1 BASIC CONCEPTS: REVIEW AND PREVIEW

IN CHAPTER 1 WE CONSIDER the nature of mathematics, where mathematics comes from, and how it is used. This chapter lays a foundation for the entire book. Section 1.1 describes how mathematical models represent real-world problems, including calculator use and approximations. Sections 1.2 and 1.3 review terminology and the properties of numbers related to ordering and absolute values. Section 1.4 introduces the ideas of graphs and their uses, both on a number line and in the plane. Section 1.5 reviews some of the techniques from elementary algebra, how these techniques relate to graphing technology, and how they allow us to find solution sets for a variety of kinds of open sentences. The final section demonstrates how to approach and formulate a number of different problems, introducing techniques that are useful throughout the rest of the book and all of the study of mathematics.

1.1 MATHEMATICS MODELS THE WORLD

> Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.
>
> *Joseph Fourier*

What Is Mathematics?

Consider these situations and note what they have in common:

1. At the edge of the Beaufort Sea, north of the Arctic Circle, a dozen adults of the Inuit people are tossing a young boy aloft on a human-powered trampoline made of a blanket.

2. From the observation deck of the Sears Tower (the world's tallest building at 1454 feet) a visitor can see nearly six miles further out into Lake Michigan than someone at the top of the John Hancock Center (1127 feet tall).
3. A pilot of a Goodyear blimp heading south over Lake Okeechobee at 5300 feet wants to estimate the time remaining before visual contact with the Orange Bowl, where a football game is to be televised.

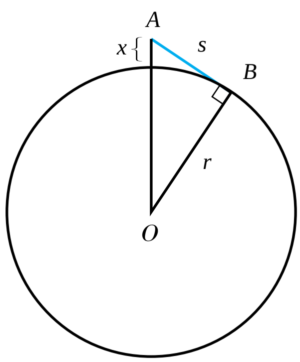


FIGURE 1
How far can you see from x miles above the earth?

Each of these situations deals with the curvature of the earth's surface and the fact that it is possible to see farther from a higher elevation. The Inuits want to get an observer high enough to see whether the pack ice is breaking up in the spring; the Sears Tower is 327 feet taller than the Hancock Center; at an elevation just over a mile, how far can the blimp pilot see?

Mathematics strips away the differences in these situations and finds one simple model to describe common key features. Figure 1 shows a cross section of the earth as a circle with center at O and radius r. Our model assumes a spherical earth, a fairly good approximation of the truth. From point A located x feet above the earth, the line of sight extends to B. (Line AB is tangent to the circle and hence perpendicular to radius OB.)

All of the situations listed above fit in this structure. The Inuit boy, the visitor to the top of a skyscraper, and the blimp pilot could each be seen as located at point A for different values of x. For any given x, applying the Pythagorean theorem (discussed in Section 1.4) to the right triangle AOB gives the corresponding distance s to the horizon.

$$\begin{aligned} r^2 + s^2 &= (r + x)^2, \text{ where } x, r, \text{ and } s \text{ are in miles.} \\ r^2 + s^2 &= r^2 + 2rx + x^2 \\ s^2 &= 2rx + x^2 \\ s &= \sqrt{2rx + x^2}. \end{aligned}$$

Part of the power of mathematics comes from its capacity to express in a single sentence truths about several seemingly diverse situations. The solution to one equation automatically applies to any other application that gives rise to the same equation. The expression for s can be used to solve any of the problems listed above. See Exercises 39–44.

Geometry really turned me on. My father taught me by giving me problems to solve. He gave me thousands of geometry problems while I was still in high school. After he gave me one and I came back with a solution, he would say, "Well, I'll give you another one." The solving of thousands of problems during my high school days—at the time when my brain was growing—did more than anything else to develop my analytic power.
George Dantzig

Mathematics and the Real World

Much of the importance and vitality of mathematics comes from its relationship with the world around us. Humans invented numbers to count our sheep; we created rules for addition and multiplication as we needed to barter or compare land holdings. As human understanding of the world grew more sophisticated, mathematical tools grew as well. Sometimes mathematical curiosity led people in unexpected directions and their explorations became important for their own sake.

Mathematics is a lively part of our intellectual heritage. Some of the most intriguing and challenging mathematical investigations grew out of attempts to answer seemingly innocuous questions or understand simple observations. The most lasting and significant human achievements are direct consequences of our desire to understand and control the world.

Mathematics and Mathematical Models

When we encounter a problem whose solution involves the use of mathematics, we must decide how much detail is essential. In the line-of-sight examples the solution assumed the earth as a perfect sphere. The differences between that mathematician's earth and the actual globe are substantial. The equation for the distance to the horizon (s miles) implies that someone could see more than 40 miles from the top of either the Sears Tower or the Hancock Center; on a clear day someone in Chicago might want to check that conclusion.

In a mountain valley ringed by peaks that rise several thousand feet, it isn't possible to see 40 miles in any direction, even from 1500 feet up. Does that invalidate our mathematical model? Of course not; we must know something about particulars when we interpret a result. Questions about the way the world works frequently require simplifying assumptions to make the problems more tractable. See the Historical Note, Mathematical Models and Gravity (p. 135).

Technology

We assume that every student has access to some kind of graphing technology that permits graphing functions. Yours may be as simple as a graphing calculator or as complex as sophisticated computer software. *We use the language of graphing calculators* in this text, but you can use any available technology to do the work. If you are working with technology that is new to you, perhaps the most important thing is to experiment freely so that you become comfortable and confident with your own tools. Verify every computation in our examples. Talk with others about what works and what doesn't. Make sure that you can produce the same kinds of pictures that we show in the text.

Each graphing calculator and computer graphing software package is different; display screens have different proportions, and commands and syntax vary. We cannot give instructions to fit every kind of machine, but it should be possible to duplicate our computations and calculator graphs on almost any kind of graphing technology you have available. In our Technology Tips we make suggestions that may be helpful. If it seems that your calculator won't do something we are describing, discuss it with your instructor, look at your owner's manual, and ask classmates. There may be another way to get around the problem.

Calculators and computers have become incredibly powerful, but they remain limited. While they can do wonders, they may still properly be called "Smart-Stupids," a name coined by Douglas Hofstadter. However amazing their computing power, the machine is not smart enough to know that we *meant* to press $\boxed{+}$ when we pressed the $\boxed{\div}$ key.

Approximate Numbers and Significant Digits

When we use mathematics to model the real world, we have to realize that measurements of physical quantities can be only approximations. A biologist may be able to count exactly the number of eggs in a bird's nest, but comparing the volume or weight of two eggs requires *approximate numbers,* since any number we use is only as good as our measuring device. We also use approximate numbers when we need a decimal form for a number such as $\sqrt{3}$.

Questions involving approximations entail decisions about the tolerable degree of error. Error tolerance decisions usually hinge on concerns other than mathematics, but all of us must make such decisions in working problems that involve measurements or when we use calculators. We need guidelines.

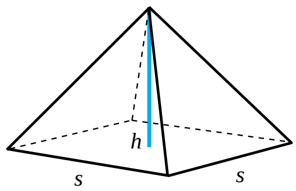


FIGURE 2
The volume of a pyramid of height h and sides s: $V = \frac{1}{3}s^2h$.

Perhaps the greatest problem in working with calculators is interpreting displayed results. When we enter data, the calculator returns so many digits so quickly and easily that we may think we have gained more information than we really have. This difficulty can be illustrated by an example from a recent calculus text. The book derives an equation for the volume of a pyramid, as shown in Figure 2, and then applies the formula to find the volume of the Great Pyramid of Cheops. The original dimensions are given (approximately) as

$$s = 754 \text{ feet} \quad \text{and} \quad h = 482 \text{ feet}.$$

When we substitute these numbers into the formula, a calculator immediately displays 91341570.67, from which the authors conclude that the volume is "approximately 91,341,571 cubic feet." In the following example, we illustrate why we are not justified in rounding to the nearest cubic foot, even if the values for s and h are measured to the nearest foot.

▶EXAMPLE 1 ***Appropriate rounding*** Assuming that the height h and side length s are measured to the nearest foot, giving 482 feet for h and 754 for s, how much variation can this leave in the computed volume, using $V = \frac{s^2h}{3}$?

Strategy: Let V_0 be the volume using the smaller values of s and h, while V_1 is the volume using the larger values of s and h. Compute V_0 and V_1, and then compare the results.

Solution

To say the linear measurements are correct to the nearest foot means that they satisfy the inequalities

$$753.5 < s < 754.5 \quad \text{and} \quad 481.5 < h < 482.5.$$

Using the smaller values for s and h gives

$$V_0 = \frac{(753.5)^2(481.5)}{3} \approx 91{,}125{,}841.12.$$

The upper values for s and h yield

$$V_1 = \frac{(754.5)^2(482.5)}{3} \approx 91{,}557{,}631.87.$$

The difference between V_1 and V_0 is

$$V_1 - V_0 \approx 431{,}790.75.$$

The computed and actual volumes could differ by nearly *half a million cubic feet!* See Example 3. ◀

The world of mathematics is an *ideal* world, dealing with exact numbers and precise relationships, but mathematics also says much about the inexactitude and fuzziness of the physical world. In applying mathematics, we create a precise model to mirror an imprecise reality. Whenever mathematics delivers an answer for an applied problem, we must ask what the numbers mean and what degree of significance they have for the original problem.

What Do the Digits Mean?

What is the diagonal of a square that measures a mile on each side? The mathematical model of a square of side 1 has a diagonal of exactly $\sqrt{2}$. Our calculator displays 1.414213562 for $\sqrt{2}$. For a mile, what does each of these decimal places measure? We must consider what degree of accuracy makes sense in the real world. If the sides are measured only to the nearest 10 feet, it makes no sense to pretend to have an accuracy indicated by enough digits to measure the thickness of a blade of grass! The following box shows something of the meaning of each decimal place when we talk of the decimal parts of a mile.

Decimal parts of a mile:	
.1	Two football fields
.01	Width of a city street
.001	Height of a 5′3″ person
.0001	6″ (less than a handspan)
.00001	Width of a finger
.000001	$\frac{1}{16}$″ (smallest mark on a ruler)
.0000001	Thickness of 1.5 sheets of paper
.00000001	Half the thickness of a human hair

Significant Digits, Precision, and Scientific Notation

When multiplying and dividing approximate numbers, we consider **significant digits,** the digits that indicate measured accuracy. Normally zeros that serve only to locate the decimal point are not significant. These numbers each have four significant digits:

400.5 ft. 0.002596 mm. 1.032 km. 93,410,000 mi.

In scientific applications, special notation makes it easy to identify the significant digits. By moving the decimal point as needed, any positive real number can be written as a product of a number between 1 and 10 and some power of 10. A number written as such a product is said to be in **scientific notation.** We would write the four numbers above in scientific notation as follows.

$4.005 \times 10^2 \qquad 2.596 \times 10^{-3} \qquad 1.032 \qquad 9.341 \times 10^7$

Usually we do not write 10^0 for a number that is already between 1 and 10.

In addition and subtraction, our concern is *precision,* the *question of which decimal places have meaning*. If we are told that an ancient tree is 3000 years old, we consider that 3000 as a less precise number than the age of a 17 year old.

Guidelines for computation with approximate numbers

In multiplication and division with approximate numbers, round off final results to the *least number of significant digits* in the data used. Results are no more accurate than the least accurate data; *record no more significant digits than occur in* any *of the given data.*
In addition and subtraction with approximate numbers, round off final results to the *least level of precision* in the data used.

►***EXAMPLE 2*** ***Significant digits*** Determine which digits are significant, and write the number in scientific notation.

(a) 325.6 **(b)** 28.40 **(c)** 205,000 **(d)** 0.00640

Solution

(a) All digits are significant: $325.6 = 3.256 \times 10^2$.
(b) The last 0 does not locate the decimal point; all digits are significant: $28.40 = 2.480 \times 10$.

(c) Without additional information, we can only assume that the first three digits are significant: $205{,}000 = 2.05 \times 10^5$. If we had some reason to believe that 205,000 represented a measurement accurate to the nearest hundred, then four digits would be significant and we would write $205{,}000 = 2.050 \times 10^5$.

(d) The first three zeros just locate the decimal point, but the last zero is significant: $0.00640 = 6.40 \times 10^{-3}$. ◀

Strategy: Since both s and h are given to three significant digits, use the formula and round off the result to the same accuracy.

►*EXAMPLE 3* ***Rounding off*** Use the formula $V = \frac{s^2h}{3}$ to calculate the volume of the Great Pyramid of Cheops, where $s = 754$ feet and $h = 482$ feet.

Solution

Follow the strategy.

$$V = \frac{(754)^2(482)}{3} \approx 91{,}341{,}570.67 \approx 91{,}300{,}000.$$

Hence the volume is approximately 91,300,000 cubic feet. As we would expect, the result lies well between the extreme values of V_0 and V_1 in Example 1. ◀

►*EXAMPLE 4* ***Precision*** Simplify, assuming that the numbers are approximate measurements:

(a) $2.483 + 15.4$ **(b)** $7200 - 1720 + 32$

Solution

(a) Since 15.4 (measured to the nearest tenth) is less precise than 2.483 (measured to the nearest thousandth), round off the sum to the precision of the less precise number.

$$2.483 + 15.4 = 17.883 \approx 17.9.$$

(b) The least precise of these numbers is 7200, so round off the sum to the same level of precision, to the nearest 100:

$$7200 - 1720 + 32 = 5512 \approx 5500. \quad ◀$$

The Number Pi

The number pi, denoted by the Greek letter π, pops up in the most unexpected places in mathematics, several of which we encounter in this book. See the Historical Note, "The Number π." Most of us first meet pi in connection with circles through its historical definition as the ratio of the circumference to the diameter of a circle. Scientific calculators have a key labeled $\boxed{\pi}$, which approximates π:

$$\pi \approx 3.141592654.$$

►*EXAMPLE 5* ***Rounding off*** A rectangular garden measures 23 feet by 36 feet (to the nearest foot). What is the distance c between its opposite corners? (See Figure 3.)

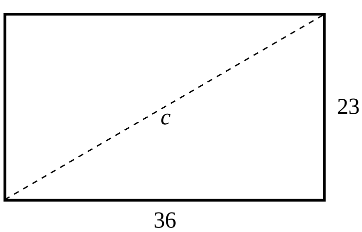


FIGURE 3
Find the distance between opposite corners.

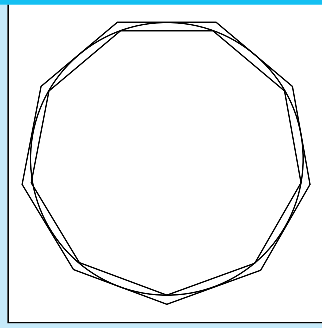
Solution

Using the Pythagorean theorem,

$$c = \sqrt{23^2 + 36^2} \approx 42.72001873.$$

Our rule suggests stating the result to two significant digits, so the diagonal distance is 43. ◀

HISTORICAL NOTE THE NUMBER π



The number π (which represents the ratio of the circumference to the diameter of a circle) has fascinated people since antiquity. The Babylonians, Chinese, and Hebrews all knew that the value of π was near 3. The Egyptians computed the area of a circle by squaring $\frac{8}{9}$ of the diameter, implying a value for π of $\frac{256}{81} = 3.16$. In about 200 B.C., Archimedes used inscribed and circumscribed polygons (see figure) to get the bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$. We still use $\frac{22}{7}$ as a convenient (and fairly good) approximation.

More than 1700 years passed before a Frenchman, Viète, significantly improved on the efforts of the Greeks. Real progress accompanied the invention of calculus. Sir Isaac Newton calculated π to 15 decimal places and confessed to a colleague, "I am ashamed to tell you to how many figures I carried these calculations, having no other business at the time." By 1706 Machin in England correctly computed π to 100 digits.

Through all this time, people were looking for a repeating pattern of digits. It wasn't until 1761 (2000 years after the Pythagoreans proved that $\sqrt{2}$ is not rational) that Lambert (from Germany) finally proved that π is *irrational,* so the pattern of digits will never repeat.

►EXAMPLE 6 ***Area of circle*** The radius of a circle measures 24.5 cm. What is its area?

Solution

The equation for the area A of a circle in terms of the radius is $A = \pi r^2$. Replacing r with 24.5, and rounding off to three significant digits,

$$A = \pi(24.5)^2 \approx 1890.$$

The area is approximately 1890 cm^2. ◄

EXERCISES 1.1

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. If x is any real number, then x^2 is positive.
2. If x is a number such that $\frac{1}{x} < 1$, then x must be greater than 1.
3. For all nonnegative numbers x and y, $\sqrt{x + y} \geq \sqrt{x} + \sqrt{y}$.
4. Without additional information, we must assume that the zeros in the numbers 45,000 and 0.0045 are not significant digits.
5. All of the zeros in the numbers 3.005 and 4.720 are significant digits.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. In the decimal representation of the quotient $\frac{5}{7}$, the digit in the fourth decimal place is ______ .
7. The number of significant digits in
   (a) 10.2 is ______. **(b)** 1200 is ______.
   (c) 0.12 is ______.
8. If the dimensions (in inches) of a cereal box are measured to be $3.1 \times 7.2 \times 8.0$, then the diagonal of the box can be calculated and the number of meaningful significant digits is ______ .
9. Of the three numbers π, $\frac{333}{106}$, $\frac{355}{113}$, the largest one is ______ .
10. If the radius of a circle is doubled, then its area increases by a factor of ______ .

Develop Mastery

Exercises 1–15 ***Calculator Evaluation*** *The purpose of these exercises is to give you practice with your calculator. Many of the exercises are simple enough to solve in your head. Their real value comes from the effort to make your calculator do all the necessary steps and agree with the result in brackets. Some answers are rounded off to three decimal places.*

1. $(6 + 3) \cdot 8$ [72]

2. $6 \cdot 3 + 3 \cdot 8$ [42]

3. $2 \cdot 3^2 + 3 \cdot 4^2$ [66]

4. $\dfrac{1/2 - 3}{4}$ [−0.625]

5. $\dfrac{2/3 + 3/4}{7/8}$ [1.619]

6. $(2 \cdot 3)^2 + (3 \cdot 4)^2$ [180]

7. $\dfrac{(4.5 - 3.1)^2}{5.6}$ [0.35]

8. $4.5^2 - \dfrac{(3.1)^2}{5.6}$ [18.534]

9. $\dfrac{(4.5)^2 - (3.1)^2}{5.6}$ [1.9]

10. $\sqrt{2 + 3}$ [2.236]

11. $\sqrt{2} + \sqrt{3}$ [3.146]

12. $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}}$ [1.284]

13. $\dfrac{1}{\sqrt{2} + \sqrt{3}}$ [0.318]

14. $\dfrac{1 + \sqrt{3^2 - 1}}{3}$ [1.276]

15. $\sqrt{\dfrac{4 - \sqrt{2}}{2}}$ [1.137]

Exercises 16–17 ***Scientific Notation*** *Write in scientific notation and tell which digits are significant.*

16. **(a)** 406 **(b)** 40600 **(c)** 406.0 **(d)** 0.0406

17. **(a)** 807 **(b)** 8070 **(c)** 807.0 **(d)** 0.008070

Exercises 18–19 ***Significant Digits*** *Tell which digits are significant, and then express in standard decimal notation.*

18. **(a)** 3.2×10^3 **(b)** 5.06×10^{-3} **(c)** 8.400×10^{-2} **(d)** 3.40×10^3

19. **(a)** 6.4×10^3 **(b)** 7.06×10^{-3} **(c)** 3.470×10^{-2} **(d)** 5.60×10^4

Exercises 20–21 Round off to two significant digits.

20. **(a)** 3254 **(b)** 4.32 **(c)** 0.05642 **(d)** 357894

21. **(a)** 80.5 **(b)** 0.35501 **(c)** 0.03618 **(d)** 247631

Exercises 22–23 ***How Big Is a Trillion?***

22. A stack of 250 dollar bills is about 1 inch high. How high would a stack of
(a) 4 million **(b)** 4 billion **(c)** 4 trillion dollar bills reach? Make a guess before you do any calculations, such as: 250 feet, $\frac{1}{4}$ mile, the distance from Boston to New York (230 miles), the distance from the earth to the moon (240,000 miles), the distance from the earth to the sun. (The federal debt is well over 5 trillion dollars.)

23. Would you classify as a baby, a teenager, an adult, an old person, or otherwise a person who is
(a) a million seconds old?
(b) a billion seconds old?
(c) a trillion seconds old?

Exercises 24–25 Give decimal approximations rounded off to three significant digits.

24. **(a)** $\sqrt{7}$ **(b)** $\dfrac{2\pi}{5}$ **(c)** $\sqrt{4\pi}$ **(d)** $\sqrt{25 - \sqrt{5}}$

25. **(a)** $\sqrt{3}$ **(b)** $\dfrac{\pi}{5}$ **(c)** $\sqrt{25\pi}$ **(d)** $\sqrt{17 + \sqrt{47}}$

Exercises 26–32 ***Rounding Off*** *Consider all data as measured numbers. Round off each computation to an appropriate number of significant digits.*

26. $x = 33.7$, $y = 2.35$, $z = 0.431$. Find
(a) xy **(b)** yz **(c)** $\dfrac{y}{z}$.

27. Evaluate
(a) $32.51 + 63.2$ **(b)** $65.1 - 23.18 + 2.407$
(c) $\sqrt{3.82^2 + 2.63^2}$.

28. Find the length of a diagonal of a rectangle with sides of 31.4 feet and 16.3 feet.

29. The radius of a circle is 3.64 feet. What is its
(a) circumference? **(b)** area?

30. The legs of a right triangle measure 2.4 meters and 5.8 meters. Find the
(a) hypotenuse **(b)** perimeter **(c)** area

31. What is the volume of a sphere whose radius measures 31.4 inches? See inside cover.

32. The length of a side of a square is 2.4 yards and the radius of a circle is 4.1 feet. Which has the greater area, the square or the circle? By how much?

Exercises 33–38 ***Circular Motion*** *Consider the following.*

(a) Nicole rides the Sky Scraper, a gigantic Ferris wheel at Lagoon.
(b) The Galapagos islands, located near the equator, rotate with the earth.

(c) Minneapolis, located near 45° latitude, rotates with the earth.
(d) A space capsule orbits the earth.
(e) The earth orbits the sun.

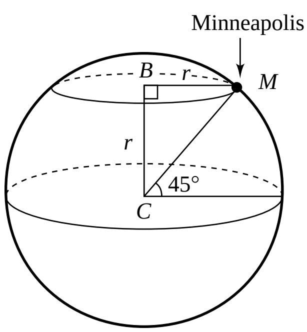
Each situation can be modeled mathematically as an object traveling in a circular orbit of radius r at a fixed speed. The distance traveled in one revolution is $2\pi r$ (the circumference of the circle). The time T for one rotation and the rotational speed V are related by the equation $VT = 2\pi r$. (*Hint*: If N is the number of rotations per unit of time, then $T = \frac{1}{N}$.)

33. The Sky Scraper carries its riders to a height of nearly 150 feet, has a wheel diameter of 137 feet, and has two speeds, 1.30 or 1.60 rotations per minute. At the slower speed, determine Nicole's speed in
(a) feet per second **(b)** feet per minute
(c) miles per hour.

34. Determine Nicole's speed, as in Exercise 33, when the Sky Scraper rotates 1.60 times per minute.

35. How fast are the giant tortoises of the Galapagos islands moving about the center of the earth (in miles per hour)? Take the radius of the earth to be 3960 miles.

36. How fast is a baseball player standing at first base in Minneapolis moving about the axis of the earth (in miles per hour)? See the diagram.



37. If the space capsule is 270 miles above the surface of the earth and makes a complete orbit in 1.70 hours, how fast is it traveling due to its rotation? The radius of the earth is 3960 miles.

38. How fast are Nicole, the Galapagos tortoises, and the entire baseball team in Minneapolis traveling (in miles per hour) about the sun? Use 93 million miles as the distance from the earth to the sun.

Exercises 39–44 ***Distance to Horizon*** *Use the model discussed at the beginning of this section (Figure 1), with the radius of the earth as* 3.960×10^3 *miles. If h is the height in feet, then* $x = \frac{h}{5280}$ *miles.*

39. How far can the Inuit boy see if he is tossed 15 feet high?

40. Compare the distances that can be seen from
(a) the Sears Tower ($h = 1454$ feet)
(b) John Hancock Center ($h = 1127$ feet).

41. If the air is clear, how far should the pilot of the Goodyear blimp be able to see from an elevation of 5300 feet?

42. From the top of Lagoon's Sky Scraper ride (see Exercise 33), how far should Nicole be able to see over the Great Salt Lake?

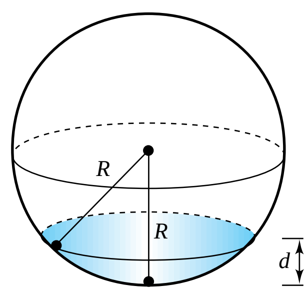
43. Sailors follow a rule of thumb that they can see as many miles to the horizon as the square root of their height above the waterline, so a lookout in a crow's nest 64 feet up should be able to see about 8 miles. How does this estimate compare with the figure given by the model in this section? Which do you think is more accurate? Why? See Develop Mastery Exercise 45, Section 7.1.

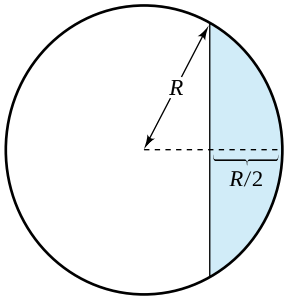
44. If $s = \sqrt{2rx + x^2}$, as in this section, write an equation giving x in terms of s. A lighthouse is to be built on Cape Cod on the shore of the Atlantic Ocean. How high above the ocean must the observation platform be to allow the operator to see a ship 12 miles from shore?

45. The distance between the earth and the sun is sometimes given as 93 million miles. Actually, the distance varies, from the nearest point (perihelion), about 91.4 million miles, to the furthest (aphelion), about 94.4 million miles. The speed of light is approximately 186,000 miles per second.
(a) How long does it take light from the sun to reach the earth when the earth is at perihelion? At the aphelion? Give answers in seconds and also in minutes rounded off.
(b) What is the difference between the times in part **(a)**?

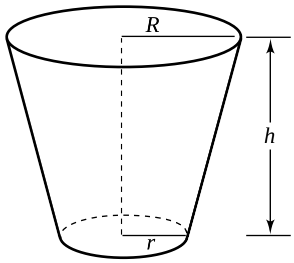
Exercises 46–51 ***Applying Geometric Formulas*** *Use the formulas for the following diagrams.*

(a) Section cut from a sphere of radius R, of depth d:
$$V = \pi d^2(R - d/3)$$





(b) Segment of a circle of radius R, depth $R/2$:
$A = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)R^2$



(c) Frustum of cone:
$V = \frac{1}{3}\pi h(R^2 + Rr + r^2)$

Conversion between fluid ounces and cubic inches:
1 quart = 32 ounces = 57.75 cubic inches

46. The height h and diameter d of a cylindrical can of pineapple juice are measured: $h = 6\frac{3}{4}$ inches, $d = 4\frac{1}{8}$ inches. Find the volume in cubic inches and its equivalent in fluid ounces. Use the formula for frustum of a cone with $r = R$. The label on the can indicates 46 ounces of pineapple juice. What is the difference between your answer and 46 ounces? Explain.

47. For a soft drink cup that is supposed to hold 44 ounces, the top diameter is $4\frac{3}{8}''$ and the bottom diameter is $3\frac{3}{8}''$. The height of the cup is measured as $6\frac{3}{4}''$. If all measurements are accurate to the nearest $\frac{1}{8}''$, find the largest and smallest possible values for the volume. Is it reasonable to call the cup as 44-ounce cup?

48. A soft drink cup is made in the shape of a frustum of a cone. If the cup is to have an upper diameter of $4''$ and the lower diameter of $3''$, what should the height be if it is to hold 32 ounces?

49. A direct mail catalog features an Oriental wok in the shape of a section of a sphere. The catalog gives dimensions that indicate $R = 6$ in., $d = 3$ in. and claims that the wok holds $2\frac{1}{2}$ qts. Assuming that the measurements are accurate to the nearest $\frac{1}{8}$ in., find the volume corresponding to
(a) $R = 5\frac{7}{8}$ in. $\quad d = 2\frac{7}{8}$ in.
(b) $R = 6\frac{1}{8}$ in. $\quad d = 3\frac{1}{8}$ in.
On the basis of your results in parts **(a)** and **(b)**, is the catalog claim of $2\frac{1}{2}$ qts reasonable? Explain.

50. A metal barrel $18''$ in diameter and $30''$ long is cut in half to make a trough $9''$ deep and $30''$ long.
(a) Find the volume (in cubic inches) of the resulting trough.
(b) If the diameter and length are measured accurate to the nearest quarter-inch, find the largest and smallest possible values for the volume (see Example 1).

51. Suppose the trough in Exercise 50 is cut down to make a trough of depth $4.5''$. What percent of the volume of the original is now in the shallower trough?

52. The box "Decimal Parts of a Mile" gives some familiar comparison measurements for decimal parts of a mile. Complete a similar chart for decimal parts of a kilometer.

0.1 km	
0.01 km	
0.001 km	
0.0001 km	
0.00001 km	
0.000001 km	

1.2 REAL NUMBERS

> The complexities of modern science and modern society have created a need for scientific generalists, for men (and women as well) trained in many fields of science. The habits of mind and not the subject matter are what distinguish the sciences.
>
> *Mosteller, Bode, Tukey, Winsor*

Numbers occur in every phase of life. It is impossible to imagine how anyone could function in a civilized society without having some familiarity with numbers. We recognize that you have had considerable experience working with numbers, and we also assume that you know something about the language and notation of sets.

Subsets of Real Numbers

We denote the set of real numbers by R. We make no attempt to develop the properties and operations of R; this is reserved for more advanced courses. Several subsets of the set of real numbers are used so frequently that we give them names. Most of these sets are familiar. The set of **natural numbers** is also called the set of **positive integers** or **counting numbers.** A **prime** is a positive integer greater than 1 that is divisible only by 1 and itself. The table lists the most commonly encountered subsets of R.

Subsets of R

Subset	*Symbol and Elements*
Natural numbers	$N = \{1, 2, 3, \ldots\}$
Whole numbers	$W = \{0, 1, 2, 3, \ldots\}$
Integers	$I = \{\ldots, -1, 0, 1, 2, 3, \ldots\}$
Even integers	$E = \{\ldots, -2, 0, 2, 4, 6, \ldots\}$
Odd integers	$O = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$
Prime numbers	$P = \{2, 3, 5, 7, 11, 13, \ldots\}$
Rationals	$Q = \{\frac{p}{q} \mid p, q \in I, q \neq 0\}$
Irrationals	$H = \{x \mid x \in R \text{ and } x \notin Q\}$

I had such an amazingly deprived high school education. There wasn't a useful math book in the library.
Bill Gosper

Figure 4 indicates schematically that some of the sets listed are subsets of others. For example, $P \subset N$, $N \subset W$, and $W \subset I$. The sets E and O together make up I, so we can write $E \cup O = I$. Further, for any $p \in I$, since $p = \frac{p}{1}$, every integer is also a rational number, so $I \subset Q$.

The existence of some irrational numbers has been known since at least the time of the ancient Greeks, who discovered that the length of the diagonal of a

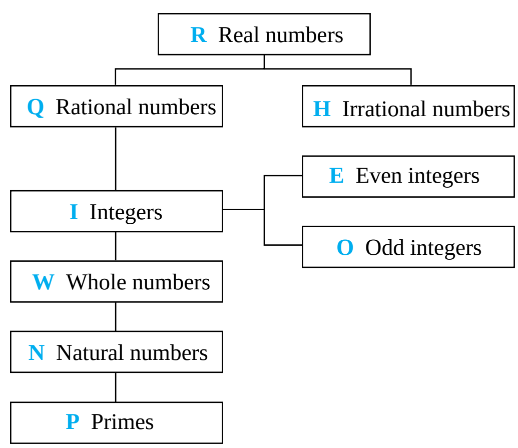


FIGURE 4
Subsets of the real numbers.

square is not a rational multiple of the length of the sides (see Develop Mastery Exercise 38). The length of the diagonal of a unit square is the irrational number $\sqrt{2}$, and we recognize many others such as $\sqrt{3} - 1$ and $2 + \sqrt[3]{7}$ and π. The ratio of the circumference of any circle to its diameter is the number π (pi), approximately 3.1416. (See the earlier Historical Note, "The Number Pi.")

Although most of this book (and most of calculus as well) involves only real numbers, we also make use of the set of *complex numbers* (see Section 1.3), especially in Chapters 3 and 7.

▶EXAMPLE 1 ***Set notation*** Determine whether the statement is true.

(a) $N \subset Q$ **(b)** $I \cap H = \emptyset$ **(c)** $\sqrt{5} \in Q$
(d) $\sqrt{64} \in H$ **(e)** $41 \in P$ **(f)** $87 \notin P$

Solution

(a) True; every natural number is rational.
(b) True; every integer is rational and hence not in H.
(c) False; $\sqrt{5}$ is an irrational number.
(d) False; $\sqrt{64} = 8$ and is not irrational.
(e) True; 41 is a prime number.
(f) True; $87 = 3 \cdot 29$, so 87 is not a prime number. ◀

▶EXAMPLE 2 ***Union and intersection*** Simplify:

(a) $P \cap N$ **(b)** $W \cap Q$ **(c)** $Q \cup H$

Solution

Strategy: Think about the meaning of each set (in words). For given numbers, decide if each fits the description of the indicated set.

(a) $P \cap N = P$; every prime number is also a natural number.
(b) $W \cap Q = W$; every whole number is also a rational number.
(c) $Q \cup H = R$; every real number is rational or irrational. ◀

Decimal Representation of Numbers

Every real number also has a decimal "name." For instance, the rational number $\frac{3}{4}$ can also be written as 0.75, which is called a **terminating decimal.** To get the decimal representation for the rational number $\frac{5}{11}$, we divide 5 by 11 and get the **repeating** (nonterminating) decimal 0.454545. . . , which we write as $0.\overline{45}$. The bar notation indicates that the block under the bar, in this instance *45*, repeats forever. A terminating decimal can also be considered as repeating. For instance, $\frac{3}{4}$ can be named by 0.75, or by $0.75\overline{0}$, or even by $0.74\overline{9}$ (see Example 3).

An irrational number such as $\sqrt{2}$ has a **nonterminating** and **nonrepeating** decimal representation. The distinction between repeating and nonrepeating decimals distinguishes the rational numbers from the irrationals.

Approximating Pi

As indicated in Section 1.1, the important number π occurs in problem-solving applications as well as theoretical mathematics. In recent years sophisticated techniques have allowed computer evaluation of π to *billions* of decimal places, but there is still no way to express the decimal representation of π exactly. See the Historical Note, "Approximating the Number π."

HISTORICAL NOTE APPROXIMATING THE NUMBER π

People continued to be fascinated by π even after it was shown to be irrational. In 1844 Johann Dase, who could multiply 100 digit numbers in his head, took months to compute π to 205 digits. The champion at hand calculating must be William Shanks, who spent 20 years to grind out 707 digits. His record stood until 1945, when D. W. Ferguson used a mechanical calculator to find an error in Shanks' 528th digit.

No further search for accuracy can be justified for practical purposes of distance or area computation. An approximation to 45 digits would measure the circumference of a circle encompassing the entire universe with an error less than the radius of a single electron. People have found many other reasons, in addition to the sheer fascination of knowing, for computing the digits of π.

Computers brought a new era. In 1949, a machine called ENIAC, composed of rooms full of vacuum tubes and wires, in 70 hours computed 2037 digits of π.

More recent milestones are listed below. Remarkably, the last record was achieved on a *home-built* super computer. You can read more in "Ramanujan and Pi," *Scientific American* (Feb. 1988), and in "The Mountains of Pi," *The New Yorker* (Mar. 12, 1992).

π = 3.14159 26535 89793 23846 26433
83279 50288 41971 69399 37510
58209 74944 59230 78164 06286
20899 86280 34825 34211 70679
82148 08651 32823 06647 09384
46095 50582 23172 53594 08128
48111 74502 84102 70193 85211
05559 64462 29489 54930 38196
44288 10975 66593 34461 28475
64823 37867 83165 27120 19091
45648 56692 34603 48610 45432
66482 13393 60726 02491 41273

A computer can calculate these first 300 digits of π in a fraction of a second. The same calculation by hand requires months of work.

1973	Jean Guilloud, M. Bouyer	CDC7600	1 million digits
1985	R. William Gosper	Symbolics	17 million digits
1986	David H. Bailey	Cray-2	29 million digits
1987	Yasumasa Kanada	NEC SX-2	134 million digits
1989	D. and G. Chudnovsky		480 million digits
1990	Yasumasa Kanada	NEC SX-2	1 billion digits
1991	D. and G. Chudnovsky	M Zero	2.26 billion digits

The rational number $\frac{22}{7}$ is sometimes used as an approximation to π, but it is important to understand that π is not equal to $\frac{22}{7}$. Other rational number approximations of π include $\frac{333}{106}$, $\frac{355}{113}$, and $\frac{208,341}{66,317}$ (see Develop Mastery Exercises 39 and 40).

Characterizing real numbers

A real number is **rational** if and only if its decimal representation repeats or terminates.

A real number is **irrational** if and only if its decimal representation is nonterminating and nonrepeating.

From Decimal Representations to Quotient Form

Finding a decimal representation for a given rational number is simply a matter of division; going the other way is more involved but is not difficult. Some graphing calculators have built-in routines to convert decimals to fractions. Such programs are limited because calculators must work with truncated (cut-off, finite) decimals.

There is no way to tell a calculator that a given decimal repeats (infinitely). If we know that a given number x has a repeating decimal representation, these steps will give the desired rational number as a quotient.

1. Multiply x by an appropriate power of 10 to move the decimal point to the beginning of the repeating block.
2. Multiply x by another power of 10 to move the decimal point to the beginning of the next block.
3. The difference between these two multiples of x is an integer, which allows us to solve for x.

►EXAMPLE 3 ***From decimal to quotient*** Express each number as a quotient of integers in lowest terms.

(a) 0.74 **(b)** $0.\overline{74}$ **(c)** $0.74\overline{9}$

Solution

(a) From the meaning of decimal notation, $0.74 = \frac{74}{100}$, which reduces to $\frac{37}{50}$. Thus 0.74 represents the rational number $\frac{37}{50}$.

(b) With a repeating block, we follow the procedure outlined above. Let $x = 0.\overline{74}$. The decimal point is already at the beginning of the block, so multiply by 100 to move the decimal point to the beginning of the next block.

$$\begin{aligned} 100x &= 74.\overline{74} \\ x &= 0.\overline{74} \\ \hline 99x &= 74, \text{ from which } x = \frac{74}{99}. \end{aligned}$$

Thus $0.\overline{74}$ represents the rational number $\frac{74}{99}$. You may wish to verify this by dividing 74 by 99.

(c) Let $y = 0.74\overline{9}$, multiply by 1000, then by 100, and take the difference:

$$\begin{aligned} 1000y &= 749.\overline{9} \\ 100y &= 74.\overline{9} \\ \hline 1900y &= 675, \text{ from which } y = \frac{675}{900} = \frac{3}{4}. \end{aligned}$$

Hence $0.74\overline{9}$ represents the rational number $\frac{3}{4}$, which says that $\frac{3}{4}$ has *two different* decimal names, $0.74\overline{9}$ and 0.75. Actually, every rational number that can be written as a terminating decimal has two representations. ◄

Note that the procedure outlined above involves subtracting repeating decimals as if they were finite decimals. We justify such operations in Section 8.3.

Exact Answers and Decimal Approximations

When we use a calculator to evaluate a numerical expression, in most cases the answer is a **decimal approximation** of the exact answer. When we ask for a four decimal place approximation, we mean round off the calculator display to four decimal places.

►EXAMPLE 4 ***Calculator evaluation*** Use a calculator to get a four decimal place value. Is the value exact or an approximation?

(a) $\frac{3}{4} + \frac{1}{8}$ **(b)** $\frac{1}{5} + \frac{2}{3}$ **(c)** $\sqrt{2}$

Solution

(a) $\frac{3}{4} + \frac{1}{8} = 0.8750$; exact decimal value.
(b) $\frac{1}{5} + \frac{2}{3} \approx 0.866666667 \approx 0.8667$; approximation.
(c) $\sqrt{2} \approx 1.414213562 \approx 1.4142$; approximation. ◀

Square Roots and the Square Root Symbol

There are two numbers whose square is 2. That is, the equation $x^2 = 2$ has two roots. We reserve the symbol $\sqrt{2}$ for the **positive root,** so the roots of the equation are $\sqrt{2}$ and $-\sqrt{2}$, which we often write as $\pm\sqrt{2}$. For every positive x, the calculator will display a positive number when we press $\boxed{\sqrt{x}}$, and we use $\sqrt{x}$ to denote the positive number whose square is x.

▶EXAMPLE 5 ***Calculators and rounding off*** Find an approximation rounded off to four decimal places.

(a) $1 + \sqrt{3}$ **(b)** $\sqrt{1 + \sqrt{3}}$

Solution

(a) Using a calculator, we get $1 + \sqrt{3} \approx 2.7321$.
(b) After evaluating $1 + \sqrt{3}$, take the square root to get

$$\sqrt{1 + \sqrt{3}} \approx 1.6529.$$ ◀

EXERCISES 1.2

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. The number π is equal to $\frac{22}{7}$.
2. The integer 119 is a prime number.
3. The intersection of the set of rational numbers and the set of irrational numbers is the empty set.
4. The set of prime numbers is a subset of the set of odd numbers.
5. The sum of any two odd numbers is an odd number.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. The product of two odd numbers is an ______ number.
7. When $\frac{5}{7}$ is expressed as a repeating decimal, the eighth digit after the decimal is ______.
8. Of the numbers $\frac{5}{71}$, $-\frac{24}{17}$, $\sqrt{25 - 9}$, $\sqrt{64 - 14}$, the one that is irrational is ______.
9. Of the four numbers π, $\sqrt{64 + 16}$, 0.564, $\frac{\sqrt{5}}{2}$, the one that is rational is ______.
10. Of the four numbers $\frac{8}{11}$, $\frac{5}{7}$, 0.714, $0.\overline{714}$ the smallest one is ______.

Develop Mastery

Exercises 1–8 ***Subsets of Real Numbers*** *Determine whether each statement is true or false. Refer to the subsets of R listed in this section.*

1. **(a)** $0 \in N$ **(b)** $17 \notin P$
2. **(a)** $-5 \notin N$ **(b)** $-5 \in I$
3. **(a)** $\{-4, 3\} \subseteq I$ **(b)** $\{7, 81\} \subset P$
4. **(a)** $\{\sqrt{4}, \sqrt{5}\} \subset H$ **(b)** $\{0.5, 0.7\} \subset Q$
5. **(a)** $I \cup N = I$ **(b)** $I \cap W = W$
6. **(a)** $P \cap I = P$ **(b)** $Q \cup I = Q$
7. **(a)** $Q \subseteq H$ **(b)** $H \cup I = H$
8. **(a)** $P \cup Q = Q$ **(b)** $I \cap Q = I$

Exercises 9–10 Indicate which of the subsets P, N, I, O, E, Q, and H contain each number. For instance, 17 belongs to P, N, I, O, and Q.

9. **(a)** $\frac{29}{3}$ **(b)** $\sqrt{16}$ **(c)** $\sqrt{32}$ **(d)** $\frac{2^5}{2^3}$
10. **(a)** $3.\overline{27}$ **(b)** 29 **(c)** $\frac{0.13}{1.27}$ **(d)** $2\pi - 1$

Exercises 11–14 ***Fraction to Decimal*** *Express each as a terminating decimal, or as a repeating decimal using the bar notation.*

11. **(a)** $\frac{5}{8}$ **(b)** $\frac{5}{12}$

12. **(a)** $\frac{73}{40}$ **(b)** $\frac{25}{33}$

13. **(a)** $\frac{37}{45}$ **(b)** $\frac{10}{13}$

14. **(a)** $\frac{16}{35}$ **(b)** $\frac{48}{65}$

Exercises 15–18 ***Decimal to Fraction*** *Express each as a fraction (quotient of integers) in lowest terms.*

15. **(a)** 0.63 **(b)** $0.\overline{63}$

16. **(a)** 1.45 **(b)** $1.\overline{45}$

17. **(a)** $0.8\overline{3}$ **(b)** $0.\overline{83}$

18. **(a)** $1.3\overline{6}$ **(b)** $0.\overline{621}$

Exercises 19–21 *Give a decimal approximation rounded off to three decimal places.*

19. **(a)** $\frac{67}{195}$ **(b)** $\frac{\sqrt{17}}{12}$

20. **(a)** $\frac{1142}{735}$ **(b)** $\sqrt{1+\sqrt{2}}$

21. **(a)** $\frac{343}{110}$ **(b)** $\frac{11(4-\sqrt{3})}{8}$

Exercises 22–30 ***Decimal Approximations*** *Give decimal approximations rounded off to six decimal places. Do the numbers appear to be equal?*

22. $\sqrt{8};\ 2\sqrt{2}$

23. $\sqrt{48};\ 4\sqrt{3}$

24. $1+\sqrt{2};\ \frac{1}{\sqrt{2}-1}$

25. $\frac{\sqrt{3}+1}{2};\ \frac{1}{\sqrt{3}-1}$

26. $\sqrt{6}+\sqrt{2};\ 2\sqrt{2+\sqrt{3}}$

27. $\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}};\ \sqrt{10}$

28. $\sqrt{6+4\sqrt{2}};\ 2+\sqrt{2}$

29. $\sqrt{8+2\sqrt{15}};\ \sqrt{5}+\sqrt{3}$

30. $\sqrt{6-2\sqrt{5}};\ 1-\sqrt{5}$

31. What is the smallest nonprime positive integer greater than 1 that has no factors less than 12?

32. What is the smallest prime number that divides $3^7 + 7^{11}$?

Exercises 33–34 *True or False.* *Give reasons.*

33. **(a)** The sum of any two odd numbers is an odd number.
(b) The product of any two odd numbers is an odd number.
(c) The product of any two consecutive positive integers is an even number.

34. **(a)** The sum of three consecutive even numbers is an odd number.
(b) If a positive even integer is a perfect square, then it is the square of an even number.
(c) If the sum of two integers is even, then both must be even.

35. Give an example of irrational numbers for x and y that satisfy the given condition.
(a) $x + y$ is irrational. **(b)** $x + y$ is rational.
(c) $x \cdot y$ is rational. **(d)** $\frac{x}{y}$ is rational.

36. If $x = \sqrt{1.5+\sqrt{2}} + \sqrt{1.5-\sqrt{2}}$, determine whether x is rational or irrational. (*Hint:* Evaluate x^2.)

37. If $x = \sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}$, determine whether x is rational or irrational. (*Hint:* Evaluate x^2.)

38. Prove that $\sqrt{2}$ is not a rational number. (*Hint:* Suppose $\sqrt{2} = \frac{b}{c}$, where $b, c \in N$ and $\frac{b}{c}$ is in lowest terms. Then $b^2 = 2c^2$. Explain why b must be even. Then also explain why c must be even. This would contradict the assumption that $\frac{b}{c}$ is in lowest terms.)

Exercises 39–40 ***Approximations for π*** *Refer to the number π, whose decimal form is nonterminating and nonrepeating. Rounded off to 24 decimal places,*

$$\pi \approx 3.1415\ 92653\ 58979\ 32384\ 62643.$$

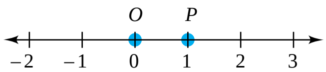
39. The following rational numbers are used as approximations of π. Use your calculator to evaluate and compare each result with the given decimal approximation of π.
(a) $\frac{22}{7}$ **(b)** $\frac{333}{106}$ **(c)** $\frac{355}{113}$

40. The rational number $\frac{208,341}{66,317}$ is an excellent approximation of π. Evaluate it to at least 12 decimal places and compare the result with the approximation given above.

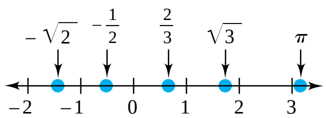
41. In 1991 the Chudnovsky brothers used a supercomputer they built to compute more than 2.26 billion digits of π. Count the number of symbols in an average line of this book and estimate how long a line of type (measured in miles) the Chudnovsky result would give. (See the Historical Note, "Approximating the Number π.")

1.3 REAL NUMBER PROPERTIES; COMPLEX NUMBERS

> . . . one of the central themes of science [is] the mysterious power of mathematics to prepare the ground for physical discoveries which could not have been foreseen by the mathematicians who gave the concepts birth.
>
> *Freeman Dyson*



(a) Integers on a number line.



(b) Real numbers on a number line.

FIGURE 5

> ***T****he mathematics course at San Diego High School was standard for that time: plane geometry in the tenth grade, advanced algebra in the eleventh, and trigonometry and solid geometry in the twelfth. . . . After plane geometry, I was the only girl still taking mathematics.*
>
> Julia Robinson

Real Number Line

One of the great ideas in the history of mathematics is that the set of real numbers can be associated with the set of points on a line. We assume a one-to-one correspondence that associates each real number with exactly one point on a line, and every point on the line corresponds to exactly one real number.

We frequently identify a number with its point, and vice versa, speaking of "the point 2" rather than "the point that corresponds to 2." Figure 5 shows a few numbers and the corresponding points on a number line.

Order Relations and Intervals

The number line also represents the ordering of the real numbers. We assume that the ideas of less than and greater than, and the following notation are familiar:

Order relations for real numbers

Notation	*Terminology*	*Meaning*
$b < c$	b is less than c.	$c = b + d$, for some positive number d
$c > b$	c is greater than b	$b < c$
$b \le c$	b is less than or equal to c.	$b < c$ or $b = c$
$c \ge b$	c is greater than or equal to b.	$b \le c$

We also need notation for sets of all numbers between two given numbers, or all numbers less than or greater than a given number. Such sets are called **intervals.**

Definition: intervals

Suppose b and c are real numbers and $b < c$:

Name	*Notation*	*Number-line diagram*
Open interval	$(b, c) = \{x \mid b < x < c\}$	
Closed interval	$[b, c] = \{x \mid b \le x \le c\}$	
Half-open interval	$[b, c) = \{x \mid b \le x < c\}$	
Half-open interval	$(b, c] = \{x \mid b < x \le c\}$	
Infinite intervals	$(b, \infty) = \{x \mid x > b\}$	
	$[b, \infty) = \{x \mid x \ge b\}$	
	$(-\infty, b) = \{x \mid x < b\}$	
	$(-\infty, b] = \{x \mid x \le b\}$	

In these definitions, the symbol ∞ (infinity) *does not represent a number,* and we never use a closed bracket to indicate that ∞ is included in an infinite interval.

Absolute Value and Distance

We have no difficulty in finding the absolute value of specific numbers, as in

$$|2| = 2, \qquad |0| = 0, \qquad |-1.375| = 1.375.$$

There are always *two numbers* having the same nonzero absolute value, as

$$|2| = |-2| = 2, \qquad \text{and} \qquad \left|\frac{-\pi}{3}\right| = \left|\frac{\pi}{3}\right| = \frac{\pi}{3}.$$

Following the pattern of the above examples, *the absolute value of a positive number is itself; the absolute value of a negative number is its opposite.*

In working with an expression like $|2x - 3|$, the quantity $2x - 3$ is neither positive nor negative until we give a value to x. All we can say is that $|2x - 3|$ is either $2x - 3$ or its opposite, $-(2x - 3)$. Thus

$|2x - 3| = 2x - 3$ for all the x-values that make $2x - 3$ positive $(2, \frac{9}{5}, 5\pi, \ldots)$,

$|2x - 3| = -2x + 3$ for all the x-values that make $2x - 3$ negative $(1, \frac{\pi}{3}, 0, \ldots)$.

Looking at a number line, the numbers satisfying $|x| = 3$ are 3 and -3, the two numbers *whose distance from 0 is 3*. More generally, the numbers located two units from 7 are the numbers 5 and 9, and $|7 - 5| = 2$, and $|7 - 9| = 2$.

These examples lead to two ways of looking at absolute values, both of which are useful, so we include them in our definition.

Definition: absolute value

For any expression u, the ***absolute value of u,*** denoted by $|u|$, is given by

$$|u| = \begin{cases} u & \text{if} \quad u \geq 0 \\ -u & \text{if} \quad u < 0 \end{cases}$$

If a and b are any real numbers, then ***the distance between a and b*** is given by

$$|a - b| = |b - a|.$$

It follows that $|a| = |a - 0|$ is ***the distance between a and 0.***

Calculators and Absolute Value

In finding the absolute value of any particular number, we shouldn't have to rely on a calculator, but a calculator can be helpful nonetheless.

We know, for example, that $|\pi - \sqrt{10}|$ is either $\pi - \sqrt{10}$ or $\sqrt{10} - \pi$, depending on which is positive. We do not need to use the calculator function $\boxed{\text{ABS}}$; in fact, if we were to try (see the Technology Tip in Section 1.5 for suggestions about how to enter $\boxed{\text{ABS}}$), we would find only that

$$\text{ABS}(\pi - \sqrt{10}) \approx 0.020685.$$

This is true, but not helpful in deciding whether $|\pi - \sqrt{10}|$ is equal to $\pi - \sqrt{10}$ or $\sqrt{10} - \pi$. If, however, we use the calculator to learn that $\pi - \sqrt{10}$ is negative, about -0.020685, then we immediately know that $|\pi - \sqrt{10}| = \sqrt{10} - \pi$.

▶EXAMPLE 1 *Absolute value*

(a) If $t = 1 - \sqrt{3}$, show both t and $-t$ on a number line and express $|t|$ in exact form without using absolute values. **(b)** Find all numbers x such that $|2x - 3| = 1$.

Solution

(a) Since t is negative ($t \approx -.0732$), $|t|$ is the opposite of t:

$$|t| = |1 - \sqrt{3}| = -(1 - \sqrt{3}) = \sqrt{3} - 1.$$

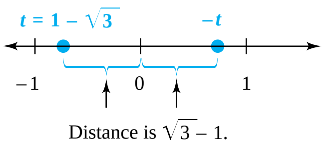


FIGURE 6

Both t and $-t$ are shown on the number line in Figure 6.

(b) The two numbers whose absolute value is 1 are 1 and -1. Thus, if $|2x - 3| = 1$, we have either

$$2x - 3 = 1 \qquad \text{or} \qquad 2x - 3 = -1.$$

Solving each, we have

$$x = 2 \qquad \text{or} \qquad x = 1. \quad ◀$$

Some useful properties of absolute values

Suppose x and y are any real numbers.

1. $\|x\| \geq 0$	**2.** $\|x\| = \|-x\|$
3. $\|x \cdot y\| = \|x\| \cdot \|y\|$	**4.** $\sqrt{x^2} = \|x\|$
5. $\left\|\frac{x}{y}\right\| = \frac{\|x\|}{\|y\|}$ if $y \neq 0$	**6.** $\|x + y\| \leq \|x\| + \|y\|$

▶EXAMPLE 2 *Absolute value arithmetic*

Let $x = -3$, $y = 2$, and $z = 1 - \sqrt{3}$. Evaluate the expressions. Are the values in each pair equal or not?

(a) $\sqrt{x^2}$, x **(b)** $|x \cdot y|$, $|x| \cdot |y|$ **(c)** $|y + z|$, $|y| + |z|$
(d) $|x + z|$, $|x| + |z|$

Strategy: Identify each number as positive or negative before applying a definition of absolute value.

Solution

(a) $\sqrt{x^2} = \sqrt{(-3)^2} = \sqrt{9} = 3$; $x = -3$.
(b) $|x \cdot y| = |(-3) \cdot 2| = |-6| = 6$; $|x| \cdot |y| = |-3| \cdot |2| = 3 \cdot 2 = 6$.
(c) $|y + z| = |2 + (1 - \sqrt{3})| = |3 - \sqrt{3}| = 3 - \sqrt{3} \approx 1.27$.
$|y| + |z| = |2| + |1 - \sqrt{3}| = 2 + (\sqrt{3} - 1) = 1 + \sqrt{3} \approx 2.73$.
(d) $|x + z| = |-3 + (1 - \sqrt{3})| = |-2 - \sqrt{3}| = 2 + \sqrt{3} \approx 3.73$.
$|x| + |z| = |-3| + |1 - \sqrt{3}| = 3 + (\sqrt{3} - 1) = 2 + \sqrt{3}$.

The pairs in parts (b) and (d) are equal; those in (a) and (c) are not. ◀

Complex Numbers

Although most of our work deals exclusively with real numbers, sometimes we must expand to a larger set, the set of **complex numbers.** We need complex numbers mostly in two settings: for the solutions of polynomial equations, and for some trigonometric applications (Chapter 7). For the time being, all the information we need is simple complex-number arithmetic and how to take square roots. We include a picture of the complex plane and some properties of complex numbers for reference.

HISTORICAL NOTE — GROWTH OF THE NUMBER SYSTEM

Boethius (left), using written Arabic numerals, triumphs over Pythagoras and his abacus in a mathematical contest. The goddess Arithmetica presides over the competition.

The ancient Greeks believed numbers expressed the essence of the whole world. Numbers to the Pythagorean philosophers meant whole numbers and their ratios—what we would call the *positive rational numbers*. It was extremely distressing to some when they discovered that something as simple as the diagonal of a square cannot be expressed rationally in terms of the length of the side of the square. Pythagoras (ca. 550 B.C.) is said to have sacrificed 100 oxen in honor of the discovery of irrational numbers. Nonetheless, irrational numbers were called *alogos* in Greek, carrying the double meaning that such numbers were not ratios and also that they were not to be spoken.

Hundreds of years passed before mathematicians became comfortable with the use of numbers like $\sqrt{2}$. Even we are reluctant to accept new numbers, as our language reflects. We equate rational with reasonable, and dislike irrational or negative concepts.

Not until the Middle Ages did mathematicians become secure with fractions and negative numbers. Also at that time, they recognized that irrationals have negatives, so two numbers have the square 2, namely $\sqrt{2}$ and $-\sqrt{2}$.

Complex numbers have a history somewhat shorter than that of irrational numbers. Cardan made the first public use of complex numbers in 1545 when he showed how to find two numbers with a sum of 10 and a product of 40, giving the result as $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. Although he observed that the product equals $5^2 - (-15)$ or 40, he considered such expressions no more real than negative numbers describing lengths of line segments. In 1777 Euler first used the symbol i to denote $\sqrt{-1}$. As with $\sqrt{2}$, people first considered only one root of -1, but then recognized that i and $-i$ both satisfy the equation $x^2 + 1 = 0$.

We have come gradually to recognize that our number system is much larger and richer than childhood experience conceives. We extend our counting numbers to accommodate subtraction and division and then to solve simple equations such as $x^2 = 2$ and $x^2 = -1$. Other extensions are possible, and we hope to be open to accepting whatever is useful for solving new problems.

We are familiar with the fact that whenever we take the square of a nonzero real number, we always get a *positive* number. There is no real number that satisfies the simple equation $x^2 + 1 = 0$. Accordingly, we extend the real number system by introducing a new number i, whose distinguishing characteristic is that its square equals -1, $i^2 = -1$, which is sufficient to define the set C of complex numbers. See the Historical Note, "Growth of the Number System."

Definition: the set of complex numbers

$$C = \{c + di \mid c \text{ and } d \text{ are real numbers, and } i^2 = -1\}.$$

The set of complex numbers is really an extension of the set of real numbers because for any real number a, $a = a + 0i$. This observation gives the conclusion:

Every real number is also a complex number.

The Quadratic Formula, Complex Numbers, and Principal Square Roots

The roots of a quadratic equation may or may not be real numbers. For solving such an equation, we rely on another familiar tool from introductory algebra, the **quadratic formula.**

Quadratic formula

The roots of the equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ is called the **discriminant** of the equation and determines the nature of the roots.

If $b^2 - 4ac = 0$, there is *only one root,* given by $x = \dfrac{-b}{2a}$.
If $b^2 - 4ac > 0$, the quadratic formula gives *two real roots.*
If $b^2 - 4ac < 0$, there are *two nonreal complex roots.*

To apply the quadratic formula in the case with a negative discriminant, we need to extend the idea of square roots from our definition in Section 1.2 to the following.

Definition: principal square root

Suppose p is a positive real number. Then the **principal square roots** of p and $-p$ are given by:

$\sqrt{p}$ is the *nonnegative number whose square is p,* as $\sqrt{4} = 2$.
$\sqrt{-p}$ is the *complex number* $\sqrt{p}i$, as $\sqrt{-9} = 3i$.

For example, if $x^2 - 4x + 5 = 0$, the discriminant is negative and the quadratic formula gives the roots as $x = \dfrac{4 \pm \sqrt{-4}}{2} = \dfrac{4 \pm 2i}{2} = 2 \pm i$. The nonreal complex roots given by the quadratic formula always occur in what are called **conjugate pairs,** in this case $2 + i$ and $2 - i$. In general, for the complex number $z = c + di$, the **conjugate** of z, denoted by $\overline{z}$, is given by $\overline{z} = c - di$.

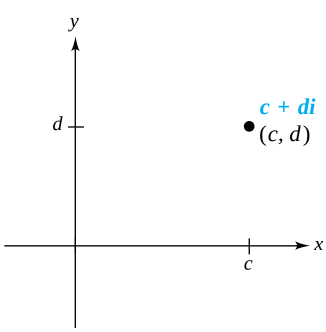


FIGURE 7
The complex number $c + di$ is identified with the point (c, d).

The Complex Plane

Just as we identify each real number with a point on a number line, we identify each complex number $c + di$ with a point in the plane having coordinates (c, d). See Figure 7. In this correspondence, the x-axis is the real number line and all real multiples of i are located on the y-axis.

The **standard form** for a complex number is $c + di$, where c and d are real numbers. The number c is called the **real part** and d is the **imaginary part.** If the imaginary part is nonzero, then we call the complex number $c + di$ a **nonreal-complex number.**

Complex Number Arithmetic

When are two complex numbers equal? How do we add, subtract, multiply, and divide complex numbers? Given complex numbers z and w in standard form, say $z = a + bi$ and $w = c + di$, we treat these numbers as we would any algebraic expressions, combining like terms in the usual fashion, with one exception. In multiplication we replace i^2 by -1 wherever it occurs. For division, $\frac{z}{w}\,(w \neq 0)$, we multiply numerator and denominator by $\overline{w}$ as follows:

$$\begin{aligned}\frac{z}{w} &= \frac{a+bi}{c+di} = \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}\\ &= \frac{ac-adi+bci-bdi^2}{c^2-d^2i^2} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\end{aligned}$$

Definitions: complex number arithmetic

Suppose $z = a + bi$, $w = c + di$, where a, b, c, and d are real numbers.

Equality: $z = w$ if and only if $a = c$ and $b = d$.
Addition: $z + w = (a + c) + (b + d)i$
Subtraction: $z - w = (a - c) + (b - d)i$
Multiplication: $z \cdot w = (ac - bd) + (ad + bc)i$
Division: $\dfrac{z}{w} = \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}i,$
where $c^2 + d^2 \neq 0$.

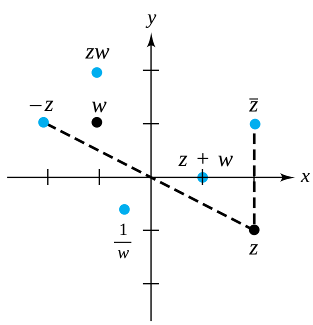


FIGURE 8
Points in the complex plane.

