent the amount in the account as a **function of more than one variable:** $A(P, r, t) = Pe^{rt}$. Note that to evaluate this function, we need inputs for each of the **independent variables** P,r, and t.
- The air resistance on a wing in a wind tunnel depends on the shape of the wing, the speed of the wind, the wing's orientation (pitch, yaw, and roll), plus a myriad of other things that I can't begin to describe.
- The amount of your television cable bill depends on which basic rate structure you have chosen and how many pay-per-view movies you ordered.

Since the real world is so complicated, we want to extend our calculus ideas to functions of several variables. If $X_1, X_2, X_3, ..., X_n$ are real numbers, then $(X_1, X_2, X_3, ..., X_n)$ is called an $n-$tuple. This is an extension of ordered pairs and triples. A function of n variables is a function whose domain is some set of $n-$tuples and whose range is some set of real numbers. For much of what we do here, everything would work the same if we were working with 2, 3, or 47 variables. Because we're trying to keep things a little bit simple, we'll concentrate on functions of two variables.

A Function of Two Variables

A function of two variables is a function – that is, to each input is associated exactly one output.

The inputs are ordered pairs, (x, y). The outputs are real numbers. The domain of a function is the set of all possible inputs (ordered pairs); the range is the set of all possible outputs (real numbers).

The function can be written $z = f(x, y)$.

Functions of two variables can be described numerically (a table), graphically, algebraically (a formula), or in English.

We will often now call the familiar $y = f(x)$ a function of one variable.

Example 26.1. The cost $C(d, m)$ in dollars for renting a car for d days and driving it m miles is given by the formula $C(d, m) = 40d + 0.15m$

(a) What is the cost of renting a car for 3 days and driving it 200 miles?

(b) What is $C(100, 4)$? What is $C(4, 100)$?

(c) Suppose we rent the car for 3 days. Is C an increasing function of miles?

The graph of a function of two variables is a surface in three-dimensional space. Let's start by looking at the 3–dimensional rectangular coordinate system, how to locate points in three dimensions, and distance between points in three dimensions.

In the 2–dimensional rectangular coordinate system we have two coordinate axes that meet at right angles at the origin, and it takes two numbers, an ordered pair (x, y), to specify the rectangular coordinate location of a point in the plane (2 dimensions). Each ordered pair (x, y) specifies the location of exactly one point, and the location of each point is given by exactly one ordered pair (x, y). The x and y values are the coordinates of the point (x, y).

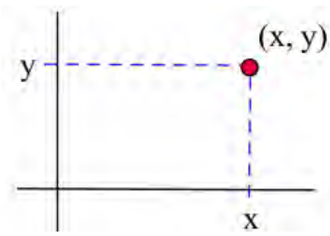


Figure 26.1

The situation in three dimensions is very similar. In the 3–dimensional rectangular coordinate system we have three coordinate axes that meet at right angles, and three numbers, an ordered triple (x, y, z), are needed to specify the location of a point. Each ordered triple (x, y, z) specifies the location of exactly one point, and the location of each point is given by exactly one ordered triple (x, y, z). The x, y and z values are the coordinates of the point (x, y, z).

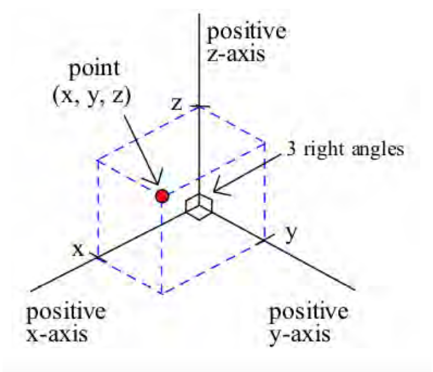


Figure 26.2

Figure 26.3 is an example of the plot of the locations of the following points on the 3–dimensional rectangular coordinate system:
$P = (0, 3, 4)$
$Q = (2, 0, 4)$
$R = (1, 4, 0)$
$S = (3, 2, 1)$ and $T = (-1, 2, 1)$