►EXAMPLE 3 Complex number arithmetic If $z = 2 - i$ and $w = -1 + i$, write each expression as a complex number in standard form and locate each on a diagram of the complex plane:

(a) $z + w$ **(b)** $\overline{z}$ and $-z$ **(c)** zw **(d)** $\dfrac{1}{w}$

Solution

(a) $z + w = (2 - i) + (-1 + i) = 1 + 0i = 1.$
(b) $\overline{z} = 2 + i$, and $-z = -2 + i$.
(c) $zw = (2 - i)(-1 + i) = -2 + 2i + i - i^2 = -2 + 3i - (-1)$
$= -1 + 3i.$
(d) $\dfrac{1}{w} = \dfrac{1\overline{w}}{w\overline{w}} = \dfrac{1(-1-i)}{(-1+i)(-1-i)} = \dfrac{-1-i}{1-i^2} = \dfrac{-1-i}{1-(-1)} = -\dfrac{1}{2} - \dfrac{1}{2}i.$

The points are shown in Figure 8. ◄

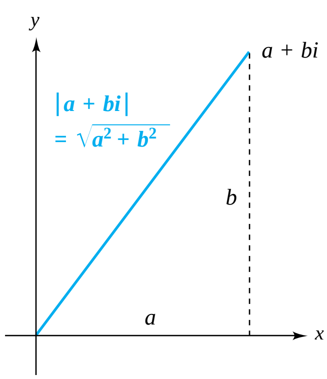


FIGURE 9
The absolute value of a complex number is the distance to the origin in the complex plane.

Absolute Value of a Complex Number

We define the absolute value of a real number x as the *distance* between the point x and the origin 0. In a similar manner, we define the absolute value of $a + bi$ as the distance in the plane between (a, b) and the origin $(0, 0)$. The diagram in Figure 9 and the Pythagorean theorem give a distance of $\sqrt{a^2 + b^2}$.

Definition: absolute value of a complex number

Suppose z is the complex number $a + bi$. The *absolute value of* z, denoted by $|z|$, is $\sqrt{a^2 + b^2}$, and we write $|z| = \sqrt{a^2 + b^2}$.

Many of the properties of absolute values of complex numbers are the same as those for real numbers. From the definition of distance in the next section, we can also observe that $|z - w|$ is the distance between the complex numbers z and w.

Properties of absolute value of a complex number

If z and w are any complex numbers, then:

1. $|z| \geq 0$ **2.** $|\bar{z}| = |-z| = |z|$

3. $|z \cdot w| = |z| \cdot |w|$ **4.** $\sqrt{z \cdot \bar{z}} = |z|$

5. $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ if $w \neq 0$ **6.** $|z + w| \leq |z| + |w|$

►EXAMPLE 4 ***Absolute values of complex numbers*** Suppose $z = 2 - i$ and $w = -1 + i$ (the complex numbers of Example 3). Verify that the properties of absolute values hold for each pair.

(a) $|z| + |w|, |z + w|$ **(b)** $|\bar{z}|, |-z|$ **(c)** $|zw|, |z| \cdot |w|$

Solution

(a) $|z| + |w| = |2 - i| + |-1 + i| = \sqrt{2^2 + (-1)^2} + \sqrt{(-1)^2 + 1^2}$
$= \sqrt{5} + \sqrt{2}.$

$|z + w| = |1 + 0i| = \sqrt{1^2 + 0^2} = 1.$

Since $1 < \sqrt{5} + \sqrt{2}$, $|z + w| < |z| + |w|$ (Property 6).

(b) $|\bar{z}| = |2 + i| = \sqrt{2^2 + 1^2} = \sqrt{5}$;
$|-z| = |-2 + i| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}.$

Thus $|\bar{z}| = |-z|$ (Property 2).

(c) $zw = -1 + 3i$, and so $|zw| = |-1 + 3i| = \sqrt{10}.$
$|z| \cdot |w| = \sqrt{5} \cdot \sqrt{2} = \sqrt{10}$. Therefore $|zw| = |z| \cdot |w|$ (Property 3). ◄

Ordering of the Complex Numbers

Real numbers are ordered by the less than relation, so that if b is any real number, then exactly one of the following is true:

$$b = 0 \qquad \text{or} \qquad b < 0 \qquad \text{or} \qquad b > 0.$$

Since R is a subset of C, any ordering of the complex numbers should be consistent with the ordering of R. Consider the nonzero complex number i. If we could extend the ordering of R to C, then we would have to have

$$i < 0 \qquad \text{or} \qquad i > 0.$$

If $i > 0$, then we can multiply both sides by i and get

$$i \cdot i > i \cdot 0 \qquad \text{or} \qquad -1 > 0,$$

which is not a true statement in R. That leaves only the possibility that i is negative, $i < 0$. If we multiply both sides by a negative number, we must reverse the direction of the inequality, and we get the same contradiction:

$$i \cdot i > i \cdot 0 \qquad \text{or} \qquad -1 > 0.$$

We are forced to conclude that *there is no consistent way to order the set of complex numbers using the less than relation.* Thus, for example, we cannot say that one of the two numbers $3 - 4i$ and $-1 + 2i$ is less than the other.

EXERCISES 1.3

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. If both b and d are negative real numbers, then $\sqrt{-b}\sqrt{-d} = \sqrt{bd}$.
2. $|3 - \sqrt{10}| = \sqrt{10} - 3$.
3. $1 - \sqrt{10} > 1 - \pi$
4. In the complex plane, $3 + 4i$ is farther from the origin than $5i$.
5. If $0 < b < 1$ and $-1 < c < 0$, then $-1 < b - c < 1$.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. The smallest integer that is greater than $3 - \sqrt{10}$ is ______.
7. The greatest integer less than $\dfrac{3 - \pi}{4}$ is ______.
8. The largest integer in the set $\{x \mid -\sqrt{3} < x < \sqrt{148}\}$ is ______.
9. The number of integers between $\sqrt{29} + 1$ and 8π is ______.
10. The number of prime numbers between $\sqrt{17} - 2$ and $\sqrt{83} + 8$ is ______.

Develop Mastery

Exercises 1–6 ***Number Line*** *Show the set on a number line.*

1. $\{x \mid x > -2 \text{ and } x < 2\}$
2. $\{x \mid x \geq -1 \text{ and } x \leq 1\}$
3. $\{x \mid x < 1 \text{ or } x > 3\}$
4. $\{x \mid x \leq -1\} \cup \{x \mid x > 4\}$
5. $\{x \mid x > 0\} \cap \{x \mid x < 3\}$
6. $\{x \mid 1 \leq x < \sqrt{7}\}$

Exercises 7–9 ***Absolute Value*** *Simplify. Express in exact form without using absolute values, and as a decimal approximation rounded off to four decimal places.*

7. **(a)** $\left|\dfrac{1}{4} - \dfrac{3}{2}\right|$ **(b)** $\left|3 - \dfrac{9}{2}\right|$
8. **(a)** $\left|1 - \dfrac{4}{7}\right|$ **(b)** $|3 - \sqrt{17}|$
9. **(a)** $|\pi - 3|$ **(b)** $\left|\pi - \dfrac{22}{7}\right|$

Exercises 10–13 Enter one of the three symbols $<$, $>$, or $=$ in each blank space to make the resulting statement true.

10. **(a)** -4 ______ -6 **(b)** $-\pi$ ______ $-\sqrt{10}$
11. **(a)** $\dfrac{5}{11}$ ______ 0.45 **(b)** $1 + \sqrt{2}$ ______ 2.9
12. **(a)** $\dfrac{47}{3}$ ______ 16 **(b)** $0.\overline{63}$ ______ $\frac{7}{11}$
13. **(a)** $|1 - \sqrt{3}|$ ______ $\sqrt{3} - 1$ **(b)** $|-5|$ ______ 4

Exercises 14–17 ***Ordering Numbers*** *Order the set of three numbers from smallest to largest. Express the result using the symbol $<$, as, for instance, $y < z < x$.*

14. $x = 5, y = -7, z = -3$
15. $x = \dfrac{16}{23}, y = \dfrac{5}{12}, z = \dfrac{7}{15}$
16. $x = 1 - \sqrt{3}, y = \sqrt{3} - 1, z = -1$
17. $x = \left|1 - \dfrac{7}{5}\right|, y = \left|1 - \dfrac{6}{5}\right|, z = \left|1 - \dfrac{1}{5}\right|$

Exercises 18–19 True or False.

18. **(a)** $\pi^2 < 10$ **(b)** $\dfrac{1}{\sqrt{2} - 1} > 2.28$
19. **(a)** $1.33 < 1.\overline{3}$ **(b)** $0.54 > \dfrac{6}{11}$

Exercises 20–25 ***Intervals on Number Line*** *Show the intervals on a number line.*

20. $(-1, 4)$
21. $(-\infty, 2)$
22. $[-2, \infty)$
23. $[1, 4] \cap (0, 5)$
24. $(-\infty, 3) \cup (3, 4]$
25. $[-3, 2] \cap [2, 5]$

Exercises 26–31 ***Verbal to Number Line*** *Set S is described verbally. Show S on the number line.*

26. Set S contains all negative numbers greater than -5.
27. Set S contains all real numbers greater than -2 and less than 3.
28. Set S consists of all integers between -3 and 8.
29. Set S consists of all prime numbers between 0 and 16.
30. Set S consists of all real numbers between $-\sqrt{3}$ and $\sqrt{5}$.
31. Set S consists of all positive real numbers less than 4.
32. What is the largest integer that is **(a)** less than or equal to -5? **(b)** less than -5?
33. What is the largest integer that is less than $1 + \sqrt{17}$?

34. What is the smallest integer that is greater than $\frac{348}{37}$?

35. What is the smallest even integer that is greater than $12 + \sqrt{5}$?

36. What is the largest prime number that is less than $\frac{23}{0.23}$?

Exercises 37–50 ***Complex Number Arithmetic*** *Perform the indicated operations. Express the result as a complex number in standard form.*

37. $(5 + 2i) + (3 - 6i)$ **38.** $(3 - i) + (-1 + 5i)$

39. $(6 - i) - (3 - 4i)$ **40.** $8 - (3 + 5i) + 2i$

41. $(2 + i)(3 - i)$ **42.** $(-1 + i)(2 + 3i)$

43. $(1 + 3i)(1 - 3i)$ **44.** $(7 - 2i)(7 + 2i)$

45. $\dfrac{1 + 3i}{i}$ **46.** $\dfrac{1 + i}{1 - i}$

47. $(1 + \sqrt{3}i)^2$ **48.** $\dfrac{2i}{(1 - i)(2 - i)}$

49. **(a)** i^2 **(b)** i^6 **(c)** i^{12} **(d)** i^{18}

50. **(a)** i^5 **(b)** i^9 **(c)** i^{15} **(d)** i^{21}

Exercises 51–55 If z is $1 - i$ and w is $-2 + i$, express in standard form.

51. $z + 3w$ **52.** $zw - 4$ **53.** $\bar{z} \cdot \bar{w}$

54. $|z + w|$ **55.** $\dfrac{z - \bar{w}}{w}$

Exercises 56–59 ***Complex Plane*** *For the given z and w, show in the complex plane:*
(a) z ***(b)*** w ***(c)*** $\bar{z}$ ***(d)*** $z + w$ ***(e)*** $z \cdot w$

56. $z = 2 - 2i$; $w = 3 + 4i$

57. $z = -3 + 2i$; $w = -2 - i$

58. $z = -1 + 2i$; $w = 3i$

59. $z = 5 - i$; $w = -1 + i$

60. **(a)** If $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$; find z^2, z^3, z^4, z^5, z^6.

(b) Evaluate each of $|z|, |z^2|, |z^3|, \ldots, |z^6|$.

61. From Exercise 60 draw a diagram showing z, z^2, z^3, z^4, z^5, and z^6 in the complex plane. Note the distance from each of these points to the origin. On what circle do these points lie?

Exercises 62–63 In Exercises 60–61, replace z with $z = \frac{1}{\sqrt{2}}(1 + i)$.

1.4 RECTANGULAR COORDINATES, TECHNOLOGY, AND GRAPHS

Creative people live in two worlds. One is the ordinary world which they share with others and in which they are not in any special way set apart from their fellow men. The other is private and it is in this world that the creative acts take place. It is a world with its own passions, elations and despairs, and it is here that, if one is as great as Einstein, one may even hear the voice of God.

Mark Kac

Rectangular Coordinates

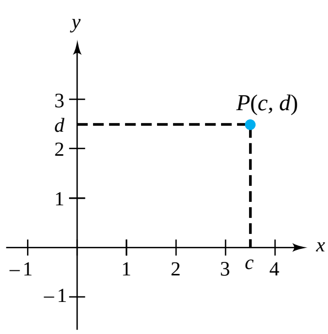


FIGURE 10

Few intellectual discoveries have had more far-reaching consequences than coordinatizing the plane by René Descartes nearly 400 years ago. We speak of **Cartesian** or rectangular coordinates in his honor.

A rectangular coordinate system uses two perpendicular number lines in the plane, which we call coordinate axes. The more common orientation is a horizontal ***x*-axis** and a vertical ***y*-axis,** but other variable names and orientations are sometimes useful.

Each point P in the plane is identified by an ordered pair of real numbers (c, d), called the **coordinates** of P, where c and d are numbers on the respective axes as shown in Figure 10. Conversely, every pair of real numbers names a unique point on the plane.

A rectangular system of coordinates provides a one-to-one correspondence between the set of ordered pairs of real numbers and the points in the plane.

I had lots of exams at school. At sixteen I took [a nationwide exam] in mathematics, physics, and chemistry, and was told that if I passed chemistry I could then drop it and do just pure math, applied math, and physics. So I did. . . I now realize that I quite enjoyed organic chemistry because that ties in somewhat with graph theory.

Robin Wilson

The axes divide the plane into four quadrants labeled I, II, III, IV, as shown in Figure 11. In the figure, points A and B are in Quadrant I, C is in II, D is in III, and E is in IV. Point F is on the x-axis while G is on the y-axis; points on the coordinate axes are not in any quadrant.

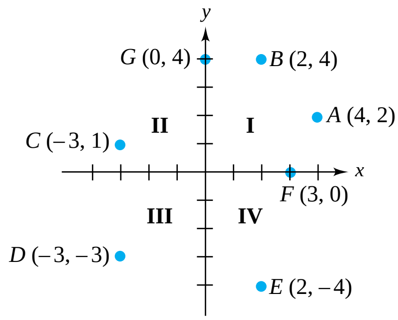


FIGURE 11

The distance $d(r, s)$ between points r and s on a number line is expressed in terms of absolute value.

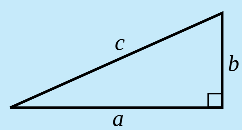
$$d(r, s) = |r - s|$$

We extend the idea of distance to the coordinate plane by means of the familiar Pythagorean Theorem.

Pythagorean theorem

Suppose a and b are the lengths of the legs of a right triangle and c is the hypotenuse. Then

$$a^2 + b^2 = c^2.$$



Conversely, suppose $a^2 + b^2 = c^2$. Then the triangle must be a right triangle with hypotenuse c.

Distance Between Points in a Plane

Suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are any two points in the plane. The distance between P and Q, denoted $d(P, Q)$ or $|\overline{PQ}|$, is defined to be the length of the line segment between P and Q. ($\overline{PQ}$ denotes the line segment from P to Q.) Figure 12 shows right triangle PTQ whose legs are given in terms of absolute values.

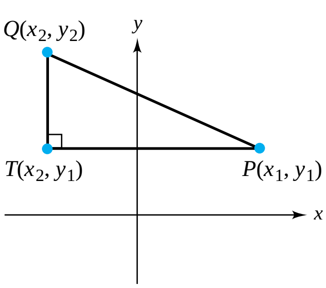


FIGURE 12

$$|\overline{PT}| = |x_1 - x_2| \qquad \text{and} \qquad |\overline{TQ}| = |y_1 - y_2|$$

Applying the Pythagorean theorem gives

$$|\overline{PQ}|^2 = |\overline{PT}|^2 + |\overline{TQ}|^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2.$$

Since $|x_1 - x_2|^2 = (x_1 - x_2)^2$ and $|y_1 - y_2|^2 = (y_1 - y_2)^2$, we have the following.

Distance formula

Suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are any two points in the plane. If $d(P, Q)$ denotes the distance between P and Q, then

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

We can also write $d(P, Q)$ as $|\overline{PQ}|$.

Midpoint of a Line Segment

Suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are any two points in the plane. To get the midpoint M of the line segment $\overline{PQ}$, we take the average of the two x-values and the average of the two y-values. M is the point

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

It is easy to show that $d(P, M) = d(Q, M)$ and that $d(P, Q) = 2 \cdot d(P, M)$.

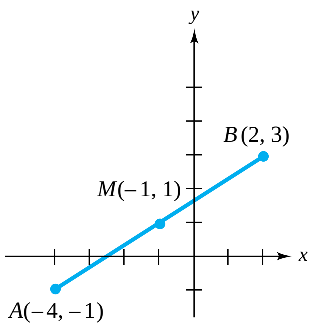


FIGURE 13

▶***EXAMPLE 1*** ***Midpoint*** Given points $A(-4, -1)$ and $B(2, 3)$, find the coordinates of the midpoint M of the segment $\overline{AB}$ and locate all three points on a diagram. In which quadrant is A? B? M? Find $d(A, B)$ and $d(A, M)$.

Solution

The coordinates of M are $\left(\frac{-4+2}{2}, \frac{-1+3}{2}\right)$, or $(-1, 1)$. Point A is in Quadrant III, M is in Quadrant II, and B is in Quadrant I, as shown in Figure 13.

$$d(A, B) = |\overline{AB}| = \sqrt{[2 - (-4)]^2 + [3 - (-1)]^2} = \sqrt{52} = 2\sqrt{13}.$$
$$d(A, M) = |\overline{AM}| = \sqrt{[-1 - (-4)]^2 + [1 - (-1)]^2} = \sqrt{13}.$$ ◀

Graphs

A coordinate plane allows us to make an algebraic relation visible in the form of a graph. We can then apply visual and geometric tools to reveal analytic properties.

Definition: graph of an equation in two variables

The graph of an equation in variables x and y is the set of points whose coordinates (x, y) satisfy the equation.

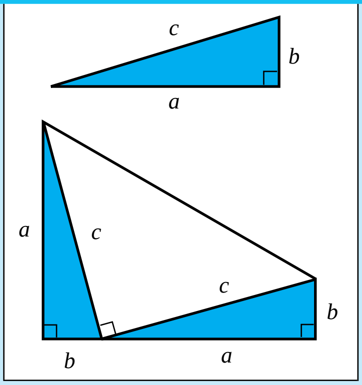
Technically, there is a difference between a graph as a set of points and a sketch or picture of a set. Here we use *graph* to refer to the set or to any representation of the set, most often a pencil sketch, a figure in the book, or a display on a graphing calculator or computer. Without the aid of technology, graphing can be a very tedious process, but graphing is one of the things a graphing calculator does best. To make the best use of a graphing calculator, we need to understand a little about how such a calculator (or computer) works.

How a Graphing Calculator Represents a Graph

A graphing calculator screen is a rectangular array of *picture elements* or *pixels*. After we have entered an equation (usually of the form $y = \ldots$), we choose the **window** through which we will view the graph. This is done by setting an x-range

HISTORICAL NOTE

A PROOF OF THE PYTHAGOREAN THEOREM



How many United States presidents have made an original contribution to mathematics? There is at least one. In 1876 while a member of Congress, four years before he became president, James A. Garfield discovered an original proof for the Pythagorean theorem, one of dozens of proofs given after Euclid's (ca. 300 B.C.).

President Garfield's proof uses two facts. First, the area of a right triangle is half the product of the legs (base × altitude). Second, the area of a trapezoid equals its base times its average height. Given any right triangle, two copies and an isosceles right triangle can be put together to form a trapezoid, as shown in the figure. The sum of the areas of the triangles is $2\frac{ab}{2} + \frac{c^2}{2}$; the area of the trapezoid is $(a+b)\left(\frac{a+b}{2}\right)$. Equating these expressions and multiplying by 2 gives

$$2ab + c^2 = a^2 + 2ab + b^2$$

or

$$c^2 = a^2 + b^2.$$

and a y-range. The calculator divides the x-range into as many pieces as there are columns of pixels (the number differs on each type of calculator, but current graphing calculators have from 94 to 130 pixel columns; see inside front cover) and computes a y-value for every column. The pixel in each column nearest the computed y-value is turned on, to make a graph we can see. In **connected mode,** the calculator turns on as many pixels in a column as needed to "connect the dots," in **dot mode,** we see at most one lighted pixel in each column.

It is essential to understand that a graph produced by any calculator or computer is obtained by computing one value for each pixel column; the calculator only *samples* a graph. Whatever happens (if anything) between pixels does not show on the screen. No matter what window we use, we see at most about a hundred points of the graph from the specified x-range.

Decimal and "Friendly" Windows

When we TRACE on a graph, coordinates are displayed on the screen. The x-coordinate of the nth pixel is given by Xmin + $(n-1)$(Xmax − Xmin)/k, where k is the number of pixel columns. Although usually we don't care about making the x-coordinates "nice" numbers, there are times when it is convenient. Most calculators either have a default window or a ZOOM or RANGE option (labeled something like ZDECM or INIT that sets a window in which x-coordinates are tenths (as 1.1, 1.2, . . .). For obvious reasons we refer to such a window as a **Decimal Window,** although there are other windows with similarly "nice" coordinates. For convenient reference, we give the settings for several calculators.

TECHNOLOGY TIP ◆ ***Decimal windows***

Calculator	*Set*	*#cols*	*x-range*	*y-range*
TI-81	*Range Values*	95	$[-4.8, 4.7]$	$[-3.2, 3.1]$
TI-82	ZOOM 4	94	$[-4.7, 4.7]$	$[-3.1, 3.1]$
TI-85	ZOOM MORE ZDECM	126	$[-6.3, 6.3]$	$[-3.1, 3.1]$
$Casio^{7700}_{9700}$	Range INIT	94	$[-4.7, 4.7]$	$[-3.1, 3.1]$
	Range INIT	126	$[-6.3, 6.3]$	$[-3.7, 3.7]$
HP-38	■PLOT ■CLEAR	130	$[-6.5, 6.5]$	$[-3.1, 3.2]$
HP-48	PLOT NXT Reset Plot	130	$[-6.5, 6.5]$	$[-3.1, 3.2]$

From the Decimal Window as outlined above, there are obvious adjustments that keep nice x-pixel coordinates. Without being more specific, we sometimes call such a window “Friendly.” For example, if we divide all range values by 2, or multiply each by 2, the result is a friendly window. We can shift a window right, left, up or down, by adding the same quantity to both ends of a range. We could reasonably call such a friendly window a “shifted decimal window.” When we suggest a window for a calculator graph in this book, we use the notation $[a, b] \times [c, d]$, where the interval $[a, b]$ is the x-range and $[c, d]$ is the y-range. More often, we encourage you to choose your window as you wish. Experiment to find the picture that is most helpful for your purposes at the moment.

Equal Scale Windows

When we sketch a graph by hand, we usually use the same scale for the two coordinate axes. The unit of distance is the same horizontally and vertically. That relationship does not hold with calculator graphs unless we take special care to set what we call an **Equal Scale Window.** The ranges for an equal scale window depend on the pixel dimensions of your calculator, and can only be approximate. If your calculator screen is 94 by 62, then the ratio of the x-range to the y-range must be approximately $\frac{94}{62}$, or about $\frac{3}{2}$; if your default screen is 130 by 63, then you need a ratio of about $\frac{2}{1}$. Experiment with your calculator. If you have a window that distorts the picture you want to see, you may be able to “square-up” the display by using a command from the ZOOM menu such as ZSQR.

We begin our graphing with two simple and important graphs, both of which we revisit in later chapters: lines and circles.

Lines: Graphs of Linear Equations

Linear equations and lines

A **linear equation** in x and y is an equation equivalent to

$$ax + by + c = 0, \tag{1}$$

where a, b, and c are real numbers and at least one of a and b is nonzero. The graph of any linear equation is a **line.** We will often identify the line with its equation, so that we will speak of “the line $ax + by + c = 0$.”

We find points on a line by substituting a value for x or y, and solving the equation for the corresponding value of the other variable. Since a line is determined by any two points, to sketch the graph of a linear equation, it is sufficient to locate any pair of points on the line. A point where a graph crosses a coordinate axis is called an **intercept point.** Intercept points are often convenient for drawing a line. If the line contains the origin (that is, if $c = 0$), then of course the x- and y-intercept points coincide, and we must find another point to enable us to draw the line.

Intercept points of a line

To find the ***y*-intercept point,**

substitute 0 for x in Equation (1) and solve for y.

To find the ***x*-intercept point,**

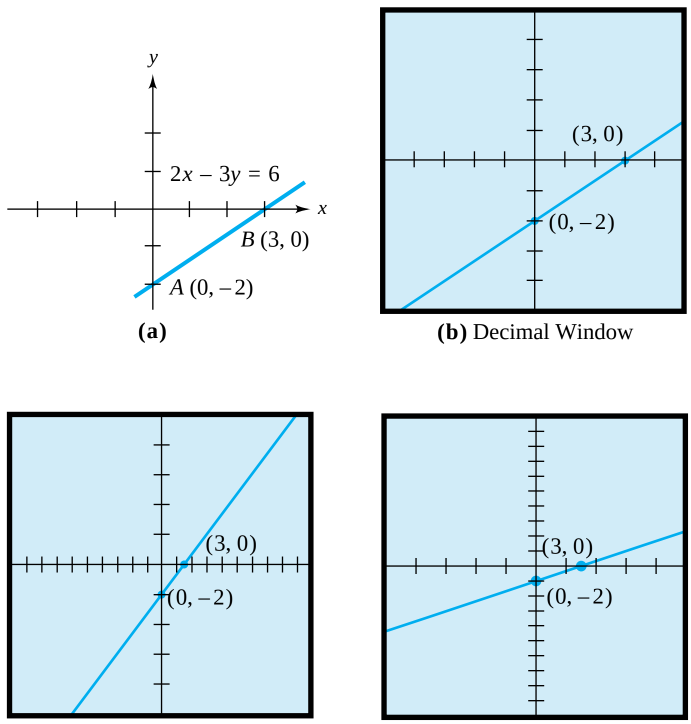
substitute 0 for y in Equation (1) and solve for x.

▶EXAMPLE 2 **Graphing a line** Given the line $2x - 3y = 6$, find the intercept points and draw a graph **(a)** by hand, and **(b)** on the graphing calculator.

Solution

Substituting 0 for x, we have $-3y = 6$, or $y = -2$, so the y-intercept point is $A(0, -2)$. Similarly, if $y = 0$, then $x = 3$, giving the x-intercept point $B(3, 0)$.

(a) Plotting the intercept points A and B and drawing the line containing them gives the graph in Figure 14a.



(a) **(b)** Decimal Window

[–20, 20] by [–10, 10]
(c)

[–10, 10] by [–20, 20]
(d)

FIGURE 14

(b) To enter an equation for graphing, we must first solve for y: $y = \frac{2x-6}{3}$, so we enter `Y = (2X − 6)/3`, and we then must select a window. In a decimal window (Figure 14b) the graph looks much like the line we drew by hand. Other windows can greatly affect the appearance of the line. For example, in the $[-20, 20] \times [-10, 10]$ window, we get something like Figure 14c, and in the $[-10, 10] \times [-20, 20]$ window, we see a much less steep line (Figure 14d). We need to keep in mind that the line remains the same; by changing the window we alter the portion of the line that appears on our (non-square) screen. ◀

TECHNOLOGY TIP ◆ Proper use of parentheses

Since equations are entered on a single line in a calculator, it is hard to overemphasize the importance of the proper use of parentheses. The equation in Example 2 must be entered as `Y = (2X − 6)/3`. See what happens if you forget parentheses by graphing `Y = 2X − 6/3` on the same screen. How would you enter $y = \frac{2}{3}x - 2$? Check by graphing.

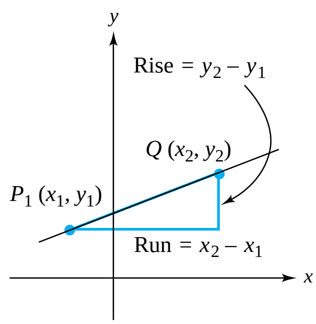


FIGURE 15
Slope: $m = \frac{y_2 - y_1}{x_2 - x_1}$

Slope of a Nonvertical Line

The intuitive idea we have of the direction of a line may not be apparent in a calculator graph, as we can see from the figures for Example 2. We make this idea more precise with the concept of **slope,** which measures how steeply a line rises or drops as we move along the line to the right. See Figure 15.

Definition: slope of a nonvertical line

If L is a nonvertical line and $P(x_1, y_1)$ and $Q(x_2, y_2)$ are any two points on L, then

$$\Delta y = y_2 - y_1 \text{ is called the } \textbf{rise} \text{ from } P \text{ to } Q,$$

$$\Delta x = x_2 - x_1 \text{ is called the } \textbf{run} \text{ from } P \text{ to } Q,$$

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \text{ is called the } \textbf{slope} \text{ of } L. \qquad \textbf{(2)}$$

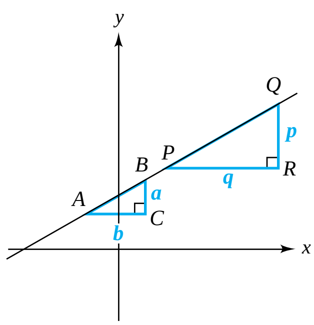


FIGURE 16
Slope: $m = \frac{a}{b} = \frac{p}{q}$

The slope of a line is independent of the two points we choose on the line, as Figure 16 shows. The slope from Equation (2) is the ratio of two sides of a triangle. Triangles ABC and PQR are similar, so ratios of corresponding sides are the same. The slope of L is either $\dfrac{a}{b}$ or $\dfrac{p}{q}$.

The slope of a line is very handy in drawing a graph. Express the slope m as a fraction (with denominator 1 if needed). The denominator is the run and the numerator is the corresponding rise. From any point on the line, move to the right for the run and up or down for the rise, to get the coordinates of another point on the line.

▶EXAMPLE 3 ***Using slopes to draw lines*** Line L_1 passes through the origin, and the y-intercept point of line L_2 is $B(0, 1)$. Their slopes are given respectively by $m_1 = 2$ and $m_2 = -\frac{1}{2}$. Draw both lines on the same set of coordinate axes.

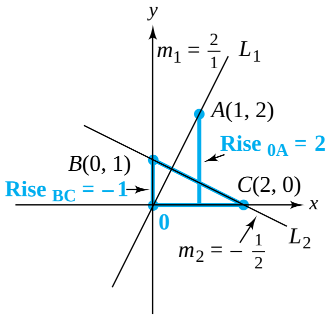


FIGURE 17

Solution

Express m_1 as a fraction, $m_1 = \frac{2}{1}$. To locate another point on L_1, from $O(0, 0)$ plot a run of 1 unit to the right and a rise of 2, giving point $A(1, 2)$. Similarly, since $m_2 = -\frac{1}{2}$, a run of 2 corresponds to a rise of -1 (so the line drops as it moves rightward). From the y-intercept point, 2 units right and 1 down gives point $C(2, 0)$. Line L_1 is determined by O and A; L_2 contains B and C. Both lines are shown in Figure 17. ◀

Horizontal and Vertical Lines

A horizontal line has slope zero because the rise from one point to any other is zero; $\Delta y = 0$. Since every point on a horizontal line has the same y-coordinate, **every horizontal line has an equation of the form $y = c$** for some constant c. On the other hand, it is impossible to define the slope of a vertical line because the run between any two points is zero; $\Delta x = 0$, and we cannot divide by 0. Every point on a vertical line has the same x-coordinate, so **every vertical line has an equation of the form $x = c$.**

Circles

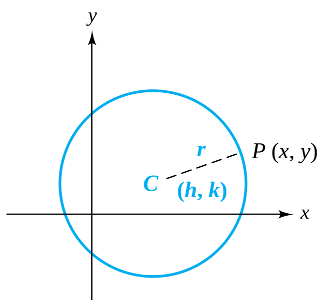


FIGURE 18
Circle with center $C(h, k)$ and radius r

A circle is defined as the set of points that are a fixed distance, called the *radius*, from a fixed point, called the *center*. If the center is point $C(h, k)$, then $P(x, y)$ is on the circle with radius r precisely when the distance $d(P, C)$ equals r (see Figure 18). Using the distance formula,

$$d(P, C) = \sqrt{(x - h)^2 + (y - k)^2} = r.$$

Since the radius r is a positive number, we may square both sides to get the standard form for an equation of a circle.

Standard form for equation of a circle

Suppose h, k, and r are given real numbers ($r > 0$). Point (x, y) lies on a circle of radius r and center (h, k) if and only if (x, y) satisfies

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \textbf{(3)}$$

▶EXAMPLE 4 Equation and graph of a circle

(a) Write an equation for the circle with center $C(2, -3)$ and radius 3. Sketch the graph.

(b) Determine which of the points $O(0, 0)$, $A(2, 0)$, and $B(4, -1)$ are inside the circle.

Solution

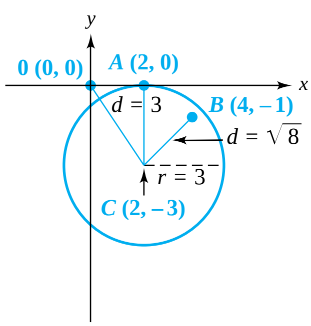


FIGURE 19

(a) Given the coordinates of the center, $h = 2$ and $k = -3$, replace h by 2, k by -3, and r by 3 in Equation 3 to get

$$(x - 2)^2 + [y - (-3)]^2 = 3^2 \qquad \text{or} \qquad (x - 2)^2 + (y + 3)^2 = 9.$$

The graph, with points O, A, and B, is shown in Figure 19.

(b) From the graph, it appears that O is outside the circle, but it is not as clear whether A and B are inside or outside. Using the distance formula,

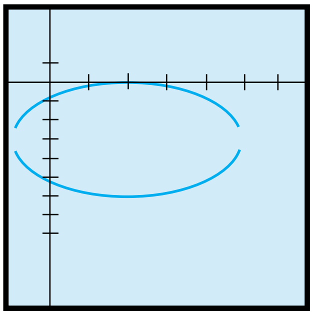
$$d(A, C) = \sqrt{(2 - 2)^2 + (0 + 3)^2} = \sqrt{9} = 3,$$
$$d(B, C) = \sqrt{(4 - 2)^2 + (-1 + 3)^2} = \sqrt{8}.$$

From the definition of a circle, A is on the circle and B is inside because $\sqrt{8} < 3$. ◀

Calculator Graphs of Circles

While there are some calculators that have special adaptations for drawing circles, most graphing calculators do not graph circles as easily as lines. At this point we are limited to entering equations in the form $y = \ldots$. For the circle in Example 4, $(x-2)^2 + (y+3)^2 = 9$, we would have to solve the equation for y by subtracting $(x-2)^2$ from both sides, taking square roots (there are two), and finally subtracting 3:

$$(y+3)^2 = 9 - (x-2)^2, \quad y + 3 = \pm\sqrt{9-(x-2)^2},$$
$$y = -3 \pm \sqrt{9-(x-2)^2}.$$



[−1, 7] by [−6, 2]

FIGURE 20

Thus we must enter *two* equations, one for each sign, Y1 = −3 + √(9 − (X − 2)²) and Y2 = −3 − √(9 − (X − 2)²) (watch the parentheses), choose a window, and graph. In the standard decimal window we see only the upper portion of the circle, so we need a larger window. Looking carefully at Figure 19, we see that our window must include at least the interval $[-1, 5]$ in the x-direction and $[-6, 0]$ in the y-direction. If we try, say, $[-1, 7] \times [-6, 2]$, the result doesn't look much like the circle in Figure 19. See Figure 20. There are two problems: We don't have an equal scale window, so the circle is "squashed," and when we trace along the curve, we see that there is no x-pixel coordinate for -1 or for 5. Thus the two pieces of the circle don't meet at the ends. Both obstacles may be overcome with some work (in this instance by simply shifting all the x-and y-values the same amount from the decimal window), but we suggest that you learn to interpret what the calculator shows. You should recognize that what appears on your display screen need not look like what you would draw yourself.

There is another way to draw circles on a graphing calculator that avoids gaps. In Chapter 6 we discuss the use of trigonometric functions to graph circles in parametric mode.

The standard form for an equation of a circle identifies the center and the radius. If we expand the squared terms on the left side of Equation (3) and collect the constants, we get a general form for an equation of a circle.

General form for the equation of a circle

For any real numbers A, B, and C, the graph of the equation

$$x^2 + y^2 + Ax + By + C = 0 \qquad \textbf{(4)}$$

is either a circle, a point, or no points.

We can show that Equation (4) is equivalent to Equation (3) by completing the square on $x^2 + Ax$ and on $y^2 + By$, as illustrated in Example 5.

Strategy: Write in the form of Equation (4) and then complete the squares. Remember to add the same quantity to both sides of the equation.

▶*EXAMPLE 5* ***Finding center and radius*** Find the center and radius. Draw a graph.

(a) $x^2 + y^2 = 4x$ **(b)** $2x^2 + 2y^2 - 4x + 6y + \frac{1}{2} = 0$

Solution

(a) Follow the strategy.

$$(x^2 - 4x) + y^2 = 0$$
$$(x^2 - 4x + 4) + y^2 = 4 \quad \text{or} \quad (x-2)^2 + (y-0)^2 = 2^2$$

The center is (2, 0) and the radius is 2.

(b) Divide by 2 and collect x- and y-terms. Complete the squares.

$$x^2 + y^2 - 2x + 3y + \frac{1}{4} = 0$$
$$(x^2 - 2x) + (y^2 + 3y) = -\frac{1}{4}$$
$$(x^2 - 2x + 1) + \left(y^2 + 3y + \frac{9}{4}\right) = -\frac{1}{4} + 1 + \frac{9}{4}$$
$$(x-1)^2 + \left(y + \frac{3}{2}\right)^2 = 3.$$

The center is $(1, -\frac{3}{2})$ and the radius is $\sqrt{3}$. Both circles are shown in Figure 21. ◀

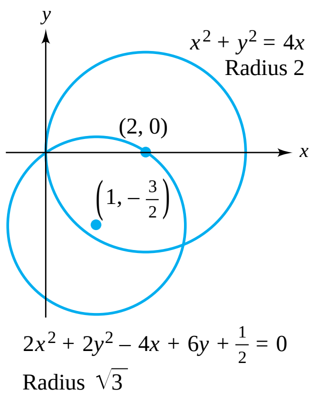


FIGURE 21

▶EXAMPLE 6 ***Finding extreme points*** Find the coordinates of the highest and the lowest points on the graph of $x^2 + y^2 - 4x + 6y - 3 = 0$.

Solution

Looking at the equation, our first observation is that the graph appears to be a circle. From the graph we should be able to locate the high and low points. Find the center and radius by completing the squares on $x^2 - 4x$ and $y^2 + 6y$.

$$(x-2)^2 + (y+3)^2 = 16.$$

The center of the circle is at (2, −3) and the radius is 4. The graph is shown in Figure 22. The graph shows the highest point 4 units above the center, at (2, 1), and the lowest point 4 units below the center, at (2, −7). ◀

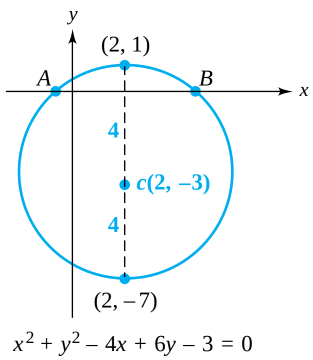


FIGURE 22

▶EXAMPLE 7 ***Intercept points*** Find the distance between the x-intercept points on the graph of $x^2 + y^2 - 4x + 6y - 3 = 0$.

Solution

This is the equation from Example 6, with the graph in Figure 22. Find the distance between points A and B. To find the x-intercept points, replace y by 0 and get

$$x^2 - 4x - 3 = 0.$$

Using the quadratic formula, the roots are $2 \pm \sqrt{7}$.

Here we have two different values of x. The graph makes it clear that the x-coordinate of A is negative, namely $2 - \sqrt{7}$ (≈ -0.65), and the x-coordinate of B is $2 + \sqrt{7}$ (≈ 4.65), so A and B are the points $(2 - \sqrt{7}, 0)$ and $(2 + \sqrt{7}, 0)$, respectively. Since A and B lie on the same horizontal line, the distance between them is given by the difference in their x-coordinates:

$$d(A, B) = \left|(2 + \sqrt{7}) - (2 - \sqrt{7})\right| = 2\sqrt{7} \approx 5.29. \quad ◀$$

►EXAMPLE 8 ***Point and imaginary circles*** Describe the graphs of

(a) $x^2 + y^2 - 2x + 4y + 5 = 0$. **(b)** $x^2 + y^2 - 2x + 4y + 6 = 0$.

Solution

(a) Completing the squares on $x^2 - 2x$ and $y^2 + 4y$ gives

$$(x - 1)^2 + (y + 2)^2 = 0.$$

This equation has the appearance of the equation of a circle with center at $(1, -2)$ and radius 0! We want the set of points whose coordinates satisfy the equation. The sum of two squares can be 0 only if both terms are 0. The graph is a single point, $(1, -2)$. This is sometimes called a *point circle*.

(b) Completing the squares on the x and y terms, we get

$$(x - 1)^2 + (y + 2)^2 = -1.$$

This is even worse; there are no points whose coordinates satisfy the given equation. No graph is associated with the equation, but, by analogy with the point circle of part **(a)**, the equation in part **(b)** is sometimes called an *imaginary circle*. ◄

EXERCISES 1.4

Check Your Understanding

Exercises 1–7 True or False. Given reasons.

1. The graph of $x^2 + y^2 + 2x + 1 = 0$ is a single point.
2. The point $(1, -2)$ is the center of the circle whose equation is $x^2 + y^2 + 2x - 4y = 0$.
3. The graph of $x^2 + y^2 - 2x = 0$ is a circle with diameter 2.
4. If the graphs of $5x - 2y = 4$ and $3x + y = 20$ are drawn simultaneously using the window $[-10, 10]$ by $[-5, 5]$, the display will show two lines intersecting at a point on the screen.
5. When the graphs of $2x + 3y = 12$ and $x^2 + y^2 = 8$ are drawn simultaneously using the window $[-9.4, 9.6]$ by $[-6.2, 6.4]$, the display will show a line and a circle intersecting at two points.
6. There is no real number c such that the point $(1, c)$ is 1 unit from $(-1, 2)$. (*Hint:* Think geometrically.)
7. There are two numbers c for which the point $(1, c)$ is 4 units from $(-1, 2)$. (*Hint:* Think geometrically.)

Exercises 8–10 Fill in the blank so that the resulting statement is true.

8. The graph of $x^2 + y^2 - 2x + 4y - 5 = 0$ is a circle having center in Quadrant ______.
9. If (a, b) is any point in the second quadrant, then (b, a) is in Quadrant ______.
10. The graph of $x + y - 1 = 0$ does not contain any points in Quadrant ______.

Develop Mastery

Exercises 1–5 ***Applying Distance Formula*** ***(a)*** *Draw a diagram showing points A and B and find the distance between them.* ***(b)*** *Find the coordinates of the midpoint M of the line segment $\overline{AB}$.* ***(c)*** *Verify that $d(A, M) = \frac{1}{2}d(A, B)$.*

1. $A(1, 3)$, $B(-2, 4)$
2. $A(-2, 3)$, $B(4, -1)$
3. $A(-\frac{1}{2}, 2)$, $B(1, -\frac{1}{3})$
4. $A(1, -2)$, $B(-\frac{1}{3}, -\frac{1}{3})$
5. $A(2\sqrt{2}, -3\sqrt{2})$, $B(-2\sqrt{2}, \sqrt{2})$

Exercises 6–11 ***Special Triangles*** *Determine whether the three points are vertices of a right triangle, an equilateral triangle, an isosceles triangle, or none of these. (Hint: For a right triangle, use the Pythagorean theorem.)*

6. $A(-1, 2)$, $B(4, -2)$, $C(8, 3)$
7. $A(4, -2)$, $B(-4, 2)$, $C(7, 4)$
8. $A(0, 0)$ $B(2\sqrt{3}, 2)$, $C(0, 4)$
9. $A(0, 0)$, $B(4, 2)$, $C(0, 4)$
10. $A(-1, -1)$, $B(4, 1)$, $C(1, 4)$
11. $A(-2, -3)$, $B(4, 6)$, $C(-6, -\frac{1}{3})$

Exercises 12–21 ***Equation of a Circle*** *Write an equation for the circle that satisfies the given conditions. First draw a diagram showing the circle. Give the result in expanded form.*

12. Center $(0, 0)$; radius 3
13. Center $(1, 1)$; radius $\sqrt{3}$
14. Center $(2, -1)$; radius $\sqrt{5}$

15. Center $(-1, 5)$; diameter 1

16. Center $(-2, -1)$; tangent to the x-axis

17. Center $(-2, -1)$; tangent to the y-axis

18. The segment from $A(-3, 4)$ to $B(1, 1)$ is a diameter.

19. The segment from $A(3, -2)$ to $B(5, 4)$ is a diameter.

20. The circle is circumscribed about the triangle having vertices $A(0, 0)$, $B(8, 0)$ and $C(8, 6)$. (*Hint:* Triangle ABC is a right triangle.)

21. The circle passes through the three points $A(2, 1)$, $B(6, 1)$ and $C(6, 4)$. (*Hint:* $\angle ABC$ is a right angle.)

Exercises 22–23 ***Extreme Points*** *An equation of a circle is given. Find* ***(a)*** *the highest and lowest points and* ***(b)*** *the points furthest to the right and left. See Example 6.*

22. $x^2 + y^2 + 4x - 4y - 8 = 0$

23. $x^2 + y^2 - 6x - 2y + 1 = 0$

Exercises 24–25 *An equation of a circle is given. Find* ***(a)*** *the x- and y-intercept points,* ***(b)*** *the distance between the x-intercept points, and* ***(c)*** *the distance between the y-intercept points. See Example 7.*

24. $x^2 + y^2 + 2x - 2y - 8 = 0$

25. $x^2 + y^2 - 6x - 2y + 1 = 0$

Exercises 26–29 ***Intercept Points*** *An equation of a line is given. Find* ***(a)*** *the x- and y-intercept points,* ***(b)*** *the distance between the intercept points,* ***(c)*** *the slope of the line.*

26. $2x - 3y = 6$

27. $3x - 4y + 12 = 0$

28. $4x + 3y = 6$

29. $3x - 4y = 6$

Exercises 30–33 ***Using Distance Formula*** *Find the value(s) of x or y so that the distance d between the two points is the given distance.*

30. $(-2, 3)$, $(x, -1)$; $d = 6$

31. $(3, -1)$, $(x, 4)$; $d = 8$

32. $(-4, 2)$, $(2, y)$; $d = 5$

33. $(3, -1)$, $(5, y)$; $d = 3$

Exercises 34–44 ***Graph of Equation*** ***(a)*** *Identify the graph of the equation as a line or a circle.* ***(b)*** *For a line, find the coordinates of the intercept points. For a circle, find the radius and the coordinates of the center.* ***(c)*** *Sketch the graph.*

34. $2x + 3y = 6$

35. $x + y = 4$

36. $x^2 + y^2 = 4$

37. $y = 3x - 2$

38. $2y = x^2 + y^2$

39. $3x^2 + 3y^2 = 21$

40. $7x + 7y = 21$

41. $x^2 + 2x + y^2 = 0$

42. $x^2 + y^2 = 2x + 4y$

43. $x^2 - 4x + y^2 + 2y + 1 = 0$

44. $2x^2 + 2y^2 + 4y = 12x + 15$

Exercises 45–55 *Draw a calculator graph of each equation in Exercises 34–44. Experiment with windows of various sizes.*

Exercises 56–59 ***Windows*** *The graphs of the given equations are perpendicular lines. Draw the graphs. Experiment with windows of different sizes until you get lines that appear to be perpendicular. Indicate the window size you use. Use* TRACE *to find the point of intersection of the lines (one decimal place).*

56. $2x - y = 4$, $\quad x + 2y - 2 = 0$

57. $3x - 2y - 6 = 0$, $\quad 2x + 3y = 12$

58. $8x - 5y = 36$, $\quad 5x + 8y = 4$

59. $3x - y - 7 = 0$, $\quad x + 3y + 5 = 0$

Exercises 60–63 ***Viewing Windows*** *In order to draw a graph of the equation that will show the x- and y-intercept points, which window would you use? Try each one.*

60. $2x + y = 25$
(i) $[-10, 10] \times [-10, 10]$
(ii) $[-15, 5] \times [-15, 15]$
(iii) $[-5, 20] \times [-5, 30]$

61. $3x - 2y + 40 = 0$
(i) $[-10, 10] \times [-10, 10]$
(ii) $[-20, 5] \times [-10, 10]$
(iii) $[-15, 5] \times [-5, 25]$

62. $(x - 2)^2 + (y + 4)^2 = 64$
(i) $[-10, 10] \times [-10, 10]$
(ii) $[-4.7, 4.8] \times [-3.2, 3.6]$
(iii) $[-13.1, 14.5] \times [-13.3, 5.6]$

63. $x^2 + y^2 - 2x + 4y - 100 = 0$
(i) $[-10, 10] \times [-10, 10]$
(ii) $[-5.5, 18.3] \times [-5.2, 18.3]$
(iii) $[-17.0, 17.3] \times [-12.5, 8.9]$

Exercises 64–65 *The graph of the equation is a circle. Which of the windows gives a graph that is nearly circular?*

64. $x^2 + y^2 = 16$
(i) $[-10, 10] \times [-10, 10]$
(ii) $[-8, 8] \times [-5, 5]$
(iii) $[-10, 10] \times [-5, 5]$

65. $(x - 1)^2 + (y + 4)^2 = 25$
(i) $[-10, 10] \times [-10, 10]$
(ii) $[-12, 12] \times [-10, 2]$
(iii) $[-8, 10] \times [-10, 2]$

Exercises 66–69 ***Half Circle Graphs*** *Draw graphs to determine the quadrant(s) in which the two half circles intersect (if any).*

66. Lower half of $x^2 + y^2 = 16$ and the lower half of $x^2 + y^2 - 12x + 11 = 0$.

67. Lower half of $x^2 + y^2 = 16$ and the lower half of $x^2 + y^2 - 12x - 28 = 0$.

68. Upper half of $x^2 + y^2 = 16$ and the upper half of $x^2 + y^2 - 12x + 20 = 0$.

69. Upper half of $x^2 + y^2 = 16$ and the upper half of $x^2 + y^2 + 12x + 20 = 0$.

Exercises 70–72 ***Lattice Points*** *For these exercises, remember that point (x, y) is a lattice point if both x and y are integers.*

70. Give an example of two lattice points in Quadrant I that define a line segment whose midpoint is not a lattice point.

71. Give an example of two lattice points in Quadrant I that define a line segment whose midpoint is also a lattice point.

72. Find a pair of lattice points, A and B, with A in Quadrant II and B in Quadrant IV for which the midpoint of segment $\overline{AB}$ is **(a)** in Quadrant I, **(b)** in Quadrant II, **(c)** in Quadrant III, **(d)** not in any quadrant.

Exercises 73–75 Find all lattice points in Quadrant I on the graph of the equation.

73. $2x + y = 6$

74. $x + 3y = 13$

75. $2x + 3y = 12$

Exercises 76–77 ***Pythagorean Triples*** *If the lengths of the three sides of a right triangle are integers, then the triplet of numbers is called a Pythagorean triple. For example, $a = 3, b = 4$, and so $3^2 + 4^2 = 25 = 5^2$, and $[3, 4, 5]$ is a Pythagorean triple. We illustrate an interesting way to find Pythagorean triples:*

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4};\ 3^2 + 4^2 = 5^2 \text{ gives } [3, 4, 5]$$

$$\frac{1}{3} + \frac{1}{5} = \frac{8}{15};\ 8^2 + 15^2 = 17^2 \text{ gives } [8, 15, 17]$$

$$\frac{1}{7} + \frac{1}{9} = \frac{16}{63};\ 16^2 + 63^2 = 65^2 \text{ gives } [16, 63, 65]$$

76. Follow the pattern suggested above and find three additional Pythagorean triples.

77. Prove that the pattern illustrated here always yields a Pythagorean triple. (*Hint:* Consider $\frac{1}{n-1} + \frac{1}{n+1}$ where n is any positive integer greater than 1.)

78. If $A(0, 3)$, $B(-1, -1)$, and $C(4, 1)$ are three vertices of a parallelogram, what are the coordinates of the fourth vertex? Draw a diagram. Is the answer unique?

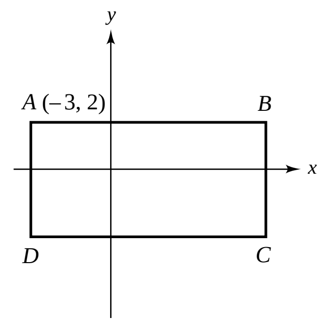
Exercises 79–80 Find the point P that is equidistant from the three points A, B, and C; that is, find P such that $|\overline{PA}| = |\overline{PB}| = |\overline{PC}|$. (Hint: First show that the three points are vertices of a right triangle and consider the circle circumscribing $\triangle ABC$.)

79. $A(8, 3)$, $B(4, 10)$, $C(2, 6)$

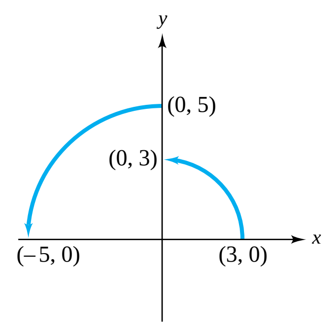
80. $A(2, 3)$, $B(8, 0)$, $C(5, 9)$

81. A rectangle has sides parallel to the coordinate axes. Two of its vertices are at $(-5, -7)$ and $(4, -2)$. Find the coordinates of the other two vertices and the length of a diagonal.

82. A rectangle has sides parallel to the coordinate axes and its upper left corner at $A(-3, 2)$ as shown in the diagram (which is not drawn to scale). The length (horizontal side) is twice the width, and the perimeter is 30. Find the coordinates of the other three vertices.



83. A 90° rotation of a plane counterclockwise about the origin moves point $(3, 0)$ to $(0, 3)$ and point $(0, 5)$ to $(-5, 0)$ (see the diagram). What is the image of each point under the same rotation? Draw diagrams.
(a) $(-4, 0)$ **(b)** $(0, -3)$
(c) $(3, 4)$ **(d)** $(-3, -4)$



84. Find the area of the region that is inside the circle $x^2 + y^2 - 2x - 3 = 0$ and outside the circle $x^3 + y^2 = 1$.
(*Hint:* First draw a diagram.)

85. What is the area of a circle if the reciprocal of its circumference equals the length of its radius?

86. The line $x + y = 3$ divides the interior of the circle $x^2 + y^2 = 9$ into two regions. If A_1 and A_2 are the areas of the larger and smaller of the two regions, respectively, find the ratio $\frac{A_1}{A_2}$.

1.5 ONE-VARIABLE SENTENCES: ALGEBRAIC AND GRAPHICAL TOOLS

> There is something very funny here. We can teach a computer to decide whether a mathematical formula is well formed or not. That's very easy. But we cannot teach a computer to talk, to form sentences. It is obviously a million times as hard. But take any kid who learns how to speak. If, as a kid, he hears two languages, he learns two languages. If he is mentally retarded, he still becomes bilingual. He will know fewer words, but he will know those words in both languages. He will form sentences. Now try to explain to him what is a well formed algebraic formula!
>
> *Lipman Bers*

In mathematics, language is the key to understanding while problem solving is the key to learning. The two are closely related since it is impossible to solve a problem without first understanding it. Learning to communicate is fundamental to all education. People express ideas in words, which they combine to form meaningful sentences. Sentences are the basic elements of communication.

> *I was definitely not the best student in my class. There were five of us who graduated, and there was one girl who was much smarter than I. There was another very bright kid. I was maybe third out of five. I went to the university thinking that I would make C's but I made A's without any trouble.*
>
> Mary Ellen Rudin

The language of mathematics is both precise and concise, often making use of symbols. However, mathematical symbols are combined together to form sentences having similar grammatical structures, including subjects and predicates, as sentences in our more familiar daily language.

The use of symbols allows us to write sentences in very compact form. For instance, in place of "The sum of 2 and 3 is 5," we write "$2 + 3 = 5$." Similarly, the symbolic sentence "$x \leq 4$" in everyday language means "x is less than or equal to 4," which is a complete sentence.

Statements and Open Sentences

One of our primary interests in mathematics is to determine the truth value of a statement, or to find all values of a particular variable that make a sentence true. Consider the following sentences.

(a) $2 + 5 = 7$ **(b)** $3 > \sqrt{16}$ **(c)** $x^2 - 2x - 3 = 0$

(d) Every even integer greater than 2 is the sum of two primes.

We can say that sentence **(a)** is true and sentence **(b)** is false (since $\sqrt{16} = 4$). Sentence **(d)** has a truth value; it is either true or false. But in more than 300 years no one has been able to prove either that it is true or to find a single counterexample to show that it is false. This famous unsolved problem is known as Goldbach's conjecture (see the Historical Note).

We cannot assign a truth value to sentence **(c)** above unless we replace x by a number. Replacing x by 2 gives $2^2 - 2 \cdot 2 - 3 = 0$, which is false. Replacing x by 3, however, gives $3^2 - 2 \cdot 3 - 3 = 0$, which is true. Sentence **(c)** is called an **open sentence;** there is no truth value until the variable x (a placeholder, an open spot) is filled. The remaining three sentences, because they are either true or false, are called **statements.**

Definition: statement

A **statement** is a sentence that has a truth value, either true or false.

HISTORICAL NOTE GOLDBACH, COUNTEREXAMPLES, AND UNSOLVED PROBLEMS

On page 38 we mentioned Goldbach's conjecture that "Every even integer greater than 2 is the sum of two primes." Goldbach made this assertion in 1742 in a letter to Euler. A quick look at the first cases shows how reasonable the conjecture seems:

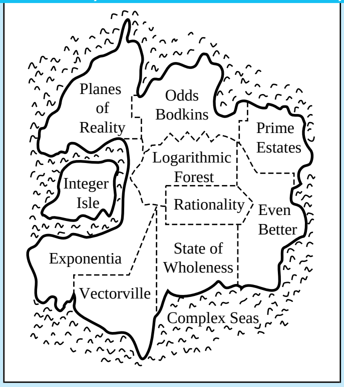
$$4 = 2 + 2, \qquad 6 = 3 + 3,$$
$$8 = 5 + 3, \qquad 10 = 7 + 3.$$

The question, of course, is how long this continues. Could we use computers to check?

Computer searches support Goldbach's assertion for *all even numbers up to a hundred million,* but forever? Who knows? A single *counterexample,* one even number that is not the sum of two primes, would prove Goldbach's conjecture false. Computers could conceivably show that Goldbach was wrong; no blind search process can ever prove him right.

An unsolved problem in mathematics does not necessarily mean there is no solution; it means that we cannot yet prove or disprove an assertion.

Each year, some long-standing questions are answered, and each answer raises more questions.



Four colors suffice to color even complicated maps.

Note: At a Cambridge University seminar in June, 1993, Professor Andrew Wiles of Princeton announced the proof of a conjecture about elliptic curves. His proof establishes that Fermat's Last Theorem is true.

Some recent milestones:

Four Color Theorem Four colors are enough to color any map; part of the proof required 1200 hours of computer time.

Classification of Simple Groups There are exactly 26 simple groups of a special type; the proof requires thousands of pages contributed by many mathematicians. The biggest group, called "The Monster," has more than 10^{53} elements.

Fermat's Last Theorem The equation $x^n + y^n = z^n$ has no solutions in integers if $n > 2$. (For $n = 2$ there are lots; see Explore and Discover in Section 5.2.) A major step was taken by a German mathematician in 1983, marking the most progress in more than a hundred years. The problem is over three hundred years old.

What makes such progress exciting is more than just the solution of an unsolved problem. Work on one problem can help us understand others, and light shed in one corner of mathematics often lights up whole new vistas whose existence we may not even have suspected previously.

Solving Open Sentences

Open sentences include equations and inequalities. Because many of the methods of solution are the same, we treat equations and inequalities together. By solving an open sentence we mean finding all the admissible replacement values for the variable that make the sentence *true*. The **domain** or **replacement set** for a variable is the set of numbers that the problem allows as replacements for the variable. Any restrictions on the domain must be clearly stated; otherwise we adopt the following convention regarding domains.

Domain convention

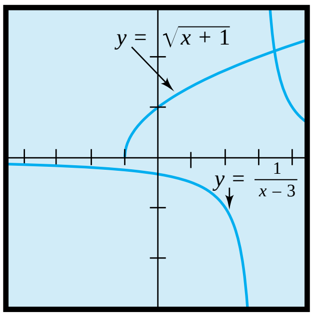
If no restrictions are stated, the domain of a variable is assumed to be the set of all real numbers that give meaningful real number statements. This excludes any division by zero or square roots of negative numbers.

▶EXAMPLE 1 ***Finding domain*** Find the domain of the variable x in the open sentence:

$$\sqrt{x + 1} \geq \frac{1}{x - 3}.$$

Solution

Algebraic Assuming the domain convention, we require $x - 3 \neq 0$ or $x \neq 3$, and we must have $x + 1 \geq 0$, or $x \geq -1$. Taking the conditions together, the domain D of the variable consists of all real numbers greater than or equal to -1, except for 3. In interval notation, D is $[-1, 3) \cup (3, \infty)$.



$[-4.5, 4.5]$ by $[-3, 3]$

FIGURE 23

Graphical Graphing calculators can be used to confirm conclusions about the domains of equations. If we graph the two expressions that appear in the above inequality, $y = \sqrt{x + 1}$ and $y = 1/(x - 3)$, we get the two graphs in Figure 23. Tracing along each curve, the calculator shows that there is no y-value for $y = \sqrt{x + 1}$ when x is less than -1. Similarly, there is no y-value for the other curve when $x = 3$. We can thus literally *see* that the domain of the open sentence consists of all real numbers greater than -1, except for 3. ◀

The **solution set** for an open sentence is the set of all numbers in the domain that yield true statements. To solve an equation or inequality means to find the solution set, and the *roots* of an equation are the numbers in the solution set.

Solving equations and inequalities is not always easy, but to simplify this work we most generally perform operations that give us equivalent open sentences, hoping to reach a sentence whose solution set is obvious. For example, $2x - 3 = 5$ is equivalent to $2x = 8$, which is equivalent to $x = 4$. The solution to $2x - 3 = 5$ is 4. Equivalent open sentences have the same solution set. The following equivalence operations on open sentences yield equivalent open sentences.

Equivalence operations

1. *Replace* any expression in the sentence by another expression identically equal to it.
2. *Add* or *subtract* the same quantity on both sides.
3. For an equation, *multiply* or *divide* both sides by the same nonzero quantity.
4. For an inequality, *multiply* or *divide* both sides by the same positive quantity, or multiply or divide both sides by the same negative quantity and reverse the direction of the inequality.

The last equivalence operation for inequalities points up one of the major differences between equations and inequalities: multiplication by a negative number reverses the direction of an inequality. To avoid the necessity of treating separate cases, we suggest that you *never* multiply an inequality by an expression involving a variable.

Linear Equations and Inequalities

A **linear open sentence** is one that is equivalent to

$$ax + b \,\square\, 0, \text{ with } \square \text{ replaced by } =, <, >, \leq, \text{ or } \geq,$$

where a and b are constants and a is not zero.

The equivalence operations allow us to find the solution set for any linear open sentence.

Strategy: **(a)** First use Equivalence Operation 2 to get all x-terms on one side and the constants on the other (i.e., subtract x and 4 from both sides).
(b) Similarly, use Operation 2 to collect the x-terms on one side and constants on the other.

▶*EXAMPLE 2* ***Solving linear open sentences*** Find the solution set.

(a) $3x + 4 = x - 1$
(b) $2 - 3x \leq 4$

Solution

Follow the strategy.

(a) $3x - x = -1 - 4, \quad \text{or} \quad 2x = -5.$

Divide both sides by 2 (Equivalence Operation 3), giving $x = -\frac{5}{2}$. The solution set is $\{-\frac{5}{2}\}$.

(b) $-3x \leq 4 - 2, \quad \text{or} \quad -3x \leq 2.$

By Equivalence Operation 4, we can divide both sides by -3 if we reverse the direction of the inequality, getting $x \geq -\frac{2}{3}$. The solution set is $\{x \mid x \geq -\frac{2}{3}\}$, or in interval notation, $[-\frac{2}{3}, \infty)$. ◀

Factorable Equations and Inequalities

An equation or inequality that can be written in the form of a product (or quotient) of *linear* factors on one side and 0 on the other side, can be solved using a variety of techniques. All methods we discuss for finding the solution set of such open sentences rely on the Signed Product Principles.

Signed product principles

Zero-product Principle A product of factors *equals zero* if and only if *at least one factor equals zero.*
Positive-product Principle A product of two factors is *positive* if and only if they have the *same sign.*
Negative-product Principle A product of two factors is *negative* if and only if they have *opposite signs.*

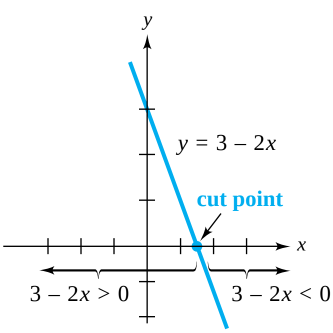


FIGURE 24

Associated with each linear factor is what we call a **cut point,** the point on the number line where *the factor equals 0.* A linear expression is always positive in one direction from its cut point and negative in the other direction. For example, $3 - 2x = 0$ when $x = \frac{3}{2}$, so $\frac{3}{2}$ is the cut point. When we replace x by any number greater than $\frac{3}{2}$ (to the right of $\frac{3}{2}$ on the number line), $3 - 2x$ is negative; for any x less than $\frac{3}{2}$, $3 - 2x$ is positive. The name *cut point* reminds us that $\frac{3}{2}$ *cuts* the number line into a piece where $3 - 2x$ is positive and a piece where $3 - 2x$ is negative. Looking at the graph of the line $y = 3 - 2x$ (Figure 24), we can see where the line cuts the x-axis, separating the portion to the left of $(\frac{3}{2}, 0)$, where the y-coordinates are positive $(3 - 2x > 0)$, from the portion to the right.

Quadratic Equations and Inequalities

A **quadratic open sentence** is one that is equivalent to

$$ax^2 + bx + c \;\square\; 0, \text{ with } \square \text{ replaced by } =, <, >, \leq, \text{ or } \geq,$$

where a, b, and c are constants and a is not zero.

▶EXAMPLE 3 ***Quadratic open sentences*** Find the solution set.

(a) $2x^2 - 3x - 2 = 0$ **(b)** $2x^2 - 3x - 2 < 0$

Solution

(a) $2x^2 - 3x - 2 = (2x + 1)(x - 2)$, and so by Equivalence Operation 1 the given equation is equivalent to

$$(2x + 1)(x - 2) = 0.$$

By the zero-product principle,

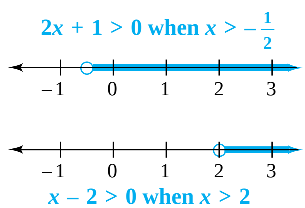
$$2x + 1 = 0 \quad \text{or} \quad x - 2 = 0.$$

Each of these equations determines a cut point for the linear factor and hence a *root of the original equation* (Check!), so the solution set is $\{-\frac{1}{2}, 2\}$.

(b) ***Algebraic*** Equivalence Operation 1 says that the inequality is equivalent to

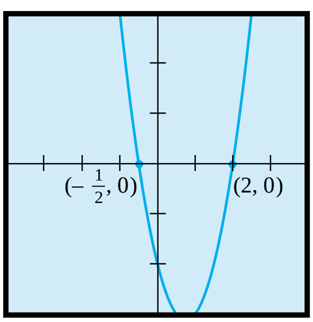
$$(2x + 1)(x - 2) < 0.$$

In the solution to part **(a)**, we found the cut points for the expression, $-\frac{1}{2}$ from $2x + 1 = 0$, and 2 from $x - 2 = 0$. We use the cut points to visualize the sign pattern for the product. The factor $2x + 1$ is positive when $x > -\frac{1}{2}$, that is, to the right of $-\frac{1}{2}$ on the number line, and the factor $x - 2$ is positive to the right of 2. We show this information on a pair of number lines.



By the negative-product principle, the desired inequality holds when the two factors have *opposite signs*. From the two number lines, we can see that the two factors have opposite signs, $2x + 1$ positive, and $x - 2$ negative, between $-\frac{1}{2}$ and 2. Thus the solution set is the open interval, $(-\frac{1}{2}, 2)$.

Graphical Having the given inequality in factored form (so we can identify the cut points), we can use a graphing calculator to see where the product is positive or negative. We enter $y = (2x + 1)(x - 2)$ and graph. See Figure 25. From the figure it is apparent that for any number x between the cut points, the y-value is negative, which is the condition we want. We can see that the solution set is the open interval, $(-\frac{1}{2}, 2)$. ◀



[−4, 4] by [−3, 3]

FIGURE 25

For quadratic expressions that cannot be factored readily, we can use the quadratic formula to find the zeros and hence to identify the cut points.

Strategy: Use the quadratic formula and determine whether or not the solutions are real numbers.

▶EXAMPLE 4 ***Using the quadratic formula*** Apply the quadratic formula to solve $2x^2 + 4x + 3 = 0$, where the domain set is

(a) the set of real numbers **(b)** the set of complex numbers.

Solution

Substituting 2 for a, 4 for b, and 3 for c in the quadratic formula gives

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(3)}}{2 \cdot 2} = \frac{-4 \pm \sqrt{-8}}{4} = \frac{-2 \pm \sqrt{2}i}{2}.$$

Since the solutions are imaginary numbers, we conclude that:

(a) The given equation has no solutions in R.
(b) In C the solutions are:

$$\frac{-2 + \sqrt{2}i}{2} \quad \text{and} \quad \frac{-2 - \sqrt{2}i}{2}. \quad ◀$$

More on Quadratic Inequalities

The solution sets for the inequality and the equation in Example 3 illustrate a general relationship that applies to a broad class of quadratic inequalities. Given an inequality $ax^2 + bx + c \ \square\ 0$, we speak of the **related equation,** $ax^2 + bx + c = 0$. If the related equation has two distinct real roots, say $r_1 < r_2$, then the solution set for the inequality consists either of

all points *between* r_1 and r_2, or all points *outside* the interval (r_1, r_2).

The numbers r_1 and r_2 are included if the inequality sign is either $\le$ or $\ge$.

Strategy: Find the roots of the related equation $-x^2 + 2\sqrt{2}x + 2 = 0$ by using the quadratic formula, with $a = -1$, $b = 2\sqrt{2}$, $c = 2$ to find r_1 and r_2. Pick a test number to see whether the solution set is inside or outside the interval (r_1, r_2).

▶EXAMPLE 5 ***Quadratic open sentence*** Find the solution set for $-x^2 + 2\sqrt{2}x + 2 \le 0$.

Solution

Follow the strategy.

$$\begin{aligned} x &= \frac{-2\sqrt{2} \pm \sqrt{(2\sqrt{2})^2 - 4(-1)(2)}}{-2} \\ &= \frac{-2\sqrt{2} \pm 4}{-2} \\ &= \sqrt{2} \pm 2. \end{aligned}$$

This gives us cut points $r_1 = \sqrt{2} - 2$ and $r_2 = \sqrt{2} + 2$ (about -0.59 and 3.41), both of which are included in the solution set S. Using either an analysis of the sign pattern or from a graph of $y = -x^2 + 2\sqrt{2}x + 2$, we find that S consists of r_1, r_2, and everything outside the interval (r_1, r_2). That is,

$$S = (-\infty, \sqrt{2} - 2] \cup [\sqrt{2} + 2, \infty). \quad ◀$$

More Applications of the Product Principles

We can use the zero-product principle whenever we have a product equal to zero. The next example illustrates some typical uses.

Strategy: Use Equivalence Operation 2 to get a zero on one side of the open sentence. **(a)** Factor as far as possible and apply the zero-product principle. **(b)** Get a single fraction and identify cut points.

▶EXAMPLE 6 ***Solving other open sentences*** Find the solution set.

(a) $x^3 = x^2 + 4x$ **(b)** $1 < \dfrac{3}{x + 1}$

Solution

(a) Subtract $x^2 + 4x$ from both sides to get zero on one side and factor.

$$x^3 - x^2 - 4x = 0 \qquad \text{or} \qquad x(x^2 - x - 4) = 0$$

By the zero-product principle, either $x = 0$ or $x^2 - x - 4 = 0$. We can use the quadratic formula to find the roots of the second equation:

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-4)}}{2} = \frac{1 \pm \sqrt{17}}{2}.$$

The solution set is

$$\left\{0, \frac{1 + \sqrt{17}}{2}, \frac{1 - \sqrt{17}}{2}\right\}.$$

(b) First, we get a zero on one side by subtraction.

$$1 - \frac{3}{x + 1} < 0.$$

Combining fractions we get $1 - \dfrac{3}{x + 1} = \dfrac{x - 2}{x + 1}$. Therefore, by Equivalence Operation 1, the given inequality is equivalent to

$$\frac{x - 2}{x + 1} < 0.$$

The sign properties for quotients are the same as for products; to be negative, the two factors, $x - 2$ and $x + 1$, must have opposite signs. Thus -1 and 2 are *cut points*. Choose a test number in each of the three intervals, $(-\infty, -1)$, $(-1, 2)$, or $(2, \infty)$, say -2, 1, and 5. Go back to the original inequality and replace x by each test number. The results are, respectively, $1 < -3$ (false), $1 < \frac{3}{2}$ (true), and $1 < \frac{3}{6}$ (false). The solution set is the interval $(-1, 2)$. ◀

WARNING: If we were to "clear fractions" by multiplying both sides of the inequality in Example 6(b) by $x + 1$, **we would not get an equivalent inequality.** In Equivalence Operation 4, the inequality may **remain or reverse,** depending on the sign of the multiplier, and if there is a variable, the sign may change.

Equations and Inequalities Involving Absolute Values

When working with an open sentence such as $|2x + 1| \leq 3$ or $|x - 1| > \sqrt{2}$, it is often easier to replace the sentence with an equivalent one without absolute values. To understand the appropriate replacements, it is helpful to see a picture.

TECHNOLOGY TIP ◆ ***Entering absolute values***

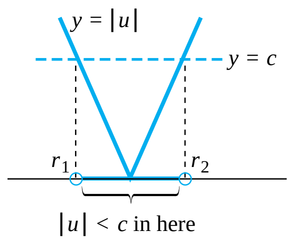
On the TI-82 and TI-81 the absolute value key is easy to spot in the left column, 2nd ABS. On the HP-38 ABS is above the $\boxed{-x}$ key. On other calculators the ABS key is hidden:

TI-85, 2nd MATH F1(NUM) F5 (Abs) .

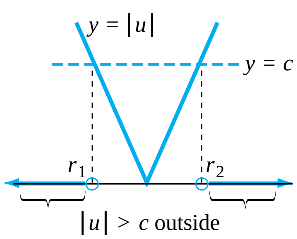
Casio, SHIFT MATH F3(NUM) F1 (Abs) .

HP-48, MTH REAL NXT ABS.

On all calculators except the HP–48, the ABS key enters the function on the home screen or on the function menu, and the HP–48 does the same thing when you write a function in tick marks.



(a)



(b)

FIGURE 27

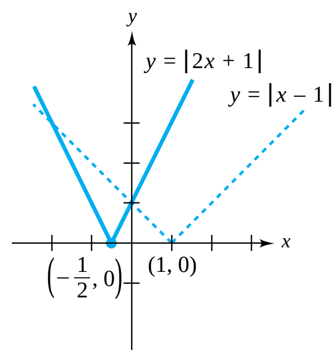


FIGURE 26

Figure 26 shows graphs of absolute value expressions $y = |2x + 1|$ and $y = |x - 1|$. The graph of any expression of the form $y = |ax + b|$ (entered on most graphing calculators as Y = ABS(AX + B)) is some sort of "vee" with its corner on the x-axis. The absolute value graph meets a horizontal line $y = c$ (for any positive number c) at two points, (r_1, c) and (r_2, c). The absolute value graph is *below the horizontal line between* r_1 and r_2, and the solution set for $|ax + b| < c$ is interval (r_1, r_2). *Outside the interval* from r_1 to r_2, the absolute value graph is *above the line* and the solution set for $|ax + b| > c$ is the union $(-\infty, r_1) \cup (r_2, \infty)$. See Figure 27. Finding the solution set for the type of open sentence inequalities we are considering is simply a matter of finding the numbers r_1 and r_2 and thinking about a graph, as illustrated in the next example.

Strategy: For **(a)**, recall that $|-3| = |3| = 3$, so if $|u| = 3$ then u must be 3 or -3.

►*EXAMPLE 7* *Absolute value inequalities*

(a) Solve the equations $|2x + 1| = 3$ and $|x - 1| = \sqrt{2}$.
(b) Use the solutions from part **(a)** to find the solution set for $|2x + 1| \le 3$ and $|x - 1| > \sqrt{2}$.

Solution

(a) Follow the strategy, $2x + 1$ must be either 3 or -3. That is, we must solve two equations,

$$2x + 1 = 3 \quad \text{or} \quad 2x + 1 = -3, \text{ from which}$$
$$x = 1 \quad \text{or} \quad x = -2.$$

Using the same reasoning, for the second equation, we have $x - 1 = \sqrt{2}$ or $x - 1 = -\sqrt{2}$. The solutions are $1 + \sqrt{2}$ and $1 - \sqrt{2}$.

(b) The solutions in part **(a)** give us the numbers r_1 and r_2 that we need for the graphs above.

From Figure 28a the absolute value graph is below the line $y = 3$, which means that $|2x + 1| \le 3$ when x is between -2 and 1. The solution set for the inequality $|2x + 1| \le 3$ is the closed interval $[-2, 1]$. *(Why are the endpoints -2 and 1 included in the solution set?)*

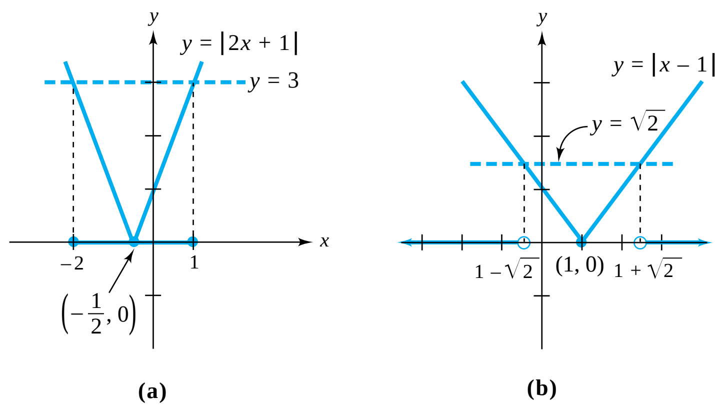


FIGURE 28

From Figure 28b, the absolute value graph is above the line $y = \sqrt{2}$ when x is any number *outside* the interval (r_1, r_2), so in interval notation, the solution set for the second inequality is $(-\infty, 1 - \sqrt{2}) \cup (1 + \sqrt{2}, \infty)$. *(Why are the endpoints $1 \pm \sqrt{2}$ not included?)* ◀

Summing up our observations, we have some guidelines for equivalent absolute value expressions and finding solution sets.

Absolute value equivalents

Suppose c is any positive number and u is an expression involving the variable x. Then

$|u| = c$ may be replaced by the two equations $u = c$ or $u = -c$,
$|u| < c$ may be replaced by the two inequalities $-c < u < c$ (the solution set consists of the numbers between r_1 and r_2),
$|u| > c$ may be replaced by the two inequalities $u < -c$ or $u > c$ (the solution set consists of the numbers *outside* $[r_1, r_2]$).

Strategy: Consider two cases: **(i)** for $x \ge 0$, replace $|x|$ by x; **(ii)** for $x < 0$, replace $|x|$ by $-x$. In each case solve the related equation to get cut points.

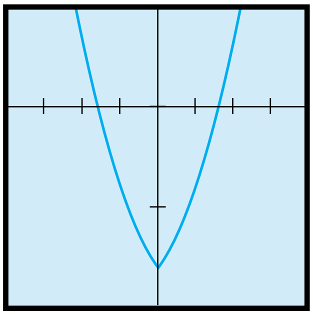
▶*EXAMPLE 8* ***Integer solutions*** What integers satisfy the inequality $x^2 + 2|x| - 8 < 0$?

Solution

Follow the strategy.

(i) $x \ge 0$: $\quad x^2 + 2x - 8 = 0, \qquad$ or $\qquad (x + 4)(x - 2) = 0.$

We have two roots, -4 and 2, but only 2 satisfies $x \ge 0$.



$[-5, 5]$ by $[-10, 5]$
$y = x^2 + 2|x| - 8$

FIGURE 29

(ii) $x < 0$: $x^2 - 2x - 8 = 0$ or $(x - 4)(x + 2) = 0$.

Again, there are two roots, 4 and -2, but only -2 satisfies $x < 0$.

The cut points for the original inequality are 2 and -2. Checking test points in the original, we find that the solution set is the interval $(-2, 2)$. The integers in the interval $(-2, 2)$ are -1, 0, and 1.

Graphical If we look at a calculator graph of $y = x^2 + 2|x| - 8$ in the window $[-5, 5] \times [-10, 5]$ (see Figure 29), we see that the graph is below the x-axis (meaning that the y-coordinates are less than zero, or that $x^2 + 2|x| - 8 < 0$) on an interval from about -2 to 2. Without a decimal window, when we trace, we may not be able to tell exactly where the graph crosses the axis. We still need to do some analysis as above to identify the endpoints precisely. From the graph we can easily see that y is negative at the integer values 0, 1, and -1, and we can evaluate $x^2 + 2|x| - 8$ to verify that $y = 0$ at 2 and -2. Thus the integer values that satisfy the given inequality are 0, 1, and -1. ◀

EXERCISES 1.5

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. The two equations $x^3 - 2x^2 - 5x = 0$ and $x^2 - 2x - 5 = 0$ have the same solution set.
2. The sum of all the integers in the set $\{x \mid -7 < 3x - 1 < 14\}$ is 14.
3. The number -2 is in the solution set for $x^2 - 3x - 5 > |x|$.
4. The solution set for $(x + 3)^2 = 1$ is the same as the solution set for $x + 3 = 1$.
5. The solution set for $x < \frac{4}{x}$ is the same as the solution set for $x^2 < 4$.
6. The solution set for $x < |x|$ is the set of negative numbers.

Exercise 7–10 Fill in the blank so that the resulting statement is true.

7. The largest prime number in the set $\{x \mid |x - 3| \le 21\}$ is ______.
8. The smallest positive integer that is not in the set $\{x \mid x^2 - 4x - 5 < 0\}$ is ______.
9. If $S = \{x \mid (x - 3)(x + 2) \le 0\}$, then the sum of all integers in S is ______.
10. If k is any positive number, then the number of real roots for $x^2 + 2x - k = 0$ is ______.

Develop Mastery

If not specified, the domain of the variable is assumed to be R.

Exercises 1–8 ***Solving Equations*** *Solve. Simplify the result.*

1. $5 - 3x = 7 + x$
2. $5x - 1 = \sqrt{3}$
3. $(x - 2)^2 = x^2 - 2$
4. $(1 - 2x)^2 = 4x^2 - x$
5. $3x^2 + 2x - 1 = 0$
6. $2x^2 + x = 10$
7. $6x + 5 = 9x^2 - 3$
8. $\sqrt{3}x - 4 = x$

Exercises 9–10 Assume the replacement set is the set of complex numbers. Solve. Simplify the result.

9. $4x^2 + 4x - 15 = 0$
10. $2x^2 + 4x + 5 = 0$

Exercises 11–26 ***Solving Inequalities*** *Solve. Use a graph as a check.*

11. $3x - 1 > 5$
12. $\frac{1 - 2x}{-3} > \frac{1}{2}$
13. $-0.1 \le 2x + 1 \le 0.1$
14. $-1 \le \frac{x + 3}{-2} \le 1$
15. $(2 - x)(1 + x) \ge 0$
16. $2x^2 - x - 3 > 0$
17. $\frac{4 - x^2}{x + 3} \ge 0$
18. $x + 1 > \frac{2}{x}$
19. $2 \le 3x - 1 \le 8$
20. $0 \le x^2 - 1 \le 8$
21. $|x| > x$
22. $x^4 + 4x^3 \ge 12x^2$
23. $|2x - 3| > 5$
24. $|x - 4| + x \le 6$
25. $|x - 2| + 2x \le 4$
26. $x^2 - 71x - 10{,}296 < 0$

Exercises 27–30 ***Solution Set*** *Find the solution set and show it on a number line. Use a graph as a check.*

27. $5x - 1 > 3 + 7x$
28. $x^2 - x > 12$
29. $\frac{x + 2}{x^2 - 9} > 0$
30. $2x + 1 > \frac{2}{x}$

Exercises 31–34 ***Absolute Value Inequalities*** *Find the solution set and express it in interval notation.*

31. $|x - 1| < 2$

32. $|2x + 1| > 3$

33. $|x| + 1 < \sqrt{2}$

34. $|1 - x| \leq 0.1$

Exercises 35–36 ***Discriminant*** *Use the discriminant to determine the number of real roots.*

35. $x^2 - 15x + 8 = 0$

36. $4x^2 + 4\sqrt{3}x + 3 = 0$

Exercises 37–46 ***Solving Equations*** *Solve. Use a graph to support your answer.*

37. $x^4 + 3x^2 - 10 = 0$

38. $2|x + 3| - 1 = 5$

39. $|5 - x| - 5 = 3$

40. $(\sqrt{x})^2 - 2\sqrt{x} - 3 = 0$

41. $|x|^2 - 2|x| = 3$

42. $\sqrt{2x + 3} = 1$

43. $\sqrt{x^2 + 4x} = x + 2$

44. $\frac{1}{x} - \frac{3}{x} = \frac{1}{2} + \frac{1}{4}$

45. $\sqrt{x^4 - 5x^2 - 35} = 1$

46. $x - 2\sqrt{x} - 8 = 0$

Exercises 47–50 *Find the solution set. Assume that the replacement set is the set of integers.*

47. $-4 \leq 3x - 2 \leq 4$

48. $|2 - 3x| < 4$

49. $2x^2 + x < 15$

50. $\sqrt{(x - 2)^2} \leq 3$

Exercises 51–52 *Determine the values of x for which the expression yields a real number.*

51. $\sqrt{-x^2 - 4x - 3}$

52. $\sqrt{x - \frac{4}{x}}$

Exercises 53–54 *Determine the values of x for which the expression yields complex nonreal numbers.*

53. $\sqrt{-x^2 - 5x - 6}$

54. $\sqrt{4 - 2|x|}$

Exercises 55–56 *Find the solution set.* (*Hint:* *Recall* $\sqrt{u^2} = |u|$.)

55. $\sqrt{(2x - 1)^2} = 5$

56. $\sqrt{x^2} = -x$

Exercises 57–58 *Find the solution set.*

57. $x + 2 < 3$ and $x + 2 > -3$

58. $2x - 3 \leq -1$ and $2x - 3 > -4$

Exercises 59–62 ***Zero-product Principle*** *Use the zero-product principle to find a quadratic equation with the pair of roots.*

59. $-2, -4$

60. $-2, \frac{1}{2}$

61. $1 + \sqrt{2}, 1 - \sqrt{2}$

62. $1 + i, 1 - i$

Exercises 63–64 ***(a)*** *Determine the number of integers in the set.* ***(b)*** *Find the sum of all the integers in the set.*

63. $\{x \mid 2x + 5 > 0 \text{ and } -3x + 16 > 3\}$

64. $\{x \mid |x - 3| < \sqrt{5}\}$

Exercises 65–66 *Use the zero-product principle to find the solution set.*

65. $(x^2 - 9)(x^2 + x - 6) = 0$

66. $(|x| - 1)(3 - |x + 1|) = 0$

67. How many prime numbers are contained in the set $\{x \mid x^2 - 15x \leq 0\}$? What is the largest one?

68. Find the largest integer k for which the equation $kx^2 + 10x + 3 = 0$ will have real roots.

69. Find the smallest integer c for which the equation $x^2 + 5x - c = 0$ will have real roots.

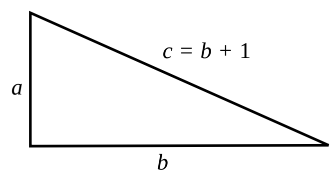
70. What is the sum of all the positive integers x for which $x^2 - 2x - 17$ is negative?

71. What is the smallest integer k such that $3x(kx - 4) - x^2 + 4 = 0$ has no real roots?

72. What is the largest integer x such that the reciprocal of $x + 4$ is greater than $x - 4$?

73. What is the sum of all prime numbers in the set $\{x \mid 3x + 4 < 5x + 7 < 4x + 15\}$?

74. In the right triangle in the diagram, $c = b + 1$, and the perimeter is 12. Find a, b, and c.



75. In a triangle having sides a, b, and c, if $a = 10$, $b = 12$, and $b^2 = a^2 + c^2 - ac$, find all possible values for c.

76. **(a)** If $x + y = 8$ and $x^2 + 4xy + 3y^2 = 48$, find $2x + 6y$. (*Hint:* Factor the left-hand side.)

(b) If $a^2b + ab^2 + a + b = 72$ and $a \cdot b = 8$, then find the sum $a + b$ and the sum $a^2 + b^2$. (*Hint:* First factor the left side, and then use $(a + b)^2 = a^2 + 2ab + b^2$.)

77. A certain chemical reaction takes place when the temperature is between 5° and 20° Celsius. What is the corresponding temperature on the Fahrenheit scale? (*Hint:* $F = \frac{9}{5}C + 32$.)

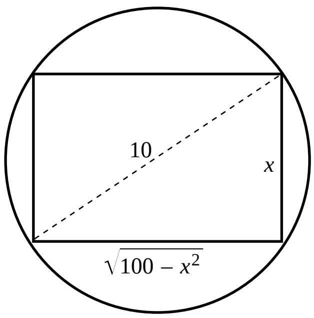
78. In 1990 the population of Newbury increased by 1600 people. During 1991 the population decreased by 12 percent and the town ended up with 56 fewer people than had lived there before the 1600 increase. What was the original population?

79. A boy walking to school averages 90 steps per minute, each step 3 feet in length. It takes him 15 minutes to reach school. His friend walks to school along the same route averaging 100 steps per minute, each step covering 2.5 feet. How long does it take for the friend to walk to school?

80. A beaker contains 100 cc of water. Suppose x cc of water are removed and replaced by x cc of pure acid. From the resulting mixture, another x cc are removed and replaced by x cc of acid. In the final mixture the ratio of water to acid is 16 to 9. Find x, and the final volume of acid.

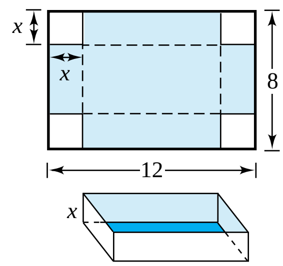
81. A farmer has 200 feet of fencing to enclose a rectangular garden. If the width of the garden is x feet, find an equation that gives the area A of the garden in terms of x. For what values of x is the equation meaningful?

82. ***Largest Area*** A rectangle is inscribed in a circle of diameter 10 inches, as shown.



(a) Using the information shown in the diagram, find an equation that gives the area A of the rectangle in terms of x.
(b) For what values of x is the equation meaningful?
(c) Use a graph to read the value of x (one decimal place) that will give the largest value of A.

83. ***Maximum Volume*** A box with an open top is to be made from a rectangular piece of tin 8 inches by 12 inches, by cutting a square from each corner and bending up the sides as shown in the diagram. Let x be the length of the sides of each square.
(a) Show that the volume V of the box is given by $V = 4x^3 - 40x^2 + 96x$.
(b) For what values of x is $V > 0$?
(c) What is the value of x (one decimal place) that gives the largest value of V? What is the maximum volume?



1.6 MODELS AND PROBLEM SOLVING

> One cadet, who had a private airplane pilot's license, was failing mathematics. When he was asked how much gas he would need to carry if he were going to fly two hundred miles at so many miles per gallon, he didn't know whether to multiply or divide. How, the officers asked, was he able to get the right answer? He replied that he did it both ways and took the reasonable answer. They felt that anybody who knew what was a reasonable answer had promise, so they gave him a second chance.
>
> *Ralph P. Boas, Jr.*

Problem solving is the key to learning mathematics. In this section we consider problems that are somewhat different from some you may have met earlier. Here we try to draw on what you already know, and to develop reasoning and strategy to attack a given problem. Always try your own approach; do not just follow an example in the book or mimic a solution from someone else. Genuine learning takes place when you think for yourself.

No single strategy applies to all problems. Here we look at several examples andthen outlineafew general guidelines. We begin with an example illustrating two different methods of solution.

►EXAMPLE 1 ***Area in a square*** The area of the square $ABCD$, shown in Figure 30 is 64. Points E, F, G, and H are midpoints of the sides, as shown. Find the area of the shaded region.

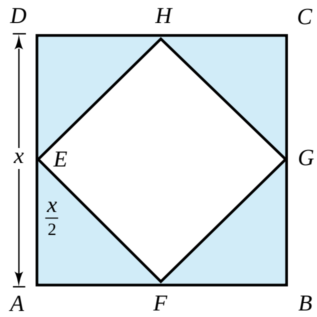


FIGURE 30

Solution 1

First note that the shaded region consists of four congruent right triangles, so the area of the shaded region is four times the area of any one of the shaded triangles. If K denotes the area of $\triangle AEF$ and M is the area of the shaded region, then M $=$ 4K. Our problem reduces to finding K.

Triangle AEF is an isosceles right triangle with le
of the square. Since the area of the square is 64, x
$\frac{1}{2}(4)(4) = 8$, and M $= 4(8) = 32$. Hence the area of the shaded region is 32.

Solution 2

Draw line segments $\overline{EG}$ and $\overline{FH}$ (see Figure 31). We have four more unshaded right triangles,eachcongruenttotheshadedtriangles.(Why?)Therefore,the areaofthe shaded region is half of the area of the square. The area of the shaded region is $\frac{64}{2}$, or 32.

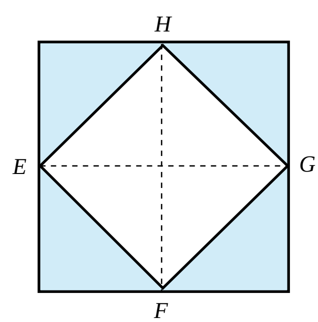


FIGURE 31

Strategy: The ready-made model for such problems is $d = r \cdot t$. Let T be the time it would take for Maria to catch up, when the distances would have to be equal. We know their rates; equate distances and solve for T.

►EXAMPLE 2 ***Time, rate, and distance*** Inichi and Maria share an apartment 2 miles from campus, where they have the same 8:45 class. Inichi leaves home at 8:00, walking at her usual 3 mph pace, while Maria is still in the shower. Maria, who has missed class three days in a row, knows that she can jog all the way at a 5 mph pace. If she gets out the door by 8:20, will that pace allow Maria to **(a)** catch up with Inichi on the way or **(b)** get to class on time?

Solution

(a) Suppose Maria can catch Inichi in T minutes. Maria's speed, 5 mph, is equal to 1 mile in $\frac{1}{5}$ of an hour (12 minutes), so in T minutes she travels $(\frac{1}{12})T$ miles. By the time Maria has jogged T minutes, Inichi has walked for $T + 20$ minutes at 3 mph (1 mile in 20 minutes), for a distance of $(\frac{1}{20})(T + 20)$ miles. Therefore T must satisfy the equation

$$\frac{1}{12}T = \frac{1}{20}(T + 20), \text{ so}$$
$$20T = 12T + 240 \qquad T = 30.$$

Maria could catch Inichi in 30 minutes, or at 8:50. However, it takes Inichi only 40 minutes ($\frac{2}{3}$ of an hour) to get to school, so she arrives at 8:40; Maria cannot catch her.

(b) It takes Maria $\frac{2}{5}$ of an hour (24 minutes) to jog 2 miles, so if she leaves home at 8:20 and doesn't have to wait for a streetlight, she can make it to class with 1 minute to spare. ◄

Strategy: Let x be the dollar amount of sales in a week, and let A and B be the amounts earned per week with the given options. **(a)** Evaluate both when you sell \$1500. **(b)** Express as an inequality to be solved for x.

►EXAMPLE 3 ***Commission options*** You are offered a job as a sales representative for a cosmetics firm. You can choose between two compensation arrangements: a straight 10 percent commission on total sales, or \$100 per week plus a 5 percent commission on your total weekly sales.

(a) How much money would you earn under each option if you sell \$1500 a week?
(b) At what weekly sales level would you earn more on straight commission?
(c) On a graphing calculator, plot your weekly earnings under both options and use the graph to answer the question in part **(b).**

Solution

Follow the strategy. From the given information we have

$$A = (0.10)x \qquad B = 100 + (0.05)x.$$

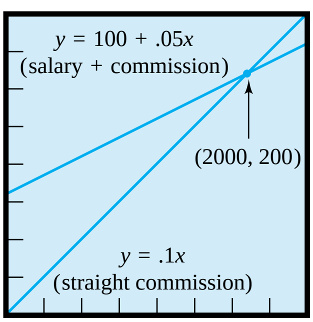
(a) When $x = 1500$, then $A = 150$ and $B = 175$. The salary plus commission option pays \$25 more.

(b) We want to find out when $A > B$, or the values of x for which

$$(0.10)x > 100 + (0.05)x.$$

Solving the inequality we get $(0.05)x > 100$ or $x > 2000$. If you can sell more than \$2000 worth of cosmetics per week, you will earn more on straight commission.

(c) We want to graph the equations Y = .1X and Y = 100 + .05X for the two options, so that the y-value gives the weekly earnings. From the calculations in parts **(a)** and **(b)**, we know that we need a window with a much larger x-range than y. Set a window $[0,2500] \times [0,250]$ and get a graph as shown in Figure 32. Trace and zoom in as necessary to verify the conclusion of part **(b)** that straight commission earnings become greater for weekly sales beyond \$2000. ◄



[0, 2500] by [0, 250]

FIGURE 32

In the next example we first show an indirect approach in solving a problem and then suggest an alternate method.

▶EXAMPLE 4 ***Altitudes of a triangle*** In the isosceles triangle ABC shown in Figure 33(**a**), we are given

$$|\overline{AB}| = 169, \; |\overline{AC}| = 169, \text{ and } |\overline{BC}| = 130.$$

Find the length of the altitude h to side $|\overline{AB}|$.

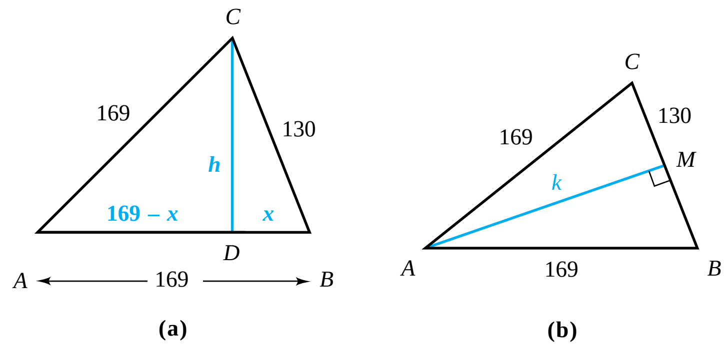


FIGURE 33

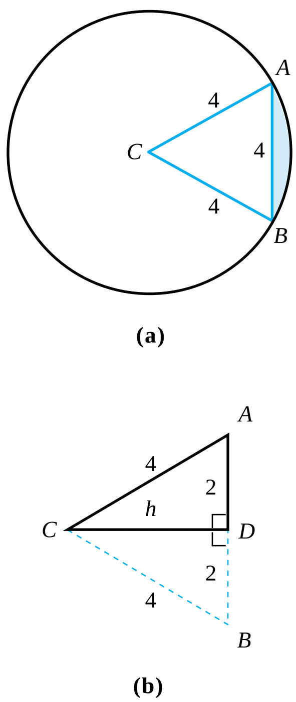


FIGURE 34

Solution 1

If we had the area K of $\triangle ABC$, we could use the relation area $= (\frac{1}{2})(\text{base} \times \text{altitude})$ and find h from $K = (\frac{1}{2})(169)h$. Let k be the altitude to side $\overline{BC}$ (Figure 33(**b**)). We can use the Pythagorean theorem to find k and then the area.

$$k^2 = 169^2 - 65^2 = 24{,}336 \qquad \text{or} \qquad k = 156.$$

Therefore the area of $\triangle ABC$ is $K = \frac{1}{2}(130)(156) = 10{,}140$. Since we also know that $K = \frac{1}{2}(169)h$, we can solve for h.

$$h = \frac{2(10{,}140)}{169} = 120.$$

Thus the length of the altitude to side $\overline{AB}$ is 120.

Solution 2

While Solution 1 is straightforward, try the following approach. In Figure 33(**a**), let x be the length of $\overline{BD}$ and $\overline{AD} = 169 - x$. Apply the Pythagorean theorem to the right triangles ADC and BCD, and get two expressions for h^2 in terms of x. Set these equal to each other, solve the resulting equation for x and then find h. See Exercise 17. ◀

▶EXAMPLE 5 ***Area of a circular segment*** Figure 34(**a**) shows an equilateral triangle ABC in which the length of each side is 4 and a circle with center C that passes through A and B. What is the area of the shaded region?

Strategy: If K_1 is the area of the circular sector and K_2 is the area of the triangle, then the area K of the shaded region is $K_1 - K_2$.

Solution

Follow the strategy. The area K_1 of the circular sector is one-sixth of the area of a circle of radius 4. (Why?) Thus

$$K_1 = \frac{1}{6}(\pi r^2) = \frac{1}{6}(\pi \cdot 4^2) = \frac{8\pi}{3}.$$

To get the area K_2 of $\triangle ABC$, draw a separate diagram (Figure 34**(b)**) and determine the length h of an altitude. Applying the Pythagorean theorem to $\triangle BCD$ gives

$$h^2 + 2^2 = 4^2, \qquad \text{or} \qquad h = \sqrt{12} = 2\sqrt{3}.$$

Therefore, for K_2 we have

$$K_2 = \frac{1}{2}|\overline{AB}|h = \frac{1}{2} \cdot 4 \cdot 2\sqrt{3} = 4\sqrt{3}.$$

Finally, the area K of the shaded region is equal to $K_1 - K_2$, so the area of the shaded region is $\frac{8\pi}{3} - 4\sqrt{3}$ (exact form), or approximately 1.45 square units. ◀

▶EXAMPLE 6 ***Squares and cubes*** The sum of two numbers is 8 and their product is 5. What is the sum **(a)** of their squares? **(b)** of their cubes?

Strategy: Let u and v denote the two numbers, so $u + v = 8$ and $u \cdot v = 5$. Find $u^2 + v^2$ and $u^3 + v^3$. If we try to find u and v, the solution gets messy, but identities for $(u + v)^2$ and for $(u + v)^3$ involve the product and sum of u and v.

Solution

Follow the strategy.

(a) From the identity $(u + v)^2 = u^2 + 2uv + v^2$ subtract $2uv$ from both sides to express $u^2 + v^2$ in terms of the sum and product of u and v:

$$u^2 + v^2 = (u + v)^2 - 2uv = 8^2 - 2 \cdot 5 = 64 - 10 = 54.$$

The sum of the squares is 54.

(b) For the sum of the cubes we try a similar approach.

$$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 = u^3 + v^3 + 3uv(u + v)$$
$$u^3 + v^3 = (u + v)^3 - 3uv(u + v) = 8^3 - 3 \cdot 5 \cdot 8 = 392.$$

The sum of the cubes of the two numbers is 392. It is instructive to compare the work we have done in this example with the work it takes to find the two numbers u and v, and then to square each and cube each, to find the sums of the squares and the cubes. See Develop Mastery Exercise 18. ◀

As should be clear from the diverse problems we have considered and the variety of approaches illustrated, no single set of methods is sufficient to solve any particular problem, but some consistent guidelines can help.

Problem solving guidelines

1. **Be certain that you understand the problem.** You may need to read it several times.
2. **Concentrate on what the problem calls for** and identify all the information given.
3. **Draw diagrams or graphs whenever appropriate.** This is extremely important in planning your solution strategy.
4. **Introduce variables** to name the quantities involved and label diagrams.

5. **Use what you have learned earlier.** For instance, if you need the area of a figure, review pertinent formulas. Be aware that many useful formulas and relations appear in this book, many of them on the inside front and back covers.
6. **Always check your results.** Don't just plug a number into a formula, but ask yourself if your results make sense in terms of the original statement of the problem. Whenever possible begin with some kind of reasonable estimate of what the result should be.
7. **Work in terms of complete sentences.** Use clearly readable sentences and precise mathematical notation to identify variables, state relationships, etc. Write your conclusion as a sentence, as well. This care will pay great dividends in clarity of thinking, in understanding how to approach a problem, and in knowing what the result means.

EXERCISES 1.6

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. The three line segments joining the midpoints of the sides of an equilateral triangle form an equilateral triangle.
2. When 31^{64} is expanded and written in usual base 10 form, the units digit is 4.
3. The isosceles triangle having sides of lengths 6, 6, and 4 has an altitude (drawn to the short side) of length $4\sqrt{2}$.
4. The area of the triangle described in Exercise 3 is equal to $16\sqrt{2}$.
5. Points $(-2, 3)$ and $(4, 1)$ lie on a circle whose center is at $(-1, -4)$.
6. When x is replaced by $\frac{-2}{3}$ in the open sentence $|3x + 1| - 2x = 1$, the resulting statement is true.

Exercises 7–10 Complete the sentence by entering "less than," "greater than," or "equal to" in the blank so that the resulting statement is true.

7. The area of a square having sides of length k is ______ the area of a circle with diameter of length k.
8. The time it takes to walk 2 miles at a rate of 4 mph is ______ the time it takes to walk 3 miles at a rate of 5 mph.
9. The distance from point $A(-1, -4)$ to point $B(-2, 3)$ is ______ the distance from A to point $C(4, 1)$.
10. If the sum of two numbers is 5 and their product is 3, then the sum of their squares is ______ 19. (*Hint:* Use $(x + y)^2 = x^2 + 2xy + y^2$.)

Develop Mastery

1. If the reciprocal of b is 12, and $\frac{b}{c} = 1$, then find c.
2. Find all values of x (if any) for which the reciprocal of $x + 1$ equals $x - 1$.
3. When a meatball mixture is molded into a spherical shape, the radius of the sphere is 4 inches. How many meatballs of radius 1 inch each can be made from the mixture?
4. ***Consecutive Integers***
   (a) Find five consecutive integers whose sum is 100.
   (b) Find eight consecutive integers whose sum is 100.
   (c) Are there six consecutive integers whose sum is 100? Explain.
5. An auto repair shop charges a \$20 shop charge plus \$25 per hour for labor. If the total charge for a repair job is \$80 plus parts, how many hours of labor did the job require?
6. Robin and Bart are 0.7 miles apart when they begin walking in a straight path toward each other. Robin walks at the rate of 3 mph and Bart at the rate of 4 mph.
   (a) How long (in minutes) will it take for the two to meet?
   (b) How far will each walk?
7. Anna enters a walkathon that covers a total distance of 20 miles. She runs part of the distance at the rate of 6 mph and walks the remaining distance at a rate of 4 mph, completing the course in 4 hours and 30 minutes.
   (a) How far did Anna run? How far did she walk?
   (b) How many hours did she run? How many did she walk?

8. Two insurance companies, Arliss and Bailey, pay sales representatives every month. Arliss pays a fixed 12 percent commission on the total amount of insurance sold, while Bailey pays \$250 per month plus 7 percent commission on the total sold. For a month's sales of x dollars, let A denote the amount Arliss pays, and let B denote the amount Bailey pays for the same sales.
  (a) Find formulas for A and B in terms of x.
  (b) For what volume of sales will Arliss pay more than Bailey?
  (c) As a check draw graphs as in Example 3(c). Give the window dimensions you are using.

9. The diameter of a circle is 6 times the reciprocal of the circumference. Find the area of the circle.

10. Find the area of a circle if the reciprocal of the circumference equals the length of the radius.

11. If $\dfrac{1}{a} - \dfrac{1}{c} = \dfrac{1}{a + c}$ then find the value of the ratio $\dfrac{a}{c}$.

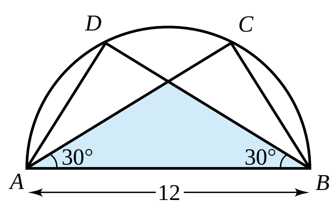
12. ***Antifreeze Solution*** The radiator of a car has a capacity of 6 quarts and is filled with a 30 percent mixture of antifreeze.
  (a) How many quarts of antifreeze are in the radiator?
  (b) If you drain a quarter of the mixture in the radiator and replace it with pure antifreeze, what is the percentage of antifreeze in the resulting mixture?
  (c) How many quarts of the original mixture should you drain and replace with pure antifreeze to get a mixture that is 51 percent antifreeze?

13. Is the expression $\dfrac{1 + x^2}{\sqrt{1 + x^2}} - \sqrt{1 + x^2}$ equal to zero for every real number x? Give an algebraic explanation. Check graphically.

14. A square is inscribed in a circle and then a circle is inscribed in the square.
  (a) What is the ratio of the area of the larger circle to the area of the smaller circle?
  (b) If the larger circle has a radius of 16 cm, what is the area of the ring-shaped region between the two circles, and what is the area of the square?

15. A chord of a circle is the perpendicular bisector of a radius of length 4. How long is the chord?

16. Two triangles are inscribed in a semicircle as shown in the diagram, where $|\overline{AB}| = 12$ and $\angle BAC = \angle ABD = 30°$. What is the area of the shaded triangular region common to triangles ABC and ABD?



17. Carry out the details suggested in Solution 2 of Example 4.

18. **(a)** If $u = 4 + \sqrt{11}$ and $v = 4 - \sqrt{11}$, show that $u + v = 8$ and $u \cdot v = 5$.
  (b) Evaluate $u^2 + v^2$ and $u^3 + v^3$. Compare with Example 6.

19. Given that the side of one square is the diagonal of a second square. If A_1 is the area of the first square and A_2 is the area of the second square, then find the value of the ratio $\dfrac{A_1}{A_2}$.

20. ***Average Speed*** If you drive 160 miles at an average speed of 50 miles per hour, and then return along the same route at a more leisurely speed of 30 miles per hour, what is your average speed for the round trip?

21. If you drive d miles at an average speed of 50 mph and return along the same route at an average speed of 30 mph, what is your average speed for the round trip?

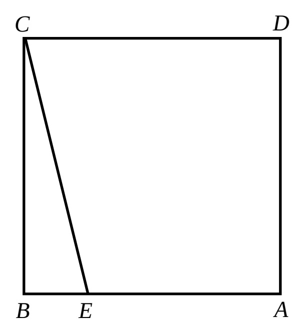
22. If you drive d miles at an average speed of v_1 mph and return along the same route at an average speed of v_2 mph, what is your average speed for the round trip?

23. A race car driver must average 150 mph for four separate laps to qualify for a race. Because of a minor engine problem the car averages only 120 mph for the first two laps. What average speed is required on the final two laps to qualify for the race?

24. How many ounces of a 60 percent solution of acid must be added to 20 ounces of a 30 percent solution to get a 40 percent solution?

25. A circle is inscribed in an equilateral triangle of side length 4. Find the area of the circle.

26. In rectangle $ABCD$ shown in the diagram, $|\overline{AE}|$ is $\frac{3}{4}$ of $|\overline{AB}|$, and the area of triangle BEC is 24 cm^2. What is the area of the rectangle?



Exercises 27–30 ***Circles*** *The Center C and diameter d of a circle are given.* ***(a)*** *Determine the dimensions of a calculator window that will show the graph as nearly a circle (rather than an ellipse) as possible.* ***(b)*** *Give formulas for the upper half and the lower half of the circle.* ***(c)*** *Draw a graph.*

27. $C(-3, 2)$, $d = 10$

28. $C(3, 5)$, $d = 12$

29. $C(2, 4)$, $d = 16$

30. $C(-2, -4)$, $d = 15$

Exercises 31–32 ***Viewing Window*** ***(a)*** *Determine the dimensions of a window so that the display shows the two circles (not ellipses) intersecting at two points.* ***(b)*** *Give formulas you would use to graph.* ***(c)*** *Draw the graphs.*

31. $(x + 3)^2 + (y - 2)^2 = 25$, $x^2 + y^2 = 25$

32. $(x + 2)^2 + (y + 4)^2 = 25$, $x^2 + y^2 = 25$

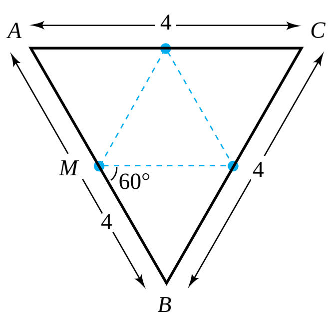
33. Denote the two x-intercept points of the graph of $x^2 + y^2 = 16$ by A and B, and the two y-intercept points by C and D. What is the area of the quadrilateral with vertices A, B, C, and D?

34. Two candles of the same length are made of different materials and hence burn at different rates. One burns down completely at a uniform rate in 4 hours while it takes the other 5 hours. If both candles are lit at 2:00 P.M., at what time will one be half as long as the other?

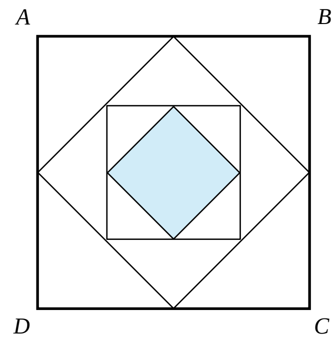
35. Given points $A(-2, 0)$ and $B(4, 0)$, find a point C with both coordinates positive integers and such that the area of $\triangle ABC$ is a minimum. Is the answer unique? What is the area of $\triangle ABC$?

36. Point P is 6 units from the center of a circle of radius 10. How many chords having integer length can be drawn through P? (*Hint:* First draw a diagram.) The longest chord is a diameter. What is the shortest chord?

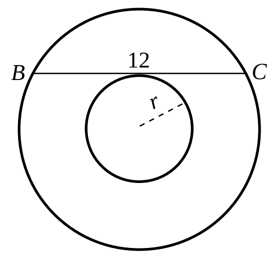
37. Three mirrors, each of length 4 feet, are placed to form an equilateral triangle ABC. A light source is placed at the midpoint M of side $\overline{AB}$, as shown in the diagram, and is aimed at an angle of 60° so that the light will be reflected to follow the dotted line. How far does the light travel before returning to M?



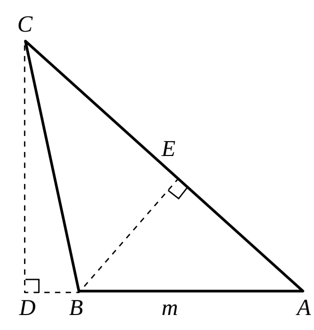
38. In the diagram square $ABCD$ has an area of 16. The vertices of each inscribed square are midpoints of the sides of the square in which it is inscribed. Find the area of the shaded square.



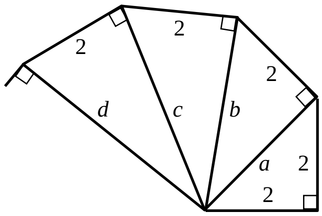
39. The diagram shows two concentric circles. Segment $\overline{BC}$ is tangent to the inner circle and is a chord of length 12 cm in the outer circle. Suppose r is the radius of the smaller circle. Find the area of the region between the circles when **(a)** $r = 4$, **(b)** $r = 8$, **(c)** $r = 15$, **(d)** Guess a formula for the area A where r is any positive number. Prove that your guess is valid.



40. In the diagram $|\overline{AC}| = 36$, $|\overline{CD}| = 24$, $|\overline{BE}| = 12$, and $|\overline{AB}| = m$. Determine m. (*Hint:* Consider the area of triangle ABC.)



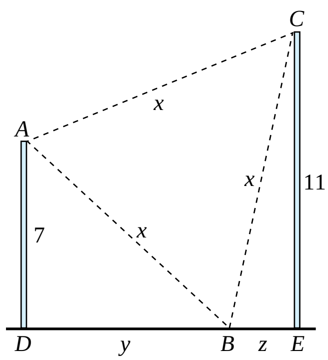
41. ***Pattern*** The diagram at the top of the next page starts with an isosceles right triangle with legs of length 2 and hypotenuse of length a, then adds successive right triangles each having one leg of length 2 and the other leg as the hypotenuse of the preceding triangle.



(a) Find lengths a, b, c, and d.

(b) If we continue constructing right triangles and labeling the length of each hypotenuse with successive letters of the alphabet, what letter corresponds to the hypotenuse of length 6?

42. ***Patio Design*** For the patio in the diagram, where vertical walls $\overline{AD}$ and $\overline{CE}$ are 7 and 11 feet high, respectively, how far apart must the walls be to allow the lower vertex of an equilateral triangle ($\triangle ABC$) to touch the floor? (*Hint:* First find x, then find y and z.)



43. Given three distinct lines l_1, l_2, and l_3 in a plane, if l_1 intersects the parallel lines l_3 and l_2, how many points in the plane are equidistant from all three lines? (*Hint:* Draw a diagram.)

44. ***Working with Large Numbers*** The Andromeda galaxy is approaching our galaxy at a speed of about 100 kilometers per second.

(a) How fast is Andromeda approaching us in miles per hour? (1 mile = 1.609 km)

(b) How far does Andromeda travel toward us each year?

(c) How long would it take an object moving at Andromeda's speed to travel from the sun to the earth (93 million miles)?

(d) How long would it take an object moving at Andromeda's speed to travel 1 light year (the distance light, moving at 186,000 miles per second, travels in a year)?

(e) The estimated distance between the Milky Way (our galaxy) and Andromeda is 2 million light years. Assuming Andromeda continues to approach us at its current speed, when will our galaxies meet?

CHAPTER 1 REVIEW

Test Your Understanding

Exercises 1–41 True or False. Give reasons.

1. The largest real number is ∞.

2. There is only one even prime number.

3. There is no smallest positive real number.

4. There is no greatest negative integer.

5. There is no smallest positive rational number.

6. $2^{16} + 3^{65}$ is an odd number.

7. $3^{17} + 7^{27}$ is an odd number.

8. The imaginary number $3i$ is greater than $2i$.

9. There is no real number x for which $\sqrt{-x} > 0$.

10. $x \geq \dfrac{1}{x}$ for every positive real number x.

11. No real number x satisfies the equation $\sqrt{-x} = \sqrt{x}$.

12. For every real number x, $|x - 4| + 3 > 0$.

13. It is not true that the solution set for the equation $|x - 1| + 1 = 0$ is the empty set.

14. $\sqrt{x^2} + |x| = 2|x|$ for every real number x.

15. The inequalities $x^2 < 4$ and $x < \dfrac{4}{x}$ have the same solution set.

16. If $\dfrac{a}{b} = x$, where $b \neq 0$ and $a \neq b$, then $\dfrac{a+b}{a-b} = \dfrac{x+1}{x-1}$.

17. If x is any positive real number, then $\sqrt{x} < x$.

18. There is no positive real number x for which $\sqrt{-x^2 - x}$ will be a real number.

19. If $x = \dfrac{1}{\sqrt{2} - 1}$ and $y = \sqrt{2} + 1$, then $x = y$.

20. The intervals $(-2, 2]$ and $(2, 3]$ are disjoint.

21. The intersection of the intervals $(0, 1.\overline{3})$ and $(1.3, 2)$ is the empty set.

22. The sum of any two irrational numbers is an irrational number.

23. The product of any two irrational numbers is an irrational number.

24. If n is any positive integer, then $n(n+1)$ will be an even positive integer.

25. The product of any two different prime numbers is greater than 5.

26. There is no point (x, y) in Quadrant I that is on the line $x + y + 1 = 0$.

27. The graph of $x - y = 1$ does not pass through Quadrant II.

28. The graph of $x^2 + y^2 = 2x$ passes through the point $(1, 1)$.

29. For every real number x, $\sqrt{x^2 + 1} = x + 1$.

30. The solution set for the equation $(x+1)^2 - 1 = x^2 + 2x$ is the set of real numbers.

31. If x and y are real numbers where $x < y$, then $|x - y| = y - x$.

32. If both x and y are negative numbers, then $|x + y| = -x - y$.

33. If x and y are any real numbers, then $|x + y| = |x| + |y|$.

34. If x is positive and y is negative, then $|x - y| = x - y$.

35. A triangle with sides 3, 4, and 5 is a right triangle.

36. A triangle with sides 1, 2, and $\sqrt{3}$ is a right triangle.

37. The graph of $x^2 + y^2 - 2x = 0$ is a circle with center at $(-1, 0)$.

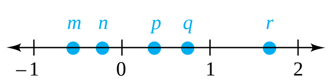
38. The graph of $2x - 3y = 6$ is a line passing through $(3, 2)$.

39. The solution set for $x^2 - 2x - 3 = 0$ is $\{-1, 3\}$.

40. The solution set for $\dfrac{2x - 4}{x - 5} = 0$ is $\{2, 5\}$.

41. The graph of $(x - y)(x^2 + y^2 - 1) = 0$ consists of a line and a circle.

Exercises 42–44 From the diagram showing m, n, p, q, and r on the number line, determine the truth value.



42. **(a)** $mr < 0$ **(b)** $\dfrac{1}{r} < 1$ **(c)** $pq < q$

43. **(a)** $|m + n| = n + m$ **(b)** $|n - p| = n - p$
(c) $|p - q| = p - q$

44. **(a)** $\dfrac{m}{n} < 0$ **(b)** $\dfrac{m}{n} > 1$ **(c)** $r - p > r - q$

Exercises 45–50 Fill in the blank so that the resulting statement is true.

45. The graphs of $5x + 3y + 21 = 0$ and $x - y - 15 = 0$ intersect in Quadrant ______.

46. The graphs of $x - 2y + 12 = 0$ and $x + 2y + 4 = 0$ intersect in Quadrant ______.

47. The graphs of $x + y = 2$ and $y + \sqrt{4 - (x-2)^2} = 0$ intersect in Quadrant ______.

48. The graphs of $x^2 + y^2 = 4$ and $x^2 + y^2 + 4x = 0$ intersect in Quadrant(s) ______.

49. The number of points of intersection of the graphs of $x^2 + y^2 - 6x = 0$ and $2x - y = 3$ is ______.

50. The number of points of intersection of the graphs of $x^2 + y^2 - 8x = 0$ and $x^2 + y^2 + 2x = 8$ is ______.

Review for Mastery

1. Is $\sqrt{9 - 4\sqrt{5}}$ equal to $2 - \sqrt{5}$? Explain.

2. Is $\sqrt{23 - 8\sqrt{7}}$ equal to $\sqrt{7} - 4$? Explain.

3. Is $0.\overline{54}$ equal to $\frac{5}{11}$? Explain.

4. Which number, π, $\frac{22}{7}$, or $\frac{355}{113}$, is the smallest? Which is the largest?

5. Express each of the following in exact form without using absolute value.

(a) $\left|3 - \dfrac{22}{7}\right|$ **(b)** $|\sqrt{8} - 3|$ **(c)** $|0.36 - 0.\overline{36}|$

6. Express as a fraction of two integers in lowest terms.
(a) 1.36 **(b)** $1.\overline{36}$ **(c)** $0.\overline{45} - 0.45$

7. Enter one of the symbols $<$, $>$, or $=$ in the blank so that the resulting statement is true.
(a) -5 ______ -7 **(b)** $\sqrt{3} - 1$ ______ 0.732
(c) $|\sqrt{2} - \sqrt{8}|$ ______ $\sqrt{2}$

8. Subsets of real numbers are given in interval notation. Show each on a number line.
(a) $(-1, 4)$ **(b)** $[-1, 0] \cup [2, 4]$
(c) $[-3, 1) \cap (0, 3]$ **(d)** $(-\infty, -2] \cup [2, \infty)$

9. Show the subset of real numbers on a number line.
(a) A is the set of all prime numbers less than 8.
(b) B is the set of all real numbers greater than 2 and less than 5.
(c) C is the set of all integers greater than -3 and less than 4.

10. For what values of x is $|x + 3| = x + 3$?

11. For what values of x is $|x - 3| = 3 - x$?

12. How many real numbers are in the set $\{x \mid x^2 - 2 = 0\}$?

Exercises 13–24 ***Solution Set for Equation*** *Find the solution set.*

13. $3x - 5 = 3$

14. $2x^2 - 3x = 0$

15. $|x + 1| - 1 = 0$

16. $2|x + 1| - 3 = 0$

17. $2x^2 - 4x - 5 = 0$

18. $\sqrt{3}x = x + 1$

19. $3 - 2x - x^2 = 0$

20. $\sqrt{2x - 3} = 3$

21. $x - 3 = \dfrac{4}{x}$

22. $(3 - 2x)(x^2 - 5x) = 0$

23. $\sqrt{x^2} - 2|x| + 3 = 0$

24. $\sqrt{(x + 3)^2 = 4}$

Exercises 25–28 ***Complex Number Arithmetic*** *Express in $a + bi$ form where a and b are real numbers.*

25. $\dfrac{5 + 10i}{2 - i}$

26. $i^2 - i^3 + i^4$

27. $\sqrt{-3}\,\sqrt{-8}$

28. $(1 + i)^2 + 3i^2$

Exercises 29–40 ***Solving Open Sentences*** *Determine the solution set.*

29. $3x - 4 < 5$

30. $4x - 3 \le x + 7$

31. $2x^2 > 2 - 3x$

32. $\dfrac{x^2 - 1}{x + 2} \ge 0$

33. $2x - 3 > \dfrac{5}{x}$

34. $x < \dfrac{1}{x}$

35. $|x + 1| - 2 \le 0$

36. $|x^2 - 1| < 2$

37. **(a)** $|x| - x = 2$ **(b)** $|x| - x < 2$

38. **(a)** $x - 2 = \dfrac{8}{x}$ **(b)** $x - 2 \ge \dfrac{8}{x}$

39. **(a)** $|x| = x$ **(b)** $|x| > x$

40. **(a)** $x^2 - 1 = x + 1$ **(b)** $x^2 - 1 < x + 1$

41. For what values of x is $\sqrt{5 - 4x - x^2}$ a real number?

42. For what values of x is $\sqrt{x - \frac{4}{x}}$ a nonreal complex number?

43. Find an equation for the circle with center at $(-3, 2)$ and radius 1.

44. Find the center and radius of the circle given by $x^2 + y^2 - 2x + 4y + 1 = 0$.

Exercises 45–50 ***Graphs and Intercepts*** ***(a)*** *Draw a graph. Give the coordinates of any* ***(b)*** *x-intercept points,* ***(c)*** *y-intercept points.*

45. $3x - 2y = 6$

46. $4x + 3y + 6 = 0$

47. $(x - 3)^2 + (y + 1)^2 = 4$

48. $x^2 + y^2 + 2x + 4y + 1 = 0$

49. $\sqrt{3}x + y = 3$

50. $x^2 + y^2 = 4x$

Exercises 51–54 ***Points of Intersection of a Half Circle and a Line***

(a) *Draw graphs of the half circle and the line L on the same screen. Determine dimensions of a window so that the circular graph appears to be a half circle (not an ellipse).*

(b) *Give formulas that you are using for the half circle and the line.*

(c) *In what quadrant do the graphs intersect?*

51. Upper half of $x^2 + y^2 = 9$, L: $2x - y = 0$

52. Upper half of $x^2 + y^2 - 4x + 6y - 3 = 0$, L: $2x - y = 7$

53. Lower half of $x^2 + y^2 - 4x + 6y - 3 = 0$, L: $2x - y = 12$

54. Lower half of $x^2 + y^2 = 9$, L: $2x + y = 0$

55. **(a)** Draw a graph of the circle $(x - 3)^2 + (y - 2)^2 = 4$.