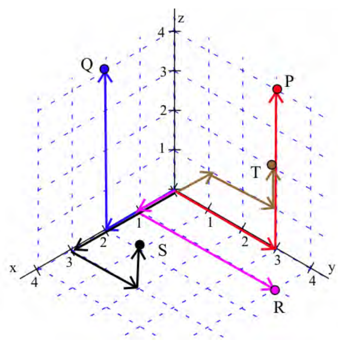


Figure 26.3

Once we can locate points, we can begin to consider the graphs of various collections of points. By the graph of "$z = 2$" we mean the collection of all points (x, y, z) which have the form "$(x, y, 2)$". Since no condition is imposed on the x and y variables, they take all possible values. The graph of $z = 2$ is a plane parallel to the xy–plane and 2 units above the xy–plane. Similarly, the graph of $y = 3$ is a plane parallel to the xz–plane , and $x = 4$ is a plane parallel to the yz–plane. (Note: The planes have been drawn as rectangles, but they actually extend infinitely far.)

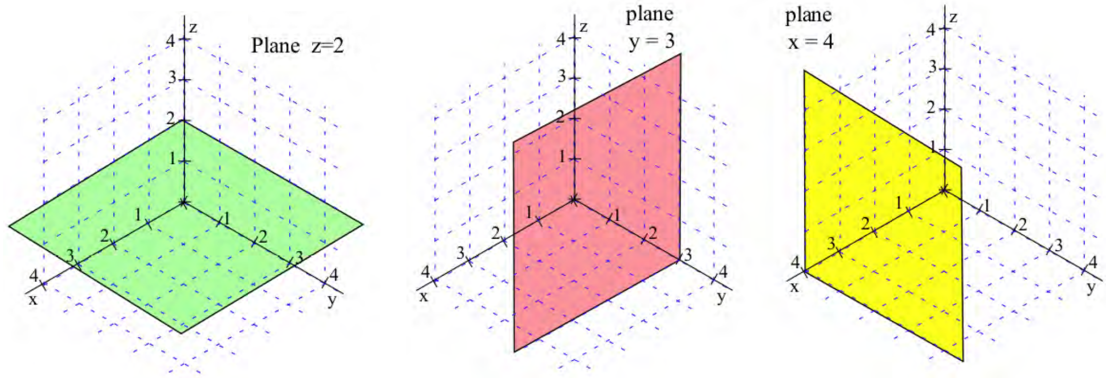


Figure 26.4

Of course, when we move beyond the case of a single independent variable, we lose the luxury of examining a graph of a function in the $x - y$ coordinate system. As such, the relationships between the **independent variables** and the value of the function are more difficult to visualize.

Imagine a surface, the graph of a function of two variables. Imagine that the surface is smooth and has some hills and some valleys. Concentrate on one point on your surface. What do we want the derivative to tell us? It ought to tell us how quickly the height of the surface changes as we move. But, which direction do we want to move? This is the reason that derivatives are more complicated for functions of several variables – there are so many directions we could move from any point. As such, the idea of fixing one variable and watching what happens to the function as the other changes is the key to extending the idea of derivatives to more than one variable. This idea is the concept of **partial derivatives** that will provide us useful information about the relationships between the **independent variables** and the value of the function.

Partial Derivatives

Suppose that $z = f(x, y)$ is a function of two variables.

The **partial derivative of f with respect to x** is the derivative of the function $f(x, y)$ where we think of x as the only variable and act as if y is a constant.

The **partial derivative of f with respect to y** is the derivative of the function $f(x, y)$ where we think of y as the only variable and act as if x is a constant.

The "with respect to x" or "with respect to y" part is really important – you have to know and tell which variable you are thinking of as THE **independent variable**.

Notation for the Partial Derivative

Suppose that $z = f(x, y)$ is a function of two variables.