(b) For points $A(5, 1)$, $B(2, 3)$, and $C(3, 0)$, determine which are inside the circle, outside the circle, or on the circle.

56. A ball is dropped from the top of a building 256 feet high. Its position at t seconds after being dropped is given by $s = 256 - 16t^2$, where s is its distance from the ground.

(a) How long will the ball take to drop halfway to the ground?

(b) What values of t are meaningful in the given formula?

57. You mix 2 quarts of antifreeze with 3 quarts of water.

(a) What percentage antifreeze is the mixture?

(b) How much more antifreeze should be added to get a mixture that is 60 percent antifreeze?

58. The campus bookstore is having a 25 percent off sale. Hilary purchases a book and, after a 5 percent sales tax is added, she pays a total of \$28.98. What is the original price of the book?

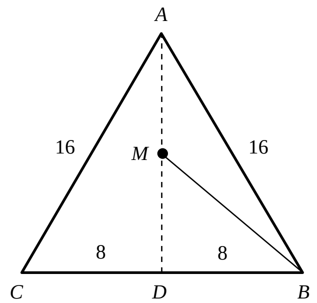
59. A car and truck are traveling along a highway in the same direction. The car is 20 feet long and is traveling at a speed of 60 mph (88 feet per second), while the truck is 46 feet long and its speed is 45 mph (66 feet per second). How many seconds will elapse from the instant the car reaches the truck until the car is completely past the truck?

60. Suppose in Exercise 59 we are not given the speed of the truck, but we know that it takes 4 seconds for the car to pass it. How fast is the truck traveling?

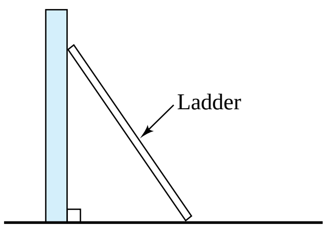
61. An equilateral triangle is inscribed in a circle with a radius of length 8. Find the area of the region inside the circle and outside the triangle.

62. If $u + v = 10$ and $uv = 7$, find $u^2 + v^2$, $u^3 + v^3$, $u^4 + v^4$. (*Hint:* See Example 6 of Section 1.6.)

63. In an equilateral triangle ABC, where each side has length 16, let D be the foot of the perpendicular from A to $\overline{BC}$, and let M denote the midpoint of $\overline{AD}$. See the diagram. What is the length of $\overline{BM}$?



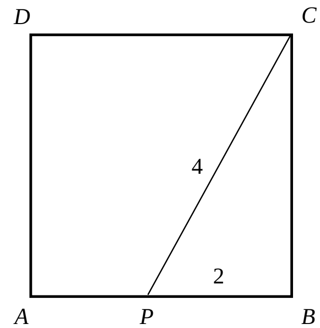
64. A ladder is resting vertically against a wall. When the bottom of the ladder is pulled horizontally out from the wall a distance of 15 feet, the top of the ladder slides down the wall a distance equal to $\frac{2}{5}$ of the length of the ladder. What is the length of the ladder?



65. In the diagram, $ABCD$ is a square and point P is located on side $\overline{AB}$ so that $|\overline{PB}| = 2$ ft. and $|\overline{PC}| = 4$ ft.

(a) What is the perimeter of the square?

(b) What is the area of the square?



2 FUNCTIONS

A COMMON THREAD RUNNING THROUGH MATHEMATICS is the notion of *function.* This chapter introduces the basic ideas, notation, and terminology for functions in general, and for some key classes of functions. Historically, functions have developed in different contexts. Each context has contributed a definition, and each definition can deepen understanding. In the first two sections of the chapter we give definitions and show how to visualize functions by means of graphs. Graphs will then become central to every aspect of our study, for both theoretical understanding and problem solving.

Lines are graphs of linear functions (Section 2.4) and parabolas are graphs of quadratic functions (Section 2.5). We use these two types of functions and their graphs to solve problems in Section 2.5. Section 2.6 shows how we combine functions in analysis, particularly composition, which is central to the study of calculus. Also vital for calculus (and for much of this book) is the study of inverse functions (Section 2.7). Section 2.8 gives just a taste of the incredible variety of ways mathematicians use functions to describe the world.

2.1 THE WORLD OF FUNCTIONS

> This is a very common situation in mathematics; a required quantity is unknown to us, but we do know certain relationships in which it stands to other quantities. From these relationships we may be able to find out the value of the unknown quantity.
>
> *Rózsa Péter*

Definition of Function

We all make daily use of the idea of correspondences. We assign a number to a person, or a street address to a house, or a number to another number (as, 11 is the fifth prime). The area of a square depends on (is a function of) the side length. In words, the area is the square of the length of a side; in symbols, $A = s^2$.

The most common rules of mathematical correspondence are given by equations. For example, if $y = x^2 - 1$, for each selected x value, we get the corresponding y value by squaring the x value and then subtracting 1. If, however, for each x value we subtract 1 and then square the result, we have a very different correspondence, given by the equation $y = (x - 1)^2$.

Computer (or calculator) terminology provides rich language to describe functions. We think of x values as *input* values for the function, each with a corresponding *output*. For the square root function, $y = \sqrt{x}$, when we enter 4, the output is 2. For the input 3, the display is something like 1.732050808; -2 is not an acceptable input and the calculator gives an error message. For any given function, the set of acceptable inputs is the **domain** of the function, and the set of outputs is the **range.**

Definition: function, domain, and range

A **function** $\boldsymbol{f}$ is a correspondence between the elements of two non-empty sets D and R, established by a rule that assigns to each element of D exactly one element of R.

The set D is called the **domain** of f; the set R is called the **range** of f.

On a graph of $y = f(x)$, the *domain* is the **set of $\boldsymbol{x}$-values** of the points of the graph; the *range* is the **set of $\boldsymbol{y}$-values** of the points of the graph.

Functional Notation

For any element x in the domain of function f, the element that corresponds to x is denoted $f(x)$, which is read "f of x" or "the value of f at x." If the rule of correspondence is given by an equation such as $y = x^2 - 1$, then we may say "the function $f(x) = x^2 - 1$," or "the function f given by $f(x) = x^2 - 1$." Other letters can designate functions and variables, as for example,

$$g(u) = \sqrt{u^2 - 2u - 3} \quad \text{or} \quad h(x) = \frac{3x - 2}{4x}.$$

When we use notation such as $y = x^2 - 1$, we say that y depends on x, so y is a **dependent variable,** and in this case x is the **independent variable.** In general, the independent variable comes from the domain and the dependent variable comes from the range. In the equation defining the area of a square, $A = s^2$, the dependent variable is A and the side length s is the independent variable.

I think I always had a fascination for numbers. When [my grandfather] drove along in a car, he would factorize every car number [three-digit license plate] coming along. He once actually drove into a brick wall while multiplying out car numbers in his head. When I was a teenager, I used to sing in our church choir and the sermons used to go on a bit, so I used to do things like multiply the numbers on the hymn board, or square all the numbers up to a hundred. I enjoyed playing with numbers and puzzles.

Robin Wilson

Many functions that mathematicians need frequently have standard names, including exponential, logarithmic, and trigonometric functions. Calculator keys often use these names as labels, as $\boxed{\text{log}}$, $\boxed{\sqrt{\ }}$, or $\boxed{\text{sin}}$.

The rule of correspondence for a given function may be specified by an equation, or it may be stated in words or presented graphically, or as a table of data. Since a function pairs a range element to each domain element, the function may be described as a set of ordered pairs.

►EXAMPLE 1 ***Domain and range*** Suppose the domain D of function f is $\{-1, 3, 5\}$ and f assigns to each number in D its square. Evaluate $f(x)$ for each x in D and find the range of f.

Solution

$$f(-1) = (-1)^2 = 1 \qquad f(3) = 3^2 = 9 \qquad f(5) = 5^2 = 25$$

The range of f consists of the outputs, so $R = \{1, 9, 25\}$. ◄

Domain Convention

The definition of a function must include its domain. The domain can be given explicitly (as in Example 1), or it may be clear from context, as in the function for the area of a square where only positive side lengths have meaning. Whenever the domain of a function is not explicitly stated, the following domain convention applies.

> ***Domain convention***
>
> If the domain D of a function f is not explicitly stated, assume D is the set of all real numbers x for which $f(x)$ is also a real number.

To determine the domain of a function f, begin by weeding out all unacceptable input numbers for f. The easiest things to look for are division by zero and square roots of negative numbers.

►EXAMPLE 2 ***Domain and range from a graph*** Use a calculator graph to determine the domain and range of the function given by $f(x) = \sqrt{3 - x^2}$. Find the domain and range exactly and compare.

Strategy: What kinds of numbers have (real) square roots? Exclude all values of x that make $3 - x^2$ negative.

Solution

Graphical If we start with a window such as $[-10, 10] \times [-10, 10]$, we see a small oval centered near the origin. To get a better view, try a decimal window (which is an equal scale window). The graph now appears to be most of the top half of a circle. In Section 1.5, we observed that, because of pixel coordinate limitations, it isn't always possible to get a calculator graph that shows all of the picture we might like to see. We can trace to see that the graph is defined for all x-values from about -1.7 to 1.7, with y-values from near 0 to about 1.73. If we zoom in on the right end of the semicircle, we can get closer to the right end-point, which, to two decimal places, appears to be 1.73, giving us a domain of $[-1.73, 1.73]$ and a range of $[0, 1.73]$ (although, of course, we cannot read the graph accurately enough to tell whether either is a closed interval).

Algebraic We know that the function is defined whenever $3 - x^2 \geq 0$, and that the maximum value occurs when $x = 0$. Solving the inequality, we find $-\sqrt{3} \leq x \leq \sqrt{3}$. The domain is the interval $[-\sqrt{3}, \sqrt{3}]$. Since $f(0) = 3$ and $f(\pm\sqrt{3}) = 0$, the range is the interval $[0, \sqrt{3}]$. Note that the algebraic approach clarifies what we were seeing on our graphs; the decimal approximation for $\sqrt{3}$ begins 1.73. ◀

Functions of Algebraic Expressions

We have so far applied function rules to input numbers to get output numbers. We often need to allow algebraic expressions as input, as well. If $f(x) = 2x + 4$, then f doubles the input and adds 4, whether the input is a number or an algebraic expression.

$$f(-3) = 2(-3) + 4 = -2 \quad \text{and} \quad f(x - 1) = 2(x - 1) + 4 = 2x + 2.$$

In calculus the definition of a derivative involves a *difference quotient* expressed in terms of a function of an expression, as illustrated in the next example.

►EXAMPLE 3 ***Function evaluation*** If $f(x) = x^2 - 2x$, then evaluate or simplify:

(a) $f(2)$ **(b)** $f(0)$ **(c)** $f(a + 1)$ **(d)** $\dfrac{f(x + h) - f(x)}{h}$.

Solution

(a) $f(2) = 2^2 - 2 \cdot 2 = 0$.
(b) $f(0) = 0^2 - 2 \cdot 0 = 0$.
(c) $f(a + 1) = (a + 1)^2 - 2(a + 1) = (a^2 + 2a + 1) - 2a - 2$
$= a^2 - 1$
(d) $$\frac{f(x + h) - f(x)}{h} = \frac{[(x + h)^2 - 2(x + h)] - [x^2 - 2x]}{h} = \frac{2xh + h^2 - 2h}{h} = 2x + h - 2. \quad ◀$$

In many cases, the rule for a function cannot be expressed by a single equation. When different equations apply for different portions of the domain, as in Example 4, the function is defined **piecewise** (in pieces). Other function rules are best given verbally, as in Example 5.

►EXAMPLE 4 ***Piecewise-defined functions*** For the function

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

evaluate **(a)** $f(-3)$, **(b)** $f(1)$, and **(c)** $f(\sqrt{3})$.

Solution

(a) and **(b)** Since $-3 < 1$ and $1 \leq 1$, the top piece of the function definition applies:

$$f(-3) = (-3)^2 = 9 \qquad \text{and} \qquad f(1) = 1^2 = 1.$$

(c) $\sqrt{3} > 1$, so the bottom part of the definition gives $f(\sqrt{3}) = 2 - \sqrt{3}$. ◀

Piecewise-defined functions occur in many different kinds of applications and are used throughout calculus. While we should be able to graph such functions by considering each piece separately, graphing calculators give us another convenient tool to get a good picture of what is happening. The following Technology Tip describes the process for the specific function in Example 4.

TECHNOLOGY TIP ◆ ***Graphing piecewise-defined functions***

The parts of the function in Example 4 are defined on limited domains, and then the pieces are added or two functions are graphed.

TI: The function is entered as

$$\text{Y1} = (x^2)(x \le 1) + (2 - x)(x > 1),$$

where the inequality signs come from the TEST menu. We have one function, $y = x^2$, on the limited domain where $x \le 1$, added to another function, $y = 2 - x$, on another limited domain.

Casio: Each function is entered separately with its limited domain. The second is added by overwriting or by putting in a Return ($\boxed{\text{SHIFT}}\boxed{\text{ENT}}$) at the end of the first line (both end-points of intervals are needed):

$$\text{Graph Y} = \text{X}^2, [-5, 1]$$

$$\text{Graph Y} = 2 - \text{X}, [1, 5]$$

HP: The function is entered as an algebraic expression using the abbreviation IFTE (for IF, THEN, ELSE):

$$\text{'IFTE (X} \le 1, \text{X\^{}2}, 2 - \text{X)'}.$$

Strategy: Follow the rule for the function and count the number of primes less than x.

▶EXAMPLE 5 ***Function defined verbally*** Function f is stated "$f(x)$ is the number of prime numbers less than x." The domain of f is the set of positive numbers. Evaluate

(a) $f(2)$, **(b)** $f(12)$, and **(c)** $f(5\sqrt{2})$.

Solution

(a) The value of $f(2)$ is the number of prime numbers less than 2. Since there are no primes less than 2, $f(2) = 0$.
(b) There are five primes less than 12, namely 2, 3, 5, 7, and 11, so $f(12) = 5$.
(c) $5\sqrt{2} \approx 7.071$, so there are four primes less than $5\sqrt{2}$; $f(5\sqrt{2}) = 4$.

The function f is well-defined for every positive number, but to evaluate something like $f(14{,}732)$, we would need an extensive table of prime numbers. For the curious reader, $f(14{,}732) = 1{,}724$. ◀

The next function introduces some useful notation. First, $\min(a, b)$ denotes the *minimum* of the two numbers a and b. Similarly, $\max(a, b)$ is the *maximum* of a and b. If $a = b$, then $\min(a, b) = \max(a, b)$.

▶EXAMPLE 6 ***"Max" and "min" functions*** $g(x) = \min(x + 2, 6 - x)$

(a) Evaluate $g(-1)$, $g(2)$, and $g(\sqrt{5})$.
(b) Write a formula for g in piecewise form and draw a graph.

Solution

(a) $g(-1) = min[-1 + 2, 6 - (-1)] = min(1, 7) = 1$

$g(2) = min[(2 + 2, 6 - 2)] = min(4, 4) = 4$

$g(\sqrt{5}) = min[\sqrt{5} + 2, 6 - \sqrt{5}] = 6 - \sqrt{5}$,

because $6 - \sqrt{5} \approx 3.8$ and $\sqrt{5} + 2 \approx 4.2$.

(b) Because $g(x)$ is always the smaller of $x + 2$ and $6 - x$, we know that $g(x) = x + 2$ when $x + 2 \leq 6 - x$, and $g(x) = 6 - x$ otherwise. Solving the inequality $x + 2 \leq 6 - x$, we find that $x \leq 2$. Therefore,

$$g(x) = \begin{cases} x + 2 & \text{if} \quad x \leq 2 \\ 6 - x & \text{if} \quad x > 2 \end{cases}.$$

Because $x + 2 = 6 - x$ when $x = 2$, we could have the equality with either piece of the definition of g. The graph is shown in Figure 1. ◀

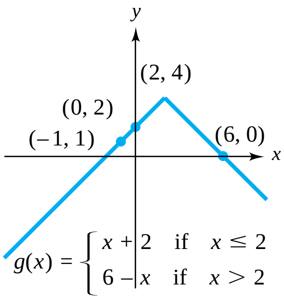


FIGURE 1

EXERCISES 2.1

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. There is no function with domain $\{0, 1\}$ and range $\{3\}$.
2. There is no function with domain $\{1, 2\}$ and range $\{3, 4, 5\}$.
3. If $f(x) = x + 1$, then $f(\sqrt{2} + \sqrt{3}) = f(\sqrt{2}) + f(\sqrt{3})$.
4. In interval notation, the domain of $f(x) = \dfrac{x}{\sqrt{1 - x}}$ is $(-\infty, 1)$.
5. If $f(u) = \dfrac{1 + u}{1 - u}$ then $f(-u) = \dfrac{1}{f(u)}$.

Exercises 6–10 Fill in the blank so that the resulting statement is true, where $f(x) = 2x + 3$.

6. The largest prime number less than $f(11)$ is ______.
7. The number of primes between $f(6)$ and $f(10)$ is ______.
8. The smallest even number greater than $f(7)$ is ______.
9. $f(3) + f(-3) =$ ______.
10. The domain of f is ______.

Develop Mastery

Exercises 1–4 Write the range (in set notation) of the function f, where D is the domain.

1. $f(x) = 4x$
   $D = \{-1, 0, 1\}$
2. $f(x) = x^2 + 1$
   $D = \{-1, 0, 1, 2\}$
3. $f(x) = \dfrac{x^2}{x^2 + 2}$
   $D = \{-3, -2, 2, 3\}$
4. $f(x) = \sqrt{x^2 + 4x}$
   $D = \{-4, 0, 4\}$

*Exercises 5–12 **(a)** Evaluate f as indicated. **(b)** Write the domain of f in set notation using the domain convention. **(c)** Find the zero(s) of f.*

5. $f(x) = 3x + 4$; $f(-1)$
6. $f(x) = \dfrac{x}{x - 2}$; $f(-2)$
7. $f(x) = \dfrac{x + 1}{x^2 + 1}$; $f(4)$
8. $f(x) = x + \sqrt{x + 1}$; $f(3)$
9. $f(x) = \dfrac{\sqrt{1 - x}}{x + 2}$; $f(-3)$
10. $f(x) = \dfrac{\sqrt{x + 1}}{\sqrt{4 - x}}$; $f(1)$
11. $f(x) = \sqrt{x^2 + 3x - 4}$; $f(-5)$
12. $f(x) = \sqrt{4 - 3x - x^2}$; $f(-4)$

Exercises 13–16 Evaluate the indicated expression.

13. $f(x) = \dfrac{x + 1}{x^2 - 2x + 1}$; $f(x + 1)$
14. $f(x) = 1 + \sqrt{-x}$; $f(-4x^2)$
15. $g(x) = \dfrac{x}{\sqrt{x^2 + 4}}$; $g(-x)$
16. $g(x) = x^2 + x$; $g(x + 1) - g(x)$

Exercises 17–20 Evaluate f at the indicated numbers. If the result is a rational number, leave it in exact form; otherwise approximate it to two decimal places.

17. $f(x) = 5x + \sqrt{3}$; $f(-4)$, $f\left(\dfrac{-\sqrt{3}}{5}\right)$

18. $f(x) = \sqrt{x^2 + 4}$; $f(\sqrt{3})$, $f(\pi)$

19. $g(x) = \sqrt{25 - x^2}$; $g(-4)$, $g(1.3)$

20. $g(x) = \dfrac{x}{x + 1}$; $g(-3)$, $g(\sqrt{3})$

Exercises 21–26 ***Difference Quotient*** *Express the difference quotient,* $\dfrac{f(x + h) - f(x)}{h}$, *in simplest form.*

21. $f(x) = 3x - 4$

22. $f(x) = 2 - 3x$

23. $f(x) = x^2 - 2x$

24. $f(x) = -x^2 + x + 3$

25. $f(x) = \frac{1}{x}$

26. $f(x) = 1 - \frac{3}{x}$

Exercises 27–30 ***Describe Verbally*** *Express the rule of correspondence for f as a verbal statement, that is, translate the rule into English.*

27. $f(x) = 3x + 4$

28. $g(x) = 9 - x^2$

29. $f(x) = 4 - \sqrt{x}$

30. $f(x) = \sqrt{4 + x^2}$

Exercises 31–33 ***Piecewise Function*** *Evaluate the function as indicated in exact form.*

31. $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ $f(3)$, $f(-4)$, $f(2 - \sqrt{5})$

32. $g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$ $g(-\sqrt{2})$, $g(3)$, $g(\sqrt{17} - 4)$

33. $f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer} \\ -1 & \text{if } x \text{ is not an integer} \end{cases}$ $f(-3)$, $f(\frac{3}{5})$, $f(\sqrt{7})$

Exercises 34–37 ***Domain*** *Use a graph to help you find the domain of f. Give results to one decimal place when needed. (Hint: Draw a graph of the expression under the square root and see for what values it is non-negative.)*

34. $f(x) = \sqrt{x^2 - 2x - 10}$

35. $f(x) = \sqrt{10 + 2x - x^2}$

36. $f(x) = \sqrt{4 - x - 2|x|}$

37. $f(x) = \sqrt{4 + x - 2|x|}$

Exercises 38–42 ***Verbal Rule*** *Evaluate as indicated.*

38. The function $f(x)$ is the greatest integer that is less than x.
(a) $f(3)$ **(b)** $f(4.3)$ **(c)** $f(-1.5)$
(d) $f(2 - \sqrt{5})$

39. When f is applied to any quantity, it squares that quantity and then subtracts 4 from the result.
(a) $f(3)$ **(b)** $f(-\sqrt{3})$ **(c)** $f(x^2)$
(d) $f(1 - x)$

40. When g is applied to any quantity, it subtracts the square root of that quantity from 4 and then divides the result by 3.
(a) $g(16)$ **(b)** $g(3)$ **(c)** $g(1.69)$ **(d)** $g(x^2)$

41. The rule for the function f is: $f(x)$ is equal to the smallest prime number greater than or equal to x.
(a) $f(0)$ **(b)** $f(3.4)$ **(c)** $f(2 + \sqrt{7})$
(d) $f(43)$

42. The domain of f is the set of positive integers; $f(x)$ is the remainder when x is divided by 3.
(a) $f(21)$ **(b)** $f(2)$ **(c)** $f(5)$
(d) $f(4736)$ **(e)** What is the range of f?

43. Find all values of x (if any) for which $f(x) = 4$.
(a) $f(x) = 3x + 2$ **(b)** $f(x) = \dfrac{1}{2x - 1}$
(c) $f(x) = \sqrt{x + 4}$ **(d)** $f(x) = 2x^2 - x - 1$

44. If f is a function and $f(2x - 1) = 4x^2 - 2x - 6$, find
(a) $f(2)$ **(b)** $f(-3)$ **(c)** a formula for $f(u)$.

45. If f is a function and $f(x + 1) = 2x^2 + 3x - 2$, find
(a) a formula for $f(u)$ **(b)** what are the zeros of f?

Exercises 46–47 *Evaluate. Then write the equation for g in piecewise form.*

46. $g(x) = \min(2x - 3, 6 - x)$
(a) $g(-2)$ **(b)** $g(0)$ **(c)** $g(3)$ **(d)** $g(5)$

47. $g(x) = \max(1 - 2x, 2x + 3)$
(a) $g(-3)$ **(b)** $g(-1)$ **(c)** $g(0)$ **(d)** $g(\sqrt{2})$

Exercises 48–52 *Write an equation that expresses the dependent variable as a function of the independent variable.*

48. The radius r of a circle depends upon its
(a) diameter d **(b)** circumference C.

49. The perimeter P of a square depends upon its side length s.

50. The hypotenuse of a right triangle is 4 and one leg has length x. The area A of the triangle depends upon x. What is the domain of the area function?

51. The area of a square depends upon its perimeter P.

52. The area A of a circle depends upon its circumference C.

53. A rectangle having one side of length x is inscribed in a circle of radius 5.
(a) Draw diagrams for $x = 1, 5, 9$. In each case compute the area of the rectangle.
(b) Find a formula that gives the area A of the rectangle as a function of x.

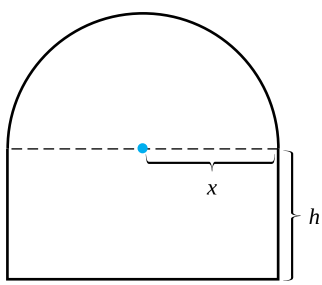
54. Point $P(x, 0)$ is on the x-axis, and point $A(0, 1)$ is on the y-axis.
(a) Draw a diagram showing point A and points P when x is 1, when x is -3, and for an arbitrary x.
(b) What is the distance from A to P when x is 2?
(c) Determine an equation giving the distance d from A to P as a function of x. What is the domain of this function?
(d) Evaluate d when x is 0, 2, -4.

55. **(a)** Give an equation that describes the distance d between points $A(0, 1)$ and $Q(x, 1)$ as a function of x.
(b) Evaluate the distance function when x is $1, -3$.
(c) What is the domain of the distance function?

56. Two cars leave at noon from the same point, one traveling east at 50 mph, the other traveling due north at 60 mph.
(a) How far apart are they at 1 PM? at 1:30 PM?
(b) If t is the number of hours after noon, find an equation expressing the distance between the cars as a function of t.

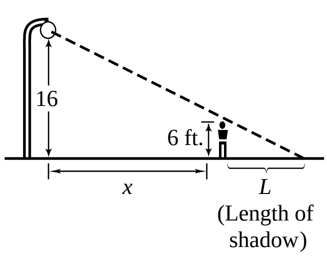
57. ***Overtime*** An auto mechanic is paid \$18 per hour when he works no more than 40 hours a week. When he works more than 40 hours a week, he earns time-and-a-half for each additional hour.
(a) How much does he earn in a 32-hour week? a 48-hour week?
(b) Find an equation expressing his wages W as a function of the number of hours x in a work week.

58. A window has the shape indicated in the diagram. The perimeter of the window (total distance around) is fixed at 16 feet. Let x denote the radius of the semicircle and h the height of the rectangle.



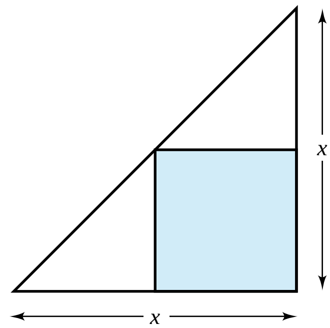
(a) Find an equation giving h as a function of x. What is the domain of the function?
(b) Express the area A of the window as a function of x.

59. ***Length of Shadow*** A man 6 feet tall is walking away from a streetlight 16 feet high so that his shadow is directly in front of him. Let x denote his distance from the base of the light as shown in the diagram.



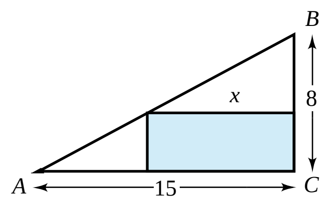
(a) Express the length L of his shadow as a function of x.
(b) How far away from the lamp must he be to cast a shadow 6 feet long?

60. A square is inscribed in an isosceles right triangle as shown in the diagram. Let x denote the length of each leg of the triangle. Express as a function of x:

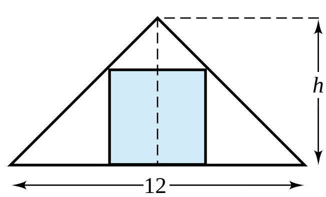


(a) The perimeter P of the triangle.
(b) The area A of the triangle.
(c) The area K of the square.

61. A rectangle is inscribed in a right triangle ABC as shown in the diagram, where $|\overline{AC}| = 15$, $|\overline{BC}| = 8$, and x is the length of one side of the rectangle. Express the area A of the rectangle as a function of x. What is the domain of this function?



62. A square is inscribed in an isosceles triangle of base 12 as shown in the diagram. Let h denote the length of the altitude drawn to the base of the triangle. Express as a function of h:
(a) The perimeter P of the triangle.
(b) The perimeter S of the square.
(c) The area A of the square.



2.2 GRAPHS OF FUNCTIONS

> One can envisage that mathematical theory will go on being elaborated and extended indefinitely. How strange that the results of just the first few centuries of mathematical endeavor are enough to achieve such enormously impressive results in physics.
>
> *P.W.C. Davies*

Graph of a Function

A function f assigns a range element to each domain element, so it is often useful to think of the function as pairing numbers (if the domain and range are sets of numbers). The function is completely determined by these ordered pairs. In mathematical notation,

$$f = \{(x, f(x)) \mid x \in D\}, \text{ where } D \text{ is the domain of } f.$$

When the function is defined by an equation, $y = f(x)$, then

$$f = \{(x, y) \mid y = f(x), x \in D\}.$$

The ordered pairs that define f look like coordinates of points in the plane, and we can use this feature to define the graph of a function.

Definition: graph of a function

If f is a function with domain D, then the graph of f is the set of points with coordinates (x, y) such that $x \in D$ and $y = f(x)$.

An accurate graph of a function makes both the domain and the range of the function apparent. The domain is the set of x values of points on the graph, and the range is the set of y values.

Function Properties and Graphs

> *I think that starting mathematics early had given me a certain self-reliance. I felt you didn't learn anything in class, you just figured it out yourself.*
>
> Paul Cohen

It is natural to think of drawing a graph by plotting points and connecting them in an appropriate way to get a curve. This is precisely how a computer or graphing calculator shows graphs on a screen. Without the capability to compute hundreds of function values quickly, however, pencil and paper techniques are time consuming and tedious. With or without access to the tools of technology, some additional tools can help us draw graphs and understand the properties of functions.

In this section, we examine symmetry properties of graphs and introduce the notion of even and odd functions. Also, certain core graphs are given. Knowing a single core graph and how it is affected by simple changes, we can draw graphs of a whole family of related functions.

Core Graphs

There are a few graphs that should be familiar to every precalculus student. We will do lots of graphing with calculators and the aid of technology, but you should be able to sketch each of the following core graphs without any help. Knowing the properties of these simple functions and some of their key features will make discussions of all kinds of functional behavior more meaningful. Use your graphing calculator as needed to help absorb the ideas, but make yourself confident that you know the graphs of the functions in Figure 2.

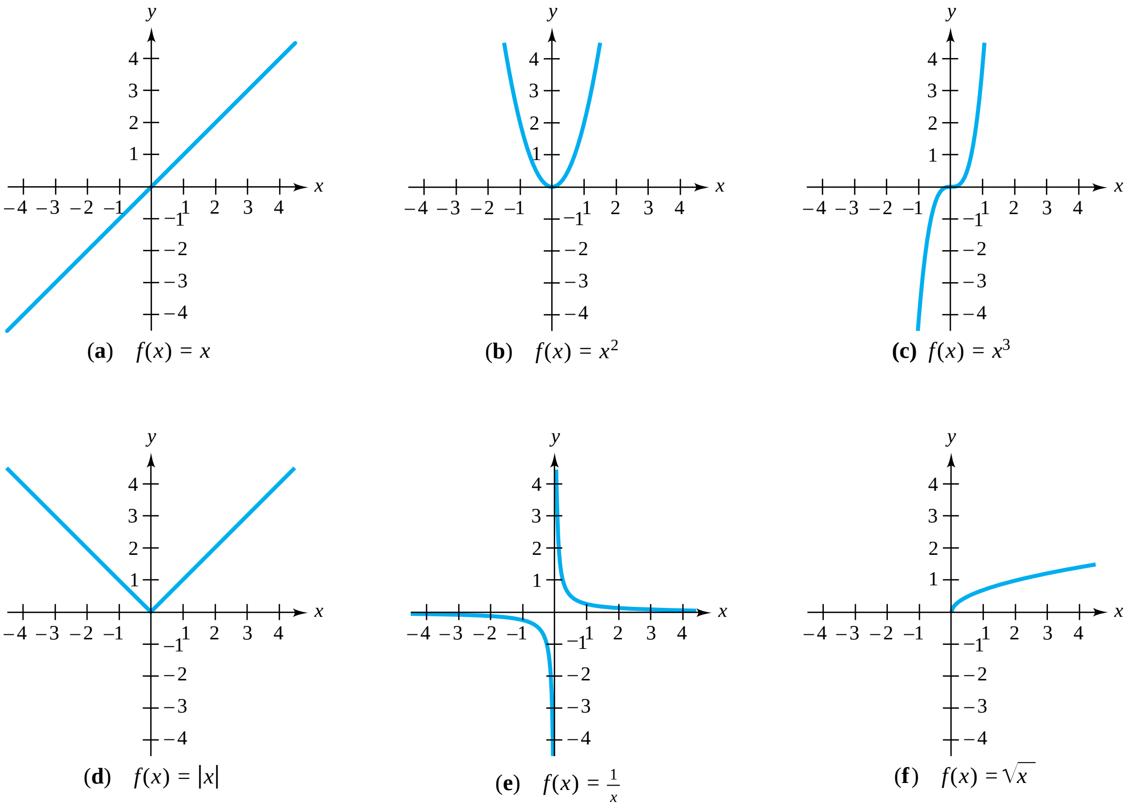


FIGURE 2
Catalog of core graphs

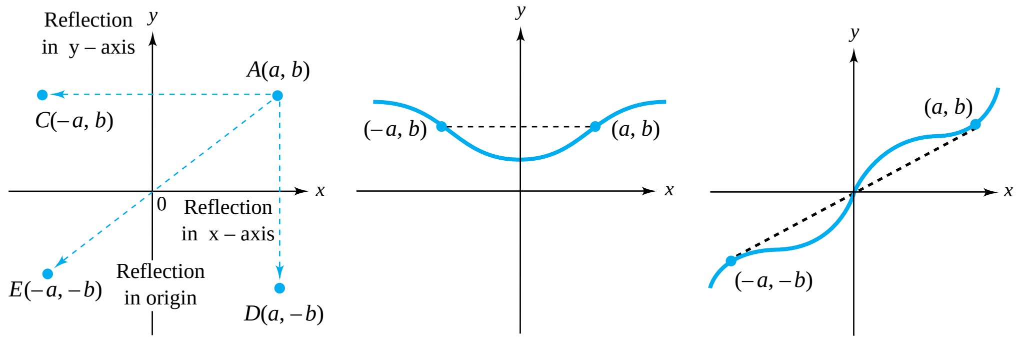
Intercepts, Symmetry, Even and Odd Functions

We are always interested in the points where a graph meets the coordinate axes. If 0 is in the domain of f, then $f(0)$ is called the *y-intercept* and the point $(0, f(0))$ is the *y-intercept point*. A graph need not meet the x-axis, but if it does, any points where it does are called *x-intercept points,* and if $f(c) = 0$, then the number c is called an *x-intercept*.

Given a point $A(a, b)$, three symmetrically located points can help in graphing. We can reflect A in the y-axis to the point $C(-a, b)$, in the x-axis to the point $D(a, -b)$ or in the origin to the point $E(-a, -b)$. Figure 3a shows a first-quadrant point A and its reflections.

The graph of a function may have all sorts of symmetry properties (or none). Suppose that for every arbitrary point (a, b) on the graph of a function, the graph also contains the reflection of (a, b) in the y-axis. That is, whenever (a, b) is on the graph, $(-a, b)$ is also on the graph. Then we say that the graph is **symmetric about the *y*-axis** and that the function is an **even function.** In functional notation, since to say that (a, b) is on the graph means that $f(a) = b$, a function is even when $f(-x) = f(x)$. See Figure 3b.

If, whenever (a, b) is on the graph of f, $(-a, -b)$ is also on the graph, then the graph is **symmetric about the origin** and the function is an **odd function.** In functional notation, a function is odd if $f(-x) = -f(x)$ for every x in D. See Figure 3c.



(a) Point Reflections **(b)** Even; Symmetric about y–axis **(c)** Odd; Symmetric about origin

FIGURE 3

The catalog of core graphs in Figure 2 has examples that should help you remember the distinction between even and odd functions. The parabola $y = x^2$ and the absolute value function $y = |x|$ are even; the graphs are clearly symmetric about the y-axis. The line $y = x$, the cubic $y = x^3$, and the reciprocal $y = \frac{1}{x}$ are all symmetric about the origin; the functions are odd. The square root function $y = \sqrt{x}$ is neither odd nor even. We sum up in the following.

Definition: even and odd functions, symmetry properties

Suppose f is a function with domain D. If, for every x in D, we have

$f(-x) = f(x)$, then f **is an** *even* **function;**

$f(-x) = -f(x)$, then f **is an** *odd* **function.**

The graph of an **even** function is **symmetric about the y-axis.**
The graph of an **odd** function is **symmetric about the origin.**

In addition to just looking at a single graph, graphing calculators can be used to see whether or not $f(-x) = f(x)$ or $f(-x) = -f(x)$; compare the graph of $y1 = f(x)$ with the graph of $y2 = f(-x)$, where we replace each x in $f(x)$ by $-x$.

▶EXAMPLE 1 ***Even and odd functions*** Sketch the graph and determine, both graphically and algebraically, whether the function is odd or even.

(a) $f(x) = x^2 + 1$ **(b)** $g(x) = x^3 - x$

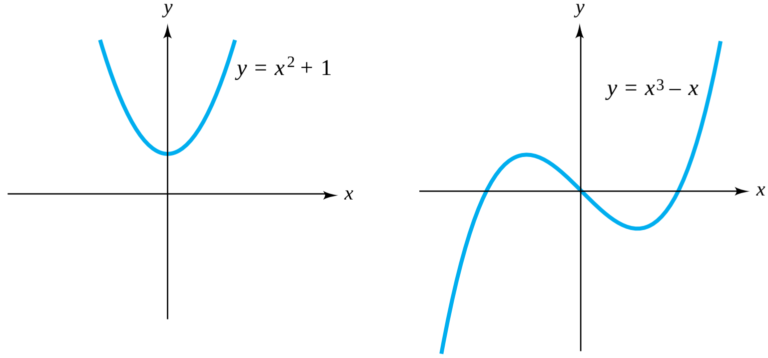
Solution

(a) ***Algebraic*** $f(-x) = (-x)^2 + 1 = x^2 + 1$, so $f(-x) = f(x)$, and by definition, f is an even function.

Graphical The graph, shown in Figure 4(a), is symmetric about the y-axis and is clearly the graph of an even function.

(b) ***Algebraic*** For the function g, we have $g(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -g(x)$, and so from the definition, g is an odd function.

Graphical The calculator confirms that the graph is symmetric about the origin, and hence the graph of an odd function. See Figure 4b. ◀



(a) Even Function

(b) Odd Function

FIGURE 4

A function may be neither even nor odd, as the next example shows.

▶EXAMPLE 2 ***Neither even nor odd*** Show that the function is neither even nor odd, and sketch its graph

(a) $F(x) = 2x - 4$ **(b)** $G(x) = \sqrt{x}$

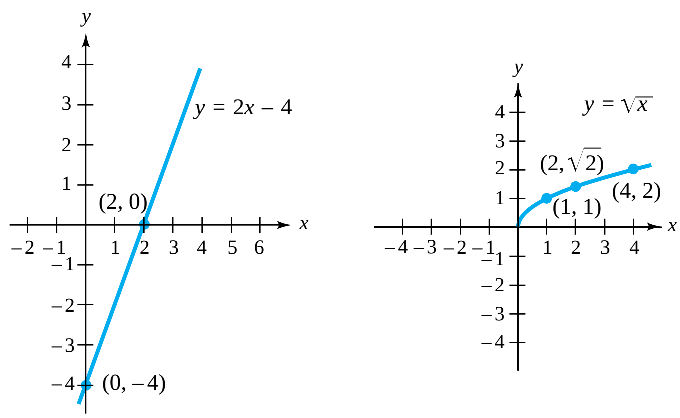
Solution

Apply the definitions for even and odd functions.

$$F(-x) = 2(-x) - 4 = -2x - 4 \qquad -F(x) = -2x + 4$$
$$G(-x) = \sqrt{-x} \qquad -G(x) = -\sqrt{x}$$

Since $F(-x) \neq F(x)$ and $F(-x) \neq -F(x)$, F is neither even nor odd, and similarly, G is neither even nor odd.

To graph the two functions, plot points as in Figure 5. ◀



(a)
$F(x) = 2x - 4$ is neither even nor odd.

(b)
$G(x) = \sqrt{x}$ is neither even nor odd.

FIGURE 5

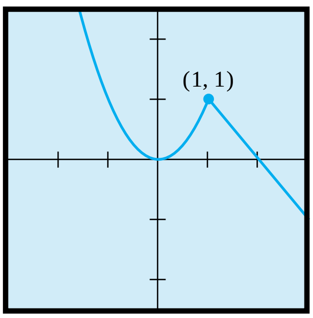
Piecewise, Step Functions, and Calculator Graphs

We have already seen some examples of an important class of functions called **piecewise** functions because they are defined in pieces (with different rules for different portions of their domains). The first example we encountered is the absolute value function, even though we did not recognize it when we first introduced absolute values. Among the properties of absolute values in Section 1.3 we listed the equality $|x| = \sqrt{x^2}$, which can be justified by considering values, but it may also be helpful to get calculator reinforcement. Graph both Y = ABS(X) and Y = $\sqrt{X^2}$ and see that the graphs are identical. The function can also be written in pieces:

$$y = |x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The graph consists of part of the line $y = -x$ and part of the line $y = x$.

You may want to refer again to the Technology Tip in Section 2.1 to make sure that you know how to graph piecewise-defined functions on your calculator.



$[-3, 3]$ by $[-2.5, 2.5]$

FIGURE 6

▶EXAMPLE 3 ***Calculator graphs of piecewise functions*** Draw a calculator graph of the piecewise function from Example 4 of Section 2.1,

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

Solution

The graph is shown in Figure 6. You may wish to experiment with different windows to see how changing the view alters the shape of the pieces we see. Whatever the window, however, the graph of f consists of two pieces that meet at the point $(1, 1)$. ◀

Greatest Integer Function

Another piecewise-defined function that is useful in many different contexts is the *greatest integer function,* which is programmed into most graphing calculators, defined as *the largest integer that is less than or equal to* x. Any real number x is either an integer or it lies between two integers. If $n \leq x < n + 1$, the largest integer less than or equal to x is n, so for such an x, $Int(x) = n$. The function is denoted by $Int(x)$ on graphing calculators, or sometimes $Floor(x)$, or, in older books, by $[x]$.

Definition: The greatest integer function

The **greatest integer** function of x, denoted by $Int(x)$, is *the largest integer that is less than or equal to* x.

As examples,

$Int(1) = 1$, and for any integer n, $Int(n) = n$,

$Int(0.3) = 0$, and for any number x between 0 and 1, $Int(x) = 0$,

$Int(\pi) = 3$ because 3 is the largest integer less than π,

$Int(-\pi) = -4$ because $-\pi$ is between -4 and -3.

Postal rates are examples of functions defined piecewise, where the definition can make use of $Int(x)$. Mailing cost is a function of the weight of a letter. It remains constant for a while and then suddenly jumps to a new value.

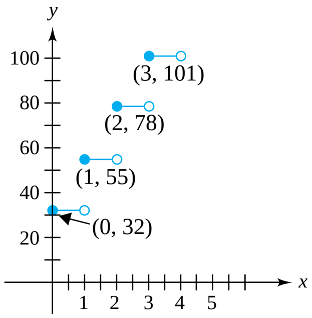


FIGURE 7
$C(W) = 32 + 23\ Int\ (W)$

▶EXAMPLE 4 ***The postage function*** In 1995, postal rates for first class letters delivered within the United States were set as follows: the cost is 32 cents for anything less than 1 ounce; for each additional ounce (up to 11 ounces) the cost increases in increments of 23 cents. Express the cost C (in cents) of first-class postage as a function of the weight W (ounces) and draw a graph.

Solution

In mathematical language, we can express the cost C as a function of the weight W either piecewise, or by using the greatest integer function.

$$C = 32 + 23\ Int(W),\ 0 < W < 11.$$

The graph of the cost function is shown in Figure 7. ◀

TECHNOLOGY TIP ◆ ***Graphing in dot mode***

In connected mode, a calculator graph of a piecewise-defined function can make it appear as if there is a vertical piece of the graph connecting pieces that should not be connected. Don't be fooled.

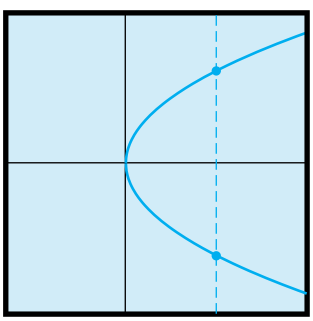
There can be a "jump" from one piece to another *between pixels.* When y-values differ in adjacent pixel columns, the calculator connects pixels in vertical columns joining separated points.

It is often helpful to change to a non-connected format to see jumps in graphs of piecewise functions. In calculator graphs, *remember that unless the calculator is in dot mode, the calculator connects separated pixels in adjacent columns.*

Not all equations define functions, and some familiar graphs are not graphs of functions. A function assigns to each domain element exactly *one* element of the range. This implies that for any function f and any given domain number c, there is exactly one point, $(c, f(c))$, on the graph of f. A vertical line can meet the graph of f in at most *one* point, which is the basis for the following handy test.

Vertical line test

For a given graph, if at each number c of the domain, the vertical line $x = c$ intersects the graph in exactly one point, then the graph represents the graph of a function. If some vertical line meets a graph in more than a single point, then the graph is not the graph of a function.



[−2, 3] by [−2, 2]

FIGURE 8
Vertical line meets graph in two places.

▶EXAMPLE 5 ***Vertical line test*** Use a calculator to sketch a graph of all points that satisfy the equation $y^2 = x$. Use the vertical line test to verify that the graph is not that of a function.

Solution

In function mode, we cannot enter equations except in the form of Y = . . . , so when we solve the given equation for y, we get $y = \pm\sqrt{x}$. On the same screen we graph Y1 = √X and Y2 = √X. The two graphs together form a parabola opening to the right, as shown in Figure 8. Since any vertical line through the positive x-axis meets the graph twice, the vertical line test tells us that the graph is not the graph of a function. ◀

EXERCISES 2.2

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. The graph of a function cannot have more than one y-intercept point.
2. The graph of $y = \frac{x}{|x|}$ is identical to the graph of $y = \frac{|x|}{x}$.
3. For the greatest integer function $f(x) = Int(x)$, **(a)** $f(-2.5) = -f(2.5)$ **(b)** $f(-3) = -f(3)$.
4. The distance between the x- and y-intercept points of the graph of $y = 1 - x$ is $\sqrt{2}$.
5. For any function f, the function $g(x) = f(x^2)$ is an even function.
6. For any even function f, if $(-2, 4)$ is on the graph of f, then $(-2, -4)$ must also be on the graph of f.

Exercises 7–10 Fill in the blank so that the following statement is true. "If you draw a graph of function f using $[-10, 10] \times [-10, 10]$, *then the number of x-intercept points shown in the display is ________."*

7. $f(x) = 0.3x^2 - 4x - 5$
8. $f(x) = 0.5x^2 + 4x + 4$
9. $f(x) = 2|x| - 3|x - 3|$
10. $f(x) = 3|x - 6| - 2|x - 1|$

Develop Mastery

Exercises 1–4 ***Isolated Points*** *A function is given along with its domain. Draw a graph of the function. The graph consists of isolated points. State the range of the function.*

1. $f(x) = 2x - 1$; $D = \{-1, 2, 3\}$
2. $f(x) = 4 - x^2$; $D = \{-1, 0, 1, 2\}$
3. $f(x) = x^3 - x$; $D = \{-2, -1, 0, 1, 2\}$
4. $f(x) = \sqrt{x}$; $D = \{1, 2, 4\}$

Exercises 5–8 Make a table of several (x, y) ordered pairs that satisfy the equation. Plot the points in your table and draw a graph.

5. $y = 2x - 4$
6. $y = 4 - 2x$
7. $y = x^2 - x$
8. $y = 2x^2 + 4x$

Exercises 9–12 Find the value of x or y so that point P is on the graph of f.

9. $f(x) = x^2 - 4x - 3$; $P(2, y)$
10. $f(x) = \sqrt{1 - 4x}$; $P(-3, y)$
11. $f(x) = 3x - 2$; $P(x, 4)$
12. $f(x) = x^2 - 2x - 8$; $P(x, -5)$

Exercises 13–16 ***Odd, Even*** *Determine whether function f is odd, even, or neither. Do the same for g. First draw a graph then use algebra to support your conclusion.*

13. $f(x) = x^4 - 3x^2$, $g(x) = \sqrt{x - 1}$
14. $f(x) = x - x^3$, $g(x) = x^2 + 2|x| - 3$
15. $f(x) = x^3 - 1$, $g(x) = x^3 - 2x$
16. $f(x) = (x + 1)(x - 1)$, $g(x) = 3 - |x|$

Exercises 17–28 ***Graphs*** *Draw a graph of f. Give the coordinates of the x- and y-intercept points.*

17. $f(x) = (x + 2)^2$
18. $f(x) = x^3 - 2$
19. $g(x) = \sqrt{x + 2}$
20. $g(x) = \sqrt[3]{-x}$
21. $f(x) = |x - 1| - 1$
22. $g(x) = \frac{1}{2}\sqrt{16 - x^2}$
23. $g(x) = (x + 1)^2 - 2$
24. $g(x) = -2\sqrt{4 - x^2}$
25. $f(x) = -(x + 2)^2$
26. $g(x) = \sqrt{x + 2} + 1$
27. $g(x) = -|x| + 2$
28. $f(x) = \sqrt{8 + 2x - x^2}$

Exercises 29–30 ***Calculator Graph*** *Suppose you are interested in using a graph to help you get information about the zeros of f. Which of the given windows would you use?*

29. $f(x) = 2x^2 + 47x - 75$
    (i) $[-10, 10] \times [-10, 10]$
    (ii) $[-20, 10] \times [-100, 100]$
    (iii) $[-40, 10] \times [-400, 200]$
30. $f(x) = x - 5|x| + 100$
    (i) $[-10, 10] \times [-10, 10]$
    (ii) $[-16, 20] \times [-5, 30]$
    (iii) $[-20, 30] \times [-5, 100]$

Exercises 31–34 ***Window Dimensions*** *The graph of f has two x-intercept points and either a highest or lowest point. Give the dimensions of a window for which the graph of f will show this information. (Answers may vary.)*

31. $f(x) = x^2 - 25x + 150$
32. $f(x) = 40 - 18x - x^2$
33. $f(x) = x - 4|x| + 45$
34. $f(x) = 2|x + 8| - x - 40$

Exercises 35–38 ***Limited Domain*** ***(a)*** *Draw a graph of f, with the indicated domain.* ***(b)*** *Find the range of f.*

35. $D = \{x \mid x \geq 0\}$; $f(x) = 2x - 3$
36. $D = \{x \mid x < 0\}$; $f(x) = 1 - x$
37. $D = \{x \mid x > 1\}$; $f(x) = x^2$
38. $D = \{x \mid x \leq 2\}$; $f(x) = x^2$

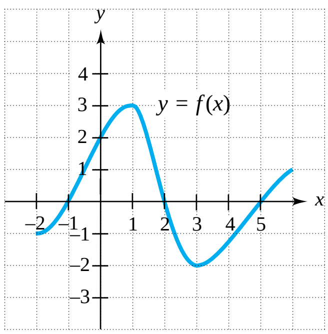
Exercises 39–41 ***Piecewise Graph*** *Draw a graph of the given function and determine the range.*

39. $f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ 2 - x, & \text{if } x < 0 \end{cases}$

40. $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ 1 + x & \text{if } x > 0 \end{cases}$

41. $f(x) = \begin{cases} x^2 + 2x & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases}$

Exercises 42–48 *Refer to the function f whose graph is shown in the diagram with domain* $[-2, 6]$.



42. From the graph, give $f(-2)$, $f(0)$, and $f(4)$.

43. Order the following numbers from smallest to largest: $f(-1), f(\frac{1}{2}), f(3), f(\frac{9}{2})$.

44. **(a)** What is the maximum value (the largest value) of $f(x)$?
(b) What is the minimum value (the smallest value) of $f(x)$?
(c) What is the range of f?

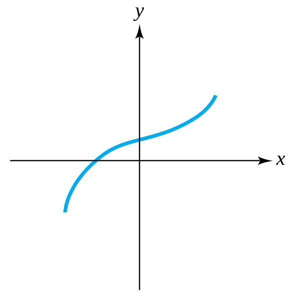
45. Give the coordinates of the highest and the lowest point on the graph.

46. Give the coordinates of the y-intercept point and the x-intercept points.

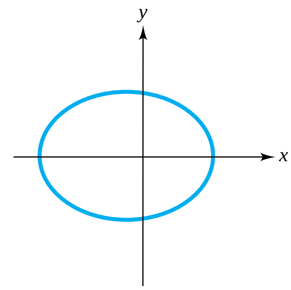
47. **(a)** For what values of x is $f(x)$ negative?
(b) For what values of x is $f(x)$ positive?

48. True or false. Explain.
(a) $f(-2)$ is less than $f(3)$.
(b) $f(4.3)$ is a negative number.
(c) $f(-1) - f(\sqrt{3})$ is a negative number.
(d) There are three x-intercept points for the graph of f.

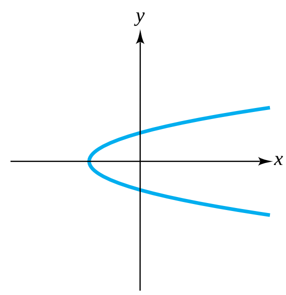
49. Use the vertical line test to determine which of these graphs are graphs of functions that have x as the independent variable.



(a)



(b)



(c)

Exercises 50–51 ***Interesting Functions***
(a) *Give the domain for f and for g. Evaluate f at* $x = 0$, 4,10, 20, 40, 56.
(b) *Draw graphs of f and g on separate screens.*
(c) *Look at the graphs and describe any interesting features.*
(d) *Are functions f and g identical? Explain.*
(e) *What is the solution set for* $f(x) = 3$? $f(x) = 4$?

50. $f(x) = \dfrac{1}{2}\left(\sqrt{x} + \sqrt{x + 64 - 16\sqrt{x}}\right)$

$g(x) = \dfrac{1}{2}\left(\sqrt{x} + |\sqrt{x} - 8|\right)$

51. $f(x) = \dfrac{1}{5}\left(\sqrt{x + 200} + \sqrt{x + 600 - 40\sqrt{x + 200}}\right)$

$g(x) = \dfrac{1}{5}\left(\sqrt{x + 200} + |\sqrt{x + 200} - 20|\right)$

Exercises 52–53 ***Functions Involving Abs***
(a) *Draw a graph of f. Use a decimal window.*
(b) *Use the graph to find the solution set for* $f(x) \leq 0$.

52. $f(x) = x - |x + 3| + |x - 4| - 1$

53. $f(x) = |x + 4| - |x - 3| - x - 1$

Exercises 54–55 ***(a)*** *Evaluate f at* $x = -5, -2, 0, 3.5$. ***(b)*** *Give a formula for f in piecewise form.* ***(c)*** *Use part (b) to draw a graph of f.*

54. $f(x) = \min(2x - 3, 6 - x)$

55. $f(x) = \max(x - 2, -2x + 7)$

Exercises 56–59 ***Graphs Involving Int*** *Draw a graph of f. Use dot mode and a decimal window.* ***(a)*** *What is the range of f?* ***(b)*** *What is the solution set for $f(x) = 5$?*

56. $f(x) = Int(x - 2)$

57. $f(x) = Int(x + 2) - 3$

58. $f(x) = Int(0.5x) + 3$

59. $f(x) = Int(2x) - Int(x)$

Exercises 60–63 ***Equation Involving [x]*** *Draw a graph of f. Use dot mode and a decimal window. Find the solution set for the given equation. Note: [x] is the same as Int(x).*

60. $f(x) = [x] - 1; f(x) = 2$

61. $f(x) = [x - 1]; f(x) = 2$

62. $f(x) = (-1)^{[x]}; f(x) = -1$

63. $f(x) = 0.5x - [0.5x]; f(x) = 0$

Exercises 64–67 ***Solution Set*** *Find the solution set for the open sentence.*

64. $2(Int(x))^2 - 5Int(x) - 12 = 0$. *(Hint: Factor.)*

65. $3^{[x]} - 1 = 0$

66. $Int(\sqrt{p}) = 4$ where p is a prime.

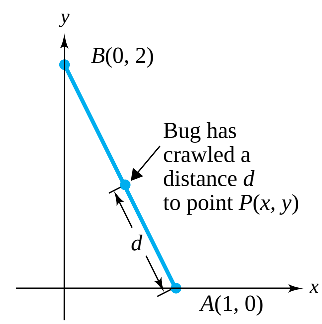
67. $Int(x) - 3 \leq 0$

68. ***Postage Costs*** This section discussed postage charges. When a parcel or letter exceeds 11 ounces, a different rule applies for determining mailing cost as a function of weight. The rule depends upon mailing zones as well as weight and is given in tabular form. For example, when mailing from Zone 1 to Zone 8, the table below lists charges where w is the weight (not exceeding the number of pounds) and c is the cost in dollars.

(a) Draw a graph of c as a function of w.
(b) What is the cost of mailing a package that weighs 2 pounds 5 ounces? 4 pounds 3 ounces?
(c) If the cost of mailing a package is $6.00, what do we know about its weight?

w	c
Pounds	*Dollars*
1	3.00
2	3.00
3	4.00
4	5.00
5	6.00
6	8.00
7	9.80
8	11.60

69. ***Bug on a Ladder*** A bug starts at point (1, 0) and travels along the line segment $\overline{AB}$ toward point (0, 2) as shown in the diagram. If $P(x, y)$ denotes the location of the bug when it has traveled a distance d from (1, 0), express the coordinates x and y as functions of the distance d.



Exercises 70–71 ***Parking Costs***

70. A parking garage charges $2.00 for parking up to one hour and $0.50 for each additional hour (or fraction thereof), with a maximum of $8.00 if you park 12 hours or longer. Suppose x denotes the number of hours you park and y (dollars) the corresponding cost. Then y is a function of x given in piecewise form:

$$y = \begin{cases} 2 + 0.50 \cdot Int(x) & \text{if } x < 12 \\ 8 & \text{if } x \geq 12 \end{cases}$$

(a) Draw a graph of this function. Use dot mode.
(b) If you have only $5.00, how long can you park?

71. A parking garage charges $3.00 for parking up to one hour and $0.30 for each additional fifteen minutes (or fraction thereof), with a maximum of $10. Suppose x denotes the number of hours you park and y (dollars) the corresponding cost. Then y is a function of x given in piecewise form:

$$y = \begin{cases} 3 & \text{if } x < 1 \\ 2.10 + 0.30 \cdot Int(4x) & \text{if } 1 \leq x < 6.75 \\ 10 & \text{if } x \geq 6.75 \end{cases}$$

(a) Use several values of x to check this formula.
(b) What is the cost for 4 hours and 20 minutes?
(c) For what values of x is the cost $4.50? Check using the graph. Use dot mode and a decimal window.

72. ***Overtime Pay*** A carpenter earns \$20 per hour when he works 40 hours or fewer per week, and time-and-a-half for the number of hours he works above 40. Let x denote the number of hours he works in a given week and y (dollars) the corresponding pay.

(a) Write a piecewise formula giving y as a function of x where $0 \le x \le 168$.

(b) If his pay for the week is \$1070, how many hours did he work?

73. ***Telephone Call Cost*** Suppose a telephone company charges 60 cents for a call up to one minute, and 50 cents for each additional minute (or fraction thereof). Let x denote the number of minutes you talk and y (dollars) the corresponding cost. Then y is a function of x given by

$$y = 0.60 + 0.50\mathit{Int}(x).$$

(a) Check several values of x to see that this formula gives what you would expect.

(b) Suppose you do not want to spend more than \$5.00 for a call. How long can you talk?
Use a graph in dot mode with a decimal window.

2.3 TRANSFORMATIONS OF GRAPHS

> Formal logic is an impoverished way of describing human thought, and the practice of mathematics goes far beyond a set of algorithmic rules.... Mathematics may indeed reflect the operations of the brain, but both brain and mind are far richer in their nature than is suggested by any structure of algorithms and logical operations.
>
> *F. David Peat*

> *[At thirteen] it was hard for me to imagine original mathematics, thinking of something that no one else had thought of before. When I went to college...I thought I might become a biologist. I was interested in many different things. I studied psychology and philosophy, for instance. We didn't have grades, but we did have written evaluations. And I kept getting the message that my true talents didn't lie in subject X but in mathematics.*
>
> William Thurston

Relationships among graphs will be used throughout precalculus and calculus. Whole families of graphs can be related to each other through a few transformations. When we understand the properties of the graph of one particular function f, we can immediately get information about domain and range, about intercepts and symmetry, for any function whose graph is the graph of f shifted up or down, right or left, reflected, squeezed or stretched.

Sometimes we work with a family related to one of the *core graphs* shown in the catalog in Figure 2, but more generally we simply ask how the graphs of two functions are related to each other.

Vertical Shifts

All of the transformations we consider can be justified algebraically. For example, the graph of a function $y = f(x)$ consists of all the points (x, y) whose coordinates satisfy the equation. If (x, y) is on the graph of $y = f(x)$, then the coordinates $(x, y - 1)$ satisfy the equation $y = f(x) - 1$. Each point $(x, y - 1)$ is one unit *below* the point (x, y), so we have an observation that applies to any graph. The graph of $y = f(x) - 1$ is obtained by *shifting the graph of* $y = f(x)$ *down 1 unit*.

The same argument applies for any positive number c.

Vertical shifts, $c > 0$

From the graph of $y = f(x)$, the graph of

$$y = f(x) + c \text{ is } \textit{shifted up } c \textit{ units},$$
$$y = f(x) - c \text{ is } \textit{shifted down } c \textit{ units}.$$

Although we can give an argument to explain the effect of each transformation, we are more concerned with having you do enough examples to see for yourself what happens for each kind of transformation we examine. Accordingly, we will show lots of graphs, but for your benefit, we strongly encourage you to use your graphing calculator to draw each graph yourself.

The first example asks for graphs of vertical shifts of two core graphs. While it is good practice to graph such graphs on your calculator, you should be able to draw these graphs without technology. Look at the equation, recognize the graph as a vertical shift, and make a rough sketch.

▶EXAMPLE 1 ***Vertical shifts*** Identify the function as a vertical shift of a core graph (Figure 2 in Section 2.2) and sketch.

(a) $y = x^2 + 1$, $y = x^2 - 2$, $y = x^2 + \frac{1}{2}$
(b) $y = |x| - 2$, $y = |x| + 1$, $y = |x| - \frac{2}{3}$

Solution

(a) Each graph is a vertical translation of the core parabola of Figure 2b. The first is shifted 1 unit up, the second 2 units down, and the third is $\frac{1}{2}$ up. The three graphs are labeled in Figure 9.

(b) Each is a vertical shift of the core absolute value graph of Figure 2d. The absolute value graphs are shown in Figure 10. ◀

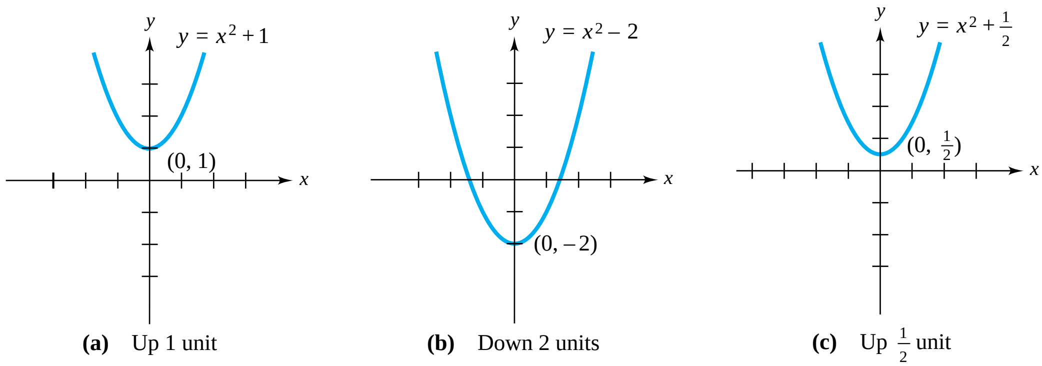


FIGURE 9
Vertical shifts of $f(x) = x^2$

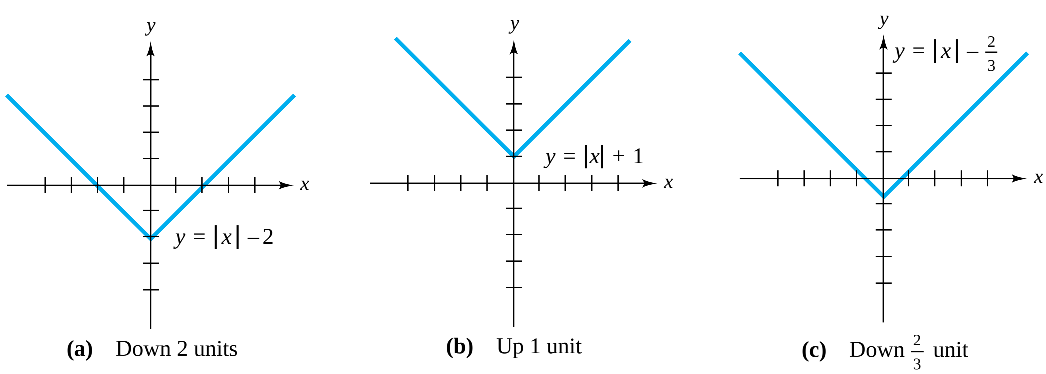


FIGURE 10
Vertical shifts of $f(x) = |x|$

Horizontal Shifts

Some operations are applied to the "outside" of a function. For example,

$$y = f(x) + 3, \quad y = -f(x), \quad y = |f(x)|.$$

The effect of such operations is to change the graph *vertically*. Other operations apply to the "inside" of the function, as

$$y = f(x+3), \quad y = f(-x), \quad y = f(|x|).$$

In the equation $y = f(u)$, u is called the **argument** of the function. In contrast to operations that affect a graph vertically, we have the following useful observation.

> ***"Outside-inside operations"***
>
> Operations applied to the "outside" of a function affect the *vertical* aspects of the graph.
>
> Operations applied to the "inside" (argument) of a function affect the *horizontal* aspects of the graph.

▶EXAMPLE 2 ***Horizontal shifts*** Sketch graphs of

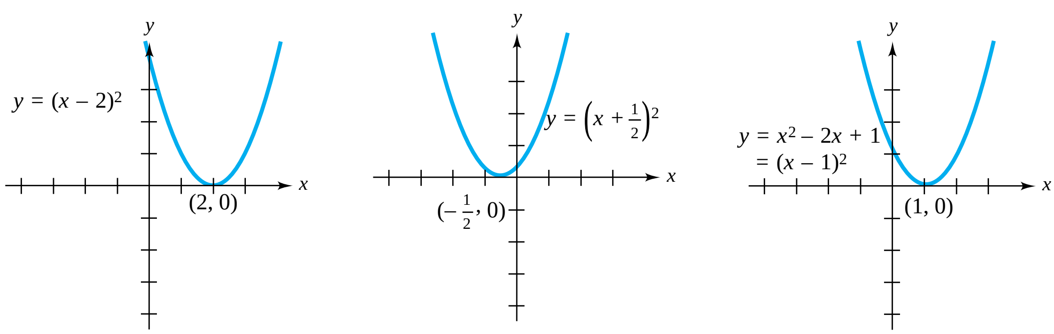
(a) $y = (x-2)^2$, $y = (x + \frac{1}{2})^2$, $y = x^2 - 2x + 1$

(b) $y = \sqrt{x+1}$, $y = \sqrt{x - \frac{2}{3}}$, $y = \sqrt{x+3}$

Solution

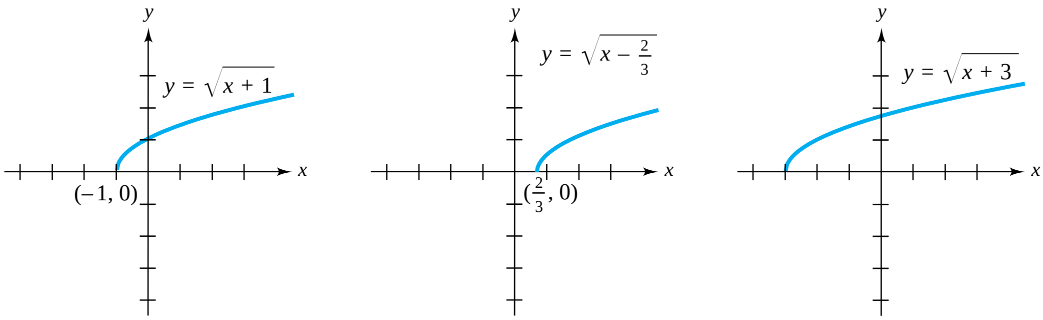
(a) The first two are obviously horizontal shifts of the core parabola $y = x^2$, 2 units right and $\frac{1}{2}$ unit left, respectively. For the third function, we must recognize that $x^2 - 2x + 1 = (x-1)^2$, and so shift the graph of $y = x^2$ right 1 unit. We have the three graphs labeled in Figure 11. We should note that the calculator will provide the same graph, whether written $y = x^2 - 2x + 1$ or $y = (x-1)^2$, and we might recognize the graph as a shifted parabola only after seeing the graph.

(b) Be careful with parentheses; note the difference between Y = √X + 1 (a vertical shift), and Y = √(X + 1) (a horizontal shift). Each graph in this part is a horizontal shift of the core square root function $y = \sqrt{x}$. See Figure 12. ◀



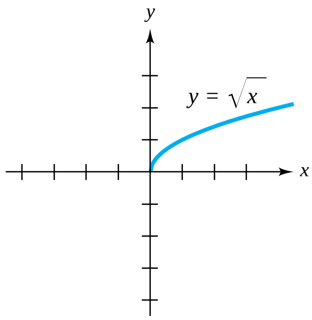
(a) Right 2 units **(b)** Left $\frac{1}{2}$ unit **(c)** Right 1 unit

FIGURE 11
Horizontal shifts of $f(x) = x^2$

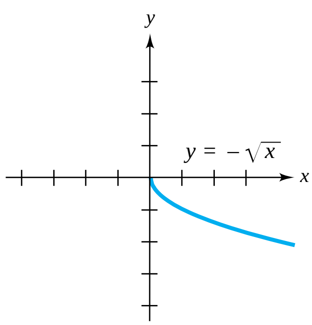


(a) Left 1 unit **(b)** Right $\frac{2}{3}$ unit **(c)** Left 3 units

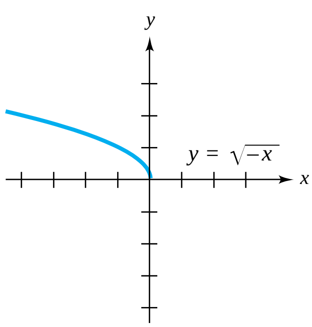
FIGURE 12
Horizontal shifts of $f(x) = \sqrt{x}$



(a) Core graph



(b) Vertical reflection



(c) Horizontal reflection

FIGURE 13
Reflections of $f(x) = \sqrt{x}$

There are some important observations we must make in looking at the horizontal shifts in Example 2. While the graph of $y = x^2 - 2$ shifts *down* from the graph of $y = x^2$, the graph of $y = (x - 2)^2$ is shifted to the *right,* the opposite direction from what some people expect. It may help to remember that the low point on the parabola $y = x^2$ occurs when $x = 0$, and on the parabola $y = (x - 2)^2$, $y = 0$ when $x = 2$. However you choose to remember the relationships, we have the following.

Horizontal shifts, $c > 0$

From the graph of $y = f(x)$, the graph of

$y = f(x + c)$ is *shifted left c units,*
$y = f(x - c)$ is *shifted right c units.*

Reflections

Comparing the graphs of $y = f(x)$ and $y = -f(x)$, it is clear that for any point (x, y) on the graph of $y = f(x)$, the point $(x, -y)$ belongs to the graph of $y = -f(x)$. That is, the graph of $y = -f(x)$ is obtained from the graph of $y = f(x)$ by “tipping it upside down,” or, in more mathematical terms, “reflecting in the x-axis.” Since multiplying a function by -1 reflects the graph vertically, we would expect multiplication of the argument by -1 to reflect the graph horizontally, as the next example shows.

▶EXAMPLE 3 ***Horizontal and vertical reflections*** Sketch graphs of

$$y = \sqrt{x}, \qquad y = -\sqrt{x}, \qquad y = \sqrt{-x}.$$

Solution

With a graphing calculator we see essentially the graphs shown in Figure 13. The graph of $y = \sqrt{x}$ is the top half of a parabola. More important at the moment are the relations with the other graphs. From the graph $y = \sqrt{x}$, the graph of $y = -\sqrt{x}$ is a reflection in the x-axis, while the graph of $y = \sqrt{-x}$ is a reflection in the y-axis, as expected. ◀

Horizontal and vertical reflections

From the graph of $y = f(x)$, the graph of

$y = -f(x)$ is *a vertical reflection (in the x-axis),*
$y = f(-x)$ is *a horizontal reflection (in the y-axis).*

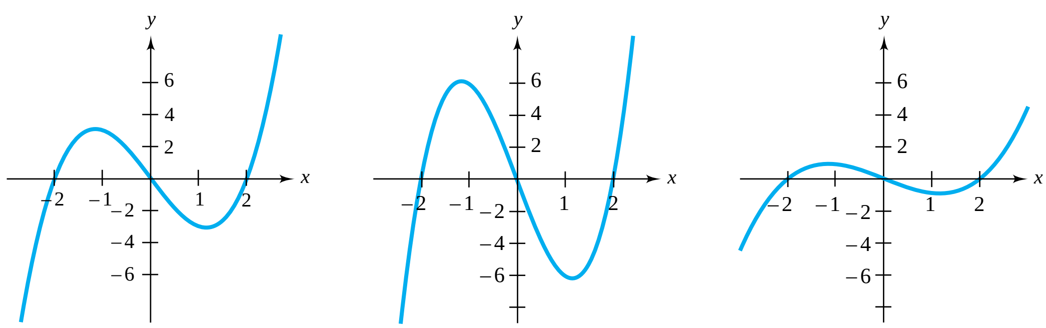
Dilations: Stretching and Compressing Graphs

Multiplying a function by a constant greater than 1 has the effect of *stretching the graph vertically:* if the point (x, y) belongs to the graph of $y = f(x)$, then the point (x, cy) is on the graph of $y = cf(x)$. If the positive constant c is smaller than 1, then the number cy is smaller than y, so the graph of $y = cf(x)$ is a *vertical compression toward the x-axis.* In a similar fashion, it can be seen that multiplying the argument has the effect of compressing or stretching the graph horizontally, toward the y-axis. A stretching or compression is called a **dilation** of the graph.

►EXAMPLE 4 Vertical dilations For the function $f(x) = x^3 - 4x$, describe how the graphs of $y = 2f(x)$ and $y = 0.3f(x)$ are related to the graph of $y = f(x)$.

Solution

Using a graphing calculator for $y = x^3 - 4x$, we get the graph shown in Figure 14(a), with x-intercept points where $x = -2, 0, 2$. Tracing along the curve, we see that the left "hump" is just a little higher than 3, where $x \approx -1.2$, and the low point is located symmetrically through the origin (the graph is clearly the graph of an odd function). For the graphs of the other two, the shape is similar, and the x-intercept points are the same, but the graph of $y = 2(x^3 - 4x)$ rises to a left hump well above 6, and the low point is below -6, *twice as far away from the x-axis* as the graph of $y = f(x)$. The graph of $y = 0.3(x^3 - 4x)$ is "squashed" vertically toward the x-axis, and the high and low points we see are less than 1 unit away from the axis. ◄



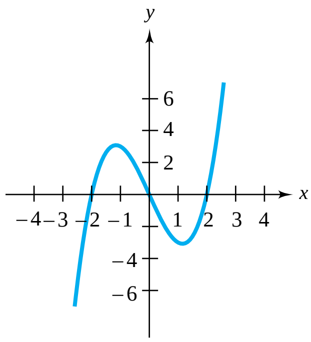
(a) $y = (x^3 - 4x)$ **(b)** $y = 2(x^3 - 4x)$ **(c)** $y = (0.3)(x^3 - 4x)$

FIGURE 14
Vertical dilations of $f(x) = x^3 - 4x$

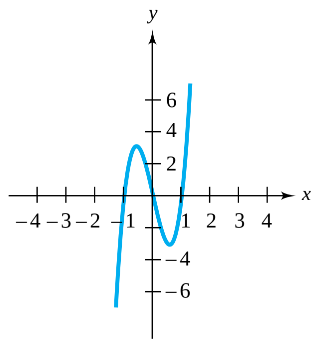
►EXAMPLE 5 Horizontal dilations For the function $f(x) = x^3 - 4x$, sketch graphs of $y = f(x)$, $y = f(2x)$, and $y = f(0.5x)$.

Solution

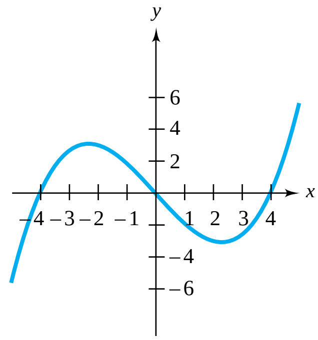
The graph of $y = f(x)$ is the one from the previous example, and is shown again in Figure 15(a). For the graph of $y = f(2x)$, we must replace each x by $2x$, so we enter `Y = (2X)^3 − 4(2X)`, and similarly for $y = f(0.5x)$.



(a) $y = x^3 - 4x$



(b) $y = (2x)^3 - 4(2x)$



(c) $y = (0.5x)^3 - 4(0.5)x$

FIGURE 15
Horizontal dilations of $f(x) = x^3 - 4x$

The graph of $y = f(2x)$ has the same vertical rise and fall to the turning points, but the x-intercepts have been squeezed together; each is *twice as close to the origin* as for $y = f(x)$. The graph of $y = f(0.5x)$ is stretched horizontally. The x-intercept point that was at (2, 0) has been moved outward to where $x = 2/0.5 = 4$; the x-intercept points are $(\pm 4, 0)$. ◀

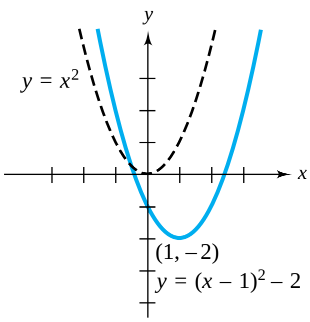


FIGURE 16
Translation of $f(x) = x^2$

Dilations, $c > 0$

From the graph of $y = f(x)$, the graph of

$y = cf(x)$ is a *vertical stretch if $c > 1$ (by a factor of c), vertical compression if $c < 1$ (by a factor of c);*

$y = f(cx)$ is a *horizontal compression if $c > 1$, horizontal stretch if $c < 1$.*

Combining Transformations

All the transformations we have considered can be combined, and if we are careful, we can predict the effect of several transformations on a graph of a function. In most instances, we take the operations "from the inside out," looking first at anything that affects the argument of the function.

▶EXAMPLE 6 ***Vertical and horizontal shifts*** Predict the effect on the graph of the function $f(x) = x^2$ in graphing

(a) $y = f(x - 1) - 2$ **(b)** $y = -f(x + 2)$.

Then check your prediction with a calculator graph.

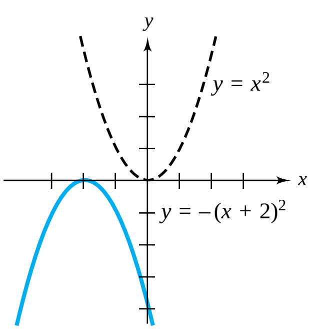


FIGURE 17
Translation and reflection of $f(x) = x^2$

Solution

(a) $y = f(x - 1) - 2 = (x - 1)^2 - 2$. From a parabola $y = x^2$, the graph of $y = (x - 1)^2$ is a shift 1 unit right. Then for $y = (x - 1)^2 - 2$, shift the graph down 2 units. The result is a parabola whose low point is at $(1, -2)$. A calculator graph shows the solid parabola in Figure 16.

(b) The graph of $y = (x + 2)^2$ is a parabola shifted 2 units left. Then multiplying by -1 reflects the graph in the x-axis, tipping it upside down. We have the solid parabola opening downward in Figure 17. ◀

▶EXAMPLE 7 ***Identifying transformed graphs*** The graph of a function f is given, together with three transformed graphs. Describe the transformations needed to get the given graph and write an equation for the function whose graph is shown. Check by graphing your function.

(a) $f(x) = |x| = abs(x)$ (Figure 18) **(b)** $f(x) = \sqrt{x}$ (Figure 19)

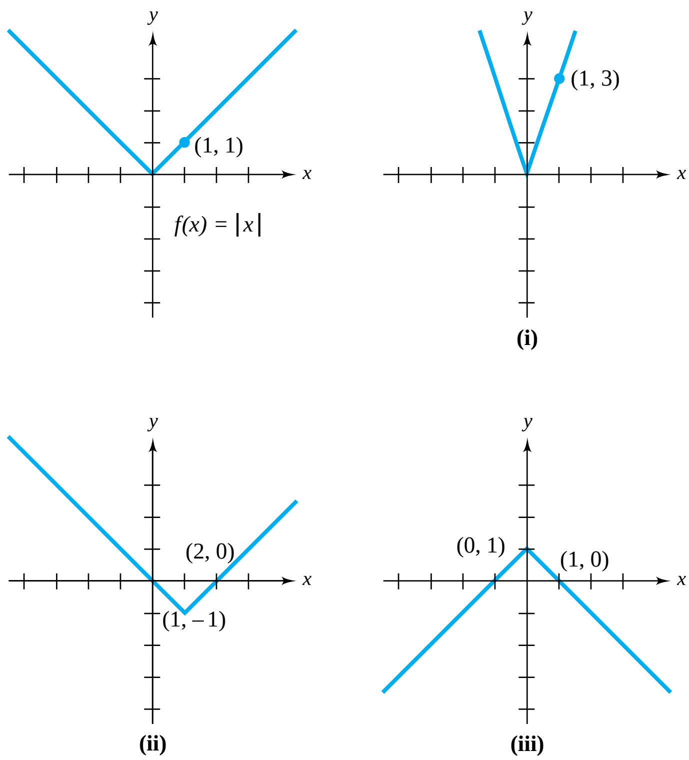


FIGURE 18
Graphs for Example 7a

Solution

(a) The graph in Figure 18(i) is a vertical stretch by a factor of 3, since (1, 1) is sent to (1, 3), so an equation for the transformed graph is $y = 3|x|$. If we use a decimal window, we can trace on the graph of $y = 3|x|$ to see that (1, 3) is on our graph, as desired. We note that in this instance, we could just as easily have obtained the transformed graph by compressing toward the y-axis, for which an equation would be $y = |3x|$. Since $3|x| = |3x|$, the function can be described either way.

For the graph in Figure 18(ii), the absolute value graph is shifted 1 unit right (replace the argument x by $x - 1$), and 1 unit down. An equation is $y = |x - 1| - 1$, which we graph to check.

In Figure 18(iii) the graph is tipped upside down (reflected in the x-axis) and shifted up 1 unit. An equation is $y = -|x| + 1$.

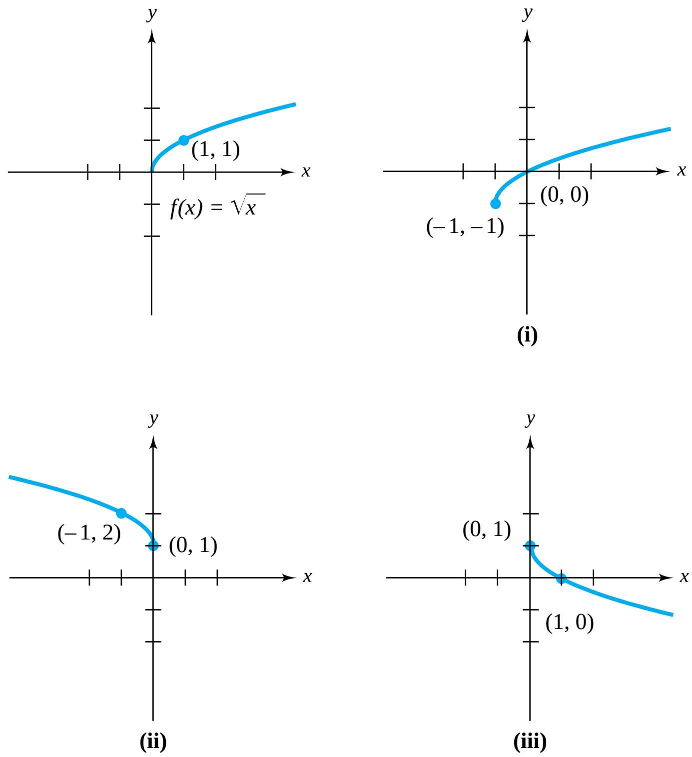


FIGURE 19
Graphs for Example 7b

(b) For panel (i) in Figure 19, shift the graph of $y = \sqrt{x}$ left 1 unit and down 1 unit, so an equation is $y = \sqrt{x + 1} - 1$. We trace on the graph to verify the location of the given points.

For the graph in panel (ii), reflect the graph of $y = \sqrt{x}$ in the y-axis, (replace x by $-x$), and shift up 1 unit. An equation is $y = \sqrt{-x} + 1$.

For the third panel, reflect the graph of $y = \sqrt{x}$ in the x-axis and shift up 1 unit, so an equation is $y = -\sqrt{x} + 1$. Verify by graphing this equation. ◀

Summary of Basic Transformations

We list here the basic transformations we have introduced in this section.

Basic transformations of the graph of $y = f(x)$, $c > 0$

The transformations that affect a graph vertically are applied "outside" the function; transformations that change horizontal aspects are applied "inside" the function (to the argument).

Vertical		*Horizontal*	
$y = f(x) + c$,	shift up	$y = f(x + c)$,	shift left
$y = f(x) - c$,	shift down	$y = f(x - c)$,	shift right
$y = -f(x)$,	reflect in x-axis	$y = f(-x)$,	reflect in y-axis
$y = cf(x)$,	dilate vertically	$y = f(cx)$,	dilate horizontally

EXERCISES 2.3

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. If the function f has a positive zero and $g(x) = f(x - 2)$, then g must have a positive zero.
2. If the function f has a zero between 1 and 2 and $g(x) = f(x + 2)$, then g must have a negative zero.
3. If the graphs of $y = f(x)$ and $y = g(x)$ intersect in Quadrants I and III, then the graphs of $y = f(-x)$ and $y = g(-x)$ must intersect in Quadrants II and IV.
4. If the graphs of $y = f(x)$ and $y = g(x)$ intersect in Quadrants II and IV, then the graphs of $y = -f(x)$ and $y = -g(x)$ must intersect in Quadrants I and III.
5. If the graph of $y = f(x)$ contains points in Quadrants III and IV, then the graph of $y = f(x) - 2$ must also contain points in Quadrants III and IV.

Exercises 6–10 Fill in the blank so that the resulting statement is true. If calculator graphs of f and g are drawn using $[-10, 10] \times [-10, 10]$, *then the display will show the graphs intersecting in Quadrant(s)* ______.

6. $f(x) = x^2 - 2x - 7$, $g(x) = -f(x) - 5$
7. $f(x) = x^2 - 4x - 4$, $g(x) = f(x - 4)$
8. $f(x) = x^2 - 2|x| - 3$, $g(x) = -f(x) + 3$
9. $f(x) = 2x - 5$, $g(x) = f(-x) + 15$
10. $f(x) = |x| - 2$, $g(x) = -f(x) + 3$

Develop Mastery

Exercises 1–6 ***Related Graphs*** *The graph of a function f contains the points* $P(-2, 4)$ *and* $Q(4, -5)$. *Give the coordinates of two points on the graph of the function* ***(a)*** *g,* ***(b)*** *h.*

1. $g(x) = f(x - 1)$; $h(x) = f(x + 2)$
2. $g(x) = f(x) - 3$; $h(x) = f(x) + 4$
3. $g(x) = f(2x)$; $h(x) = f(0.5x)$
4. $g(x) = -f(x)$; $h(x) = f(-x)$
5. $g(x) = f(x - 2) + 3$; $h(x) = 4 - f(x)$
6. $g(x) = -f(-x)$; $h(x) = 1 + f(-x)$

Exercises 7–12 ***Related Graphs*** *For the function* $f(x) = x^2 - x - 2$

(a) *Determine a formula for g and simplify.*
(b) *Draw calculator graphs of f and g on the same screen.*
(c) *Write a brief statement describing how the graphs of f and g are related.*

7. $g(x) = f(x + 2)$
8. $g(x) = f(-x)$
9. $g(x) = f(x - 3)$
10. $g(x) = f(x) + 2$
11. $g(x) = f(-x) + 2$
12. $g(x) = -f(x + 3)$

Exercises 13–14 ***Related Line Graphs*** *The graph of f is a line through points P and Q. Draw graphs of the given function. It is not necessary to find an equation for any of the functions before drawing a graph.*
(a) $y = f(x)$ ***(b)*** $y = f(x - 2)$ ***(c)*** $y = f(-x)$.

13. $P(1, -3)$ and $Q(3, 1)$
14. $P(-3, 4)$ and $Q(1, -2)$

Exercises 15–16 ***Line Segment Graphs*** *The graph of f is a line segment joining P and Q. Draw a graph and give the domain and range of*
(a) $y = f(x)$ ***(b)*** $y = f(x - 3)$ ***(c)*** $y = -f(x)$.

15. $P(2, -2)$ and $Q(4, 2)$
16. $P(-2, 3)$ and $Q(0, -3)$

Exercises 17–18 ***Line Segment Graphs*** *The graph of f is two line segments PQ and QR. Draw a graph and give the domain and range of* ***(a)*** $y = f(x + 2)$ ***(b)*** $y = f(-x)$ ***(c)*** $y = -f(x)$.

17. $P(-3, -1)$, $Q(-2, 2)$, and $R(3, 0)$
18. $P(-4, -2)$, $Q(-1, 3)$, and $R(4, -1)$

Exercises 19–22 ***Verbal Description*** *Give a verbal description of how you would draw a graph of g from the graph of f. Check by drawing the graphs.*

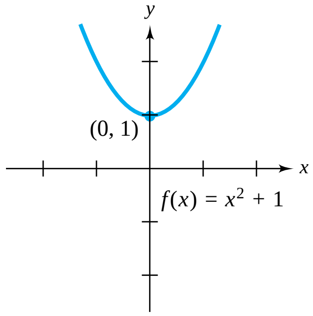
19. $f(x) = x^2 + 1$, $g(x) = (x + 2)^2 - 1$
20. $f(x) = \sqrt{x}$, $g(x) = -\sqrt{x + 1}$
21. $f(x) = x^2 - 3x$, $g(x) = 2(3x - x^2) + 1$
22. $f(x) = |x|$, $g(x) = |0.5x| + 2$

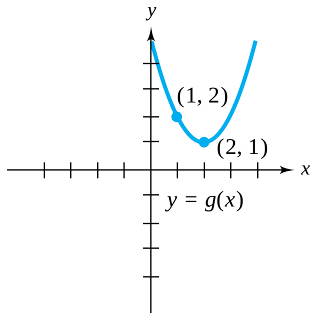
Exercises 23–26 ***Verbal to Formula*** *A verbal description of transformations of the graph of* $y = f(x)$ *is given, resulting in a graph of function g. Give a formula that describes the function g. Confirm by drawing graphs of f and g on the same screen.*

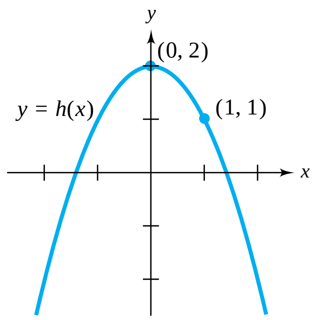
23. $f(x) = x^2 - 2x$. Translate the graph of f to the left 2 units and then reflect about the x-axis.
24. $f(x) = 2x - 4$. Translate the graph of f to the left 3 units and then reflect about the y-axis.
25. $f(x) = x^2 + 1$. Stretch the graph of f vertically upward by a factor of 2, then translate downward 3 units.
26. $f(x) = x^2 + 1$. Compress the graph of f vertically downward by a factor of 0.5, then reflect about the y-axis.

Exercises 27–30 ***Graph to Verbal and Formula*** *The graph of function f is shown along with graphs of transformed functions g and h.* ***(a)*** *Give a verbal description of the transformations that will give the graphs of g and h from the graph of f.* ***(b)*** *Give a formula for g. Do the same for h.* ***(c)*** *As a check, draw graphs of your formulas in part* ***(b)*** *and see if they agree with the given graphs.*

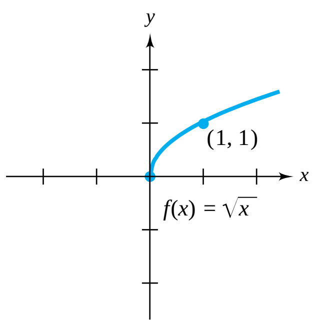
27.

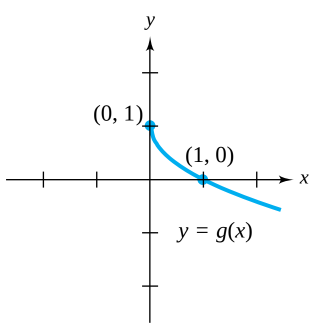

y
x
(0, 1)
f(x) = x² + 1

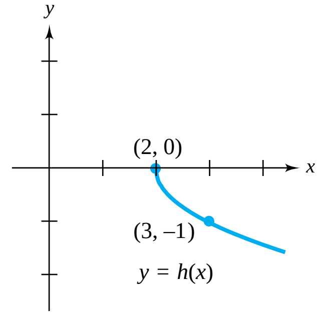

y
x
(1, 2)
(2, 1)
y = g(x)


y
x
(0, 2)
y = h(x)
(1, 1)

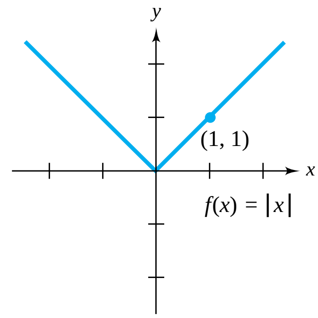
28.

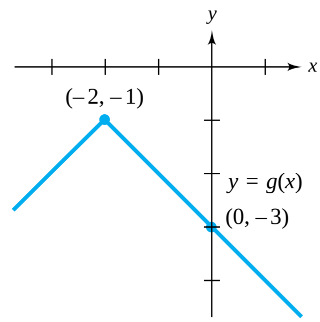

y
x
(1, 1)
f(x) = √x

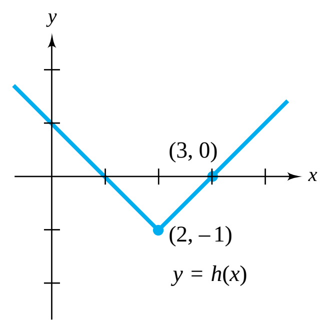

y
x
(0, 1)
(1, 0)
y = g(x)


y
x
(2, 0)
(3, –1)
y = h(x)

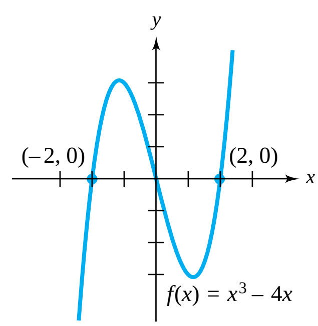
29.

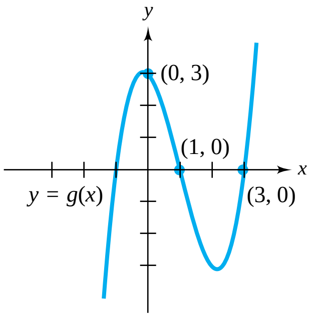

y
x
(1, 1)
f(x) = |x|

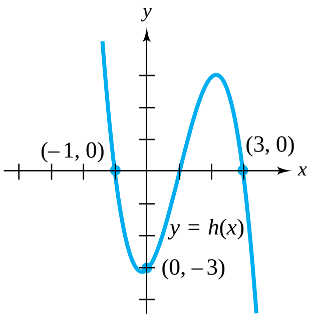

y
x
(–2, –1)
y = g(x)
(0, –3)


y
x
(3, 0)
(2, –1)
y = h(x)

30.


y
x
(–2, 0)
(2, 0)
f(x) = x³ – 4x





Exercises 31–34 ***Domain and Range*** *The domain D and range R of function f are given in interval notation. Give the domain and range of the function **(a)** g, **(b)** h.*

31. $D = [-2, 6]$, $R = [4, 8]$;
$g(x) = f(x - 1)$, $h(x) = f(x + 2)$

32. $D = (-4, 5)$, $R = [-3, 5]$;
$g(x) = f(x) + 2$, $h(x) = f(x) - 3$

33. $D = (-\infty, 4]$, $R = [-4, 6]$;
$g(x) = -f(x)$, $h(x) = f(-x)$

34. $D = [-2, \infty)$, $R = (-4, 8)$;
$g(x) = -f(x)$, $h(x) = f(-x)$

Exercises 35–38 ***Related Domain and Range*** *The domain D and range R of function g are given. Find the domain and range of function f.*

35. $g(x) = f(x + 2)$; $D = [-2, 4]$, $R = [-3, 5]$

36. $g(x) = f(x - 1) + 3$; $D = (-\infty, 4]$, $R = [-2, \infty)$

37. $g(x) = f(2x)$; $D = [3, 6]$, $R = [-1, 4]$

38. $g(x) = f(-x)$; $D = [-2, 8]$, $R = [4, \infty]$

Exercises 39–42 ***Related Intercept Points*** ***(a)*** *Determine the coordinates of the x-intercept points of the graph of function f.* ***(b)*** *Find the x-intercept point(s) of the graph of function g.* ***(c)*** *What is the y-intercept point for g?* ***(d)*** *Draw graphs as a check.*

39. $f(x) = x^2 - 3x - 4$; $g(x) = f(x - 3)$

40. $f(x) = 4 + 3x - x^2$; $g(x) = f(x + 2)$

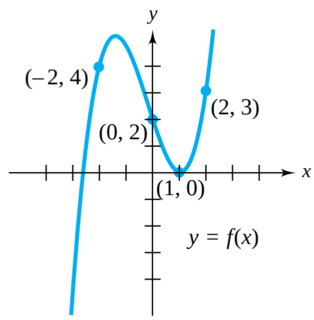
41. $f(x) = x^2 - 4$; $g(x) = f(2x)$

42. $f(x) = x^2 - 3x - 4$; $g(x) = f(2x)$

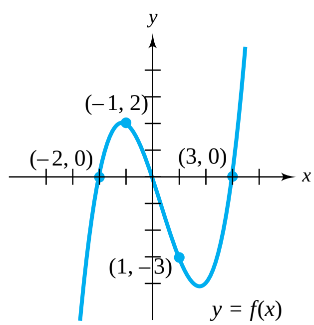
43. ***Explore*** For each number k draw a calculator graph of $y = k\sqrt{4 - x^2}$ on the same screen. From what you observe write a brief paragraph comparing the graphs for different values of k.
(a) $k = 1$ **(b)** $k = 2$ **(c)** $k = 0.5$

44. Follow instructions of Exercise 43 for $y = k\sqrt{16 - x^2}$.

45. Use the graph of f shown to sketch the graph of $y = -f(x - 1)$. Label the coordinates of four points that must be on your graph.



46. Use the graph of f shown to sketch the graph of $y = f(-x) + 1$. Label the coordinates of four points that must be on your graph.



47. ***Points on Related Graphs*** Points $P(-4, 3)$ and $Q(2.4, 5.6)$ are on the graph of $y = f(x - 1)$. Give the coordinates of two points that must be on the graph of **(a)** $y = f(x)$ **(b)** $y = f(x) + 3$.

2.4 LINEAR FUNCTIONS AND LINES

> The concept of linearity has played a central role in the development of models in all the sciences.
>
> *B. J. West*

> *And then [my father] showed me a proof of the Pythagorean theorem. He also taught me about Cartesian coordinates and showed me how to solve two linear equations by seeing where the lines intersect. And this seemed to me the most beautiful thing in the world.*
>
> Lipman Bers

Definition: linear function

A **linear function** is a function with an equation equivalent to

$$f(x) = ax + b, \tag{1}$$

where a *and* b are real numbers.

Unless there is some restriction, the domain of a linear function is R, the set of all real numbers. For a linear function we may write $y = ax + b$, or $ax - y + b = 0$. This is an equation that in Section 1.4 we called a **linear equation.** Recall from Section 1.4 that the graph of a linear equation is a line, so it follows that the graph of a linear function is a line as well. Further, since $f(0) = b$ and $f(1) = a + b$, the y-intercept point of a linear function is $(0, b)$, and the line contains the point $(1, a + b)$. We can use these two points to determine the slope m of the line:

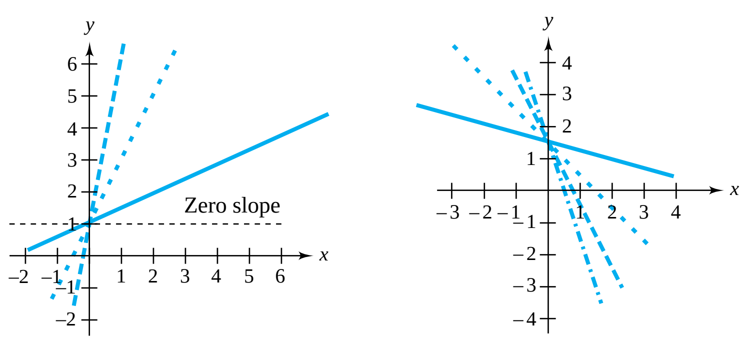
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(a + b) - b}{1 - 0} = a.$$

Thus we immediately have considerable information about the graph of Equation (1).

Graph of a linear function

The graph of the linear function $f(x) = ax + b$ is a line with slope a and y-intercept b.

If a line has positive slope, then the function value increases with x and the graph slants upward as it moves to the right (see Figure 20a). The graph of a constant function, $f(x) = b = 0x + b$, has slope 0; it consists of all points of the form (x, b) and hence is a horizontal line. A line with negative slope slants downward as it moves to the right, as in Figure 20b. A vertical line consists of all points that have the same x-coordinate, and hence can be described by an equation of the form $x = c$. The slope of a vertical line is undefined. By the vertical line test, a vertical line is not the graph of a function.



(a) Lines with nonnegative slope **(b)** Lines with negative slope

FIGURE 20

▶EXAMPLE 1 ***Points to graphs*** For the point $P(-1, 1)$, sketch the line through P and the given point. Find the slope.

(a) $A(2, -3)$ **(b)** $B(4, 1)$ **(c)** $C(0, 2)$ **(d)** $D(-1, 3)$

Solution

The lines are drawn in Figure 21. By the slope formula,

(a) $m_1 = \dfrac{-3 - 1}{2 + 1} = -\dfrac{4}{3}$ **(b)** $m_2 = \dfrac{1 - 1}{4 + 1} = 0$

(c) $m_3 = \dfrac{2 - 1}{0 + 1} = 1$ **(d)** $m_4 = \dfrac{3 - 1}{-1 + 1} = \dfrac{2}{0}$ (undefined slope)

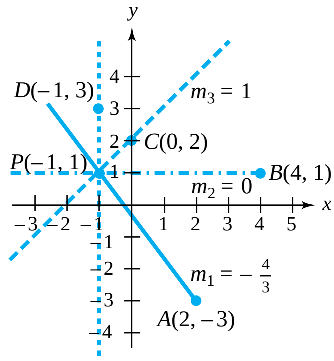


FIGURE 21

The figure illustrates lines with negative, positive, and zero slope, and a vertical line, for which no slope can be defined. ◀

Equations of a Line

We can easily draw the graph of a linear function by plotting two points, but we want to be able to find the linear function associated with a given line as well. There are several convenient forms for equations of lines.

Equation (1) identifies the slope and y-intercept for the line, and hence is called the **slope-intercept equation** for the line. A line is determined by either **(a)** a pair of points, or **(b)** a point and the slope. From the coordinates of two points that do not lie on a vertical line, we can immediately get the slope m:

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

so in either case **(a)** or case **(b)**, we can assume that we have the coordinates of a point on the line and its slope.

Let l be the line that contains the point $P(x_0, y_0)$ with slope m. Any other point $Q(x, y)$ belongs to line l if and only if the slope determined by P and Q is the number m (see Figure 22):

$$\frac{y - y_0}{x - x_0} = m, \quad \text{or, multiplying by } x - x_0,$$

$$y - y_0 = m(x - x_0). \tag{2}$$

Equation (2) is called the **point-slope equation** for the line l.

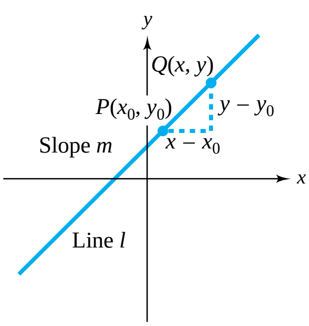


FIGURE 22
Q is on line l if and only if $\dfrac{y - y_0}{x - x_0} = m$.

A vertical line cannot be described by either the slope-intercept equation form or the point-slope form because a vertical line has an undefined slope. As we noted above, however, $x = x_0$ is an equation for the vertical line through the point $P(x_0, y_0)$.

Each of the equations above is equivalent to an equation of the form

$$Ax + By = C, \tag{3}$$

which is called a **standard equation** for a line. The box gives a summary.

Equations of a line

A nonvertical line l has an equation in any of the following forms:

1. Slope-intercept: $y = mx + b$, l has slope m *and* y-intercept b.
2. Point-slope: $y - y_0 = m(x - x_0)$, l has slope m and contains the point (x_0, y_0).
3. Standard: $Ax + By = C$, $B \neq 0$.

A vertical line has an equation of the form $x = c$.

Strategy: First determine the slope of the line through P and Q, then use the point-slope form, taking either P or Q for (x_0, y_0).

▶EXAMPLE 2 Equation from two points Find an equation for the line l that contains points $P(-1, 2)$, and $Q(3, 4)$.

Solution

Following the strategy,

$$m = \frac{4 - 2}{3 - (-1)} = \frac{1}{2}.$$

Using the point-slope form with point P, l has equation

$$y - 2 = \frac{1}{2}[x - (-1)] \quad \text{or} \quad y = \frac{1}{2}x + \frac{5}{2}.$$

The last equation is in slope-intercept form, and we can obtain a standard form equation for l by multiplying by 2 and rearranging terms: $x - 2y = -5$. ◀

Strategy: **(a)** First find the slope-intercept form.

▶EXAMPLE 3 Finding slope and intercepts An equation for line l is $3x + 2y = 6$. Find **(a)** the slope of l, **(b)** the intercept points.

Solution

(a) To write the equation for l in slope-intercept form, solve for y:

$$y = -\frac{3}{2}x + 3$$

This indicates that l has slope $-\frac{3}{2}$ and y-intercept point $(0, 3)$.

(b) To find the x-intercept point, substitute 0 for y and solve for x. This gives x as 2, so the x-intercept point is $(2, 0)$. ◀

Parallel Lines

Lines that do not intersect are **parallel.** Given a nonvertical line l, any vertical translation of l is parallel to l, and conversely, every line parallel to l can be obtained from l by vertical translation, up or down.

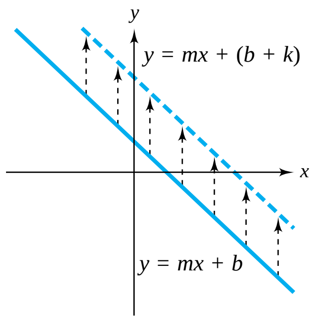


FIGURE 23
A vertical translation of a nonvertical line gives a parallel line.

A line with slope m is a graph of a linear function

$$f(x) = mx + b,$$

and any vertical translation of f has the form

$$f(x) + k = (mx + b) + k = mx + (b + k),$$

which is still an equation of a line with slope m (see Figure 23). It is also true that all vertical lines are parallel to each other.

Parallel lines

Two lines are parallel if and only if their slopes are equal, or both lines are vertical.

Perpendicular Lines

Slopes give a convenient way to tell when lines are perpendicular (intersect at right angles). To make the relationship easier to see, we use two lines that intersect at the origin, $L_1 : y = m_1 x$, and $L_2 : y = m_2 x$. The same argument could be applied at the intersection point of any two lines.

Moving 1 unit right from the origin O, we have points $A(1, m_1)$ and $B(1, m_2)$ on the two lines (see Figure 24). The two lines are perpendicular to each other if and only if $\triangle OAB$ is a right triangle, or, by the Pythagorean theorem, if and only if $a^2 + b^2 = c^2$, where a, b and c are as labeled in the diagram. By the distance formula,

$$\begin{aligned} a^2 + b^2 &= 1 + m_1^2 + 1 + m_2^2 = 2 + m_1^2 + m_2^2, \text{ and} \\ c^2 &= (m_1 - m_2)^2 = m_1^2 - 2m_1m_2 + m_2^2. \end{aligned}$$

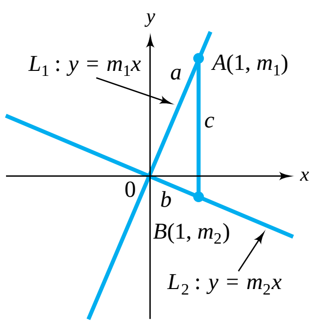


FIGURE 24
Perpendicular lines have slopes whose product is -1.

Setting these values equal and simplifying, lines L_1 and L_2 are perpendicular if and only if

$$2 + m_1^2 + m_2^2 = m_1^2 - 2m_1m_2 + m_2^2, \quad \text{or} \quad m_1m_2 = -1.$$

If either l_1 or l_2 has no slope, then one must be vertical; lines that are perpendicular to vertical lines are horizontal.

Perpendicular lines

Two lines are perpendicular if and only if the product of their slopes is -1, or one line is vertical and the other is horizontal.

▶EXAMPLE 4 ***Parallel lines*** Find an equation for the line L that passes through the point $(1, -3)$ and is parallel to the line $3x - y = 5$.

Solution

To find the slope of the given line, solve for y to get $y = 3x - 5$. The slope of the given line is 3, the coefficient of x. Since L is parallel to the given line, its slope is also 3. Hence L has slope 3 and passes through $(1, -3)$. Substituting into the point slope form gives

$$y - (-3) = 3(x - 1).$$

Solving for y and simplifying gives the slope-intercept form for L:

$$y = 3x - 6. \quad ◀$$

►EXAMPLE 5 ***Perpendicular bisector*** Find an equation for the line L that is the perpendicular bisector of the line segment $\overline{AB}$, for the points $A(-1, 4)$, and $B(3, 2)$.

Strategy: First find the coordinates of the midpoint M or $\overline{AB}$ and the slope m of the line through A and B. The perpendicular bisector is the line with slope $\frac{-1}{m}$ that contains M.

Solution

Follow the strategy. The coordinates of the midpoint M of $\overline{AB}$ are given by

$$x = \frac{-1 + 3}{2} = 1 \quad \text{and} \quad y = \frac{4 + 2}{2} = 3.$$

Hence M is the point $(1, 3)$. The slope of the line through A and B is given by

$$m = \frac{4 - 2}{-1 - 3} = -\frac{1}{2}.$$

Therefore the slope of L is 2, the negative reciprocal of $-\frac{1}{2}$. L is the line that passes through $(1, 3)$ with slope 2. Substituting into the point-slope form gives

$$y - 3 = 2(x - 1) \quad \text{or} \quad y = 2x + 1. \quad ◄$$

Lines and Circles

In geometry some lines have significant relationships with certain circles. A line that meets a circle in just one point is called a **tangent line** and the point of intersection is a **tangent point.** Recall also from geometry that the line that contains the center of the circle and the tangent point is perpendicular to the tangent line, as illustrated in the next example.

►EXAMPLE 6 ***Lines related to circles*** A circle with center C has the equation $(x + 1)^2 + (y - 2)^2 = 25$.

Strategy: **(a)** Show that the coordinates of P satisfy the equation of the circle.
(c) The slope of l_2. is $-\frac{1}{m_1}$.
(d) Substitute the value of y from the line into equation of circle, and solve for x.

(a) Show that point $P(3, 5)$ is on the circle.
(b) Find the slope m_1 and equation for the line l_1 that contains points P and C.
(c) Find an equation for the line l_2 that is perpendicular to l_1 and contains P.
(d) Show that the line l_2 intersects the circle at only one point.

Solution

Follow the strategy. It is always a good idea to draw a sketch to help visualize what is needed. The center of the circle is at $(-1, 2)$ and the radius is 5. See Figure 25.

(a) Substituting 3 for x and 5 for y, $(3 + 1)^2 + (5 - 2)^2 = 16 + 9 = 25$, so P is on the circle.

(b) $m_1 = \frac{5 - 2}{3 + 1} = \frac{3}{4}$. Using P as the point and $\frac{3}{4}$ as slope, l_1 has the equation

$$y - 5 = \frac{3}{4}(x - 3) \quad \text{or} \quad y = \frac{3}{4}x + \frac{11}{4}.$$

(c) As in the strategy, $m_2 = \frac{-4}{3}$, so using P again, line l_2 has equation

$$y - 5 = -\frac{4}{3}(x - 3) \quad \text{or} \quad y = -\frac{4}{3}x + 9.$$

(d) Substitute $\frac{-4}{3}x + 9$ for y in the equation of the circle, and simplify (check our algebra).

$$(x + 1)^2 + \left[\left(-\frac{4}{3}x + 9\right) - 2\right]^2 = 25 \quad \text{or} \quad x^2 - 6x + 9 = 0.$$

The only root of the last equation is 3, from which y is 5. The line l_2 intersects the circle only in the point $P(3, 5)$. ◄

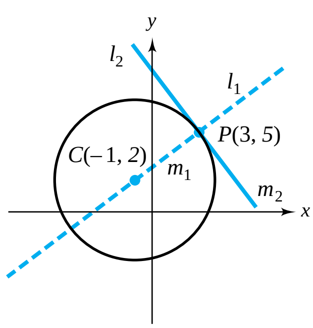


FIGURE 25

When we use a graphing calculator to graph circles and perpendicular lines, as in Figure 25, we need to keep in mind the limitations imposed by windows and pixel coordinates. Repeating our observations from Section 1.5, without an equal scale window, perpendicular lines do not appear perpendicular, and unless the window has pixel columns with x-coordinates for the right- and left-most points of a circle, the calculator graph of the circle will not close up.

TECHNOLOGY TIP ◆ *Lines and circles*

To graph a nonvertical line on a graphing calculator, we must solve the equation for y, so that we graph `Y = AX + B`.

For a circle as in Example 6, solving for y yields *two* functions. If $(x+1)^2+(y-2)^2=25$, then $(y-2)^2=25-(x+1)^2$, and $y-2=\pm\sqrt{25-(x+1)^2}$. We graph `Y1 = 2 + √(25 − (X + 1)²)`, and `Y2 = 2 − √(25 − (X + 1)²)`.

Linear Depreciation

Linear functions are useful models for many real-world phenomena. For instance, the value of business equipment decreases over time. The loss in value is called **depreciation** and is considered a tax-deductible business expense. A standard model of depreciation considers it a linear function of time. Suppose that some office furniture costs \$1000 and the time allowed for complete depreciation is ten years; each year it loses one-tenth of its original value. For tax purposes, the value of the furniture four years after its purchase is

$$V(4) = 1000 - 4\left(\frac{1}{10}\cdot 1000\right) = 600.$$

Linear depreciation formula

Let C denote the original value of a piece of equipment. If the equipment depreciates linearly over a period of n years, its value $V(t)$ after t years is described by the equation

$$V(t) = C - \frac{t}{n}C = C\left(1-\frac{t}{n}\right).$$

►*EXAMPLE 7* ***Linear depreciation*** The manager of a fish cannery has a large processing machine installed that costs \$180,000. Assuming the machine depreciates linearly over 30 years, find its tax value after seven years.

Solution

Using the linear depreciation model with $C = 180{,}000$ and $n = 30$, $V(t) = 180{,}000\left(1-\frac{t}{30}\right)$. When $t = 7$, the value is

$$V(7) = 180{,}000\left(1-\frac{7}{30}\right) = 138{,}000.$$

After seven years the machine is valued at \$138,000 (for tax purposes only; the real value to the cannery or for resale purposes may be quite different). ◄

The kinds of lines we have used in our examples are usually defined by equations with integer coefficients or, at worst, fractions. When dealing with measurements or real-world data, however, the numbers are seldom so "nice." For example, suppose that a small business has been assessed taxes as follows over a six year period:

Year	1990	1991	1992	1993	1994	1995
Tax	\$3712	\$4381	\$5045	\$5730	\$6392	\$7058

In addition to noting that taxes always increase, we may observe that there is a distressing regularity in the increase. The increases are as follows:

\$669 \$664 \$685 \$662 \$666.

If the increase were always the same, say \$670, then we could anticipate the change by a linear function, $T = 3712 + 670x$, where x is the number of years after 1990. These data are not precisely linear, but the linear function that most nearly fits the given data is approximately

$$T = 3711.476 + 669.943x.$$

EXERCISES 2.4

Check Your Understanding

Exercises 1–4 True or False. Give reasons.

1. If $f(x) = 2|x| - 3$, then f is a linear function in x.

2. The slope of the line $3x - 6y = 2$ is 2.

3. The point $(2, -4)$ is on both the lines $2x + y = 0$ and $3x - y = 10$.

4. There is no linear function whose graph contains points in quadrants QI and QIII, but no points in QII or QIV.

Exercises 5–6 Fill in the blank so that the resulting statement is true.

5. Any line that has negative slope must contain points in Quadrant(s) ______.

6. Every line with a positive slope must contain points in Quadrant(s) ______.

Exercises 7–10 Use your calculator to draw graphs of the equations on the same screen. The graphs intersect in Quadrant(s)______.

7. $3x - 4y = 12, \quad 4x + 3y = 7$

8. $x - y + 3 = 0, \quad 2x + y + 2 = 0$

9. $2x - y - 1 = 0, \quad 1.5x + y = 6, \quad 0.8x + y = 4$

10. $1.5x - y + 8.5 = 0, \quad 1.5x + y + 0.5 = 0, \quad 0.4x + y + 5 = 0$

Develop Mastery

Exercises 1–6 Find the slope of the line that contains the two points.

1. $P(-2, 4)$, $Q(0, 1)$

2. $P(3, 5)$, $Q(-4, 1)$

3. $A(-4, -3)$, $B(5, 2)$

4. $A(2, -3)$, $B(4, -5)$

5. $C(1, -2)$, $D(4, -2)$

6. $C(-3, 4)$, $D(-3, 1)$

Exercises 7–12 Find an equation in slope-intercept form for the line with slope m that contains P.

7. $P(-3, 4)$; $m = -2$

8. $P(1, 0)$; $m = -1$

9. $P(0, 0)$; $m = -\dfrac{2}{3}$

10. $P\left(-\dfrac{1}{2}, \dfrac{5}{4}\right)$; $m = 3$

11. $P\left(\dfrac{3}{4}, -\dfrac{3}{2}\right)$; $m = 0$

12. $P(\sqrt{3}, 1)$; $m = \sqrt{3}$

Exercises 13–18 Find the slope and intercept points.

13. $3x + 2y = 6$

14. $x - 3y = 3$

15. $2x + y = 4$

16. $3x - 3y = 4$

17. $6x + 2y = 12$

18. $2x - 2y = 4$

Exercises 19–24 ***(a)*** *Draw a graph showing P and the line l whose equation is given.* ***(b)*** *Find an equation for the line l_1 that contains P and is parallel to l.* ***(c)*** *Find an equation for the line l_2 that contains P and is perpendicular to l.*

19. $P(-1, 3)$; $x - 2y = 4$

20. $P(0, 0)$; $3x - 4y = 6$

21. $P(0, -2)$; $2x + 3y - 5 = 0$

22. $P(-2, 4)$; $3x - 4y = 8$

23. $P(-1, 2)$; $y + 4 = 0$

24. $P(2, 4)$; $x + 3 = 0$

Exercises 25–28 ***Collinear Points*** *Determine whether or not points A, B, and C are collinear (lie on the same line). (Hint: Consider slopes.)*

25. $A(0, 0)$, $B(1, 2)$, $C(-3, -5)$

26. $A(2, -2)$, $B(5, 2)$, $C(-1, -6)$

27. $A(0, -4)$, $B(6, 0)$, $C(3, -2)$

28. $A(0, 2)$, $B(4, 0)$, $C(5, -1)$

Exercises 29–32 Determine the quadrants through which the line passes. (Hint: Draw a graph.)

29. $2x - 3y = 6$ **30.** $x + 3y = 4$

31. $3x + 2y = 0$ **32.** $x - 2y = 0$

Exercises 33–34 ***Your Choice***

33. Draw a graph of a linear function that contains no points in
(a) QII **(b)** QIII **(c)** QII or QIV.

34. Write an equation for a linear function whose graph contains no points in **(a)** QII **(b)** QI or QIII.

Exercises 35–38 ***Midpoint*** ***(a)*** *Draw a diagram showing points P and Q, and find the midpoint M of line segment $\overline{PQ}$.* ***(b)*** *Find an equation for the perpendicular bisector of segment $\overline{PQ}$ and draw it on your diagram.*

35. $P(-2, 3)$, $Q(2, 5)$ **36.** $P(1, 4)$, $Q(-3, 2)$

37. $P(3, 0)$, $Q(-3, 4)$ **38.** $P(-3, -1)$, $Q(1, 4)$

Exercises 39–42 Let $P(x, y)$ be any point that is equidistant from points A and B. Find an equation that must be satisfied by x and y. (Hint: Draw a diagram.)

39. $A(1, -3)$, $B(3, 5)$ **40.** $A(-2, 3)$, $B(4, -1)$

41. $A(0, 0)$, $B(-2, 6)$ **42.** $A(1, -4)$, $B(1, 2)$

Exercises 43–44 For the three given values of $f(x)$, can f possibly be a linear function? Explain.

43. $f(-1) = -2, f(0) = 0, f(3) = 6$

44. $f(-1) = -5, f(\frac{1}{2}) = -2, f(3) = 2$

Exercises 45–46 For the three given values of $f(x)$, can the graph of $y = f(x)$ be a line? Explain.

45. $f(1) = 1, f(3) = 5, f(-2) = -4$

46. $f(-1) = 5, f(0) = 3, f(3) = -3$

47. If k is a positive number and the line $x + y = k$ is tangent to the circle $x^2 + y^2 = k$, then find k.

48. **(a)** Show that the three points $A(8, 3)$, $B(4, 10)$, and $C(2, 6)$ are the vertices of a right triangle.
(b) Find the equation of the circle that contains points A, B, and C. (*Hint:* Use a theorem from geometry concerning a right triangle inscribed in a circle.)

49. Repeat Exercise 48 for three points $A(6, 6)$, $B(8, 8)$ and $C(0, 12)$.

50. **(a)** Is point $P(5, 6)$ inside, outside, or on the circle $x^2 + y^2 - 2x - 6y - 15 = 0$?
(b) If P is inside, find the distance between the x-intercept points; if P is outside, find the distance between the y-intercept points; if P is on the circle, find an equation for the line passing through P that is tangent to the circle.

51. ***Car Rental*** The cost of renting a car is \$15 a day, plus 20¢ per mile.
(a) If you rent a car for four days and drive x miles, express the total cost C as a function of x.
(b) If you cannot afford more than \$100 for your four days, how many miles can you drive?

52. ***Related Temperatures*** Assume that the Celsius and Fahrenheit temperature scales have a linear relationship and that $C = 0$ when $F = 32$ and $C = 100$ when $F = 212$.
(a) Express C as a linear function of F. (*Hint:* Let $C = aF + b$, and find values a and b.)
(b) Express F as a linear function of C.

53. ***Manufacturer Profit*** A firm that manufactures calculators has a fixed daily cost for salaries and plant operation of \$1200; in addition, it costs \$10 to produce each calculator.
(a) Find an equation for the total daily cost C as a function of the daily production of x calculators.
(b) If the wholesale price of a calculator is \$16, express the total daily revenue R as a function of x.
(c) If the daily profit P equals $R - C$, express P as a function of x. For what values of x is $P > 0$?

54. ***Speeds*** You are traveling at a speed of 88 feet per second (60 mph) along a highway that runs parallel to railroad tracks on which a train is traveling in the same direction at 73 feet per second (≈50 mph).
(a) If your car is 20 feet long and the length of the train is x feet, express the time T that it takes to pass the train as a function of x.
(b) If the train is 400 feet long, how many seconds does it take the car to pass it?

55. Find b such that the graphs of $x + 2y = 3$ and $bx - 2y + 5 = 0$ intersect at right angles. Check graphically using a decimal window.

56. ***Depreciation*** After being depreciated in a linear fashion for four years of a 12 year depreciation schedule, a car is valued at \$5760. What was the initial cost?

57. ***Depreciation*** In starting a new business the manager has the office equipped with new furniture that costs \$80,000. Assuming linear depreciation over a 20 year period,
(a) what is the value of the furniture after 4 years?
(b) how much can the firm deduct for tax purposes at the end of the first year?

58. If $f(x) = 2x + 3$ and the domain D of f is given by $D = \{x \mid x^2 + x - 2 \le 0\}$, then
(a) draw a graph of f, and
(b) find the range of f. (*Hint:* What values of y occur on the graph?)

59. The length L of a metal rod is a linear function of its temperature T, where L is measured in centimeters and T in degrees Celsius. The following measurements have been made: $L = 124.91$ when $T = 0$, and $L = 125.11$ when $T = 100$.
(a) Find a formula that gives L as a function of T.
(b) What is the length of the rod when its temperature is 20° Celsius?
(c) To what temperature should the rod be heated to make it 125.17 cm long?

60. ***Store Prices*** The owner of a grocery store finds that, on average, the store can sell 872 gallons of milk per week when the price per gallon is \$1.98. When the price per gallon is \$1.75, sales average 1125 gallons a week. Assume that the number N of gallons sold per week is a linear function of the price P per gallon.
(a) Find a formula that gives N as a function of P.
(b) If the price per gallon is \$1.64, how many gallons should the store owner expect to sell per week?
(c) To sell 1400 gallons per week, what price should the store set per gallon?