The **partial derivative of $y = f(x)$ with respect to x** is written as $f_x(x, y)$, or z_x simply f_x

The Leibniz notation is $\frac{\partial f}{\partial x}$, or $\frac{\partial z}{\partial x}$

We use an adaptation of the $\frac{\partial z}{\partial x}$ notation to mean "find the partial derivative of $f(x, y)$ with respect to x:" $\frac{\partial}{\partial x}(f(x, y)) = \frac{\partial f}{\partial x}$

To compute a partial derivative from a formula:

If $f(x, y)$ is given as a formula, you can find the partial derivative with respect to x algebraically by taking the ordinary derivative thinking of x as the only variable (holding y fixed).

Similarly, everything here works the same way if we are trying to find the partial derivative with respect to y–just think of y as your only independent variable and act as if x is constant.

Example 26.2. The cost, C, of materials for a rectangular enclosure (warehouse, office building, etc.) depends on the dimensions of the enclosure: length (x), width (y), height (z), and the materials cost for the top, sides, and bottom of the enclosure. Suppose the cost function for such a structure is given by $C(x, y, z) = 6xy + 10xz + 10yz$. Currently, the plan for the dimensions calls for $x = 50$ feet, $y = 80$ feet, and $z = 20$ feet.

(a) Find the **rate** at which the cost is expected to change (marginal cost) if we decide to increase only the height by 1 foot. That is, compute the **partial derivative** of C with respect to z, which we denote by $\frac{\partial C}{\partial z}$, and evaluate at the given values for x, y, and z. Interpret the result.

(b) Repeat the analysis above if we decide only the width by 1 foot and interpret the result.

Example 26.3. Certain tests of kidney performance need to have the results adjusted based on the body surface area, A, of the patient. For a person who weighs W pounds and has height H inches, the following function is used to approximate the person's body surface area: $A(W, H) = 0.1091 W^{0.425} H^{0.725}$ square feet.

(a) Determine the approximate body surface area for a 10-year old patient who weighs 100 pounds and is 48 inches tall.

(b) Determine the **rate** at which the patient's body surface area is changing with respect to a 1 pound increase in weight with the height being held constant.

(c) Determine the **rate** at which the patient's body surface area is changing with respect to a 1 inch increase in height with the weight being held constant.

Short Answers to Examples

26.1 **(a)** $C(3, 200) = \$150$.

(b) $C(100, 4)$ represents the cost of renting a car for 100 days and driving it for 4 miles. $C(100, 4)$ represents the cost of renting a car for 4 days and driving it for 100 miles. Notice that $C(100, 4)$ makes more sense!

(c) If d is fixed to be $d = 3$, $C(d, m)$ becomes $C(3, m) = 120 + 0.15m$ which is simply a linear function with a positive slope of 0.15. With the positive slope, this is an increasing function.

26.2 **(a)** $\dfrac{\partial C}{\partial z} = 10x + 10y$; $\left.\dfrac{\partial C}{\partial z}\right|_{x=50, y=80, z=20} = 10(50) + 10(80) = \$1300$ per 1 foot increase in height.

(b) $\dfrac{\partial C}{\partial y} = 6x + 10z$; $\left.\dfrac{\partial C}{\partial y}\right|_{x=50, y=80, z=20} = 6(50) + 10(20) = \$500$ per 1 foot increase in width.

26.3 **(a)** $A \approx 12.79$ sq.ft.

(b) $\dfrac{\partial A}{\partial W} = 0.046 \cdot \dfrac{H^{0.725}}{W^{0.575}}$ sq.ft.per pound; $\left.\dfrac{\partial A}{\partial W}\right|_{W=100, H=48} = 0.0543$ sq.ft.per pound; actual change ≈ 0.054187

(c) $\dfrac{\partial A}{\partial H} = 0.079 \cdot \dfrac{W^{0.425}}{H^{0.275}}$ sq.ft.per inch; $\left.\dfrac{\partial A}{\partial H}\right|_{W=100, H=48} = 0.193$ sq.ft.per inch; actual change ≈ 0.1883.

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[1] Jay Abramson. *College Algebra*. URL: https://openstax.org/details/books/college-algebra.

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[3] Dale Hoffman. *Contemporary Calculus I*. URL: http://scidiv.bellevuecollege.edu/dh/Calculus_all/Calculus_all.html.

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