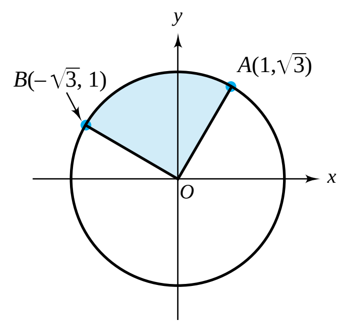
61. ***Equilateral Triangle*** For what value(s) of m will the triangle formed by the three lines $y = -2$, $y = mx + 4$, and $y = -mx + 4$ be equilateral?

62. ***Perimeter*** Find m such that the three lines $y = mx + 6$, $y = -mx + 6$, and $y = 2$ form a triangle with a perimeter of 16.

63. ***Area*** Given the three lines $y = m(x + 4)$, $y = -m(x + 4)$, and $x = 2$, for what value of m will the three lines form a triangle with an area of 24?

64. **(a)** Is the point $(-1, 3)$ on the circle $x^2 + y^2 - 4x + 2y - 20 = 0$?
(b) If the answer is yes, find an equation for the line that is tangent to the circle at $(-1, 3)$; if the answer is no, find the distance between the x-intercept points of the circle.

65. **(a)** Show that points $A(1, \sqrt{3})$ and $B(-\sqrt{3}, 1)$ are on the circle $x^2 + y^2 = 4$.
(b) Find the area of the shaded region in the diagram. (*Hint:* Show that segments $\overline{OA}$ and $\overline{OB}$ are perpendicular to each other.



66. The horizontal line $y = 2$ divides $\triangle ABC$ into two regions. Find the area of the two regions for the triangle with these vertices:
(a) $A(0, 0)$, $B(4, 0)$, $C(0, 4)$
(b) $A(0, 0)$, $B(4, 0)$, $C(8, 4)$

67. The vertices of $\triangle ABC$ are $A(0, 0)$, $B(4, 0)$, $C(0, 4)$. If $0 < k < 4$, then the horizontal line $y = k$ will divide the triangle into two regions. Draw a diagram showing these regions for a typical k. Find the value of k for which the areas of the two regions are equal.

2.5 QUADRATIC FUNCTIONS, PARABOLAS, AND PROBLEM SOLVING

> It is only fairly recently that the importance of nonlinearities has intruded itself into the world of the working scientist. Nonlinearity is one of those strange concepts that is defined by what it is not. As one physicist put it, "It is like having a zoo of nonelephants."
>
> *B. J. West*

Much of this course, and calculus courses to follow, deals with other inhabitants of the "zoo" of nonlinear functions, including families of polynomial, exponential, logarithmic, and trigonometric functions. All are important, but quadratic functions are among the simplest nonlinear functions mathematicians use to model the world.

I had mathematical curiosity very early. My father had in his library a wonderful series of German paperback books . . . . One was Euler's Algebra. I discovered by myself how to solve equations. I remember that I did this by an incredible concentration and almost painful and not-quite conscious effort. What I did amounted to completing the square in my head without paper or pencil.
Stan Ulam

Definition: quadratic function

A **quadratic function** is a function with an equation equivalent to

$$f(x) = ax^2 + bx + c, \tag{1}$$

where a, b, and c are real numbers and a is not zero.

For example, $g(x) = 5 - 4x^2$ and $h(x) = (2x - 1)^2 - x^2$ are quadratic functions, but $F(x) = \sqrt{x^2 + x + 1}$ and $G(x) = (x - 3)^2 - x^2$ are not. (In fact, G is linear.)

Basic Transformations and Graphs of Quadratic Functions

The graphing techniques we introduced in Section 2.3 are collectively called **basic transformations.** The graph of any linear function is a line, and we will show that that graph of any quadratic function can be obtained from the core parabola, $f(x) = x^2$, by applying basic transformations. We apply terminology from the core parabola to parabolas in general. The point $(0, 0)$ is called the **vertex** of the core parabola, and the y-axis is the **axis of symmetry.** The axis of symmetry is a help in making a hand-sketch of a parabola. Whenever we locate a point of the parabola on one side of the axis of symmetry, we automatically have another point located symmetrically on the other side.

We will derive a **transformation form** for a general quadratic function, an equation that identifies the vertex and axis of symmetry of the graph, but to graph any particular quadratic, you may not need all of the steps. Indeed, because a graphing calculator graphs any quadratic function, we could ask why we need the transformation form at all. Because a calculator graph is dependent on the window we choose and the pixel coordinates, we need an algebraic form from which we can read more exact information.

To begin, we factor out the coefficient a from the x-terms, and then add and subtract the square of half the resulting x-coefficient to complete the square on x.

$$\begin{aligned} f(x) = ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left[\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2}\right] + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right). \end{aligned}$$

The final equation has the form

$$f(x) = a(x - h)^2 + k \tag{2}$$

which we recognize as a core parabola shifted so that the vertex is at the point (h, k) and the axis of symmetry is the line $x = h$.

Parabola Features

Looking at the derivation of Equation (2), we can make some observations about the graphs of quadratic functions.

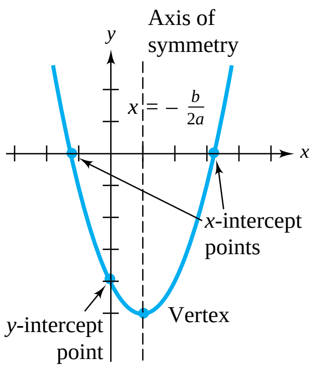


FIGURE 26

Graphs of quadratic functions

For the quadratic function $f(x) = ax^2 + bx + c$: The graph is a parabola with **axis of symmetry** $x = \dfrac{-b}{2a}$.

The parabola opens *upward if* $a > 0$, *downward if* $a < 0$.

To *find the coordinates of the vertex,* set $x = \dfrac{-b}{2a}$. Then the y-coordinate is given by $y = f\left(\dfrac{-b}{2a}\right)$.

The graph of every quadratic function intersects the y-axis (where $x = 0$), but it need not have any x-intercept points. To find any x-intercepts, we solve the equation $f(x) = 0$. By its nature, every quadratic function has a *maximum* or a *minimum* (depending on whether the parabola opens down or up) that occurs at the vertex of the parabola. See Figure 26.

▶EXAMPLE 1 ***Locating the vertex of a parabola*** The graph of the quadratic function is a parabola. Locate the vertex in two ways: **(i)** by writing the function in the form of Equation (2) and **(ii)** by setting $x = -\dfrac{b}{2a}$. Sketch the graph.

(a) $f(x) = x^2 - 2x - 3$ **(b)** $f(x) = 2x^2 + 4x - 1$

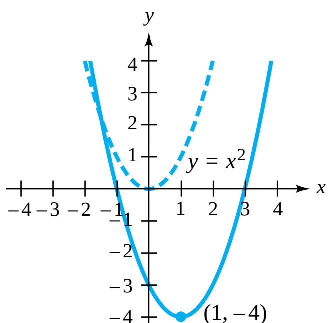
Solution

(a) **(i)** Complete the square on the x-terms.

$$f(x) = x^2 - 2x - 3 = (x^2 - 2x + 1) - 1 - 3$$
$$= (x - 1)^2 - 4,$$

From this form, the graph is the core parabola shifted 1 unit right and 4 units down; the vertex is at $(1, -4)$.

(ii) From $f(x) = x^2 - 2x - 3$, $-\frac{b}{2a} = -\frac{-2}{2 \cdot 1} = 1$. Substituting 1 for x, $f(1) = 1 - 2 \cdot 1 - 3 = -4$, so again the vertex is the point $(1, -4)$. The graph is shown in Figure 27(a).



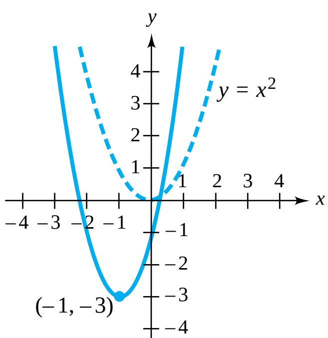
(a) $f(x) = x^2 - 2x - 3$
$= (x - 1)^2 - 4$

(b) **(i)** Before completing the square, we factor out 2 from the x-terms:

$$f(x) = 2x^2 + 4x - 1 = 2(x^2 + 2x) - 1$$
$$= 2(x^2 + 2x + 1 - 1) - 1 = 2(x + 1)^2 - 3.$$

The graph is obtained by shifting the core parabola 1 unit left, stretching by a factor of 2, and translating the stretched parabola down 3 units; the vertex is at $(-1, -3)$.

(ii) From $f(x) = 2x^2 + 4x - 1$, $-\frac{b}{2a} = -\frac{4}{4} = -1$, and $f(-1) = 2(-1)^2 + 4(-1) - 1 = -3$, and the vertex is at $(-1, -3)$. The graph appears in Figure 27(b). ◀



(b) $f(x) = 2x^2 + 4x - 1$
$= 2(x + 1)^2 - 3$

FIGURE 27

▶EXAMPLE 2 ***Graph, maximum or minimum, and intercept points*** Find the maximum or minimum value of f and the intercept points both from a calculator graph and algebraically.

(a) $f(x) = x^2 - 2x - 3$ **(b)** $f(x) = 2x^2 + 4x - 1$

Solution

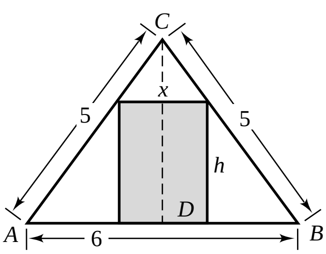
(a) From Example 1(a) and the graph in Figure 27(a), the minimum value of f is the y-coordinate of the vertex, -4. For the y-intercept, $f(0) = -3$, so the y-intercept point is $(0, -3)$. Solving the equation $x^2 - 2x - 3 = 0$, we have $(x - 3)(x + 1) = 0$, so the x-intercept points are $(3, 0)$ and $(-1, 0)$. While a calculator graph in any window shows that the x-intercept points are near $x = 3$ and $x = -1$, in trace mode, we do not see the x-intercepts exactly (where the y-coordinate equals 0) unless we have a decimal window. We can zoom in as needed, though, to get as close to $(3, 0)$ and $(-1, 0)$ as desired.

(b) From Example 1(b) and the graph in Figure 27(b), the function has a minimum value of -3. Since $f(0) = -1$, the y-intercept point is $(0, -1)$. Tracing along a calculator graph, we find that the x-intercepts are near 0.2 and -2.2. We can get closer approximations by zooming in, but the equation $2x^2 + 4x - 1 = 0$ does not factor with rational numbers, so we cannot read exact coordinates of the x-intercept points on any calculator graph. We use the quadratic formula to solve the equation and find the x-intercept points exactly: $\left(\frac{-2 + \sqrt{6}}{2}, 0\right)$ and $\left(\frac{-2 - \sqrt{6}}{2}, 0\right)$. ◄

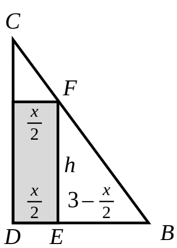
Quadratic Functions with Limited Domain

According to the domain convention the domain of any quadratic function is the set of all real numbers unless there is some restriction. Many applications place natural restrictions on domains, as illustrated in the next two examples.

Strategy: **(a)** Draw a separate diagram (Figure 28b) to show a right triangle formed by altitude CD. $\triangle CDB$ is similar to $\triangle FEB$. Use ratios of the sides to relate h and x.



(a)



(b)

FIGURE 28

►EXAMPLE 3 ***Limited domain*** A rectangle is inscribed in an isosceles triangle ABC, as shown in Figure 28a, where $|\overline{AB}| = 6$ and $|\overline{AC}| = |\overline{BC}| = 5$. Let x denote the width, h the height, and K the area of the rectangle. Find an equation for **(a)** h as a function of x, **(b)** K as a function of x. **(c)** Find the domain of each.

Solution

(a) Following the strategy, the ratios $\frac{|\overline{CD}|}{|\overline{DB}|}$ and $\frac{|\overline{FE}|}{|\overline{EB}|}$ are equal. $|\overline{DB}| = 3$, $|\overline{FE}| = h$, and $|\overline{EB}| = 3 - \frac{x}{2}$. For $|\overline{CD}|$ we can apply the Pythagorean theorem to $\triangle CDB$: $|\overline{CD}| = \sqrt{5^2 - 3^2} = 4$. Therefore

$$\frac{|\overline{FE}|}{|\overline{EB}|} = \frac{|\overline{CD}|}{|\overline{DB}|} \quad \text{or} \quad \frac{h}{3 - \frac{x}{2}} = \frac{4}{3} \quad \text{or} \quad h = 4 - \frac{2x}{3}.$$

(b) The area K is the product of x and h:

$$K = x \cdot h = x\left(4 - \frac{2x}{3}\right) = 4x - \frac{2}{3}x^2.$$

(c) From the nature of the problem, there is no rectangle unless x is a positive number less than 6. Hence the domain of both h and K is $\{x \mid 0 < x < 6\}$. ◄

▶EXAMPLE 4 ***Minimizing area*** Find the dimensions (x and h) for the rectangle with the maximum area that can be inscribed in the isosceles triangle ABC in Figure 28a.

Strategy: Graph K and find the highest point on the graph (the vertex).

Solution

Follow the strategy. In Example 3, we found the area K as a quadratic function of x:

$$K(x) = -\frac{2}{3}x^2 + 4x, \ 0 < x < 6.$$

Graphing K as a function of x, we get *part* of a parabola that opens down. The graph of K is shown in Figure 29. The maximum value of K occurs at the vertex of the parabola, where

$$x = -\frac{b}{2a} = \frac{-4}{2(-2/3)} = 3.$$

When $x = 3$, $h = 4 - \frac{2x}{3} = 4 - (\frac{2}{3})3 = 2$, and $K(3) = 6$. Therefore the inscribed rectangle with the largest area has sides of lengths 3 and 2, and area 6. ◀

Solving Quadratic Inequalities

In Section 1.5, we solved quadratic inequalities by factoring quadratic expressions. Here we look at the more general situation of solving quadratic inequalities with the use of graphs, as illustrated in the following example.

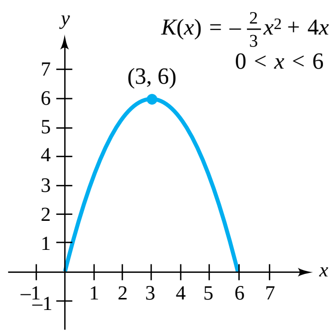


FIGURE 29

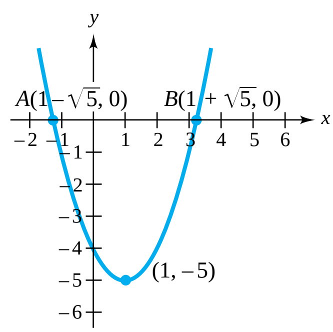


FIGURE 30

▶EXAMPLE 5 ***Using a graph*** Find the domain of the function $f(x) = \sqrt{x^2 - 2x - 4}$.

Strategy: Use a graph of $y = x^2 - 2x - 4$ to see where the y-values are non-negative.

Solution

Follow the strategy. Let $y = x^2 - 2x - 4$ and draw a graph. This gives a parabola that opens upward, with vertex at $(1, -5)$. To find the x-intercept points, solve the equation

$$x^2 - 2x - 4 = 0$$

by the quadratic formula to get $x = 1 \pm \sqrt{5}$. Thus the intercept points are $A(1 - \sqrt{5}, 0)$ and $B(1 + \sqrt{5}, 0)$, as shown in Figure 30. Use the graph to read off the solution set to the inequality that defines the domain of f:

$$\begin{aligned} D &= \{x \mid x \le 1 - \sqrt{5} \quad \text{or} \quad x \ge 1 + \sqrt{5}\} \\ &= (-\infty, 1 - \sqrt{5}] \cup [1 + \sqrt{5}, \infty). \end{aligned}$$ ◀

Maximum and Minimum Values of a Function

When we use mathematical models to answer questions about an applied problem, we frequently need to determine the maximum or minimum values of a function. The general problem can be very difficult, even using the tools of calculus, but when quadratic functions are involved one can simply read off a maximum or minimum value from a graph.

Definition: maximum or minimum value of a function

Suppose f is a function with domain D.

If there is a number k in D such that $f(k) \geq f(x)$ for every x in D, then $f(k)$ is the **maximum value of f.**
If there is a number k in D such that $f(k) \leq f(x)$ for every x in D, then $f(k)$ is the **minimum value of f.**

Strategy: First draw a graph, being aware of the given domain. From the graph, read off the minimum or maximum values of y.

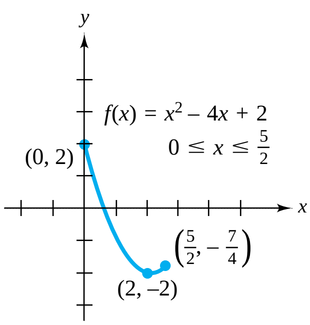


FIGURE 31

▶EXAMPLE 6 ***A function with limited domain*** For the function $f(x) = x^2 - 4x + 2$, with domain $D = \{x \mid 0 \leq x \leq \frac{5}{2}\}$ find **(a)** the maximum and minimum values of f, and **(b)** the set of values where $f(x) > 0$.

Solution

(a) The graph of f is the part of the parabola $y = x^2 - 4x + 2$ on the interval $[0, \frac{5}{2}]$. A calculator graph may not allow us to read all needed points in exact form, so we first locate the vertex and then evaluate the function at the ends of the domain.

Using $x = \frac{-b}{2a} = \frac{4}{2} = 2$, we have $f(2) = -2$, so the point $(2, -2)$ is the vertex. At the endpoints of the domain, $f(0) = 2$, and $f(\frac{5}{2}) = -\frac{7}{4}$, so the graph has endpoints $(0, 2)$ and $(\frac{5}{2}, -\frac{7}{4})$.

We get the graph shown in Figure 31. The maximum value of f is 2, which occurs at the left end of the graph, and the minimum value is -2, at the vertex of the parabola.

(b) To solve the inequality $f(x) > 0$, we need the x-intercept point between 0 and 1. From the quadratic formula, $f(x) = 0$ when $x = 2 \pm \sqrt{2}$. Since the function is only defined on the interval $[0, \frac{5}{2}]$, $f(x) > 0$ on the interval $[0, 2 - \sqrt{2}]$. ◀

▶EXAMPLE 7 ***Ranges of transformed functions*** For the function $f(x) = x^2 - 4x + 2$, with domain $D = \{x \mid 0 \leq x \leq \frac{5}{2}\}$, find the range of

(a) $y = f(x) + 2$ **(b)** $y = -f(x)$ **(c)** $y = \dfrac{3}{2}f(x)$.

Solution

(a) The graph of $y = f(x) + 2$ is shifted 2 units up from the graph in Figure 31 and is shown in Figure 32a. The range of f is the interval $[-2, 2]$, so the range of $y = f(x) + 2$ is also shifted 2 units up, to $[0, 4]$.

(b) To get the graph of $y = -f(x)$, we reflect the graph of f in the x-axis, as in Figure 32b. The range is still $[-2, 2]$.

(c) Stretching the graph of f vertically by a factor of $\frac{3}{2}$, we get part of another parabola, $y = \frac{3}{2}x^2 - 6x + 3$, as shown in Figure 32c. The range of $y = \frac{3}{2}f(x)$ is $[-3, 3]$. ◀

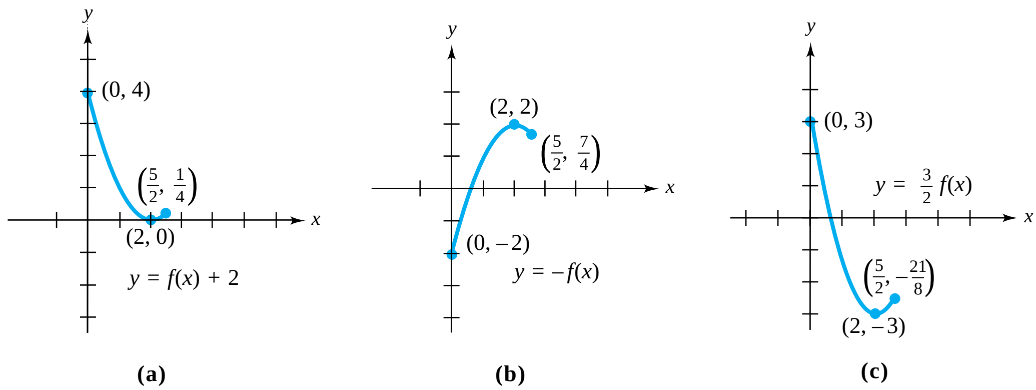


FIGURE 32

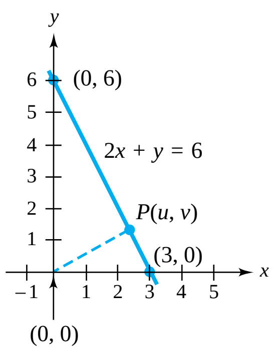


FIGURE 33

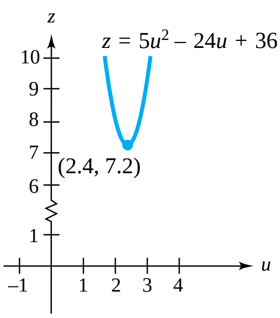


FIGURE 34

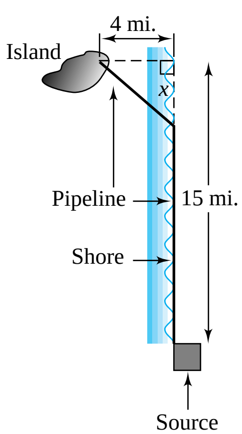


FIGURE 35

▶EXAMPLE 8 Distance from a point to a line

(a) Find the minimum distance from the origin to the line L given by $2x + y = 6$.
(b) What are the coordinates of the point Q on L that is closest to the origin?

Solution

First draw a diagram that will help formulate the problem (Figure 33). Since $P(u, v)$ is on L, then $2u + v = 6$, or $v = 6 - 2u$.

$$d = \sqrt{(u-0)^2 + (v-0)^2} = \sqrt{u^2 + v^2} = \sqrt{u^2 + (6-2u)^2}$$
$$= \sqrt{5u^2 - 24u + 36}.$$

(a) The minimum value of d will occur when the expression under the radical is a minimum. Determine the minimum of the function

$$z = 5u^2 - 24u + 36,$$

whose graph is shown in Figure 34. The lowest point on the parabola occurs where

$$u = -\frac{b}{2a} = -\frac{-24}{10} = 2.4$$

When $u = 2.4$, $z = 7.2$. Therefore, the minimum value of z is 7.2, so the minimum distance from the origin to the line L is $\sqrt{7.2}$ (≈ 2.68).

(b) The point Q on L that is closest to the origin is given by $u = 2.4$ and $v = 6 - 2(2.4) = 1.2$. Thus, Q is point $(2.4, 1.2)$. ◀

Looking Ahead to Calculus

Not all problems lead to quadratic functions. The applied problem in the next example requires calculus techniques to find an exact solution. With a graphing calculator, however, we can find an excellent approximation from a graph of a cost function.

▶EXAMPLE 9 ***Reading a solution from a graph*** A freshwater pipeline is to be built from a source on shore to an island 4 miles offshore as located in the diagram (Figure 35). The cost of running the pipeline along the shore is \$7500 per mile, but construction offshore costs \$13,500 per mile.

(a) Express the construction cost C (in thousands of dollars) as a function of x.

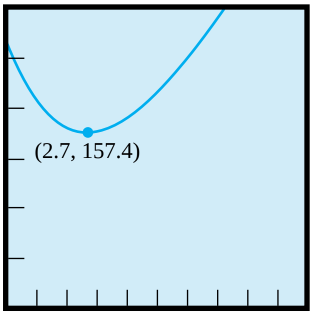
(b) Select an appropriate window and graph $y = C(x)$ to find the approximate distance x that minimizes the cost of construction.

Solution

(a) If we begin the underwater construction x miles from the point nearest the island, as in Figure 35, then we have $(15 - x)$ miles along the coast, at a cost of $(7500)(15 - x)$ dollars. The distance offshore is then $\sqrt{x^2 + 4^2}$, so the offshore construction cost is $(13{,}500)\sqrt{x^2 + 16}$ dollars.
The total cost, in thousands of dollars, is given by

$$C(x) = 7.5(15 - x) + 13.5\sqrt{x^2 + 16}.$$

(b) Because of the greater cost of offshore construction, it appears that x should be less than 10, so we might try an x-range of $[0, 10]$. We can try a value for x to evaluate C, say $x = 5 : C(5) \approx 161.4$, just over \$161,000. To safely bracket that value, let's try a y-range of $[140, 170]$. We get a graph as shown in Figure 36. Tracing to find the low point on the graph, we get a minimum of about 157.4 near $x = 2.67$. Furthermore, we observe that the graph is quite flat near the low point, that the cost differs only by a few dollars for x near 2.7. We conclude that we should build along the coast for about 12.3 miles and then head directly toward the island. The cost will be about \$157,400. ◀



[0, 10] by [140, 170]

Cost: $C(x) = 7.5(15 - x) + 13.5\sqrt{x^2 + 16}$

FIGURE 36

EXERCISES 2.5

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. If we translate the graph of $y = x^2$ two units to the right and one unit down, the result will be the graph of $y = x^2 - 4x + 3$.
2. The y-intercept point for the graph of $y = x^2 + x - 3$ is above the x-axis.
3. The maximum value of $f(x) = 15 - 2x - x^2$ is 12.
4. The graphs of $y = x^2 - 5x - 4$ and $y = 8 + 3x - x^2$ intersect at points in Quadrants II and IV.
5. If we translate the graph of $y = x^2$ three units down it will be the graph of $y = 2x^2 - 6$.

Exercises 6–8 Fill in the blank so that the resulting statement is true. The number of points at which the two graphs intersect is ______.

6. $f(x) = x^2 - 5x - 5,\ g(x) = 8 + 3x - x^2$
7. $f(x) = 3 + 3x - x^2,\ g(x) = 8 - x$
8. $f(x) = 3 + 3x - x^2,\ g(x) = |x - 2| - 2x$

Exercises 9–10 Draw a graph of function f using a $[-10, 10]$ by $[-10, 10]$ window. The number of x-intercept points visible in this window is ______.

9. $f(x) = 0.3x^2 - 4x - 1$
10. $f(x) = 3 - 3x - 0.3x^2$

Develop Mastery

Exercises 1–12 ***Intercept, Vertex*** *Find the coordinates of the intercept points and vertex algebraically, and then draw a graph as a check.*

1. $f(x) = x^2 - 3$
2. $f(x) = -x^2 + 3$
3. $g(x) = 2(x - 1)^2$
4. $g(x) = 2(x + 1)^2$
5. $f(x) = (x + 1)^2 - 3$
6. $f(x) = (x - 3)^2 + 1$
7. $f(x) = -x^2 - 2x + 2$
8. $f(x) = x^2 + 4x + 1$
9. $f(x) = 2x^2 - 4x + 2$
10. $f(x) = -2x^2 + 8x - 5$
11. $f(x) = \dfrac{1}{2}x^2 + 2x$
12. $f(x) = -\dfrac{1}{2}x^2 - 2x - 1$
13. ***Explore*** For each real number b, the graph of $f(x) = x^2 - bx - 1$ is a parabola. Choose several values of b greater than or equal to 1 and in each case draw the corresponding graph. Describe the role that b plays. Where are the x and y-intercept points located? What about the vertex?
14. Repeat Exercise 13 for $f(x) = x^2 + bx - 1$.

15. ***Explore*** Try several values of b, positive and negative, and in each case draw a graph of $f(x) = x^2 - b|x| + 4$. What do you observe about the zeros of f?

16. ***Explore*** For $f(x) = x^2 - 4x + c$, choose several values of c and in each case draw the corresponding graph. Describe the role that c plays. How many x-intercept points are there?

Exercises 17–20 ***Find the equation (in slope-intercept form) for the line containing the vertex and the y-intercept point of the graph of f. Draw a graph of f and the line as a check.***

17. $f(x) = x^2 - 4x + 1$ **18.** $f(x) = x^2 - 3x$

19. $f(x) = -2x^2 - 8x + 3$

20. $f(x) = -3x^2 - 12x - 8$

Exercises 21–24 ***Graphs and Quadrants*** *Draw a graph and determine the quadrants through which the graph of the function passes.*

21. $y = x^2 - 4x + 3$ **22.** $y = 2x^2 + 7x + 3$

23. $y = x^2 + 4x + 5$ **24.** $y = -x^2 - 2x - 1$

Exercises 25–28 ***Distance Between Intercepts*** *Find the distance between the x-intercept points for the graph of the function. Solve algebraically and then check with a graph.*

25. $f(x) = x^2 - 4x - 3$ **26.** $f(x) = x^2 + 2x - 8$

27. $f(x) = x^2 - 4x + 1$

28. $f(x) = -2x^2 + 4x + 3$

Exercises 29–32 ***Inequalities*** *Find the solution set for the given inequality.* ***(a)*** *Draw a graph of the left side and read off the answer.* ***(b)*** *Use algebra to justify your answer.*

29. $x^2 - 4x + 3 > 0$

30. $x^2 + 5x + 4 < 0$

31. $2x^2 - x - 3 < 0$

32. $-x^2 + 2x + 4 \leq 0$

Exercises 33–34 ***Intercepts to Vertex*** *The intercept points for the graph of a quadratic function f are specified. Find the coordinates of the vertex.*

33. $(-1, 0), (3, 0)\ (0, -3)$ **34.** $(-3, 0), (2, 0), (0, 6)$

Exercises 35–38 ***Range*** *Determine the range of the function. State your answer using* ***(a)*** *set notation and* ***(b)*** *interval notation. A graph will help.*

35. $f(x) = x^2 + 3x - 4$

36. $g(x) = -2x^2 + 4x + 1$

37. $f(x) = (4 - x)(2 + x)$

38. $f(x) = 2x^2 + 4\sqrt{3}\,x$

Exercises 39–42 ***Verbal to Formula*** *Give a formula for a quadratic function f that satisfies the specified conditions. The answer is not unique.*

39. Both zeros of f are positive and $f(0) = -2$.

40. The graph of f does not cross the x-axis and $f(0) = -3$.

41. A zero of f is between -2 and -1, and the other zero is 3.

42. A zero of f is greater than 1, the other zero is less than -1, and the graph contains the point $(0, 2)$.

Exercises 43–46 ***Verbal to Formula*** *Determine the quadratic function whose graph satisfies the given conditions.*

43. The axis of symmetry is $x = 2$, the point $(-1, 0)$ is on the graph, and $(0, 5)$ is the y-intercept point. (*Hint:* Use symmetry to find the other x-intercept point, and then express $f(x)$ in factored form.)

44. The vertex is $(3, -4)$ and one of the x-intercept points is $(1, 0)$. (See the hint in Exercise 43.)

45. The graph is obtained by translating the core parabola 3 units left and 2 units down.

46. The graph is obtained by reflecting the core parabola about the x-axis, then translating to the right 2 units.

Exercises 47–50 ***Area*** *Let A be the y-intercept point and B, C be the x-intercept points for the graph of the function. Draw a diagram and then find the area of $\triangle ABC$.*

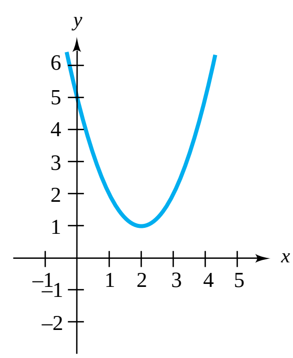
47. $f(x) = -x^2 - x + 6$ **48.** $f(x) = x^2 - 6x + 8$

49. $f(x) = -x^2 - 2x + 8$

50. $f(x) = -x^2 - 6x + 8$

Exercises 51–54 ***Graph to Verbal and Formula*** *Each of the graphs shown began with the core parabola ($y = x^2$) followed by one or more basic transformations.* ***(a)*** *Give a verbal description of the transformation used.* ***(b)*** *Give an equation for the function. Check by using a graph.*

51.



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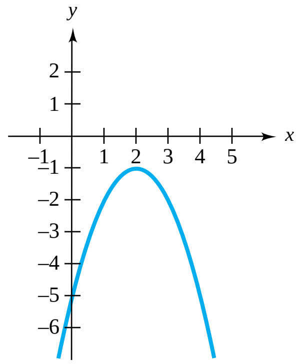

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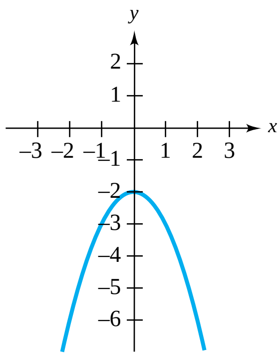

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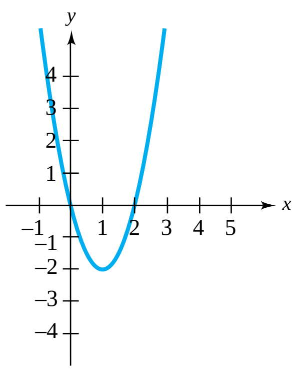


Exercises 55–58 ***Range, Limited Domain*** *A formula for a function is given along with its domain, D. Find the range of the function. Draw a graph.*

55. $f(x) = x^2 + 2x + 5;$ $D = \{x \mid -3 \le x \le 0\}$

56. $f(x) = x^2 + 2x + 5;$ $D = \{x \mid 0 \le x \le 2\}$

57. $g(x) = -x^2 - 4x + 4;$ $D = \{x \mid -3 < x < 1\}$

58. $g(x) = -x^2 + 2x + 4;$ $D = \{x \mid 0 \le x \le 3\}$

Exercises 59–64 ***Maximum, Minimum*** *Find the maximum and/or minimum value(s) of the function. A graph will be helpful.*

59. $f(x) = x^2 - 3x - 4$

60. $g(x) = -x^2 + 4x + 3$

61. $g(x) = x^2 - x, -1 \le x \le 4$

62. $g(x) = 3x - x^2, 0 < x < 4$

63. $f(x) = 6x - x^2, x \ge 0$

64. $f(x) = 12x - 3x^2, x \ge 0$

Exercises 65–66 ***Translations*** *Describe horizontal and vertical translations of the graph of f so that the result will be a graph of $y = x^2$. (Hint: Complete the square.)*

65. $f(x) = x^2 - 4x + 1$

66. $f(x) = x^2 + 4x + 5$

Exercises 67–70 ***Maximum, Minimum*** *Find the maximum and/or minimum value(s) of the function. Solve algebraically by considering the expression under the radical as a quadratic function with restricted domain. Use a graph of f as a check.*

67. $f(x) = \sqrt{3 + 2x - x^2}$

68. $f(x) = \sqrt{5 + 4x - x^2}$

69. $g(x) = \sqrt{3 + 2x + x^2}$

70. $g(x) = \sqrt{4 - 2x + x^2}$

71. ***Explore*** At the beginning of this section we noted that $F(x) = \sqrt{x^2 + x + 1}$ is not a quadratic function.

(a) In a decimal window, graph, in turn, each of the following:

$$f(x) = \sqrt{x^2 + x - 1}$$
$$F(x) = \sqrt{x^2 + x + 1}$$
$$G(x) = \sqrt{x^2 + 2x + 1}$$

(b) Write a paragraph to describe some of the differences between the graphs of f, F, and G. Note any symmetries and comment on domains and on ways each graph differs from a parabola.

(c) The graph of G should look familiar. Explain.

72. Solve the problem in Exercise 71 where the functions are

$$f(x) = \sqrt{x^2 - x - 1}$$
$$F(x) = \sqrt{x^2 - x + 1}$$
$$G(x) = \sqrt{x^2 - 2x + 1}$$

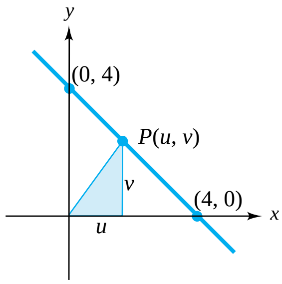
Exercises 73–74 ***Minimum Distance*** *Find the minimum distance from point P to the graph of $y = x^2 + 4x - 8$. (Hint: Let $Q(u, v)$ be any point on the parabola. Determine a formula that gives the distance d from point P as a function of u (do not simplify). Draw a graph and use* TRACE.*)*

73. $P(0, 1)$

74. $P(0, 2)$

75. ***Maximum Area*** Point $P(u, v)$ is in the first quadrant on the graph of the line $x + y = 4$. A triangular region is shown in the diagram.

(a) Express the area A of the shaded region as a function of u.



(b) For what point P will the area of the region be a maximum?

(c) What is the maximum area?

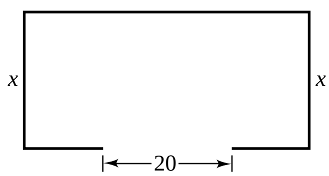
76. Repeat Exercise 75 for point $P(u, v)$ on the line segment joining (0, 3) and (4, 0).

77. **(a)** A wire 36 inches long is bent to form a rectangle. If x is the length of one side, find an equation that gives the area A of the rectangle as a function of x.

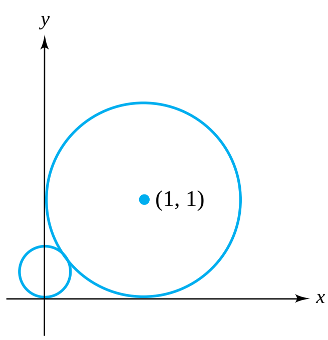
(b) For what values of x is the equation valid?

78. ***Maximum Area*** Of all the rectangles of perimeter 15 centimeters, find the dimensions (length and width) of the one with greatest area.

79. ***Maximum Area*** A farmer has 800 feet of fencing left over from an earlier job. He wants to use it to fence in a rectangular plot of land except for a 20-foot strip that will be used for a driveway (see the diagram, where x is the width of the plot). Express the area A of the plot as a function f of x. What is the domain of f? For what x is A a maximum?



80. In the diagram the smaller circle is tangent to the x-axis at the origin and it is tangent to the larger circle, which has a radius of 1 and center at (1, 1). What is the radius of the smaller circle?



81. ***Maximum Revenue*** A travel agent is proposing a tour in which a group will travel in a plane of capacity 150. The fare will be \$1400 per person if 120 or fewer people go on the tour; the fare per person for the entire group will be decreased by \$10 for each person in excess of 120. For instance, if 125 go, the fare for each will be $\$1400 - \$10(5) = \$1350$. Let x represent the number of people who go on the tour and T the total revenue (in dollars) collected by the agency. Express T as a function of x. What value of x will give a maximum total revenue? It will be helpful to draw a graph of the function.

82. ***Minimum Area*** A piece of wire 100 centimeters long is to be cut into two pieces; one of length x centimeters, to be formed into a circle of circumference x, and the other to be formed into a square of perimeter $100 - x$ centimeters. Let A represent the sum of the areas of the circle and the square.

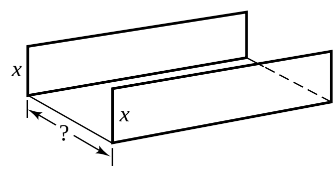
(a) Find an equation that gives A as a function of x.

(b) For what value of x will A be the smallest? What is the smallest area?

83. Solve the problem in Exercise 82 if the two pieces are to be formed into a square and an equilateral triangle.

84. Solve the problem in Exercise 82 if the two pieces are to be formed into a circle and an equilateral triangle.

85. ***Maximum Capacity*** A long, rectangular sheet of galvanized tin, 10 inches wide, is to be made into a rain gutter. The two long edges will be bent at right angles to form a rectangular trough (see diagram, which shows a cross section of the gutter with height x inches).



(a) Find a formula that gives the area A of the cross section as a function of x. What is the domain of this function?

(b) What value of x will give a gutter with maximum cross sectional area? Solve algebraically and use a graph as a check.

86. ***Minimum Time*** A forest ranger is in the forest 3 miles from the nearest point P on a straight road. His car is parked down the road 5 miles from P. He can walk in the forest at a rate 2 mi/hr and along the road at 5 mi/hr.

(a) Toward what point Q on the road between P and his car should he walk so that the total time T it takes to reach the car is the least?

(b) How long will it take to reach the car? (*Hint:* Let $|PQ| = x$, then use a graph to help you solve the problem.)

87. In Exercise 86, solve the problem if the ranger is 4 miles from P.

88. ***Looking Ahead to Calculus*** A solid has as its base the region in the xy-plane bounded by the circle $x^2 + y^2 = 4$.

(a) If every vertical cross section perpendicular to the x-axis is a semicircle, express the area K of the cross section at a distance u from the origin as a function of u.

(b) Repeat part **(a)** if each vertical cross section is an isosceles triangle with an altitude half as long as its base (not a semicircle).

(c) Repeat part **(a)** if each vertical cross section is an equilateral triangle.

(d) Repeat part **(a)** if each vertical cross section is a rectangle whose base is twice its vertical height.

2.6 COMBINING FUNCTIONS

What is proved about numbers will be a fact in any universe.

Julia Robinson

Just as we combine numbers to get other numbers, so we may combine functions to get other functions. The first four ways of combining functions give familiar sums, differences, products, or quotients, as we would expect. **Composition,** less familiar, is a key idea throughout much of what follows.

Definition: sum, difference, product, quotient functions

Suppose f and g are given functions. Functions denoted by $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$ are given by:

Sum: $(f + g)(x) = f(x) + g(x)$

Difference: $(f - g)(x) = f(x) - g(x)$

Product: $(f \cdot g)(x) = f(x) \cdot g(x)$

Quotient: $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

The domain of each combined function is the set of all real numbers for which the right side of the equation is meaningful as a real number. Use parentheses as needed for clarity.

The definitions stated here are not mere formal manipulations of symbols. For instance, the plus sign in $f + g$ is part of the name of the function that assigns to each x the sum of two numbers, $f(x) + g(x)$.

Neyman . . . interviewed me [for a job at Berkeley and] said he would let me know. . . . I didn't really expect anything to happen. I had already written 104 letters of application to black colleges. Eventually I got a letter from [Neyman] saying something like "In view of the war situation and the draft possibilities, they have decided to appoint a woman to this position." [My eventual appointment here came 12 years later.]

David Blackwell

►EXAMPLE 1 ***Combining functions*** If $f(x) = 4x - 6$ and $g(x) = 2x^2 - 3x$, write an equation for **(a)** $f - g$ and **(b)** $\frac{f}{g}$, and give the domain of each.

Solution

(a) $(f - g)(x) = f(x) - g(x) = (4x - 6) - (2x^2 - 3x) = -2x^2 + 7x - 6.$ The domain is the set of real numbers.

(b) $$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{4x - 6}{2x^2 - 3x} = \frac{2(2x - 3)}{x(2x - 3)},$$

which simplifies to $\frac{2}{x}$ for $x \neq \frac{3}{2}$. Therefore $(\frac{f}{g})(x) = \frac{2}{x}$, where the domain is $\{x \mid x \neq 0 \text{ and } x \neq \frac{3}{2}\}$. ◄

Composition of Functions

Another way to combine functions is used frequently and plays an important role in both precalculus and calculus.

Definition: composition of functions

Suppose f and g are functions. The **composition function, $f \circ g$,** read "**f of g,**" is the function whose value at x is given by

$$(f \circ g)(x) = f(g(x)).$$

Thus to write a formula for $(f \circ g)(x)$, in the rule defining f,

replace each x in $f(x)$ by $g(x)$.

The **domain** of $f \circ g$ is the set of all real numbers x such that both $g(x)$ is defined, and $f(g(x))$ is defined.

The reason for calling the composition $f \circ g$ "f of g" is that the value of the composition function at a given number c is "f of $g(c)$."

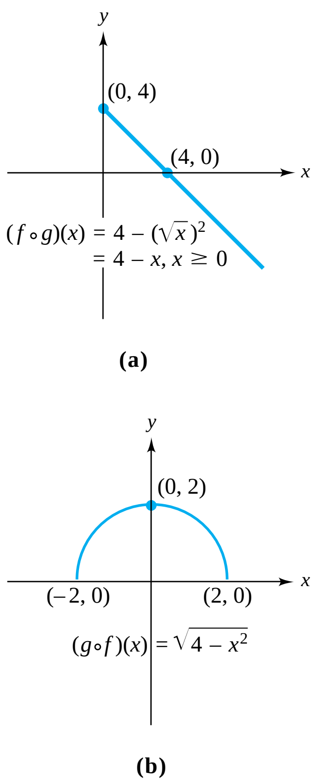


FIGURE 37

▶**EXAMPLE 2** ***Two compositions*** If $f(x) = 4 - x^2$ and $g(x) = \sqrt{x}$, **(a)** write an equation and **(b)** draw a calculator graph of ***(i)*** $f \circ g$ ***(ii)*** $g \circ f$.

Solution

(a) For each composition, we follow the procedure given in the definition.

(i) $(f \circ g)(x) = f(g(x)) = 4 - (g(x))^2 = 4 - (\sqrt{x})^2$.

For $\sqrt{x}$ to be a real number we must have $x \geq 0$, and when $x \geq 0$, we can simplify the equation for $f \circ g$:

$$(f \circ g)(x) = 4 - x, \text{ where } x \geq 0.$$

(ii) $(g \circ f)(x) = g(f(x)) = \sqrt{f(x)} = \sqrt{4 - x^2}$.

Again, the domain is limited: for $4 - x^2 \geq 0$, we have $-2 \leq x \leq 2$.

(b) With a graphing calculator we can always enter the compositions in the form we wrote above, Y1 = 4 − (√X)² and Y2 = √(4 − X²).

If your calculator has a Y = menu where you can enter several functions, there are other options. For example, having entered f and g as Y1 = 4 − X² and Y2 = √X, since $f(g(x)) = 4 - (g(x))^2$, we can enter $f \circ g$ as Y3 = 4 − Y2² and $g \circ f$ as Y4 = √Y1. Observe that for $f \circ g$ we follow the defining rule for composition functions: replace each x in $f(x)$ by $g(x)$.

The calculator graphs are shown in Figure 37. Note that the limitations on the domain are obvious from the graphs and that we can also read off the ranges. The range of $f \circ g$ is $(-\infty, 4]$, and the range of $g \circ f$ is the closed interval $[0, 2]$.

Alternate Solution Sometimes it is easier to verbalize the rules that define functions. The rules for f and g state that, for any given input, f squares the input and subtracts the result from 4, while g takes the square root of its input. Thus, suppose $\sqrt{x}$ is the input. The function f squares $\sqrt{x}$ and subtracts the result from 4: $4 - (\sqrt{x})^2$. Similarly, when g is applied to $f(x)$, g takes the square root of $f(x)$. The output is: $g(f(x)) = \sqrt{f(x)} = \sqrt{4 - x^2}$. ◀

Example 2 shows that $f \circ g$ and $g \circ f$ are not the same function. In general $f \circ g$ and $g \circ f$ are different, although there are important exceptions, as the next example demonstrates.

►EXAMPLE 3 ***Equal compositions*** If $f(x) = 3x - 8$ and $g(x) = \frac{x+8}{3}$, write an equation that gives the rule of correspondence for **(a)** $f \circ g$ **(b)** $g \circ f$.

Solution

Here the rule for f is "triple the input and then subtract 8;" for g it is "add 8 to the input and then divide the sum by 3."

(a) $$(f \circ g)(x) = f(g(x)) = f\left(\frac{x+8}{3}\right) = 3\left(\frac{x+8}{3}\right) - 8 = x.$$

(b) $$(g \circ f)(x) = g(f(x)) = g(3x - 8) = \frac{(3x-8)+8}{3} = x.$$

Thus $(f \circ g)(x) = (g \circ f)(x)$ for every number x. We say that the two functions $f \circ g$ and $g \circ f$ are equal, $f \circ g = g \circ f$. ◄

►EXAMPLE 4 ***Composition equations*** If $f(x) = x^2 - 2x$ and $g(x) = 3 - x$, solve the equations.

(a) $(f \circ g)(x) = 0$ **(b)** $(g \circ f)(x) + x^2 + 5 = 0$

Strategy: Write each equation in a more familiar form.

Solution

(a) $(f \circ g)(x) = f(g(x)) = f(3 - x) = (3 - x)^2 - 2(3 - x) = x^2 - 4x + 3$. Thus the given equation becomes

$$x^2 - 4x + 3 = 0 \qquad \text{or} \qquad (x-1)(x-3) = 0.$$

The solutions are 1 and 3.

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 - 2x) = 3 - (x^2 - 2x) = -x^2 + 2x + 3$. Replacing $(g \circ f)(x)$ by $-x^2 + 2x + 3$, the given equation becomes

$$(-x^2 + 2x + 3) + x^2 + 5 = 0 \qquad \text{or} \qquad 2x + 8 = 0.$$

The solution is -4. ◄

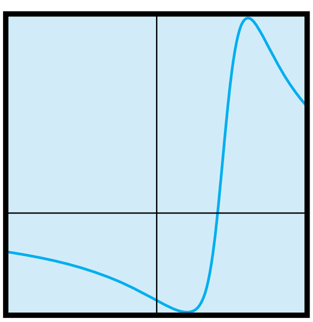
►EXAMPLE 5 ***Maxima and minima from calculator graphs*** Let F denote the composition $g \circ f$ on the limited domain $D = [-5, 5]$, where

$$f(x) = \frac{2x-4}{x^2 - 4x + 5} \qquad \text{and} \qquad g(x) = x^2 + 3x.$$

(a) Draw a calculator graph of $F(x) = g(f(x))$.
(b) From your graph, find the maximum and minimum values of F.
(c) Find the solution set for $g(f(x)) > 0$.

Solution

(a) Writing a formula for the composition $g(f(x))$ requires us to replace each x in $x^2 + 3x$ by the entire $f(x)$. The process is messy, to say the least, but some calculators are designed to make composition much easier. See the Technology Tip following this example. Not knowing the range beforehand, we may set an x-range of $[-5, 5]$ to match the domain and adjust as necessary. A calculator graph is shown in Figure 38.



$[-5, 5]$ by $[-2.1, 4.1]$

FIGURE 38
$F(x) = g(f(x))$

(b) Using the TRACE function on the graph of $y = F(x)$, we find the low point near $(1, -2)$ and the high point near $(3, 4)$. Shifting a decimal window, we confirm that the the maximum value of F is 4 and the minimum value is -2.

(c) The graph crosses the x-axis at $(2, 0)$, as is easily verified by evaluating $F(2)$, and is above the x-axis for all values of x (from the domain of F) greater than 2. Thus the solution set for $g(f(x)) > 0$ is the interval $(2, 5]$. ◀

TECHNOLOGY TIP ◆ ***Graphing compositions and defining functions***

When composing functions, let the calculator do the hard work. In Example 5, to enter $g(f(x)) = (f(x))^2 + 3f(x)$, we need *lots* of parentheses:

$$Y = ((2X - 4)/(X^2 - 4X + 5))^2 + 3((2X - 4)/(X^2 - 4X + 5)).$$

If your calculator allows you to enter a list of functions, Y1, Y2, . . . , (TI and Casio) then you can enter the composition function much more simply. First enter f as Y1 and then use Y1 to enter $g(f(x))$ as Y2 = g(Y1) :

$$Y1 = (2X - 4)/(X^2 - 4X + 5) \qquad Y2 = Y1^2 + 3Y1.$$

HP–38 Having entered functions $F1(X) = (2 * X - 4)/(X^2 - 4 * X + 5)$ and $F2(X) = X^2 + 3 * X$, write the composition as F3(X) = F2(F1(X)) and graph.

HP–48 On the home screen, enter each function as an equation, $'F(X) = (2 * X - 4)/(X^2 - 4 * X + 5)'$. Then press the DEF key (above STO). Similarly for $'G(X) = X^2 + 3 * X'$. Then on the PLOT screen enter the function as 'G(F(X))' and graph.

▶EXAMPLE 6 ***Composition inequality*** If $f(x) = x^2 - 9$ and $g(x) = 2x - 5$, find the solution for $f(g(x)) < 0$.

Strategy: Get simpler expressions for the composition function, substitute, and solve.

Solution

$f(g(x)) = f(2x - 5) = (2x - 5)^2 - 9$. Therefore the given inequality may be written as $(2x - 5)^2 - 9 < 0$. This is equivalent to

$$(2x - 5)^2 < 9 \quad \text{or} \quad -3 < 2x - 5 < 3 \quad \text{or} \quad 1 < x < 4.$$

The solution set is $\{x \mid 1 < x < 4\}$.

Alternate Solution *Graphical* We have seen often that calculator graphs allow us to read the solution set for an inequality such as $(2x - 5)^2 - 9 < 0$ or $4x^2 - 20x + 16 < 0$. We can graph $Y = (2X - 5)^2 - 9$ or we can simplify the inequality to an equivalent form, by dividing through by 4, getting $x^2 - 5x + 4 < 0$, and graph $Y = X^2 - 5X + 4$. In either case we have a parabola that crosses the x-axis at $(1, 0)$ and $(4, 0)$. See Figure 39. We read the solution set as $\{x \mid 1 < x < 4\}$. ◀

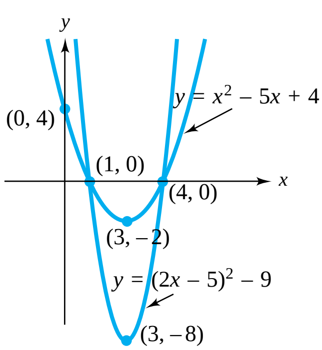


FIGURE 39

▶EXAMPLE 7 ***Applied composition*** An oil spill on a lake assumes a circular shape with an expanding radius r given by $r = \sqrt{t + 1}$, where t is the number of minutes after measurements are started and r is measured in meters.

(a) Find a formula that gives the area A of the circular region at any time t.

(b) What is the area at the beginning measurement $(t = 0)$? What is the area 3 minutes later?

Strategy: (a) Since $r = \sqrt{t + 1}$ is a function of t and $A = \pi r^2$ is a function of r, then by composing functions we can express A as a function of t.

Solution

(a) Follow the strategy.

$$A = \pi(\sqrt{t + 1})^2 = \pi(t + 1).$$

Thus A as a function of t is

$$A = \pi t + \pi.$$

When t is 0, $A = \pi \cdot 0 + \pi = \pi$ square meters. When t is 3, $A = 3\pi + \pi = 4\pi$ square meters. ◀

Calculator Evaluations

Many function evaluations by calculator actually involve composition of functions, especially with calculators that use "Reverse Polish" operations. With such a calculator, to evaluate $F(x) = \sqrt{x^2 + 1}$ when x is 3, we enter 3, square it, and add 1, after which we take the square root. This amounts to treating F as a composition $f \circ g$, where $g(x) = x^2 + 1$ and $f(x) = \sqrt{x}$. We accomplish the same thing if we have a graphing calculator using an Algebraic Operating System when we use the ANS key. Using the same example, if we evaluate $3^2 + 1$ and ENTER, the calculator displays 10. If we then evaluate √ANS, we are taking the composition of the square root function with the previously evaluated $x^2 + 1$ function.

▶EXAMPLE 8 ***Function as a composition*** If $F(x) = \frac{1}{x^2 + 1}$, express F as a composition of two functions.

Solution

Let $f(x) = \frac{1}{x}$ and $g(x) = x^2 + 1$. Then

$$f(g(x)) = f(x^2 + 1) = \frac{1}{x^2 + 1}.$$

Thus, $F(x)$ is given by $F(x) = (f \circ g)(x)$. ◀

In problems of the type discussed in Example 8, be aware that there are many different solutions. For example, we could have taken

$$f(x) = \frac{1}{x + 1} \qquad \text{and} \qquad g(x) = x^2.$$

Then

$$f(g(x)) = f(x^2) = \frac{1}{x^2 + 1}.$$

Composition with Absolute Value

Composition of functions with the absolute value function affects graphs in a consistent fashion giving us two more useful basic transformations. It is easiest to look at a specific example.

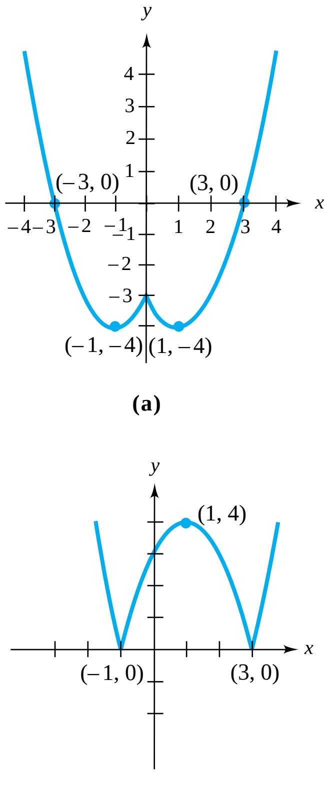


FIGURE 40

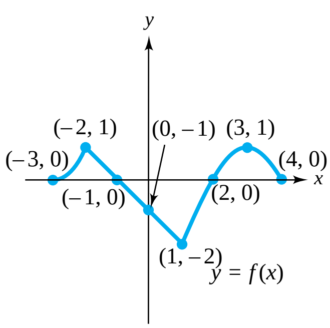


FIGURE 41

►EXAMPLE 9 ***Composing absolute value with a quadratic function*** Let $f(x) = x^2 - 2x - 3$ and $g(x) = |x|$. Write an equation and draw a calculator graph of **(a)** $f \circ g$ **(b)** $g \circ f$.

Solution

(a) $(f \circ g)(x) = f(|x|) = (|x|)^2 - 2|x| - 3 = x^2 - 2|x| - 3$, since $|x|^2 = x^2$. For a calculator graph, we enter `Y = X² − 2 abs (X) − 3`, and get a graph like that in Figure 40a. We observe that $x^2 - 2|x| - 3$ is an *even function*. The graph in Figure 40a consists of the portion of the parabola $y = x^2 - 2x - 3$ to the right of the x-axis, together with its reflection through the y-axis.

(b) $(g \circ f)(x) = g(x^2 - 2x - 3) = |x^2 - 2x - 3|$. For the graph we enter `Y = abs(X² − 2X − 3)`. The graph is shown in Figure 40b. Since the absolute value of a number cannot be negative (recall that $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$), any portion of the graph of the parabola $y = x^2 - 2x - 3$ *below the y-axis* is reflected upward through the x-axis. ◄

The effects we observe in the graphs in Figure 40 are applicable in general. For any function $f(x)$, the function $f(|x|)$ is an even function whose graph is symmetric about the y-axis. Thus the graph of $y = f(|x|)$ consists of the graph of $y = f(x)$ for $x \geq 0$, together with the horizontal reflection of this portion about the y-axis.

Similarly, since $|f(x)| \geq 0$, for the graph of $y = |f(x)|$, any part of the graph of $y = f(x)$ that lies above the x-axis is unchanged; whatever part of the graph lies below the x-axis is reflected upward through the x-axis.

These transformations are consistent with the basic transformations of Section 2.3. A transformation operation on the "outside," $|f(x)|$, affects the vertical aspects of the graph; an operation on the argument, "inside," $f(|x|)$, affects the graph horizontally.

Composition of a function with the absolute value function

From the graph of $y = f(x)$, the graph of

$y = \lvert f(x)\rvert$	$y = f(\lvert x\rvert)$
is a **vertical reflection:** the part above the x-axis is unchanged; any portion below the x-axis is reflected up, through the x-axis.	is a **horizontal reflection:** function is even; the portion to the right of the y-axis is unchanged and is also reflected to the left, through the y-axis.

►EXAMPLE 10 ***Composition with absolute value*** The graph of a function f is shown in Figure 41. If $g(x) = |x|$, draw a graph of **(a)** $f \circ g$ **(b)** $g \circ f$, identifying the points corresponding to the labeled points in Figure 41. Explain the thinking used to get each graph.

Solution

(a) $(f \circ g)(x) = f(g(x)) = f(|x|)$. From the box above, $f(|x|)$ is an even function. Knowing the graph of the function for positive x-values, the rest of the graph is obtained by taking the horizontal reflection in the y-axis. Each labeled point (a, b) is reflected to the point $(-a, b)$. The resulting graph is shown in Figure 42a.

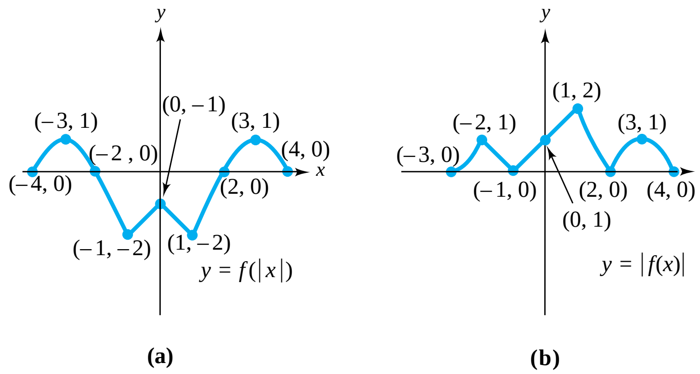


FIGURE 42

(b) $(g \circ f)(x) = g(f(x)) = |f(x)|$. Since $|f(x)|$ can never be negative, the graph can contain no points below the x-axis. Whenever the graph of f dips below the x-axis, the graph of $|f(x)|$ is reflected back up, above the axis. Any point $(a, -b)$ on the graph of f below the x-axis is reflected to the point (a, b). The graph is shown in Figure 42b. ◀

EXERCISES 2.6

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. If $f(x) = x^2$ and $g(x) = x^2 - 1$, then $g \circ f$ is a quadratic function in x.
2. If $f(x) = x^2$ and g is any function for which the domain of $g \circ f$ is not the empty set, then the function $g \circ f$ must be an even function.
3. If $f(x) = 2x - 1$, then $f(a + b) = f(a) + f(b)$ for all real numbers a and b.
4. If $g(x) = 3x$, then $g(c + d) = g(c) + g(d)$ for all real numbers c and d.
5. If $f(x) = x^2 + 1$ and $g(x) = x + 3$, then the graph of $y = (f \circ g)(x)$ contains no points below the x-axis.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. If $f(x) = 2x + 5$ and $g(x) = Int(x)$, then $(g \circ f)(\sqrt{5}) =$ ______.
7. If $f(x) = 2x + 5$ and $g(x) = Int(x)$, then $(f \circ g)(\sqrt{5}) =$ ______.
8. If $f(x) = 2x + 1$ and $g(x) = x^2$, then $(f \circ g)(-1) =$ ______.
9. If $f(x) = 2x - 1$ and $g(x) = x^2 - 3x - 4$, then the zeros of $g \circ f$ are ______.
10. If $f(x) = x^2 - 5x + 4$ and $g(x) = x^2$, then the sum of the roots of the equation $(f \circ g)(x) = 0$ is equal to ______.

Develop Mastery

*Exercises 1–4 Evaluate **(a)** $(f - g)(-1)$ **(b)** $(f \cdot g)(0.5)$.*

1. $f(x) = 2x$, $g(x) = 1 - 2x$
2. $f(x) = x^2 - 3$, $g(x) = \sqrt{x + 4}$
3. $f(x) = |x - 2|$, $g(x) = x + 1$
4. $f(x) = x^2 - x$, $g(x) = 3|1 - x|$

Exercises 5–8 ***Sum and Quotient Functions*** *Find an equation to describe the rule for **(a)** $(f + g)(x)$ and **(b)** $(\frac{f}{g})(x)$. In each case state the domain.*

5. $f(x) = x - \dfrac{1}{x}$, $g(x) = x$
6. $f(x) = x^2 - 1$, $g(x) = 1 - x$
7. $f(x) = \sqrt{x} - 2$, $g(x) = 1 - \sqrt{x}$
8. $f(x) = x - 4$, $g(x) = \dfrac{1}{x}$

Exercises 9–10 ***Composition Functions*** *Use $f(x) = x + 2$ and $g(x) = x^2 - 2x$.*

9. Evaluate **(a)** $(f \circ g)(-1)$ **(b)** $(g \circ f)(3)$ **(c)** $(f \circ f)(4)$
10. Find an equation to describe **(a)** $(f \circ g)$ **(b)** $(g \circ f)$.
11. ***Composition from Tables*** The domain of function f is $\{-3, -1, 0, 1, 3\}$ and the domain of g is $\{-1, 0, 1, 3, 5\}$. The rules for f and g are given in tabular form:

x	-3	-1	0	1	3
$f(x)$	-1	0	2	3	5

x	-1	0	1	3	5
$g(x)$	-2	-1	2	3	4

(a) Complete the following table for $g \circ f$. If an entry is undefined write U.

x	-3	-1	0	1	3
$(g \circ f)(x)$					

What is the domain of
(b) $g \circ f$? **(c)** $f \circ g$?

Exercises 12–13 ***Domain of Composition*** *Use* $f(x) = \sqrt{x}$ *and* $g(x) = x^2 - 4$.

12. Give an equation to describe $f \circ g$. State the domain.

13. Give an equation to describe $g \circ f$. State the domain.

Exercises 14–19 ***Solving Equations*** *For functions f and g,*

$$f(x) = x^2 - 2x - 3 \quad \text{and} \quad g(x) = 2x - 3,$$

solve the equation.

14. $(f + g)(x) = 10$ **15.** $\left(\frac{f}{g}\right)(x) = x + 1$

16. $(f \cdot g)(x) = 0$ **17.** $(g \circ f)(x) = 3x$

18. $(f \circ g)(x) = 5$ **19.** $(g \circ f)(x) + x^2 = 0$

Exercises 20–25 ***Solving Inequalities*** *For functions f and g,*

$$f(x) = -x^2 - x + 1 \quad \text{and} \quad g(x) = 3 - x,$$

find the solution set.

20. $(f + g)(x) < 1$ **21.** $(f - g)(x) \geq -4$

22. $(f \circ g)(x) + x \leq 1$ **23.** $(f \circ g)(x) + x^2 \geq 0$

24. $(g \circ f)(x) + 1 > 0$ **25.** $(g \circ g)(x) \geq x - 1$

Exercises 26–29 ***Graph of Compositions*** ***(a)*** *Draw a graph of the function* $y = (f \circ g)(x)$. ***(b)*** *Give the x and y intercept points. (Solve graphically and then verify algebraically.)*

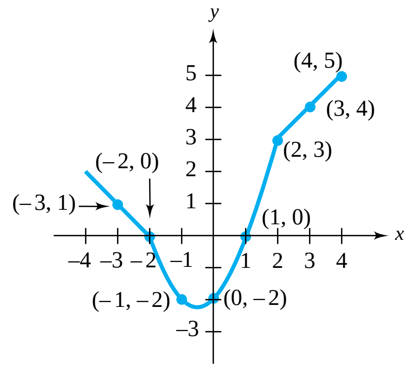
26. $f(x) = x^2 - 3x$, $g(x) = \sqrt{x}$

27. $f(x) = x^2 + x$, $g(x) = x^2 - 2x$

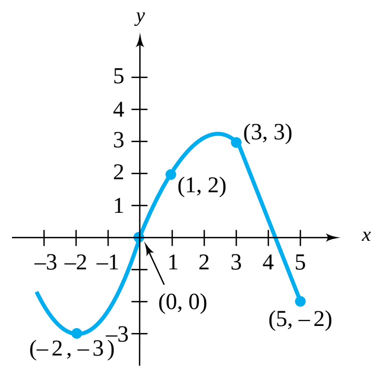
28. $f(x) = x^2 - 2x$, $g(x) = x^2 + x$

29. $f(x) = 9 - x^2$, $g(x) = x^2 - 4$

The following graphs apply to Exercises 30–36.



$y = f(x)$



$y = g(x)$

30. ***Reading Graphs*** Graphs of the functions f and g are shown. Complete the following tables.

x	-3	0	1	2	4
$(g \circ f)(x)$					

x	-2	0	1	3	5
$(f \circ g)(x)$					

Exercises 31–32 *Is the statement true or is it false?*

31. **(a)** $(f \circ g)(0.9) > 0$ **(b)** $-2 < (f \circ g)(4) < 2$

32. **(a)** $(g \circ f)(0.75) > 0$ **(b)** $2 < (f \circ g)(3.1) < 3$

Exercises 33–36 ***Related Graphs*** *Use the graphs in Exercise 30 to draw a graph of h. First give a verbal description of the strategy you plan to use. Label the coordinates of five points that must be on the graph of h. Can a graph of h be used as a check? Explain.*

33. **(a)** $h(x) = f(-x)$ **(b)** $h(x) = -g(x)$

34. **(a)** $h(x) = f(|x|)$ **(b)** $h(x) = |g(x)|$

35. **(a)** $h(x) = f(x - 2)$ **(b)** $h(x) = g(x) - 2$

36. **(a)** $h(x) = g(|x|)$ **(b)** $h(x) = |f(x)|$

Exercises 37–40 *Given functions f and g,* ***(a)*** *find equations that describe the composition functions* $f \circ g$ *and* $g \circ f$. ***(b)*** *Are the functions* $f \circ g$ *and* $g \circ f$ *equal? That is, do they have the same domain D, and is* $(f \circ g)(x) = (g \circ f)(x)$ *for every x in D?*

37. $f(x) = 3x - 1$, $g(x) = \dfrac{x+1}{3}$

38. $f(x) = 4 - 3x$, $g(x) = \dfrac{4-x}{3}$

39. $f(x) = x^2 + 1$, $g(x) = \sqrt{x-1}$

40. $f(x) = x^2 + 1$, $g(x) = \sqrt{x}$

Exercises 41–44 *For the given function f, find a function g such that* $f(g(x)) = x$ *for every value of x. (Hint: In Exercise 41,* $f(g(x)) = 2g(x) - 5$; *solve the equation* $2g(x) - 5 = x$ *for* $g(x)$.*)*

41. $f(x) = 2x - 5$

42. $f(x) = 3 - 4x$

43. $f(x) = \dfrac{2x}{x-2}$

44. $f(x) = \dfrac{-3x}{2x+3}$

Exercises 45–48 ***Evaluating Combined Functions*** *If* $f(x) = \sqrt{x}$ *and* $g(x) = \frac{x}{x-1}$, *evaluate the expression and round off the result to two decimal places.*

45. $(f + g)(\sqrt{3})$

46. $(f \circ g)(1.63)$

47. $(g \circ f)(5)$

48. $\left(\dfrac{f}{g}\right)(0.37)$

Exercises 49–52 ***Function as a Composition*** *Function F is given. Find two functions f and g so that* $F(x) = (f \circ g)(x)$. *Solutions to these problems are not unique.*

49. $F(x) = \dfrac{1}{x^2 + 5}$

50. $F(x) = \sqrt{x^2 - 3x + 5}$

51. $F(x) = |5x + 3|$

52. $F(x) = \dfrac{4}{x^2} + 1$

Exercises 53–56 ***Functions of Your Choice (a)*** *Give formulas for functions f (quadratic) and g (linear) of your choice that satisfy the specified conditions.* ***(b)*** *Determine a formula for* $f \circ g$ *and draw its graph.* ***(c)*** *What are the coordinates of the lowest or highest point on the graph of* $f \circ g$*?*

53. Function f has a positive and a negative zero; g has a zero between 1 and 3.

54. The graph of f opens downward and has no x-intercept points; the graph of g has a positive slope.

55. The graph of f contains points (0, 2) and (3, 0); the graph of g passes through $(0, -2)$ and has negative slope.

56. Function f has no real zeros and its graph contains the point $(1, -2)$; function g has a zero at -3.

57. For functions $f(x) = 2x + 5$ and $g(x) = Int(x)$, find the solution set for **(a)** $(f \circ g)(x) = 0$ and **(b)** $(g \circ f)(x) = 0$

58. For functions $f(x) = Int(x)$ and $g(x) = x^2 - x - 6$, find the minimum value of $f \circ g$.

Exercises 59–60 ***Graphing Composition Functions*** *Graph* $g \circ f$ *where* $g(x) = x^2 - 4$ *and f is the given function. For* Y1 *enter the formula for* $f(x)$, *for* Y2 *enter* $x^2 - 4$, *and for* Y3 *enter* Y1² − 4 *(that is,* $(g \circ f)(x)$*). Before drawing the graphs, make a prediction about how* Y2 *and* Y3 *are related. Review material in Section 2.3 (Operating on the "inside") if necessary. Draw the graphs of* Y2 *and* Y3 *simultaneously and see if your prediction is correct.*

59. **(a)** $f(x) = x - 2$ **(b)** $f(x) = 2x$ **(c)** $f(x) = 0.5x$

60. **(a)** $f(x) = x + 4$ **(b)** $f(x) = 1.5x$ **(c)** $f(x) = 0.3x$

Exercises 61–62 *Replace* $g(x)$ *in Exercises 59–60 with* $g(x) = x^2 - 4x - 2$.

61. **(a)** $f(x) = x - 2$ **(b)** $f(x) = x + 2$ **(c)** $f(x) = -x$

62. **(a)** $f(x) = x - 3$ **(b)** $f(x) = x + 1$ **(c)** $f(x) = -x - 1$

Exercises 63–64 ***Graphs*** *Draw graphs of* $g(x) = x^2 - 4x - 2$ *and* $f \circ g$ *for the given function f. Before you draw the graphs, predict how they are related. See Section 2.3.*

63. **(a)** $f(x) = x + 2$ **(b)** $f(x) = |x|$

64. **(a)** $f(x) = x - 2$ **(b)** $f(x) = -x$

Exercises 65–66 ***Highest and Lowest Points*** *Draw a calculator graph of* $g \circ f$. *Find the coordinates of the* ***(a)*** *highest point and* ***(b)*** *the lowest point. See Example 5. (Hint: Let* Y1 = $f(x)$ *and* Y2 = g(Y1) *and draw the graph of* Y2.*)*

65. $f(x) = \dfrac{2x}{x^2 + 1}$ $g(x) = x^2 + 3x$

66. $f(x) = \dfrac{2x}{x^2 + 1}$ $g(x) = x^2 + 3x + 1$

Exercises 67–70 Express the given function as a composition of two of these four functions

$$f(x) = x - 4 \qquad g(x) = x^2 + 1$$

$$h(x) = \frac{1}{x} \qquad k(x) = |x|.$$

67. $F(x) = |x| - 4$

68. $G(x) = \dfrac{1}{|x|}$

69. $H(x) = \dfrac{1}{x^2} + 1$

70. $K(x) = x^2 - 3$

71. If $g(x) = 2x - 3$ and $f(g(x)) = 4x^2 - x$, find $f(-5)$.

72. If $g(x) = 4 - x^2$ and $f(g(x)) = \dfrac{3 - x^2}{x^2}$, find $f(3)$.

73. If $g(x) = x - 5$ and $f(g(x)) = \sqrt{x + 1}$, find $f(3)$.

74. If $g(x) = 2x + 5$ and $f(g(x)) = x^2 + 4$, find $f(-1)$.

Exercises 75–76 ***Iteration Evaluations*** *A function f is given. New functions are denoted* $f^{(1)}, f^{(2)}, f^{(3)}, \ldots$, *where* $f^{(1)}(x) = f(x), f^{(2)}(x) = f(f(x)), f^{(3)}(x) = f(f(f(x))), \ldots$. *Observe that the notation* $f^{(n)}$ *indicates repeated composition of f, not multiplication; that is,* $f^{(n)}(x)$ *is not the same as* $(f(x))^n$.

75. $f(x) = \dfrac{-3x}{2x + 3}$

(a) Evaluate $f^{(1)}(-1), f^{(2)}(-1), f^{(3)}(-1), f^{(4)}(-1)$.

(b) Based on your observations, what is $f^{(16)}(-1)$? $f^{(23)}(-1)$?

76. $f(x) = \dfrac{2x}{x - 2}$

(a) Evaluate $f^{(1)}(3), f^{(2)}(3), f^{(3)}(3), f^{(4)}(3)$.

(b) Based on your observations, what is $f^{(24)}(3)$? $f^{(47)}(3)$?

77. ***Maximum Cost*** A manufacturer determines that the cost C (in dollars) to build x graphing calculators is described by the equation

$$C = 80 + 48x - x^2 \qquad \text{for } 0 \le x \le 40.$$

Also, it is known that in t hours, the number x of calculators that can be produced is

$$x = 4t, \text{ where } 0 \le t \le 10.$$

(a) Express C as a function of t.

(b) What is the cost when the factory operates four hours?

(c) For what time t is the cost the greatest?

78. A rock is thrown into a lake causing a ripple in the shape of an expanding circle whose radius r is given by $r = \sqrt{t}$, where t is the number of seconds after the rock hits the water and r is measured in feet.

(a) What are the radius, circumference, and area of the circle when $t = 4$?

(b) Express the circumference C and area A as functions of t.

(c) At what time t is the circumference 8 feet?

(d) At what time t is the area 36 square feet?

79. A spherical balloon is being inflated in such a way that the diameter d is given by $d = \frac{t}{2}$, where t is measured in seconds and d in centimeters.

(a) Express the volume V of the balloon as a function of t.

(b) At what time t will the volume be 20 cubic centimeters?

80. ***Cost, Revenue, Profit*** A manufacturing company sells toasters to a retail store for \$25 each plus a fixed handling charge of \$15 on each order. The retailer applies a 30 percent markup to the total price paid to the manufacturer.

(a) Suppose the order consists of 20 toasters. How much does the retailer pay for the order? What is the retailer's total revenue from the sale of the 20 toasters? How much profit per toaster does the retailer make?

(b) Suppose C is the cost to the retailer for an order of x toasters, R is the total revenue from the sale of x toasters, and P is the profit per toaster. Find formulas that give C, R, and P as functions of x.

81. A circle is shrinking in size in such a way that the radius r (in feet) is a function of time t (in minutes), given by the equation $r = f(t) = \dfrac{1}{t + 1}$. The area of the circle is given by $A(r) = \pi r^2$, so the area is also a function of time, given by $(A \circ f)(t)$.

(a) Write a formula for $(A \circ f)(t)$.

(b) What is the area at the end of one minute? Two minutes?

(c) For what value of t is the area $\dfrac{\pi}{25}$?

82. ***Number of Bacteria*** The number of bacteria in a certain food is a function of the food's temperature. When refrigerated, the number is $N(T)$ at a temperature T degrees Celsius, described by the equation

$$N(T) = 10\,T^2 - 60\,T + 800, \text{ for } 3 \le T \le 13.$$

When the food is removed from the refrigerator the temperature increases and t minutes later the temperature is $T = 2t + 3$, for $0 \le t \le 5$.

(a) Determine an equation that describes the number of bacteria t minutes after the food is removed from the refrigerator.

(b) How many bacteria are in the food three minutes after it is removed from the refrigerator?

(c) How many minutes after the food is taken out of the refrigerator will it contain 2150 bacteria? Check graphically.

83. ***Volume of Balloon*** A spherical weather balloon is being inflated in such a way that the radius is $r = f(t) = 0.25t + 3$, where t is in seconds and r is in feet. The volume V of the balloon is the function $V(r) = 4\pi r^3/3$.

(a) What is the radius when the inflation process begins?

(b) Write an equation to describe the composition $V \circ f$ that gives the volume at t seconds after inflation begins.

(c) What is the volume of the balloon 10 seconds after inflation begins?

(d) In how many seconds will the volume be 400 cubic feet? Check graphically.

2.7 INVERSE FUNCTIONS AND PARAMETRIC EQUATIONS

> Is it not a miracle that the universe is so constructed that such a simple abstraction as a number is possible? To me this is one of the strongest examples of the unreasonable effectiveness of mathematics. Indeed, I find it both strange and unexplainable.
>
> *R. W. Hamming*

In Section 2.2 we noted that a function can be considered as a set of ordered pairs in which no two different pairs have the same first component. For each first number, any ordered pair can have exactly one second number.

As an example, suppose function f is $f(x) = 2x$, where the domain is $D = \{-2, -1, 0, 1, 2\}$. In terms of ordered pairs, f may be written as

$$f = \{(-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4)\}.$$

The range R of f is given by $R = \{-4, -2, 0, 2, 4\}$. Now suppose we interchange the two entries in each of the ordered pairs of f and get a new set of ordered pairs that we denote by g.

$$g = \{(-4, -2), (-2, -1), (0, 0), (2, 1), (4, 2)\}$$

We can make several observations concerning g:

1. Since no two pairs of g have the same first numbers, g is a function.
2. The domain of g is $D' = \{-4, -2, 0, 2, 4\}$ and the range is $R' = \{-2, -1, 0, 1, 2\}$.
3. Function g is $g(x) = \dfrac{x}{2}$, where $x \in D'$.
4. The domains and ranges of f and g are interchanged; $D' = R$ and $R' = D$.

Let us consider the composition function $g \circ f$ defined by $(g \circ f)(x) = g(f(x))$ for x in D. For instance,

$$\text{when } x \text{ is } -2,\ g(f(-2)) = g(-4) = -2$$
$$\text{when } x \text{ is } -1,\ g(f(-1)) = g(-2) = -1$$

and so on. In general,

$$g(f(x)) = g(2x) = \frac{2x}{2} = x.$$

We get a similar result for $f \circ g$:

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{2}\right) = 2\left(\frac{x}{2}\right) = x.$$

The functions f and g that we have been discussing are related in a special way—one is the **inverse** of the other. Since $f(g(x)) = x$ and $g(f(x)) = x$, we may say that each of the functions "undoes" or neutralizes the other. If we start with x, apply f and get $f(x)$, and then apply g to $f(x)$, we get back to x.

Schematically, think of a function as a map that sends each element in the domain to a corresponding range element. The inverse function sends each element of the range to the original element of the domain. A diagram like Figure 43 may help clarify the relationship. Observe that the diagram may be read in either direction so that applying f and then g, or g and then f, always yields the initial input.

> *[Before I zeroed in on math], I found the whole university just fascinating. There was that rare books library that had everything in the world in it. I loved the history courses. I loved the English courses. I enjoyed physics very much. Even in my senior year I took courses all over the map—almost as much philosophy as mathematics, almost as much history as mathematics, almost as much English as mathematics, almost as much Spanish as mathematics.*
>
> Mary Ellen Rudin

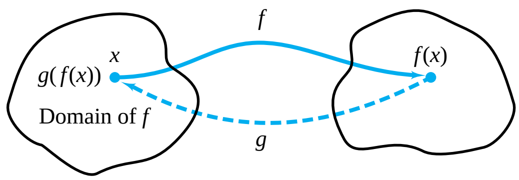


FIGURE 43
Inverse function: $g(f(x)) = x$
for every x in the domain of f.

> ***Definition: inverse functions***
>
> Suppose f and g are functions that satisfy two conditions: $g(f(x)) = x$ for every x in the domain of f, and $f(g(x)) = x$ for every x in the domain of g. Then f and g are inverses of each other.

Characterization of Inverse Functions

Suppose f is a function described as a set of ordered pairs such that no two pairs have the same second element, $f = \{(x, y) \mid y = f(x)\}$.

Let g be the set of ordered pairs obtained by interchanging the elements of each pair of f. If g is a function, then f and g are *inverses of each other*.

Notation for Inverse Functions

Suppose g is the inverse of function f. It is customary to denote g by f^{-1}. Note that f^{-1} does not mean the reciprocal of f, which we would write as $\frac{1}{f}$. Replacing g by f^{-1} in the above definition gives an important pair of identities.

> ***Inverse function identities***
>
> $$f^{-1}(f(x)) = x \text{ for each } x \text{ in the domain of } f.$$
> $$f(f^{-1}(x)) = x \text{ for each } x \text{ in the domain of } f^{-1}.$$

►EXAMPLE 1 Verify inverse

Strategy: Simply verify that $f(f^{-1}(x)) = x$ and $f^{1}(f(x)) = x$ for every real number x.

(a) Verify that

$$f(x) = 2x + 3 \qquad \text{and} \qquad f^{-1}(x) = \frac{x - 3}{2}$$

are inverses of each other.

(b) Draw graphs of $y = f(x)$ and $y = f^{-1}(x)$.

Solution

(a) Follow the strategy. For every real number x,

$$f^{-1}(f(x)) = f^{-1}(2x + 3) = \frac{(2x + 3) - 3}{2} = \frac{2x}{2} = x.$$

$$f(f^{-1}(x)) = f\left(\frac{x - 3}{2}\right) = 2\left(\frac{x - 3}{2}\right) + 3 = (x - 3) + 3 = x.$$

Therefore, the given functions are inverses of each other.

(b) The graphs of $y = 2x + 3$ and $y = \frac{x - 3}{2}$ are the lines shown in Figure 44. ◄

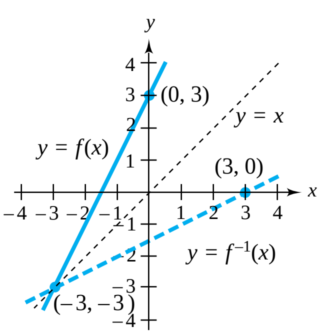


FIGURE 44

Graphs of Inverse Functions

The graphs of the functions $y = f(x)$ and $y = f^{-1}(x)$ in Figure 44 appear *symmetric with respect to the line* $y = x$, that is, each graph is a reflection of the other through that line. We want to show that for any pair of inverse functions, the graphs of f and f^{-1} are symmetric about the line $y = x$.

Suppose that (a, b) is any ordered pair in f. This implies that (b, a) is an ordered pair in f^{-1}. If we denote the line $y = x$ by L, then we must show that the points $P(a, b)$ and $Q(b, a)$ are reflections of each other in L; that they are *equidistant from* L and that *the line through* P *and* Q *is perpendicular to* L. The midpoint M of segment PQ has coordinates $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$. Direct computation verifies that M lies on line L and that P and Q are equidistant from L. Further, the line through P and Q has slope given by $m = \frac{b-a}{a-b} = -1$, and hence is perpendicular to L since the slope of L is 1. See Figure 45.

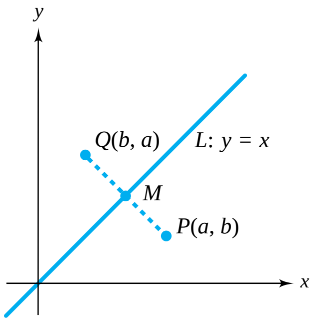


FIGURE 45

Graphs of inverse functions

For each point (a, b) on the graph of $y = f(x)$, the point (b, a) belongs to the graph of $y = f^{-1}(x)$; that is, **coordinates of every point are interchanged.**

The graph of $y = f^{-1}(x)$ is a **reflection through the line** $y = x$ of the graph of $y = f(x)$.

The domain of f becomes the range of f^{-1}, and conversely; that is, the **domains and ranges are interchanged.**

Finding Equations for Inverse Functions

It is not always easy to find an equation that describes the inverse of a particular function. In many cases, however, the inverse function does have an equation that can be found readily by a straightforward algorithm. The key is the observation that finding f^{-1} from f requires interchanging x and y in each ordered pair (x, y) of f to obtain the ordered pairs in f^{-1}.

Algorithm: finding an equation for an inverse function

Step 1. Write the equation defining f in the form $y = f(x)$.
Step 2. Interchange y and x to get $x = f(y)$.
Step 3. Solve the equation $x = f(y)$ for y, and adjust the domain as needed.
Step 4. The result is $y = f^{-1}(x)$.

►EXAMPLE 2 ***Formula for inverse*** Find a formula for the inverse of $f(x) = 2x - 1$ and verify that $f(f^{-1}(x)) = x$.

Strategy: First write $y = 2x - 1$ and then follow the steps of the algorithm.

Solution

Follow the strategy.

Step 1 $\quad y = 2x - 1$

Step 2 $\quad x = 2y - 1$

Step 3 $\quad y = \dfrac{x+1}{2}$; therefore, $f^{-1}(x) = \dfrac{x+1}{2}$.

To verify that $f(f^{-1}(x)) = x$,

$$f(f^{-1}(x)) = f\left(\frac{x+1}{2}\right) = 2\left(\frac{x+1}{2}\right) - 1 = (x+1) - 1 = x. \quad ◀$$

▶EXAMPLE 3 ***A function and its inverse*** Let $g(x) = \sqrt{3-x}$. Find the domain and range of g, and find a formula for the inverse function g^{-1}, and state its domain and range. Sketch a graph of both g and g^{-1}.

Solution

By the domain convention, g is defined when $3 - x \geq 0$, so the domain of g is the interval $(-\infty, 3]$, and since a square root is always nonnegative, the range is the interval $[0, \infty)$. To find a formula for g^{-1}, we follow the steps of the algorithm.

Steps 1 and 2. Write $y = \sqrt{3-x}$, and interchange variables, $x = \sqrt{3-y}$.

Step 3. To solve for y, we begin by squaring, $x^2 = 3 - y$. Then $y = 3 - x^2$. For the inverse function, the domain and range of f are interchanged, so the domain is the interval $[0, \infty)$, and the range is $(-\infty, 3]$.

Step 4. With the restricted domain, we have $g^{-1}(x) = 3 - x^2$, $x \geq 0$.

The graph of g is the upper half of a parabola opening to the left; the graph of g^{-1} is the right half of a parabola opening downward. See Figure 46. ◀

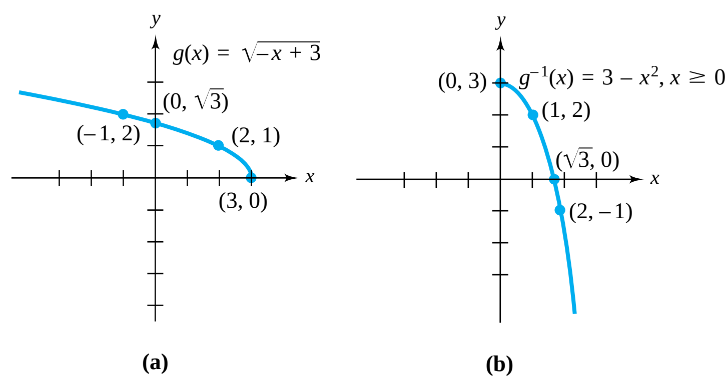


FIGURE 46

Using Parametric Mode to Graph Inverses

Graphing calculators have a mode of graphing called **parametric mode** which allows us to visualize directly what happens when we interchange the ordered pairs in a function. In parametric mode we use separate equations for the x- and y-coordinates, defining both coordinates as functions of a new variable. We study parametric equations in some detail later in the book, but we make use of the parametric mode on graphing calculator whenever it can help us to understand ideas.

To illustrate, suppose we have a pair of equations, $x = t$, $y = t^2$. Then, for each value of t, we get a pair of numbers (x, y) with the property that y is the *square* of x. For a given set of t-values, if we graph the pairs (x, y) thus determined, we get a set of points satisfying the equation $y = x^2$. That is, the two equations $x = t$, $y = t^2$, together define part (or all) of the parabola we have previously defined by the single equation $y = x^2$. The variable t is called a **parameter,** and the equations $x = t$, $y = t^2$, are called **parametric equations** for the parabola $y = x^2$.

HISTORICAL NOTE

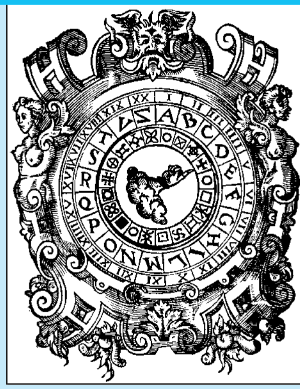
INVERSE FUNCTIONS AND CRYPTOGRAPHY

Encoding and decoding secret messages depends on functions and their inverses. Each letter is assigned a number (often, its place in the alphabet) and a coding function is applied to the number. A simple Caesar cipher is given by $f(n) = n + 5 \pmod{26}$, meaning that $n + 5$ is reduced by multiples of 26 when necessary. $S = 19$ becomes $f(19) = 24 = X$. SEND MONEY becomes XJSI RTSJD. Decoding uses the inverse function $f^{-1}(n) = n - 5 \pmod{26}$.

In a slightly more complex function, $F(n) = 3n + 5 \pmod{26}$, the letter S, which corresponds to 19, becomes

$$\begin{aligned} F(19) &= 3 \cdot 19 + 5 \pmod{26} \\ &= 62 \pmod{26} = 10. \end{aligned}$$

Therefore $F(19) = 10$, and since J corresponds to 10, S becomes J. The inverse function to decode JTUQ RXUTB is given by $F^{-1}(n) = 9n + 7 \pmod{26}$.



This Renaissance crypt-analysis instrument works by rotating the inner disk against the stationary outer disk.

Even though the coding functions have become extremely complex, involving continual modifications, until very recently *all* cryptology algorithms required the *same* work of the cryptographer and the decrypter. Knowing how to encode (which required the coding function) meant knowing how to decode (which required the inverse function). All this has changed with the invention of trapdoor codes, which have efficient algorithms for both functions and inverses, but for which inverses are effectively impossible to discover, so no one can break the code.

The most impressive are the RST codes (named for their discoverers). These depend on finding large primes whose products cannot easily be factored. A few minutes of computer time can produce 100-digit primes, but factoring the product of two such numbers would typically require millions of years.

The set of points satisfying the equation $x = y^2$ cannot be graphed as a single function on a graphing calculator, but we could graph *two functions*, $y = \sqrt{x}$ and $y = -\sqrt{x}$. Together, they form a parabola opening to the right, the graph of the inverse relation of the function $f(x) = x^2$. In parametric mode, however, it is just as easy to use the equations $y = t$, $x = t^2$ for the parabola $x = y^2$ as it is to use $x = t$, $y = t^2$ for the parabola $y = x^2$. In fact, for any function $y = f(x)$, we can use the parametric equations $x = t$, $y = f(t)$ for the function, and the parametric equations $y = t$, $x = f(t)$ define the inverse relation (which may be a function, but need not be). The fact that there can be different sets of parametric equations to define the same curve will not concern us here; we have a method to allow us to write a set of parametric equations for any given function and for its inverse.

Parametric equations for a function and its inverse

To graph a function $y = f(x)$ in parametric form, use equations

X = T, Y = f(T),

and to graph the inverse of f (which need not be a function), use equations

Y = T, X = f(T).

We discuss below conditions that will allow us to tell when the inverse of f is a function.

TECHNOLOGY TIP ◆ ***Parametric graphing and t-range***

When graphing in parametric mode, we must set a t-range as well as x-and y-ranges. If we are using equations

X = T, Y = f(T),

we can see all of the graph that appears in the window if we set the t-range to match the x-range. The tStep (or t-Pitch, or Step) determines how many points will be plotted. Usually something around 0.1 gives a reasonable graph without taking too long. Experiment with your calculator.

Limiting the t-range allows us to graph just a portion of f. For the simple example $x = t$, $y = t^2$, setting a t-range from -1 to 1 gives the "tip" of the parabola, from the point $(-1, 1)$ to the point $(1, 1)$. Similarly, for the t-range $[0, 5]$ in the decimal window, we see only the part of the parabola in the first quadrant. Again, *experiment*.

▶***EXAMPLE 4*** ***Parametric graphs*** Let $g(x) = \sqrt{3 - x}$. In parametric mode sketch a calculator graph of both g and g^{-1} on the same screen.

Solution

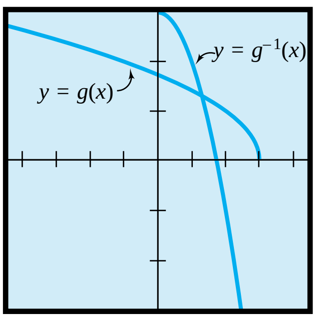
This is the same function as in Example 3.

With the calculator in parametric mode (make sure you know how to set the proper mode on your calculator), enter

X = T, Y = √(3 − T),

and to graph the inverse of g use equations

Y = T, X = √(3 − T).



[−4.5, 4.5] by [−3.1, 3.1]

FIGURE 47

For the t-range, use the same interval as the x-range. If, for example you are using a decimal window with Xmin $= -4.7$, Xmax $= 4.7$, use the same values for Tmin and Tmax and set Tstep $= .1$. When you graph, both g and g^{-1} should appear on the same screen, as in Figure 47. Note that when you TRACE, you can read the coordinates of all three variables, x, y, and t. ◀

Existence of Inverse Functions

Not every function has an inverse that is a function. For the function $y = x^2$, if we interchange x and y to get $x = y^2$ and then solve for y, we have $y = \pm\sqrt{x}$. For each $x > 0$ there are two corresponding values of y, so we do not have a function.

When we say that a function "has an inverse," we mean that its inverse is a function.

To determine whether or not a function f has an inverse, we look back at ordered pairs (x, y) of f, where $y = f(x)$. We know that for every x in the domain D of f, there is exactly one value of y. Now suppose we interchange x and y. The set of ordered pairs (y, x) will be a function only if for each y there is exactly one x. Therefore, for $y = f(x)$ to have an inverse each x must correspond to exactly one y (so f is a function) and every y to exactly one x. We call such a function a **one–one function.**

Existence of an inverse function—Part I

A function has an inverse if and only if the function is one–one.

How can we tell when a function is one–one? The best way is to draw a graph. Section 2.2 introduced the vertical line test to determine whether a graph represents a function. If we combine this test with the horizontal line test described below, we can determine whether or not a graph is that of a one–one function.

Horizontal line test

If every horizontal line intersects the graph of a function in at most one point, then that function is one–one. Therefore, it has an inverse.

Figure 48 shows graphs of three functions. The graphs in panels **(a)** and **(b)** represent one–one functions while that in panel **(c)** does not. In panel **(c)** horizontal lines like L_1 intersect the graph at one point, but lines such as L_2 intersect the graph at more than one point.

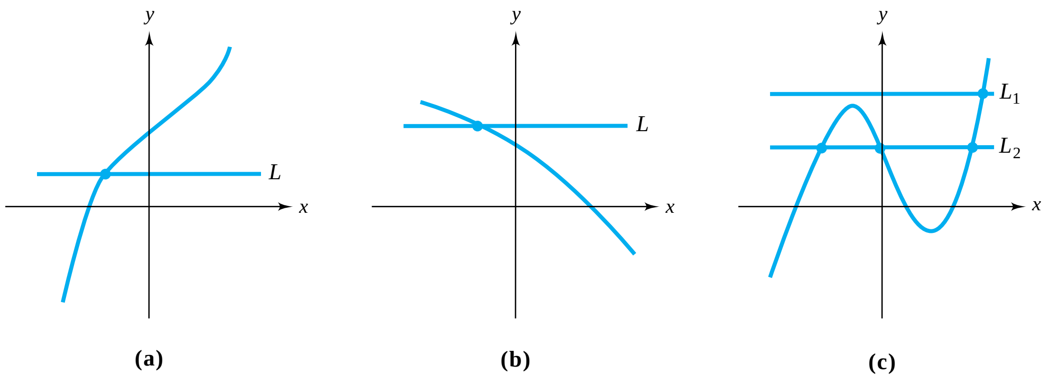


FIGURE 48

A useful criterion for determining whether the inverse of a function is itself a function comes from a property we can read from a graph. A function whose graph *rises* as we move from left to right is called an **increasing function.** Similarly, if the graph **drops** (goes down) as we move from left to right, the function is called **decreasing.** We give a formal definition, but visualizing the distinction as in Figure 49 is at least as helpful.

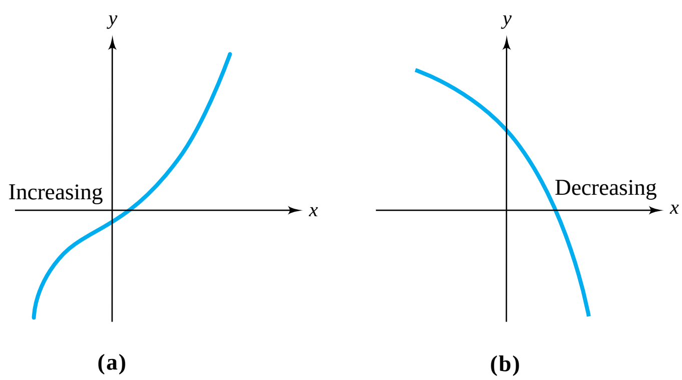


FIGURE 49

Definition: Increasing functions, decreasing functions

Suppose b and c are any numbers in the domain of a function f where $b < c$. The function f is an increasing function if $f(b) < f(c)$, and it is a decreasing function if $f(b) > f(c)$.

In terms of increasing and decreasing functions we can restate the condition for the existence of an inverse function.

Existence of an inverse function—Part II

Function f has an inverse if f is either an **increasing function** or a **decreasing function.**

▶EXAMPLE 5 ***Existence of inverse*** Determine which functions have inverses:

(a) $f(x) = 2x$ **(b)** $g(x) = \frac{1}{x}$ **(c)** $h(x) = x^2 - 2x$.

Strategy: In each case sketch a graph and then see if every horizontal line intersects the graph in at most one point.

Solution

(a) The graph of $y = 2x$ is a line as shown in Figure 50(a). Every horizontal line intersects the graph at exactly one point, so f is a one–one function and it has an inverse.

(b) The graph of $y = \frac{1}{x}$ is shown in Figure 50(b). Every horizontal line except $y = 0$ (the x-axis) intersects the graph at one point and the line $y = 0$ does not intersect at any point. Thus g is a one–one function and it has an inverse.

(c) The graph of $y = x^2 - 2x$ is a parabola, as shown in Figure 50(c). Clearly there are horizontal lines that intersect the graph in more than one point, so h does not have an inverse. ◀

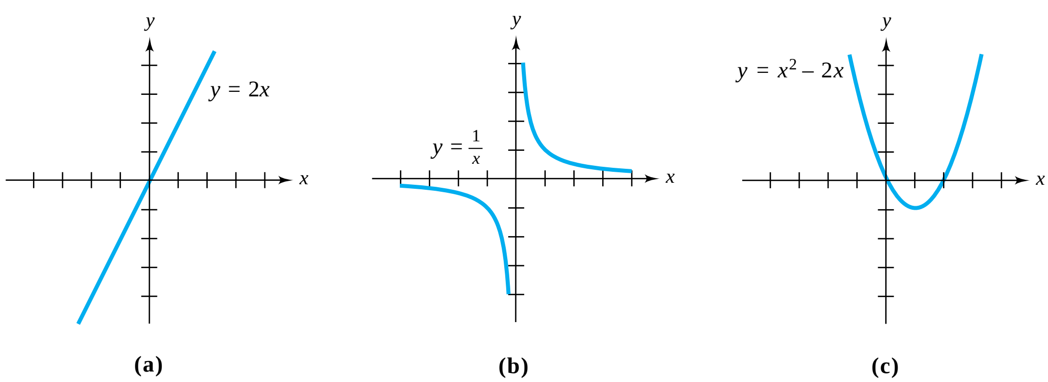


FIGURE 50

▶EXAMPLE 6 ***Increasing and decreasing functions*** Determine whether the given function is increasing, decreasing, or neither.

(a) $f(x) = x^3$ **(b)** $g(x) = -x^2$ **(c)** $h(x) = -x^2$, for $x \geq 0$

Strategy: In each case draw a graph and determine whether the function is increasing, decreasing, or neither.

Solution

Follow the strategy.

(a) The graph of $y = x^3$ is shown in Figure 51a. Clearly f is an increasing function.

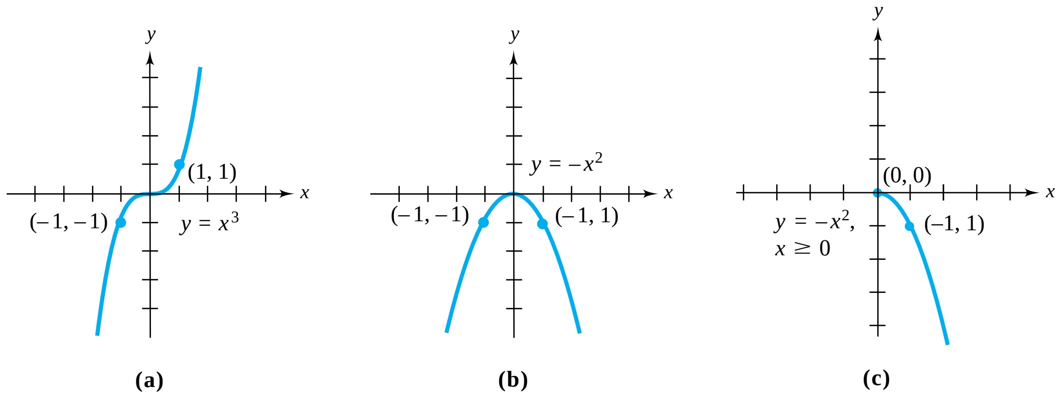


FIGURE 51

(b) The graph of $y = -x^2$ is a parabola as shown in Figure 51b. The graph shows that g is neither increasing nor decreasing.

(c) The graph of $y = -x^2$, $x \geq 0$, is the portion shown in Figure 51c. The graph indicates that h is a decreasing function. ◀

Parametric equations are ideal for graphing functions that have limited domains, as in the next example.

►EXAMPLE 7 ***Parametric graphs, functions, and inverses*** Sketch a calculator graph of the function given parametrically and give the domain and range. Then graph the inverse and determine if the inverse is a function.

(a) $f : x = t, \quad y = t^2 - 2t, \quad 0 \leq t \leq 2.$

(b) $g : x = t, \quad y = t^2 - 2t, \quad 1 \leq t \leq 3.$

Solution

(a) We put the calculator in parametric mode and enter X = T, Y = T² − 2T, set the t-range tMin = 0, tMax = 2, tStep = 0.1, and graph. We get the portion of the parabola $y = x^2 - 2x$ lying below the x-axis, with domain and range given by $D = [0, 2]$, $R = [-1, 0]$. See Figure 52a.

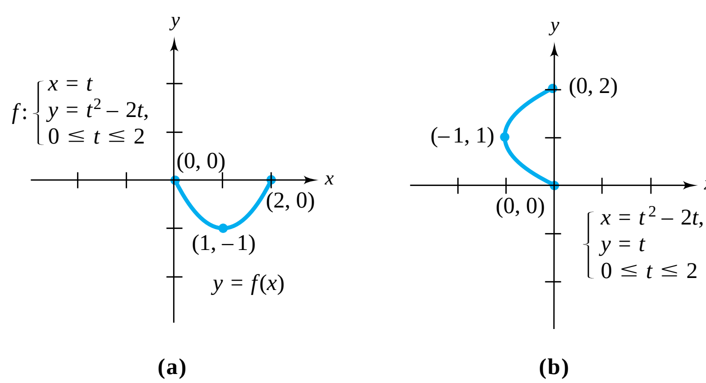
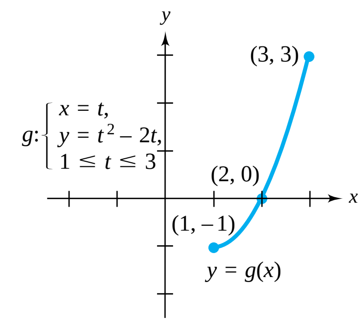


FIGURE 52

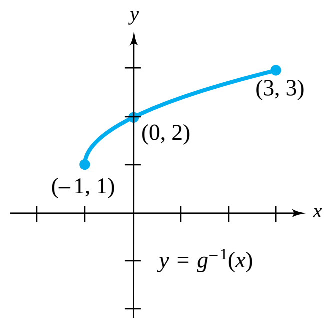
For the inverse of f, we interchange the x- and y-equations, X = T² − 2T, Y = T, keeping the same t-range, and graph. Again we have the tip of a parabola, this time opening to the right, shown in Figure 52b. We have labeled points on the two graphs that correspond when we interchange coordinates. From the graph, the domain and range of f^{-1} are given by $D = [-1, 0]$, $R = [0, 2]$. By the vertical line test, f^{-1} is not a function, as we could have foretold from the graph of f.

(b) Following the same steps for the function g, we set the t-range tMin = 1, tMax = 3, and the graph is shown in Figure 53a, part of the same parabola $y = x^2 - 2x$, from the vertex $(1, -1)$ to the point $(3, 3)$, so *g is an increasing function.* Domain and range are given by $D = [1, 3]$, $R = [-1, 3]$.

Interchanging coordinates, keeping the same t-range, the graph of g^{-1} is the reflection of the graph of g in the line $y = x$, as in Figure 53b. The domain and range of g^{-1} are obtained by interchanging the domain and range of g, and since g is increasing, g^{-1} is a function. ◀



(a)



(b)

FIGURE 53

TECHNOLOGY TIP ◆ ***t-range for inverse functions***

When graphing in parametric mode, remember the **domain and range are interchanged** for a function and its inverse.

If you **interchange defining equations,** Y = T, X = −F(T), the ***t*-range remains the same** because you are now are setting the range for the y-variable.

In some situations, particularly in applications, we are accustomed to using letters other than x and y to describe functions. Usually in such cases we want to keep the original labels when finding formulas for inverse functions. Rather than interchange variables, we simply solve for the variable we want, as illustrated in the following example.

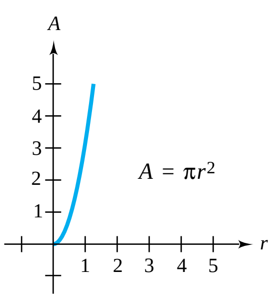


FIGURE 54
Area of a circle as a function of the radius

▶EXAMPLE 8 *Inverse function in application*

The area A of a circle is a function f of its radius r, $f(r) = \pi r^2$, or $A = \pi r^2$, where $r > 0$. Find an equation for the inverse function.

Solution

The graph of $A = \pi r^2$, shown in Figure 54, indicates that f is an increasing function, so it has an inverse. To find a formula for the inverse, it would certainly be confusing to switch the variables A and r. Therefore, simply solve the equation $A = \pi r^2$ for r and get $r = \sqrt{\frac{A}{\pi}}$. Therefore, the inverse function is given by $f^{-1}(A) = \sqrt{\frac{A}{\pi}}$, or by $r = \sqrt{\frac{A}{\pi}}$, where $A > 0$. ◀

Often, in applications like Example 8, one can omit the formalism of introducing f and f^{-1}. It is sufficient to say that the area A and the radius r are related and that the equation $A = \pi r^2$ defines A as a function of r, while $r = \sqrt{\frac{A}{\pi}}$ defines r as a function of A. The two functions are inverses of each other.

TECHNOLOGY TIP ◆ ***Built-in inverse graphs***

The TI-82 and the TI-85 both have a built-in-routine for drawing the graph of any inverse relation. If you have a calculator that draws inverses automatically, you should learn how to use it, but don't treat it as a magic key that performs a function that is not understood. Learn what inverses are and how to graph them; then take advantage of any technology that makes your work more efficient. Begin by entering a function as, say, Y1.

TI-82: 2nd DRAW 8 (DrawInv). This enters DrawInv on the home screen, and you enter the Y1 from the Y-VARS menu.

TI-85: GRAPH MORE DRAW MORE MORE DrawInv. This enters DrawInv on the home screen, and you enter Y1.

EXERCISES 2.7

Check Your Understanding

Exercises 1–5 True or False. Give reasons. For these exercises, assume that f is a function that has an inverse.

1. If the graph of f contains points in Quadrant III, then the graph of f^{-1} must also contain points in Quadrant III.
2. If $(-2, -3)$ is a point on the graph of f, then $(2, 3)$ must be a point on the graph of f^{-1}.
3. If the graph of f is a line having negative slope, then the graph of f^{-1} must be a line also with negative slope.
4. If the graph of f has a y-intercept point, then the graph of f^{-1} must have an x-intercept point.
5. The graph of any function that has an inverse cannot cross the x-axis at more than one point.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. If $f(-2) = 4$, then $f^{-1}(4) =$ ______.
7. If the graph of f^{-1} contains points in Quadrants I and II, then the graph of f must contain points in Quadrants ______.
8. If the graph of f contains points in Quadrant II, then the graph of f^{-1} must contain points in Quadrant ______.
9. If $f(2) = -5$, then a point on the graph of f^{-1} is ______.
10. For $f(x) = x^3 + x^2 + x - 4$, if you draw graphs of $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ on the same screen, then the display will show that all three graphs intersect at a point in Quadrant ______.

Develop Mastery

*Exercises 1–4 Function f is given as a set of ordered pairs. **(a)** Interchange the two entries in each ordered pair and get a new set S of ordered pairs. **(b)** Is S a function?*

1. $f = \{(0, -1), (1, 3), (2, 5)\}$
2. $f = \{(-1, 1), (0, 3), (1, 1)\}$
3. $f = \{(-3, 4), (-1, 2), (1, 1), (3, 2)\}$
4. $f = \left\{(0, 0), \left(\frac{\pi}{2}, 1\right), \left(\frac{-3\pi}{2}, -1\right)\right\}$

Exercises 5–8 Use the definition of inverse function to determine whether or not functions f and g are inverses of each other. That is, determine whether $f(g(x)) = x$ and $g(f(x)) = x$.

5. $f(x) = -1 - 2x$, $g(x) = -\dfrac{x + 1}{2}$
6. $f(x) = \dfrac{x + 1}{2x}$, $g(x) = \dfrac{1}{1 - 2x}$
7. $f(x) = 2 + \dfrac{1}{x}$, $g(x) = \dfrac{1}{2 - x}$
8. $f(x) = 4 - \dfrac{1}{x}$, $g(x) = \dfrac{1}{4 - x}$

*Exercises 9–12 **Parametric Equations** Function f has an inverse. **(a)** Using parametric equations draw graphs of f and f^{-1}. **(b)** If the graphs intersect, find the points of intersection (1 decimal place).*

9. $f(x) = x^2 + x - 2$, $x \geq 0$
10. $f(x) = 3 + 2x - x^2$, $x \leq 0$
11. $f(x) = \sqrt{4 - x}$
12. $f(x) = x^3 + 2x - 3$

Exercises 13–14 ***Is the Inverse a Function?*** *Use parametric equations to graph the inverse of f. Does the vertical line test indicate that the inverse is a function?*

13. **(a)** $f(x) = x^3 - 3x^2 + 3$
(b) $f(x) = x^3 - x^2 + 3x - 2$

14. **(a)** $f(x) = x^3 - 3x + 3$
(b) $f(x) = x^3 - x^2 + 4x - 2$

Exercises 15–24 ***Inverse Function Formulas*** *Function f has an inverse. Apply the algorithm in this section to find an equation that describes f^{-1}.*

15. $f(x) = 2x + 5$ **16.** $f(x) = 2 - 5x$

17. $f(x) = \dfrac{1 + x}{x}$ **18.** $f(x) = \dfrac{2x}{3 - x}$

19. $f(x) = \dfrac{2 - x}{x}$ **20.** $f(x) = 1 - \dfrac{3}{x}$

21. $f(x) = x$ **22.** $f(x) = -\sqrt{-x}$

23. $f(x) = x^2 - 2x + 1, x \geq 1$

24. $f(x) = x^2 - 2x - 3, x \leq 1$

Exercises 25–30 ***Horizontal Line Test*** ***(a)*** *Draw a graph of $y = f(x)$.* ***(b)*** *Use the horizontal line test to determine whether the function is one–one.* ***(c)*** *Does f have an inverse?*

25. $f(x) = 2x + 3$ **26.** $f(x) = 1 - x^2$

27. $f(x) = 4 - x^2$ **28.** $f(x) = 3 - x$

29. $f(x) = \sqrt{x}$ **30.** $f(x) = x^3$

Exercises 31–32 ***Inverse of Linear Function*** *The graph of function f is the line through points P and Q. Give a formula for* ***(a)*** *f,* ***(b)*** *f^{-1}. Draw a graph of each.*

31. $P(-2, 3), Q(2, 5)$ **32.** $P(2, -4), Q(4, 2)$

Exercises 33–34 ***Line Segment Function*** *The graph of f is the line segment having endpoints P and Q. Give a formula for* ***(a)*** *f,* ***(b)*** *f^{-1}, and give domain and range restrictions. Draw a graph of each.*

33. $P(-1, 2), Q(-3, 4)$ **34.** $P(-2, -1), Q(2, 4)$

Exercises 35–36 The graph of the inverse of function f is a line through points P and Q. Give a formula for ***(a)*** *f and* ***(b)*** *f^{-1}. Draw a graph of each.*

35. $P(-2, 4), Q(3, 1)$ **36.** $P(3, -2), Q(4, 3)$

Exercises 37–38 ***Related Functions*** *The graph of function f is a line through points P and Q. Functions g and h are given by $g(x) = f^{-1}(x)$, $h(x) = f^{-1}(x - 2)$. Find a formula for* ***(a)*** *g,* ***(b)*** *h. Give the coordinates of the intercept points for the graph of g, and h.*

37. $P(-1, 4), Q(3, 2)$ **38.** $P(2, -4), Q(4, 2)$

Exercises 39–44 ***Formula for Inverse Function*** *Function f has an inverse.* ***(a)*** *Find a formula for f^{-1}.* ***(b)*** *Use the function menu. Draw graphs of $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same screen. Are the graphs of f and f^{-1} symmetric about the graph of $y = x$?*

39. $f(x) = x + 3$ **40.** $f(x) = 2x - 4$

41. $f(x) = x^2 - 4, D = [0, \infty)$

42. $f(x) = x^2 - 1; D = (-\infty, 0]$

43. $f(x) = 0.25x^2 + 1.5x + 2; D = [-2, \infty)$

44. $f(x) = 0.25x^2 + 0.5x - 2; D = [2, \infty)$

Exercises 45–48 ***Parametric Equations*** *Given that function f has an inverse. Use parametric equations to draw a graph of f^{-1}. Trace to find $f^{-1}(c)$. The answer should be a root of $f(x) = c$. Note that we do not use the same technique as in Exercises 39–44 since these equations would be difficult to solve for y in terms of x. (Hint: $x = f(t)$, $y = t$.)*

45. $f(x) = x^3 + x - 1; c = 9$

46. $f(x) = x^3 + x - 2; c = -12$

47. $f(x) = x^3 + x^2 + x - 3; c = -9$

48. $f(x) = x^3 - x^2 + x - 1; c = 20$

Exercises 49–54 ***Increasing–Decreasing*** ***(a)*** *Draw a graph of the function and determine whether it is increasing, decreasing, or neither.* ***(b)*** *Does the function have an inverse?*

49. $f(x) = 3 - 2x$ **50.** $g(x) = 2 + 3x$

51. $f(x) = -x$ **52.** $f(x) = x^2 - 2x + 1$

53. $g(x) = \sqrt{x}$ **54.** $f(x) = -\sqrt{-x}$

55. Verify that the two points A and B are symmetric with respect to the line $y = x$. (*Hint:* You need to show that the line $y = x$ is the perpendicular bisector of $\overline{AB}$.)
(a) $A(-2, 3), B(3, -2)$
(b) $A(4, 7), B(7, 4)$
(c) $A(-3, -5), B(-5, -3)$

56. Find an equation for line L such that $A(-4, 2)$ and $B(2, 6)$ are symmetric with respect to L. (*Hint:* L must be the perpendicular bisector of the segment $\overline{AB}$.)

57. Find an equation for line L such that $A(2, -4)$ and $B(4, -2)$ are symmetric with respect to L. (*Hint:* See Exercise 56.)

58. If function f has an inverse and if $f(2) = -3$ and $f(-1) = 4$, find $f^{-1}(-3)$ and $f^{-1}(4)$.

59. ***Solving Inequalities*** Given that $f(x) = x^3 + 4x - 3$ has an inverse. Use a graph of f^{-1} and trace to find the solution set for $-2 \leq f^{-1}(x) \leq 3$.

60. If $f(x) = 2x + 1$ and $g(x) = 4x + c$, then for what value of c will $f(g(x))$ be equal to $g(f(x))$?

61. **(a)** The function $f(x) = \dfrac{-3x}{2x + 3}$ has an inverse. Show that $f^{-1}(x) = f(x)$.

(b) If c is a constant and $f(x) = \dfrac{cx}{x - 2}$, then find the value of c such that $f(x)$ will have an inverse equal to itself. That is, find c such that $f(f(x)) = x$.

62. If $f(x) = \dfrac{2x + 3}{3x - 2}$, then show that $f(f(x)) = x$ for every real number x (except $\frac{2}{3}$).

63. If $f(x) = \dfrac{ax + b}{cx - a}$, then show that $f(f(x)) = x$ for every real number x (except $\frac{a}{c}$).

Exercises 64–65 ***Line Segment Function and Inverse*** *The graph of function f consists of the line segment joining the points A(−3, −1) and B(6, 2).*

64. **(a)** Draw the graph of f and explain why f has an inverse.

(b) What are $f^{-1}(-1)$ and $f^{-1}(2)$? Draw a graph of f^{-1} and state its domain and range.

65. Draw a graph. On each graph label the intercept points. (*Hint:* Use translations.)

(a) $y = f(x - 1)$ **(b)** $y = f^{-1}(x - 1)$
(c) $y = f(x) + 2$ **(d)** $y = f^{-1}(x) + 2$

Exercises 66–68 ***Union of Two Line Segments*** *The graph of the function f is the union of the two line segments $\overline{AB}$ and $\overline{BC}$, with the points A(−4, 2), B(−2, −2), and C(4, −5).*

66. **(a)** Draw a graph and explain why f must have an inverse.

(b) Give a piecewise formula for f.

(c) If $(-3, b)$ and $(2, c)$ are on the graph of f, determine b and c.

67. What are the domain and range of

(a) f? **(b)** f^{-1}?

(c) Evaluate $f^{-1}(2), f^{-1}(-2), f^{-1}(-5)$.

68. **(a)** Give a piecewise formula for f^{-1}.

(b) Find the coordinates of the x- and y-intercept points for the graph of f^{-1}.

Exercises 69–72 ***Evaluating Inverse Function*** *Function f has an inverse. Draw a graph to support this claim. Use algebraic techniques to find $f^{-1}(c)$. (Hint: Solve $f(x) = c$. Square as needed.)*

69. $f(x) = \sqrt{x + 2}$; $c = 5$

70. $f(x) = \sqrt{x} + 2$; $c = 5$

71. $f(x) = \sqrt{x^3 + 17} - x\sqrt{x} + 5$; $c = 6$

72. $f(x) = x + 4 - \sqrt{x^2 + 8}$; $c = 2$

73. ***Volume of a Sphere*** The volume of a sphere of radius r is given by the formula $V = \frac{4}{3}\pi r^3$, defining V as a function f of r.

(a) Solve for r in terms of V to get $r = f^{-1}(V)$.

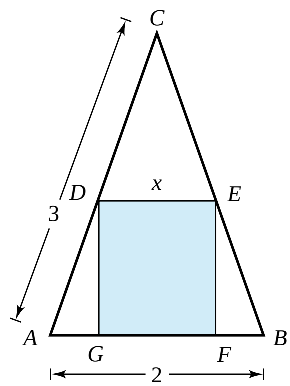
(b) Draw a graph of f^{-1} and use TRACE to find the value of r when V is 3.47, 4.83, 5.72.

74. A ball is dropped from the top of a building that is 144 feet tall. The position of the ball at any time t seconds after it is dropped is given by the formula $s = 16t^2$, where s is the distance in feet from the top of the building. This determines s as a function of t.

(a) What is the domain of this function?

(b) Solve for t in terms of s and then find the time (to one decimal place) that the ball takes to reach distances of 20 feet, 40 feet, and 80 feet from the top of the building.

75. ***Area of Inscribed Rectangle*** A rectangle $DEFG$ is inscribed in an isosceles triangle ABC as shown in the diagram, where $|\overline{AC}| = |\overline{BC}| = 3$ and $|\overline{AB}| = 2$. Let $|\overline{DE}| = x$.



(a) Find a formula that gives the area K of the rectangle as a function f of x.

(b) Give the domain and range of f.

(c) Is f a one-one function?

(d) What value or values of x will give a rectangle of area 0.5?

76. The function to convert from degrees Fahrenheit (°F) to degrees Celsius (°C) is given by $C = \frac{5}{9}(F - 32)$. Find a formula for the inverse function.

77. Suppose a cone has a fixed height of 9 inches and variable radius r. Its volume V is a function of r given by $V = 3\pi r^2$. Find an equation that describes the inverse function.

78. ***Your Choice***

(a) Give a formula for a linear function f of your choice that is increasing and has a graph that contains the point $(-2, 0)$.

(b) Give the coordinates of two points on the graph of f^{-1}.

2.8 FUNCTIONS AND MATHEMATICAL MODELS

> At one time [Conway] would be making constant appeals to give him a year, and he would immediately respond with the date of Easter, or to give him a date, so that he could tell you the day of the week or the age of the moon.
>
> *Richard K. Guy*

In this section our discussion will be limited to a few different types of problems that involve familiar functions. Applications that require other kinds of functions (such as exponential, logarithmic, or trigonometric functions) will be discussed in later chapters.

First we consider a widely used mathematical model for motion due to gravitational attraction. Theoretically our model applies to objects under the sole influence of gravity, which really implies that the object is in a vacuum, and not affected by air resistance. Although we do not live in a vacuum, for many practical applications this model closely approximates what actually occurs when an object falls. Unless we make an explicit exception, we will assume that all falling-body problems are unaffected by air resistance.

This kind of analysis of motion dates back to the time of Isaac Newton and before. We really need only two types of functions for motion due to gravity, one function of time to give the location of the object at time t (the height, usually measured from the earth's surface), and one to give the velocity as a function of t.

Begin with some terminology. *Speed* indicates how fast an object is moving, while *velocity* includes both speed and the direction of motion. Except for this distinction, we use the words *speed* and *velocity* interchangeably. The problems in this section will suppose an object moving vertically either upward or downward, so its motion is one-dimensional. *Positive speed* means the body is moving upward, while *negative speed* means the body is moving downward (toward the surface of the earth).

Formulas for Objects Moving Under the Influence of Gravity

When an object is launched, thrown, or dropped vertically at an initial speed and is then subject only to gravity, we speak of a *freely falling body*. The position of any falling body is determined by its initial velocity and initial height. The same formulas for velocity and height apply to any such body. These formulas are stated in terms of feet and seconds.

Height and speed formulas for falling bodies

The height and velocity of a falling body with initial height s_0 (feet) and initial velocity v_0 (feet per second) after t seconds are given by:

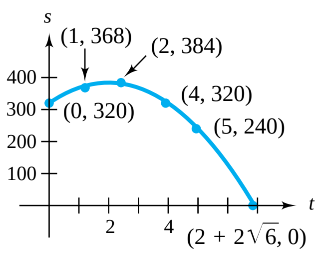
$$s(t) = s_0 + v_0 t - 16t^2 \tag{1}$$

$$v(t) = v_0 - 32t \tag{2}$$

►EXAMPLE 1 ***Ball thrown vertically*** A ball is thrown vertically upward from the top of a 320-foot high building at a speed of 64 feet per second.

(a) How far above the ground is the ball at its highest point?
(b) What is the total distance traveled by the ball in the first 5 seconds?
(c) When does the ball hit the ground?
(d) What is the velocity of the ball when t is 1? When t is 4?

Strategy: **(a)** First get a formula for s as a function of t by substituting 320 for s_0 and 64 for v_0 in Equation (1). Draw a graph of the quadratic function and find its maximum value. **(c)** To find when the ball hits the ground, set $s = 0$ and solve the equation $320 + 64t - 16t^2 = 0$.



$s(t) = 320 + 64t - 16t^2$

FIGURE 55

Solution

Follow the strategy.

$$s(t) = 320 + 64t - 16t^2.$$

(a) The graph (Figure 55) is part of a parabola that opens downward and has its highest point (vertex) where

$$t = \frac{-b}{2a} = \frac{-64}{2(-16)} = 2, \; s(2) = 320 + 64(2) - 16(2^2) = 384.$$

Note that this partial parabola is *not* the path of the ball (which goes straight up and down), but we can use it to easily read off the value of s for a given time t. For instance, at the end of 1 second, the height of the ball is 368 feet above the ground; in 2 seconds the ball is 384 feet above the ground, its maximum height. At the end of 4 seconds, $s = 320$, and so on.

Alternate Solution A physical consideration provides a different approach to finding the time when the ball reaches its highest point. At the highest point, the velocity must be zero since at that instant the ball is going neither up nor down. For this problem, $v = 64 - 32t$, so we want to find the value of t for which v is 0. Solve the equation $64 - 32t = 0$, from which t is 2, as we found above.

(b) To find the total distance traveled during the first 5 seconds, note that the ball travels upward a distance of $384 - 320 = 64$ feet during the first 2 seconds and then downward a distance of $384 - 240 = 144$ feet during the next 3 seconds. (Look at the graph.) Therefore, the total distance traveled during the first 5 seconds is $64 + 144$, or 208 feet.

(c) When the ball hits the ground, the height is 0. Setting $s(t)$ equal to 0 and solving for t gives the time when the ball reaches ground level. Solving $320 + 64t - 16t^2 = 0$ yields two values, one positive $(2 + 2\sqrt{6})$ and one negative $(2 - 2\sqrt{6})$. Since only a positive time value has physical significance in this problem, the ball must hit the ground when t is $2 + 2\sqrt{6}$, or about 6.9 seconds after being thrown.

(d) Replacing v_0 by 64 and substituting 1 for t in formula (2) gives

$$v_1 = 64 - 32 \cdot 1 = 32.$$

Hence when t is 1, the ball is moving upward at 32 feet per second. When t is 4,

$$v_4 = 64 - 32 \cdot 4 = 64 - 128 = -64.$$

The negative sign indicates that the ball is moving downward, so at the end of 4 seconds the ball is falling at a speed of 64 feet per second. ◀

In the next example we look at a slightly more involved problem.

▶*EXAMPLE 2* ***Computing distance*** A stone is dropped from the top of a building and falls past an office window below. Watchers carefully time the stone and determine that it takes 0.20 seconds to pass from the top to the bottom of the window, which measures 10 feet high. From what distance above the top of the window was the stone dropped? (That is, how far is it from the roof to the top of the window?)

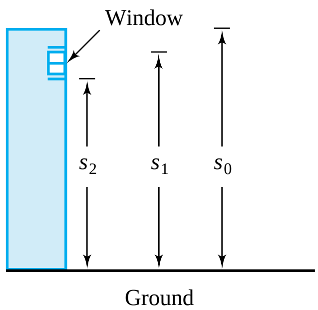


FIGURE 56

Solution

The diagram in Figure 56 identifies the distances s_0, s_1, and s_2. Given that $s_1 - s_2 = 10$ (feet), let t_1 be the time it takes for the stone to reach the top of the window and t_2 be the time to reach the bottom of the window. The problem states that $t_2 - t_1 = 0.20$, and we wish to find $s_0 - s_1$.

Equation (1) applies, where $v_0 = 0$, so

$$\begin{aligned} s &= s_0 - 16t^2 \\ s_1 &= s_0 - 16t_1^2 \\ s_2 &= s_0 - 16t_2^2 = s_0 - 16(t_1 + 0.20)^2. \end{aligned}$$

Subtracting, and using the fact that $s_1 - s_2 = 10$, gives

$$\begin{aligned} s_1 - s_2 &= -16t_1^2 + 16(t_1 + 0.20)^2 \\ 10 &= 6.4t_1 + 0.64 \\ t_1 &= \frac{9.36}{6.4} = 1.4625. \end{aligned}$$

Substituting this value of t_1 into $s_1 = s_0 - 16t_1^2$, we get

$$s_0 - s_1 = 16t_1^2 \approx 34.22.$$

Considering the precision of timing the fall past the window (two significant digits), the distance from the top of the window to the top of the building is about 34 feet. To minimize rounding error, carry out all intermediate calculations with full calculator accuracy and then round off the final result to be consistent with the accuracy of the data. ◀

Revenue Functions

We now look at a problem from the field of economics and business. The **revenue** R generated by selling x units of a product at p dollars per unit is given by the simple formula, $R = px$. The price per unit, p, is determined by a **demand function,** which is usually based on some sort of market analysis or, preferably, experience. Generally the number of units sold increases when the price goes down, and analysts often assume a linear demand function. We illustrate some of these concepts in the next example.

►EXAMPLE 3 ***Revenue problem*** The demand function for a certain product is given by

$$p = 12 - \tfrac{1}{2}x \qquad \text{for} \qquad 0 \le x \le 20,$$

where x is the number of units sold. As Figure 57 shows, the price decreases as more units are sold.

(a) Find a formula for the revenue R as a function of x.
(b) How many units should be sold to maximize revenue?
(c) What is the maximum revenue and what is the corresponding price per unit?

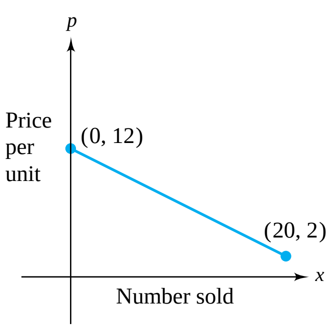
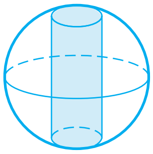


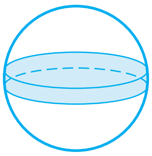
FIGURE 57

Solution

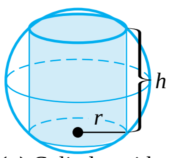
(a) $R(x) = px = (12 - \frac{1}{2}x)x = 12x - \frac{1}{2}x^2.$



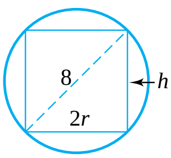
(a) Tall, skinny cylinder



(b) Wide, flat cylinder



(c) Cylinder with height h and radius r



(d) Cross section

FIGURE 58

(b) The revenue function is a quadratic function whose graph is a parabola that opens downward. To find the maximum value, locate the vertex of the parabola, which occurs when $x = \frac{-b}{2a} = \frac{-(12)}{2(-\frac{1}{2})} = 12$. Therefore the maximum revenue will be produced when 12 units are sold.

(c) The maximum revenue, which corresponds to $x = 12$, occurs when the unit price is $p(12) = 12 - \frac{1}{2}(12) = 6$ dollars per unit. The revenue is given by

$$R(12) = 12(12) - \frac{1}{2}(12)^2 = 144 - 72 = 72 \text{ dollars}$$ ◀

In the next example we look at a common type of problem in calculus.

▶***EXAMPLE 4*** ***Maximum volume of a cylinder*** What are the dimensions of the cylinder with the greatest volume that can be contained in a sphere of diameter 8?

Solution

First, get a feeling for the problem by trying to visualize various cylinders in the sphere, as in Figure 58. A tall cylinder is too skinny to have a large volume; at the other extreme, a wide flat cylinder also has a small volume. From one extreme to the other the cylinder volume first increases and then decreases, so the one with maximum volume must be somewhere in between.

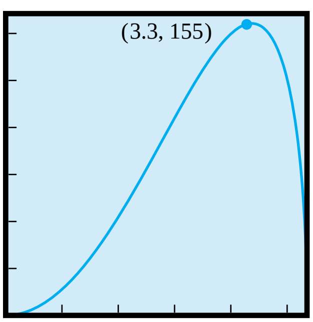
Set up the problem mathematically by expressing the volume V of the cylinder as a function of its radius r. The formula for the volume of a cylinder with radius r and height h (see inside front cover) is $V = \pi r^2 h$.

It may be easier to see in cross section, as in diagram **(d).** A right triangle has a hypotenuse of 8 (the diameter of the sphere) and legs of h and $2r$. By the Pythagorean theorem,

$$h^2 + (2r)^2 = 8^2 \qquad \text{or} \qquad h = 2\sqrt{16 - r^2}.$$

Substituting into the formula for the volume of the sphere,

$$V = \pi r^2 h = 2\pi r^2 \sqrt{16 - r^2}$$



[0, 4] by [0, 160]

FIGURE 59
$y = 2\pi x^2\sqrt{16 - x^2}$

This is another problem that requires calculus for an answer in exact form, but where technology can give a very acceptable approximation. We graph the volume as `Y = 2πX²√(16 − X²)` and look for the maximum value in an appropriate window.

From the diagram in Figure 58d, the radius must be a positive number less than 4, so we can take $[0, 4]$ for an x-range. When $r = 3$, the volume is nearly 150, so we try a y-range of $[0, 160]$. The calculator graph is shown in Figure 59. Tracing to find the maximum, we find a volume of 154.75 near $r = 3.28$, $h \approx 4.62$. If we zoom in a couple of times, we can locate the high point more precisely at about (3.266, 154.778).

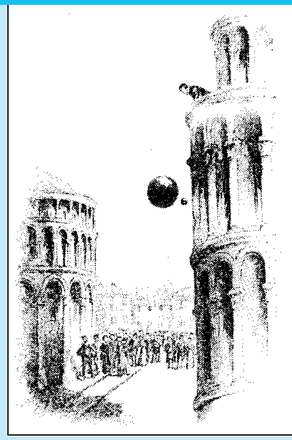
For comparison, calculus techniques show that the maximum volume is $\frac{256\pi}{3\sqrt{3}}$ when $r = \frac{4\sqrt{6}}{3}$, which to four decimal place accuracy corresponds to the point (3.2660, 154.7775). ◀

HISTORICAL NOTE MATHEMATICAL MODELS AND GRAVITY

When we write an equation or function to describe a real-world situation, we almost always need to simplify. Einstein said this well: "Everything should be made as simple as possible, but not simpler." The test of a mathematical model is its capacity to accurately describe and predict real events.

Galileo measured falling bodies and decided that the distance fallen is proportional to the square of the time (in modern terms, $f(t) = 16t^2$). His timing instrument was his pulse! We may wonder what his results might have been if his pulse had been less steady.

How good is his simple model? For heavy bodies near the earth it works beautifully. For objects like feathers or paper airplanes, the model is too simple.



Galileo's experiments with falling objects led to a mathematical model of the force of gravity.

Another example occurs in Newton's account of his discovery of the inverse square law of the force of gravity. Newton, born in 1642, the year of Galileo's death, took refuge at age 24 on an isolated farm to avoid the plague, which was then ravaging London. He devoted himself to study and within a year he had his model for the gravitational attraction between two bodies, $F = g\left(\frac{mM}{r^2}\right)$. To test it, he "compared the force requisite to keep the moon in her orb with the force of gravity at the surface of the earth, and found the answer fits 'pretty nearly.'" Newton's model was good enough to analyze the motion of the planets.

Strategy: The longest mirror is the shortest segment L touching the inside corner and both walls, forming two similar right triangles, with sides 5 and y, x and 4, as in the diagram. In the similar triangles, $\frac{y}{5} = \frac{4}{x}$.

►EXAMPLE 5 ***Getting around a corner*** Plate glass mirrors must be replaced in a dance studio. Unfortunately, the only way into the studio is down a hallway 5 feet wide and then around a corner into a hallway 4 feet wide. See the diagram in Figure 60. The question is what length mirror can be carried (vertically) around the corner. Can a 10-foot mirror be installed? 12-foot? 15-foot?

(a) Use the observation in the strategy to express L in terms of x and y, and from the relation of x and y, write L as a function of x.

(b) Use a graph to find the minimum possible length of L.

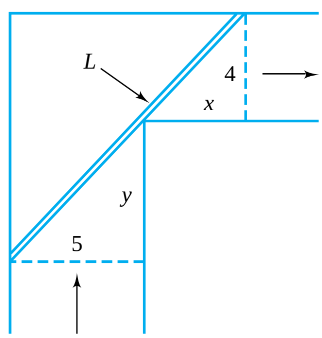


FIGURE 60

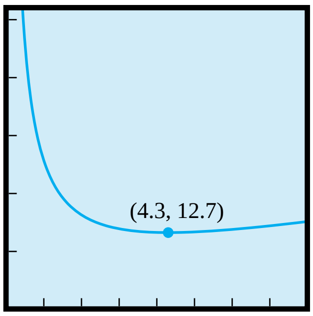
Solution

(a) Follow the strategy. L is the sum of the lengths of each hypotenuse:

$$L = \sqrt{5^2 + y^2} + \sqrt{x^2 + 4^2}.$$

From the relation of x and y in the Strategy, we can solve for $y: y = \frac{20}{x}$, and substitute into the expression for L to get a function of x alone.

$$\begin{aligned} L &= \sqrt{25 + \frac{400}{x^2}} + \sqrt{x^2 + 16} \\ &= \sqrt{\frac{25(x^2 + 16)}{x^2}} + \sqrt{x^2 + 16} \\ &= \frac{5}{x}\sqrt{x^2 + 16} + \sqrt{x^2 + 16} = \left(\frac{5}{x} + 1\right)\sqrt{x^2 + 16}. \end{aligned}$$



[0, 8] by [0, 50]

FIGURE 61

(b) We want to graph Y = (5/X + 1)√(X² + 16) in an appropriate window. It seems as if [0, 8] should be adequate for an x-range, and we sample a few values for y. In a $[0, 8] \times [0, 30]$ window we get a graph like that in Figure 61. Tracing, we find the low point near the point indicated in the figure. Thus the minimum length is about 12.7 feet, when $x \approx 4.3$ and $y \approx 4.7$. Clearly the 10-foot mirror and the 12-foot mirror can be carried around the corner (with room for fingers) but the 15-foot mirror cannot. ◀

EXERCISES 2.8

Check Your Understanding

Exercises 1–10 True or False. Give reasons.

1. A ball dropped from a height of 256 feet takes 4 seconds to hit the ground.
2. It takes twice as long for a ball to fall to the ground from a height of 64 feet than from a height of 32 feet.
3. If a ball is dropped from a height of 256 feet and at the same instant a second ball is thrown upward from ground level at a speed of 128 feet per second, the two balls will meet at a point 192 feet above the ground.
4. In Exercise 3, the two balls will meet in 2 seconds.
5. A ball rolls down a long inclined plane. It takes longer to roll down the first 10 feet than it does to roll down the next 10 feet.
6. It takes the same amount of time to travel 240 miles at 55 mph as it takes to travel the first 120 miles at 50 mph and the final 120 miles at 60 mph.
7. If a square and an equilateral triangle are inscribed in the same circle, then the square has greater area than the triangle.
8. For any rectangle with a perimeter of 16, the length of one side must be at least 4.
9. No triangle can have sides of lengths 3, 4, and 8.
10. If a sphere has diameter d, then its volume V is given by $\frac{\pi d^3}{12}$.

Develop Mastery

Exercises 1–23 Apply the formulas for motion due to gravitational attraction.

1. A stone is dropped from the top of a cliff that is 160 feet tall. How long will the stone take to hit the ground?
2. A stone is dropped from the top of a building and hits the ground 3.5 seconds later. How tall is the building?
3. A helicopter is ascending vertically at a speed of 25 feet per second. At a height of 480 feet, the pilot drops a box.
   (a) How long will it take for the box to reach the ground?
   (b) At what speed does the box hit the ground?
4. A helicopter is climbing vertically at a speed of 24 feet per second when it drops a pump near a leaking boat. The pump reaches the water 4 seconds after being dropped.

(a) How high is the helicopter when the pump is dropped?
(b) How high is the helicopter when the pump reaches the water?

5. A baseball is thrown vertically upward. When it leaves the player's fingers it is 6 feet off the ground and traveling at a speed of 48 feet per second.
(a) How high will it go?
(b) How many seconds after the ball is thrown will it hit the ground?

6. A rock is dropped from the top of a cliff 360 feet directly above a lake.
(a) State a formula that gives the height s as a function of t.
(b) What is the domain of this function?
(c) How far above the lake is the rock 2 seconds after being dropped?
(d) How far does the rock fall during the third second?

7. A rock is blasted vertically upward from the ground at a speed of 128 feet per second (about 80 mph).
(a) Find a formula that relates s and t.
(b) How far from the ground is the rock 2 seconds after the blast?
(c) How high will the rock go?

8. A vertical cliff 160 feet tall stands at the edge of a lake. A car is pushed over the edge. How many seconds will it take to hear the sound of the splash at the top of the cliff? Assume that sound travels at a speed of 1080 feet per second.

9. A rock is dropped into a deep well. It takes 4.5 seconds before the sound of the splash is heard. Assume that sound travels at a speed of 1080 feet per second. Determine s_0, the distance from the top of the well to the water level. If you measure the height s above water level, then the formula for motion due to gravity applies.
(a) Show that the rock takes $t_1 = \dfrac{\sqrt{s_0}}{4}$ seconds to reach the water.
(b) Show that the sound of the splash takes $t_2 = \dfrac{s_0}{1080}$ seconds to be heard.
(c) Since the total time that elapses before hearing the splash is 4.5 seconds, you have the equation

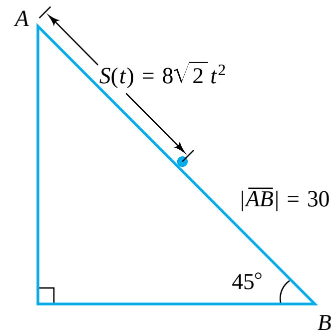
$$\frac{s_0}{1080} + \frac{\sqrt{s_0}}{4} = 4.5.$$

Clear the fractions and simplify to get

$$s_0 + 270\sqrt{s_0} - 4860 = 0,$$

which is a quadratic equation in $\sqrt{s_0}$. Use the quadratic formula to solve for $\sqrt{s_0}$ and then find s_0.

10. A ball is released from rest at point A, the top of an inclined plane 30 feet long (see the diagram). If $S(t)$ denotes the number of feet the ball rolls down the incline in t seconds after its release, then $S(t) = 8\sqrt{2}t^2$.



(a) How long does it take for the ball to reach the end of the plane?
(b) How far does the ball roll during the first 1.5 seconds?
(c) How long does it take for the ball to roll down the final 12 feet of the plane?

11. A stone is dropped from the top of New York's Empire State Building, which is 1476 feet tall.
(a) How long does it take for the stone to reach the ground?
(b) What is the speed of the stone when its hits the ground?

12. If a kangaroo jumps 8 feet vertically, how long is it in the air during the jump?

13. With what minimum vertical speed must a salmon leave the water to jump to the top of a waterfall that is 2.4 feet high?

14. A rock is thrown upward at an initial speed of 16 feet per second from the edge of a cliff 160 feet above a lake. One second later a second rock is dropped from the edge of the cliff. Which rock will hit the water first? By how many seconds?

15. ***What's Wrong?*** Major league pitches are often clocked at more than 90 miles per hour. How fast a ball can a catcher be expected to handle? In 1946, a backup catcher for the St. Louis Browns named Hank Helf caught a ball dropped from 52 stories up (a 701-foot drop to his glove). The speed of the ball was measured at 138 mph.

Use Equations (1) and (2) from this section to find (**a**) how long it takes for a ball to drop 701 feet, and (**b**) the speed of the ball when it hits the glove (in feet per second and in miles per hour). (**c**) Write a brief paragraph to discuss why the measured speed is not the same as the speed predicted by Equation (2). (*Hint:* Read the Historical Note.)

16. A diver in Acapulco leaps horizontally from a point 112 feet above the sea.
 (a) How long does it take for the diver to reach the water?
 (b) At what speed does the diver enter the water?

17. A ball player catches a ball 5 seconds after throwing it vertically upward.
 (a) At what initial speed was the ball thrown?
 (b) What was the speed of the ball when it was caught?

18. A stone is thrown vertically upward with a speed of 32 feet per second from the edge of a cliff that is 240 feet high.
 (a) How many seconds later will it reach the bottom of the cliff?
 (b) What is its speed when it hits the ground?
 (c) What is its speed when it is 120 feet above the bottom of the cliff?
 (d) What is the total distance traveled by the stone?

19. Robin, a skydiver, leaves the plane at an altitude of 1000 feet above the ground and accidentally drops her binoculars. If she descends at a constant speed of 20 feet per second, how much time elapses between the arrival of the binoculars on the ground and the time when Robin lands?

20. Frank is ballooning at an altitude of 480 feet when he turns on the burner and accidentally knocks his lunch out of the balloon. If he immediately starts to ascend at a constant speed of 4.8 feet per second, how high will he be when his lunch hits the ground?

21. A stone is dropped from the top of a building 240 feet high. It is observed to take 0.20 seconds to go past an office floor-to-ceiling window that is 12 feet high. How far is it from the bottom of the window down to the street? (*Hint:* See Example 2.)

22. A toy rocket is fired upward from ground level near an office building. Its initial velocity is 80 feet per second. An observer in one of the offices determines that the rocket takes 0.32 seconds to pass by the office window, which is 16 feet tall. How far is it from the ground to the bottom of the window?

23. The acceleration of gravity on the moon is about one-sixth of what it is on earth. The formula for a freely falling object on the moon is given by $s = s_0 + v_0 t - \frac{8}{3}t^2$. If an object is thrown upward on the moon, how much higher will it go than it would have on the earth, assuming the same initial velocity of 64 feet per second?

24. ***Maximum Revenue*** The manager of a store estimates that the demand function for calculators (see Example 3) is given by

$$p = 36 - \frac{1}{3}x \qquad 0 \leq x \leq 96,$$

where x is the number of calculators sold and p is the price of each calculator. The revenue R is given by $R = px$.
 (a) Express R as a function of x.
 (b) How many calculators should be sold to get the maximum revenue?

25. Answer the questions in Exercise 24 if the demand function is given by

$$p = 36 - (0.2)x \qquad 0 \leq x \leq 160.$$

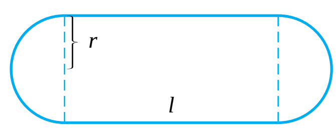
26. A car rental agency rents 400 cars a day at a rate of \$40 for each car. For every dollar increase in the rental rate, it rents 8 fewer cars per day.
 (a) What is the agency's income if the rental rate is \$40? \$42? \$45?
 (b) What rental rate will give the greatest income? What is this maximum income?

27. A car rental agency rents 200 cars a day at a rate of \$30 for each car. For every dollar increase in the rental rate, it rents 4 fewer cars per day.
 (a) What is the agency's income if the rental rate is \$30? \$35? \$40?
 (b) What rental rate will give the greatest income? What is the maximum income?

28. ***Linear Depreciation*** A computer is purchased for \$2000. After 5 years its salvage value (for tax purposes) is estimated to be \$400. Linear depreciation implies that the tax value V of the computer is a linear function of t, the number of years after purchase.
 (a) Find a formula for the linear depreciation function.
 (b) In how many years after purchase will the tax value of the computer be zero?

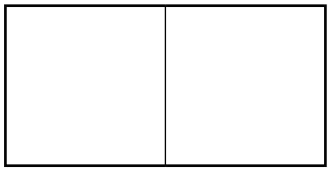
29. Repeat Exercise 28 if the original cost of the computer is \$3000 and its tax value after 8 years is \$500.

30. An indoor gymnastics arena is to be built with a rectangular region and semicircular regions on each end (see the diagram). Around the outside is a running track whose inside length is to measure 220 yards (one-eighth of a mile).

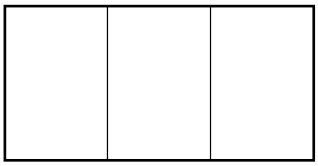


 (a) What dimensions for the rectangle will maximize the area of the rectangular region?
 (b) For the dimensions in part **(a),** what is the area of the entire region enclosed by the track?

31. A rancher has 240 feet of fencing to enclose two adjacent rectangular pens (see the diagram on the top of the next page). What dimensions will give a maximum total enclosed area?



32. In Exercise 31 suppose that the rancher wants to make three adjacent pens (see the diagram). What dimensions will give a maximum total enclosed area?

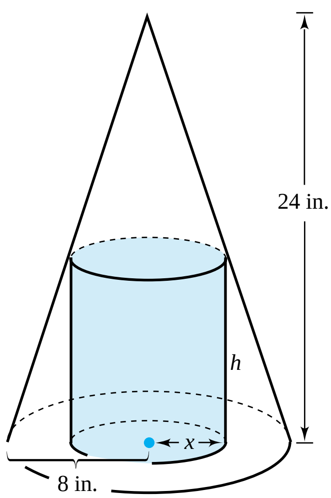


33. A pan is full of water when it springs a leak at the bottom. The volume V of water (in cubic inches) that remains in the pan t seconds after the leak occurs is given by

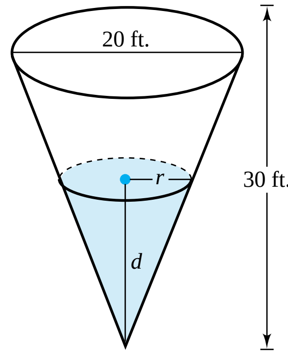
$$V = 1000 - 30t + 0.1t^2$$

(a) How much water is in the pan when the leak starts?
(b) In how many seconds will the pan be empty?
(c) What is the domain of the function?
(d) How many seconds will it take for half of the water to leak out of the pan? How long for the final half?

34. ***Looking Ahead to Calculus*** A right circular cylinder is inscribed in a right circular cone that has a height of 24 inches and a radius of 8 inches (see the diagram). Let x denote the radius of the cylinder and h denote its height.
(a) Express h as a function of x.
(b) Express the volume V of the cylinder as a function of x.
(c) Use a graph to find the value of x that gives the largest volume.

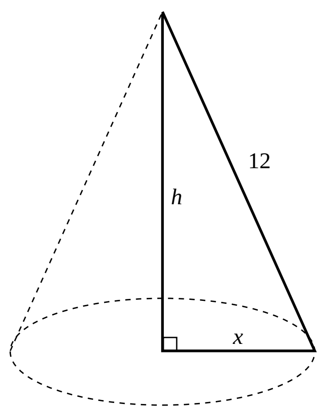


35. A water tank in the shape of an inverted circular cone is initially full of water (see the diagram for dimensions). A control valve at the bottom of the tank allows water to drain from the tank. At any depth d, the water remaining in the tank is the shape of a cone with radius r.



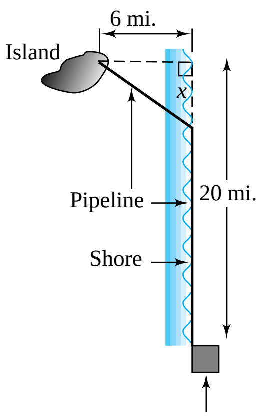
(a) Express r as a function of d and then express the volume V of water remaining as a function of d.
(b) If the depth of the water t minutes after starting to drain the tank is given by $d = 30 - 5\sqrt{t}$, then express V as a function of t.
(c) What is the volume of water that remains at the end of 16 minutes?
(d) How long will it take to empty the tank?

36. ***Looking Ahead to Calculus*** A right triangle has a fixed hypotenuse of length 12, but legs whose lengths can vary. The triangle is rotated about the vertical leg to generate a cone of radius x and height h, where x and h are the lengths of the legs of the triangle (see the diagram).



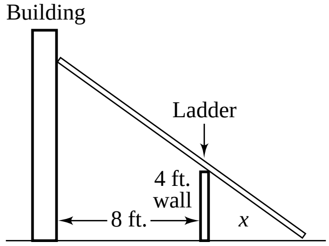
(a) Express h as a function of x.
(b) Express the volume V of the cone as a function of x.
(c) Of all such possible cones, determine x and h for the one with the largest volume. Use a graph.

37. ***Looking Ahead to Calculus*** A freshwater pipeline is to be constructed from the shore to an island located as shown in the diagram. The cost of running the pipeline along the shore is \$8,000 per mile, but construction offshore costs \$12,000 per mile.



(a) Express the construction cost C as a function of x.
(b) What is the cost when x is 3, 5, 6, 8, 10, and 15?
(c) Of all such possible pipelines, determine the value of x that will minimize the construction cost. What is the minimum cost? Use a graph.

38. ***Looking Ahead to Calculus*** A ladder of length L is placed so that it rests on the top of a 4 foot wall and leans against a building that is 8 feet from the wall. The ladder touches the ground x feet from the wall (see the diagram).



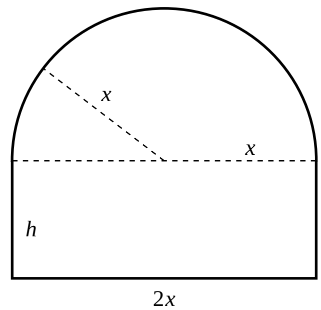
(a) Show that L can be written as a function of x as follows:

$$L = (x + 8)\sqrt{1 + \frac{16}{x^2}}.$$

(b) Evaluate L when x is 2, 3, 4, 5, 6, and 7.
(c) Of all such possible ladders, determine the value of x that will require the shortest ladder. What is the length of the shortest ladder? Use a graph.

39. ***Inscribed Rectangle***
(a) Draw a graph of $y = 4x - x^2$ and inscribe a rectangle with base on the x axis and upper vertices on the graph.
(b) Of all possible rectangles, find the dimensions of the one that has a maximum area. What is the area? (*Hint:* First get a formula giving the area A of any rectangle as a function of its height. Then use a graph.)

40. ***Maximum Light*** A so-called Norman window consists of a rectangle surmounted by a semicircle as shown in the diagram. The total perimeter of the window is to be 24 feet. What are the dimensions of the window that will admit the greatest amount of light? A graph will be helpful.

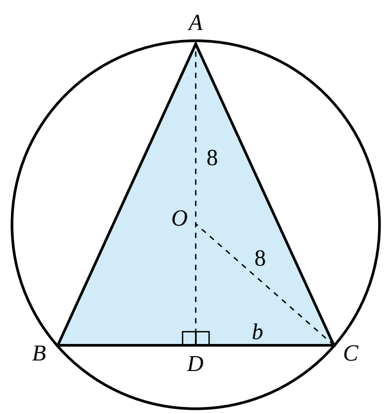


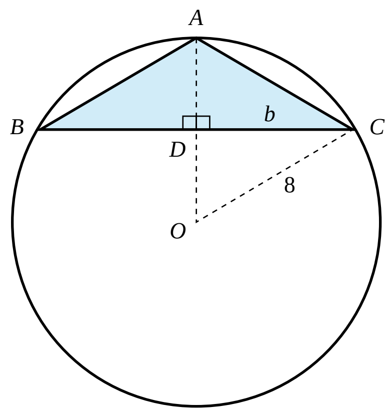
41. Solve the problem in Exercise 40 if the semicircular portion of the window is made of stained glass which admits one-half as much light as the rectangular portion.

42. ***Around a Corner*** Solve the problem in Example 5 if the hallways are 4 feet and 6 feet wide.

43. ***Maximum Volume*** Solve the problem in Example 4 if the diameter of the sphere is 12. Compare with the exact answer from calculus: The maximum volume is $96\pi\sqrt{3}$ when $r = 2\sqrt{6}$.

44. ***Maximum Area*** An isosceles triangle is inscribed in a circle of radius 8 cm. See diagrams where $|\overline{AB}| = |\overline{AC}|$, O is the center of the circle, $h = |\overline{AD}|$, and $b = |\overline{DC}|$.





(a) Show that for both diagrams, $b = \sqrt{16h - h^2}$.
(b) Express the area K of the triangle as a function of h.
(c) Of all such possible triangles, find the height and base ($|\overline{BC}|$) (1 decimal place) of the one with greatest area. What is the maximum area?

45. ***Alternative Solution*** Solve the problem in Exercise 44 by expressing the area as a function of b. For maximum area it is not necessary to consider the second figure in the diagram (why?).

CHAPTER 2 REVIEW

Test Your Understanding

True or False. Give reasons.

1. A line with slope 2 is perpendicular to any line that has a slope of -2.
2. The solution set for $|x - 1| + 1 = 0$ is the empty set.
3. The solution set for $|x - 1| + 1 \geq 0$ is the empty set.
4. The graph of $2x + y + 1 = 0$ does not pass through the first quadrant.
5. There is no function with a domain of two numbers and a range of three numbers.
6. There is no function with a domain of three numbers and a range of two numbers.
7. A line with negative slope must pass through the third quadrant.
8. A line with positive slope must pass through the third quadrant.
9. The graph of any parabola that opens upward must contain some points in the second quadrant.
10. The graph of every quadratic function must contain points in at least two of the four quadrants.
11. The graph of $y = x^2 + x + 1$ is entirely above the x-axis.
12. The graph of $y = x^2 - 3x$ does not pass through the origin.
13. If function f has an inverse, then f is a one–one function.
14. Every increasing function has an inverse.
15. The function $f(x) = x$ is one–one.
16. If $f(x) = \sqrt{x}$ and $g(x) = x^2$, then $f(g(x)) = x$ for every x in R.
17. The graph of the quadratic function $f(x) = x^2 - x + 2$ does not cross the x-axis.
18. The graph of $y = \sqrt{x^2 + 1}$ has no y-intercept point.
19. If a function f has an inverse, then every horizontal line must intersect the graph of $y = f(x)$ in exactly one point.
20. Suppose f has an inverse. If $(-2, 3)$ is on the graph of $y = f(x)$, then $(-3, 2)$ must be on the graph of $y = f^{-1}(x)$.
21. Points $(-2, 4)$ and $(4, -2)$ are symmetric with respect to the line $y = x$.
22. The graph of $y = |x| + 2$ has no x-intercept point.
23. The graph of $y = |x| + 1$ has no y-intercept point.
24. The two lines $y = 2x$ and $y = -2x$ are perpendicular to each other.
25. There is no point that is on the graphs of both $y = 2x$ and $y = 2x - 1$.
26. If f is any even function, then f cannot be one–one.
27. Every odd function has an inverse.
28. The function $f(x) = -|x|$ is an even function.
29. If f is an even function, then for every number k the function given by $g(x) = f(x) + k$ must be even.
30. If f is an odd function, then $g(x) = f(x) + k$ is an odd function for every number k.
31. The graph of $y = |x| - 1$ can be obtained by translating the graph of $y = |x|$ horizontally 1 unit to the right.
32. When the window $[-5, 15]$ by $[-5, 15]$ is used to draw a graph of $f(x) = 0.25x^2 - 4|x| + 12$, the display shows a graph that crosses the x-axis at three points.

33. When the window $[-10, 10]$ by $[-10, 10]$ is used to draw a graph of $f(x) = x^2 - 8|x| + 12$, the display shows a graph that crosses the x-axis at four points.

34. The function $f(x) = -x^2$ has no maximum value.

35. The function $f(x) = |2x + 3|$ is a linear function.

36. The function $f(x) = |x^2 - 2x - 3|$ is a quadratic function.

37. The equation $x^2 - 4|x| + 3 = 0$ has exactly two roots.

38. The equation $|2x + 3| = 1$ has only one root.

39. The graph of every parabola that passes through the origin must contain points in exactly three quadrants.

40. If the graph of an even function contains points in the third quadrant, then it must also contain points in the fourth quadrant.

41. If the graph of an odd function contains points in the second quadrant, then it must also contain points in the fourth quadrant.

Exercises 42–45 Assume that the function f has an inverse.

42. If the graph of f has an x-intercept point, then the graph of f^{-1} must have a y-intercept point.

43. If $(-2, 0)$ is an x-intercept point for the graph of f, then $(0, 2)$ is not a y-intercept point for the graph of f^{-1}.

44. If $f^{-1}(-3) = 4$, then $f(3)$ must be equal to -4.

45. The graph of f cannot have more than one x-intercept point.

46. The maximum value of the function $f(x) = -x^2 - 2x + 3$ is 4.

47. The graph of $f(x) = x - 3|x - 2| + |x + 1|$ has no lowest point.

48. The graph of $f(x) = x + 3|x + 2| - |x - 1|$ has no lowest point.

49. If $f(x) = \dfrac{x}{x^2 + 1}$ and $g(x) = x^2 - 4x$, then the graph of the function $g \circ f$ has no highest point.

50. If $f(x) = \dfrac{x}{x^2 + 1}$ and $g(x) = x^2 + 4x + 1$, then the function $g \circ f$ has no maximum value.

Review for Mastery

Exercises 1–2 Determine whether or not the set of ordered pairs represents a function. If not, explain why. Otherwise, state the domain and range.

1. $\{(-1, 2), (0, 4), (1, 6), (2, 8)\}$

2. $\{(3, -1), (4, 2), (5, -3), (3, 2)\}$

Exercises 3–8 State the domain.

3. $f(x) = 2 - 3x^2$

4. $f(x) = \sqrt{x^2 + 4}$

5. $g(x) = \sqrt{2 - x}$

6. $g(x) = \sqrt{3 - |x|}$

7. $h(x) = \dfrac{x}{x^2 - 4}$

8. $h(x) = \dfrac{\sqrt{x}}{x - 2}$

9. Find an equation for the line that passes through $(-1, 3)$ and $(2, 5)$.

10. Find an equation for the line that passes through the point $(2, 4)$ and is

(a) parallel to the line $2x - 3y + 4 = 0$

(b) perpendicular to the line $x + 2y - 3 = 0$.

11. Point P is $(-1, 2)$ and line L is given by $2x - 3y + 8 = 0$. If P is on L, then find an equation for the line that passes through P and is perpendicular to L. If P is not on L, then find an equation for the line that passes through P and is parallel to L.

12. Find an equation for the line that is the perpendicular bisector of the line segment joining $(-1, 4)$ and $(3, -2)$.

Exercises 13–18 ***Graph*** *Sketch a graph. In each case label the x- and y-intercept points.*

13. $2x - y = 4$

14. $3x + 2y = 6$

15. $y = x^2 - 4x + 3$

16. $y = x^2 - 2x + 1$

17. $y = |x - 1| + 1$

18. $y = 1 - |x + 1|$

Exercises 19–26 ***Solution Set*** *Find the solution set for the open sentence.*

19. $2x - 3 < 6 - x$

20. $|x| - 4 = 0$

21. $|x - 1| - 1 > 0$

22. $x^2 - 3x - 4 = 0$

23. $x^2 - 3x - 4 \leq 0$

24. $\sqrt{x - 1} - 2 = 0$

25. $|x| - x = 4$

26. $x(x^2 - 4) < 0$

Exercises 27–32 ***Combining Functions*** *Evaluate the combination for functions f and g.*

$$f(x) = x + 1 \quad \text{and} \quad g(x) = 2x - x^2.$$

27. $(f + g)(2)$

28. $\left(\dfrac{f}{g}\right)(-1)$

29. $(f \circ g)(-2)$

30. $(g \circ f)(3)$

31. $(f \circ g)(2)$

32. $(f - g)(2)$

Exercises 33–36 ***Equations with Combined Functions*** *Solve the equation, where $f(x) = 2 - 3x$ and $g(x) = x^2 - x$.*

33. $(f - g)(x) - 2 = 0$

34. $(f \circ g)(x) - 3x = 0$

35. $(g \circ f)(x) - 2 = 0$

36. $\left(\dfrac{f}{g}\right)(x) = \dfrac{5}{2}$

Exercises 37–40 ***Increasing, Decreasing*** *Determine whether the function is **(a)** increasing, decreasing, or neither. **(b)** Is f one–one.*

37. $f(x) = x^2 - x$

38. $f(x) = 3 - 2x$

39. $f(x) = \sqrt{x}$

40. $f(x) = |x - 1| + 1$

Exercises 41–44 Find a formula for the inverse of f. Give the domain and range of f^{-1}.

41. $f(x) = 2x - 4$

42. $f(x) = 3 - x$

43. $f(x) = \sqrt{x - 1}$

44. $f(x) = 1 + \dfrac{1}{x}$

45. ***Piecewise Function*** Function f is described by the equation

$$f(x) = \begin{cases} \sqrt{x} & \text{for } x \geq 0 \text{ and} \\ -|x| & \text{for } x < 0. \end{cases}$$

(a) Sketch a graph of $y = f(x)$.
(b) Use the horizontal line test to determine whether or not f is a one–one function.

46. ***Restricted Domain*** Function f is described by the equation

$$f(x) = x^2 - 4x$$

and the domain $\{x \mid x \leq 2\}$.
(a) Sketch a graph of $y = f(x)$.
(b) Is f an increasing function, decreasing function, or neither?
(c) Does f have an inverse?

Exercises 47–50 ***Maximum, Minimum (a)*** *Sketch a graph of $y = f(x)$. **(b)** Determine the maximum and minimum values of $f(x)$.*

47. $f(x) = x^2 - 2x + 2$

48. $f(x) = x^2 - 2x - 1,\ 0 \leq x \leq 3$

49. $f(x) = 1 + \sqrt{x}$

50. $f(x) = -x^2 - 2x,\ -3 \leq x \leq 0$

51. If $f(x) = \dfrac{3x}{x - 3}$ show that $f^{-1}(x) = f(x)$.

52. ***Related Functions*** The graph of f is the line through $P(-3, 2)$ and $Q(1, -4)$. Draw a graph of
(a) $g(x) = f(x - 1)$
(b) $h(x) = -f(x + 2)$
(c) Find formulas for g and h.

53. The graph of f consists of the line segment joining $P(-2, 3)$ and $Q(2, -3)$. Draw a graph of
(a) $g(x) = f(x + 2)$
(b) $h(x) = f(-x)$
(c) Find formulas for g and h.

54. ***Graphing Inverse*** The graph of f is the line through $P(-2, 4)$ and $Q(2, 6)$.
(a) Draw a graph of f^{-1}.
(b) Give a formula for f^{-1}.

55. The graph of f is the line segment $\overline{PQ}$ where P is $(-1, -3)$ and Q is $(2, 3)$.
(a) Draw a graph of f^{-1}.
(b) Give a formula for f^{-1}. Remember to state the restricted domain.

Exercises 56–57 ***Maximum of Composition*** *Find the maximum value of $f \circ g$.*

56. $f(x) = x^2 - 4x, \quad g(x) = \dfrac{4x}{x^2 + 2}$

57. $f(x) = x + 1, \quad g(x) = |x - 1| - 2|x + 2|$

Exercises 58–59 ***Restricted Domain and Inverse*** *Function f and its domain D are given. **(a)** Draw a graph of f and see that it is a 1–1 function. Give the range of f. **(b)** Find a formula for f^{-1}. Give the domain D and range R of f^{-1}.*

58. $f(x) = x^2 + 4x + 3; \quad D = [-1, \infty)$

59. $f(x) = x^2 - 4x + 3; \quad D = (-\infty, 2]$

60. ***Maximum Area*** A rectangle is inscribed in a circle of radius 4. If x denotes the length of one side of the rectangle, express the area A and the perimeter P as functions of x. State the domain of each function.

61. ***Shortest Ladder*** A fence 6 feet tall stands 4 feet from a tall building. What is the length of the shortest ladder that will reach from the ground outside the fence to the wall of the building?

62. Suppose a right triangle with hypotenuse of length 16 is revolved about one of its legs of length x, resulting in a cone.
(a) Draw a diagram. Give a formula for the volume V of the cone as a function of x.
(b) What are the dimensions (height and radius) of the cone having the greatest volume? What is the maximum volume?

63. ***Motion Due to Gravity*** A ball is thrown upward from a point 160 feet above ground level at an initial speed of 48 feet per second.
(a) Give a formula for the distance s of the ball from ground level as a function of time (in seconds) after the ball is thrown.
(b) How many seconds will the ball take to hit the ground?
(c) What is the highest point the ball will reach?

64. ***Discount*** The sale price of a graphing calculator after a 25 percent discount is \$60. What was the price before the discount?

65. ***Hands of a Clock*** After 12 noon, when will the hour and minute hands of a clock first point in opposite directions. Round off your answer to the nearest second.

66. ***Population Increase*** From 1970 to 1980 the population of Hazelton increased by 8 percent; from 1980 to 1990 it increased by 15 percent. What is the percentage increase in population over the 20-year period, 1970 to 1990?

67. ***Self Inverse*** If c is a constant and $f(x) = \dfrac{x}{2x + c}$, find the value of c such that $f(f(x)) = x$. That is, find c such that $f^{-1}(x) = f(x)$.

68. ***Regions of a Triangle*** If points $A(0, 0)$, $B(2, 4)$, and $C(6, 0)$ are vertices of a triangle, find the number k such that the horizontal line $y = k$ divides the triangle into two regions of equal area.

69. If points $A(0, 0)$, $B(2, 0)$, and $C(0, 4)$ are vertices of a triangle, find the number m such that the line $y = mx$ divides the triangle into two regions of equal area.

70. ***Right Triangle*** The lengths of the sides of a right triangle are given by x, $x + 2$, and $x + 4$. Find the value of x.

71. What is the smallest integer k such that the graph of $y = x^2 + kx + 5$ lies entirely above the x-axis?

72. A rock is blasted vertically upward from ground level with a velocity of 144 feet per second.

(a) How high does the rock go before it starts to fall back down?

(b) Is it going upward or downward when $t = 5$?

(c) What is its velocity when it is 200 feet above the ground on the way up?

73. ***Leaking Pail*** The volume (in cubic inches) of water remaining in a leaking pail after t seconds is given by

$$V = 1200 - 40t + 0.2t^2.$$

(a) How much water was in the pail at time $t = 0$?

(b) What is the volume of water in the pail when $t = 4$? 10? 20?

(c) How long does it take for all the water to leak out of the pail?

74. ***Height of a Cliff*** A stone dropped from the edge of a cliff takes 4.5 seconds to hit the ground. How high is the cliff?

75. How long will it take for a brick to reach the ground if it is dropped from a height of 180 feet?

76. Megan rides a bicycle up a hill at a speed of 12 feet per second and then comes back down the same hill at a speed of 24 feet per second. What is her average speed for the entire trip up and back down?

77. When traveling a distance of 150 miles, how much less time does it take at a speed of 60 mph compared to the same trip at a speed of 50 mph?

78. A car traveling at 90 kilometers per hour is 150 meters behind a truck traveling at 60 kilometers per hour.

(a) How soon will the car reach the truck?

(b) Suppose the truck is x meters ahead of the car. Let $T(x)$ be the time it takes for the car to reach the truck. Find a formula to express $T(x)$ as a function of x.

79. ***Travel Agency Maximum Revenue*** A travel agency is offering a two-week tour of the Orient, in which a group will travel in a plane of capacity 180. The fare is \$2400 per person if 100 or fewer subscribe but the cost per person will be decreased by \$15 for each person in excess of 100. For instance, if 125 go, then the cost for each is $\$2400 - 15(25) = \$2025$.

(a) Determine a formula (function) that will allow the travel agency to compute the total revenue T when x people go on the tour.

(b) What is the domain of this function?

(c) Draw a graph and use it to determine the number of people that will give the maximum revenue.

3 POLYNOMIAL AND RATIONAL FUNCTIONS

IN CHAPTER 2 WE EXPLORED a number of functions. Two of the simplest are linear and quadratic functions, whose graphs are lines and parabolas, respectively. In this chapter we examine what happens if we take products of linear and quadratic functions. The set of functions that can be built up in this way are collectively called **polynomials.**

Polynomials crop up in a diverse range of problems such as maximizing profits or efficiency in a production facility, solving a differential equation for electronic circuit analysis, or finding eigenvalues in matrix analysis to avoid resonances of rocket motors. The problem of finding roots of polynomial equations has taken on an entirely different complexion with modern technology, but technology can be applied meaningfully only when we have understanding.

Two of the features of greatest interest in the study of polynomial functions are their **zeros** (where the graph crosses the x-axis) and **local extrema** (local maximum and minimum values, where the graph has "humps" in either direction). Since both zeros and extrema have meaning in terms of graphs, graphs are an essential component of learning about polynomial functions.

Our approach to this material is influenced by new technology; our goal is an understanding of the concepts to make the technology useful. There are now routines available in most computer software, and in several graphing calculators as well, that will find all zeros of a polynomial function, real and complex, with a single keystroke. Such routines, of course, contribute nothing to our understanding of polynomial functions. Accordingly, we use (and assume) only graphing capabilities in this chapter. A major contribution of graphing calculators is that we can now appreciate polynomial behavior by looking at as many graphs as we wish, changing our view at will, to see and examine what is most significant for our purposes at the moment.

Section 3.1 begins with basic definitions and graphical concepts and gives an overview of key properties of polynomial functions. In Sections 3.2 and 3.3 we consider zeros in exact form, including some of the classical theorems, while learning something about approximations to zeros as well. The final section of the chapter builds on this material to define and discuss rational functions, or quotients of polynomials.

3.1 POLYNOMIAL FUNCTIONS

I knew formulas for the quadratic and the cubic, and they said there was a subject called Galois theory, which was a general theory giving conditions under which any equation could be solved. That there could be such a thing was beyond my wildest comprehension!

Paul Cohen

In earlier courses you learned that expressions such as

$$x^2 + 2x - 1, \qquad x^3 + 3x, \qquad -x^5 + 3x - 8.$$

are called polynomials. The following are not polynomials:

$$\frac{1-x}{x}, \qquad 2x^{-2} + 3x, \qquad |x| - 4, \qquad 4^x + 5.$$

We stated above that polynomials are functions built up as products of linear and quadratic functions. Unfortunately, polynomials seldom appear in real-world applications in factored form. Much of our work, in fact, will be devoted to finding the factors from which a given polynomial is constructed. Accordingly, we begin with the more standard definition.

Definition: polynomial function

A **polynomial function** of degree n is a function that can be written in the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{1}$$

where n is a nonnegative integer, $a_n \neq 0$, and $a_n, a_{n-1}, \ldots, a_1, a_0$ are numbers called coefficients. This course assumes that all coefficients are real numbers. The **leading term** is $a_n x^n$, the **leading coefficient** is a_n, and a_0 is the **constant term.** Equation 1 is the **standard form** for a polynomial function.

When I was thirteen, . . . I needed an emergency operation for appendicitis. I read two books in hospital. One was Jerome's Three Men in a Boat, *and the other was Lancelot Hogben's* Mathematics for the Million. *Some of it I couldn't understand, but much of it I did. I remember coming across the idea of dividing one polynomial by another. I knew how to multiply them together, but I had never divided them before. So every time my father came to visit me in hospital he brought some more polynomials that he'd multiplied out.*

Robin Wilson

It should be obvious from the definition that the domain of every polynomial function is the set of all real numbers. We already know about polynomial functions of degree 2 or less.

Degree 0: $f(x) = k,\ k \neq 0$ (constant function; the graph is a horizontal line).

Degree 1: $f(x) = ax + b$ (linear function; the graph is a line).

Degree 2: $f(x) = ax^2 + bx + c$ (quadratic function; the graph is a parabola).

For technical reasons, the zero polynomial function, $f(x) = 0$, is not assigned a degree.

Combining Polynomial Functions

It is appropriate to ask how the usual operations on functions apply to polynomial functions. What about sums, differences, products, quotients, or composition? All of these except quotients are also polynomials. The quotient of two polynomial functions is never a polynomial function unless the denominator is a constant function.

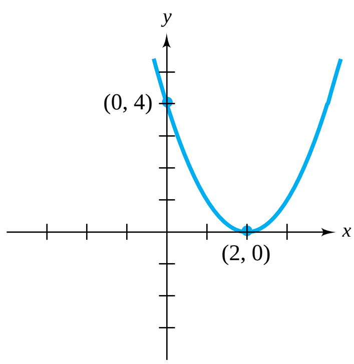
Products, Zeros, Roots, and Graphs

One consistent concern with polynomials, as in the work we did with many functions in Chapter 2, is locating their *zeros,* finding the *x-intercept* points of graphs, or finding *roots* of a polynomial equation. Every zero of a polynomial function is associated with a **factor** of the polynomial. The equivalence of these concepts for polynomials is summed up in the following box.

Roots, zeros, factors, and intercepts

Let p be a polynomial function and suppose that a is any real number for which $p(a) = 0$. Then the following are equivalent statements:

a is a **root** of the equation $p(x) = 0$	a is a **zero** of the polynomial function p
$(x - a)$ is a **factor** of the polynomial $p(x)$	$(a, 0)$ is an ***x*-intercept point** of the graph of p



$f(x) = (x - 2)^2$

FIGURE 1

We now want to compare the graphs of some simple functions with what we get if we take their products. We know that the graph of $f(x) = (x - 2)^2$ is a core parabola shifted 2 units right. The graph of f touches the x-axis at only one point, (2, 0). See Figure 1. Since the equation $(x - 2)(x - 2) = 0$ has two solutions (by the zero product principle), we say that f has a **repeated zero** or a **zero of multiplicity two** at $x = 2$.

Just as a polynomial function can be built up as a product of linear and quadratic factors, its graph can be built up in a similar fashion. To take a simple example, consider $F(x) = (x + 1)(x - 2)^2$. The zeros of F are clearly 2 (repeated) and -1. When we take values of x near 2, the factor $x + 1$ is near 3, and so we might expect the graph of F to approximate the graph of $y = 3(x - 2)^2$. The same kind of reasoning suggests that the graph of F near -1 should be something like the graph of $y = (x + 1)(-3)^2 = 9(x + 1)$. That this reasoning is valid is borne out in Example 1. To look more closely at a particular point, we may wish to zoom in.

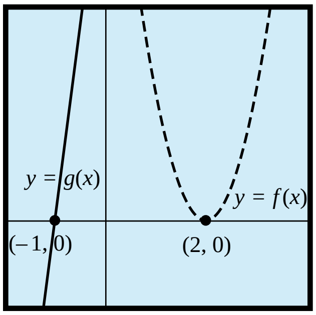
TECHNOLOGY TIP ◆ ***Zooming in on a point***

On many calculators, when we press the ZOOM IN option, we get a cursor that we move to the desired location and ENTER. On HP calculators, to zoom in on some point other than the screen center, we must first redraw the graph with the point at the center of the display window by moving the cursor to the desired point and pressing CNTR from the ZOOM menu. Then, still in the ZOOM menu, press ZIN.

▶EXAMPLE 1 Products and zeros

(a) Graph $f(x) = 3(x - 2)^2$ and $g(x) = 9(x + 1)$ in the same window.

(b) Add the graph of the product, $F(x) = (x + 1)(x - 2)^2$. Zoom in on the point $(2, 0)$ and on $(-1, 0)$. Describe in words the behavior of the product function near its zeros.

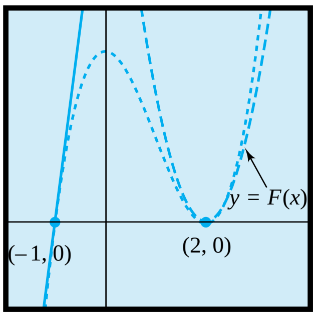


[−2, 4] by [−2, 5]

$f(x) = 3(x - 2)^2$

$g(x) = 9(x + 1)$

(a)



[−2, 4] by [−2, 5]

$F(x) = (x + 1)(x - 2)^2$

(b)

FIGURE 2

Solution

(a) The graphs look like the diagram in Figure 2a.

(b) When we add the graph of F, we get the diagram in Figure 2b. When we zoom in on the point $(2, 0)$, we see two curves that are barely distinguishable. Both graphs are tangent to the x-axis at the point $(2, 0)$. We can see the tangent behavior more clearly if we zoom in again (or several times), but the graphs are so nearly identical near $(2, 0)$ that we see only one.

Returning to the decimal window and zooming in on the point $(-1, 0)$, we see a graph (just one) that looks like a fairly steep line. Tracing, we can tell that the graphs of g and F are not identical, but they are remarkably close.

The graph of the product function near each of its zeros appears to be very closely approximated by the graph of a constant times one of the factors of F, in particular, the factor of F which shares that zero. ◀

The kind of functional behavior we observed in Example 1 is typical of products, an observation we sum up in the following.

Graphs of products near zeros

Let $F(x) = f(x)g(x)$ and suppose that a is a zero of F, where $f(a) = 0$ and $g(a) \neq 0$. Then near $(a, 0)$,

the graph of the product function F looks very much like the graph of $y = Af(x)$, where A is the constant given by $A = g(a)$.

▶EXAMPLE 2 Products, zeros, and graphs

(a) Express the function $F(x) = 2x^3 - x^4$ in factored form and identify all zeros of F with their multiplicities.

(b) For each zero a of F, find a constant A such that the graph of F near $(a, 0)$ is approximated by the graph of the form $y = Af(x)$. Check by graphing.

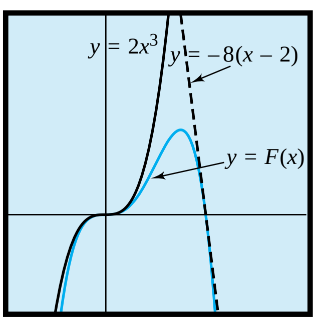
Solution

(a) If we factor out x^3, we can write $F(x) = x^3(2 - x)$. By the zero product principle, the zeros of F are 0 (of multiplicity 3) and 2.

(b) Near $x = 0$, the other factor, $(2 - x)$, is near 2, so we would expect $F(x)$ to be approximated by $y = 2x^3$ near $(0, 0)$.

For the other zero, when x is close to 2, x^3 is close to 8. F should be very nearly equal to $y = 8(2 - x) = -8(x - 2)$.

The graphs of $y = 2x^3$, $y = -8(x - 2)$, and $y = F(x)$ are all shown in Figure 3. ◀



[−2, 4] by [−2, 4]

$F(x) = x^3(2 - x)$

FIGURE 3

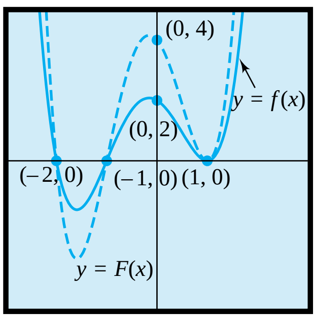
From the polynomial functions in the first two examples, it appears that we should be able to graph such functions by piecing together combinations of shifted multiples of the functions x, x^2, x^3, and so on. We have not considered the possibility of a quadratic factor with no real zeros. It turns out that there is also a close connection between the graph of a function having such a factor and the graph of a constant multiple of that factor, but we will not pursue the connection in this text. We invite the curious reader, however, to explore the graph of a function such as $f(x) = (x + 3)(x^2 + 1)$. The graph of $y = x^2 + 1$ is a parabola with vertex where $x = 0$. Compare the graph of f with the pieces $y = 3(x^2 + 1)$ and $y = 10(x + 3)$.

For each nonrepeated zero, there is a single factor, and an x-intercept point where the graph crosses the x-axis in essentially *linear* fashion. We associate a double zero, a zero with multiplicity two, with a point where the graph is tangent to the x-axis. Zeros of greater multiplicity correspond locally to translations of the graphs of $y = x^3$, $y = x^4$, and so on. We can use this observation to build product functions with any desired set of zeros. An equation for a product function can be written in factored form, or the factors can be multiplied out to obtain what is called the **expanded form.**

▶EXAMPLE 3 Polynomials with specified zeros

(a) Write an equation for a polynomial function f having zeros -1, -2, and 1 as a zero of multiplicity two.

(b) Write an equation for a polynomial function F, with the same zeros as f, whose graph contains (0, 4).



[−3, 3] by [−5, 5]

FIGURE 4
$f(x) = (x - 1)^2(x + 1)(x + 2)$
$F(x) = 2(x - 1)^2(x + 1)(x + 2)$

Solution

(a) Without specifying some additional point, there is not a unique polynomial function with the given zeros, so we build the simplest. For the repeated zero 1, 1, we need a factor $(x - 1)^2$, and we also need linear factors $x + 1$ and $x + 2$. We can write an equation for f as

$$f(x) = (x - 1)^2(x + 1)(x + 2) = x^4 + x^3 - 3x^2 - x + 2.$$

The graph is the solid curve in Figure 4.

(b) Tracing along the graph of f, we see that the y-intercept point is (0, 2), a fact that is also obvious from the expanded form, since $f(0) = 2$. For a function F with the same zeros as f such that $F(0) = 4$, we want to dilate the graph of f vertically by a factor of 2. Thus $F(x) = 2f(x) = 2(x - 1)^2(x + 1)(x + 2)$, or in expanded form, $F(x) = 2x^4 + 2x^3 - 6x^2 - 2x + 4$. Its graph is the dotted curve in Figure 4. ◀

TECHNOLOGY TIP ◆ Checking algebra

For most purposes, expanded form is not necessary, but a graphing calculator can be used to check our algebra even if it does not handle symbolic forms. To see if our expanded form of F in Example 3 is correct, we can graph both $2(x - 1)^2(x + 1)(x + 2)$ and the expanded form, $2x^4 + 2x^3 - 6x^2 - 2x + 4$, in the same screen. If the graphs show any differences, then we obviously need to check our multiplication again.

In calculus courses, techniques are developed to find maximum and minimum values of a function. Important as these techniques are, a graphing calculator can be used to get excellent approximations for such values. It is handy to have some terminology and definitions. We assume that the graph of f contains no isolated points.

Definition: local extrema and turning points

Suppose c is in the domain D of a function f.

If $f(x) \geq f(c)$ for all x in D in some open interval containing c, then $f(c)$ is called a **local minimum** of f.

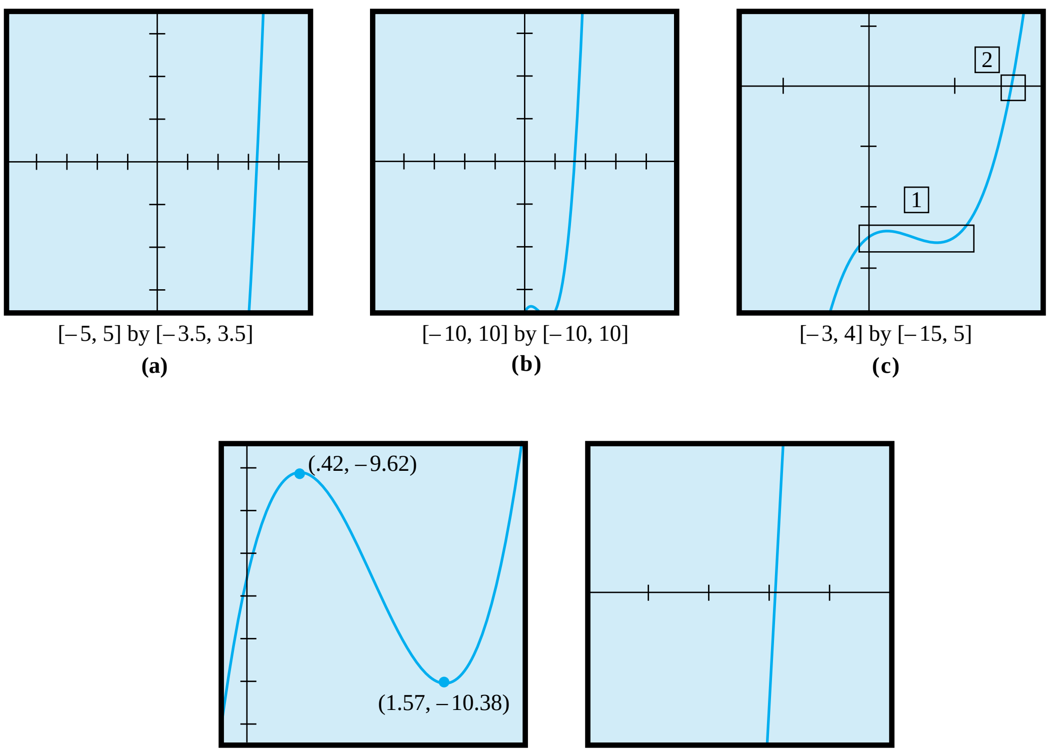
If $f(x) \leq f(c)$ for all x in D in some open interval containing c, then $f(c)$ is called a **local maximum** of f.

Local maxima and minima are called **local (or relative) extrema.** If the above inequalities hold for every x in D, then $f(c)$ is called an **absolute minimum (or maximum).**

If $f(c)$ is a local extremum, then the point $(c, f(c))$ is called a **turning point** of the graph.

When we want to find zeros and local extrema of polynomials, the choice of viewing windows is critical, as illustrated in the next example.

▶EXAMPLE 4 ***Windows and graphs*** Draw graphs of $y = x^3 - 3x^2 + 2x - 10$ to locate all zeros and local extrema.



[−5, 5] by [−3.5, 3.5]
(a)

[−10, 10] by [−10, 10]
(b)

[−3, 4] by [−15, 5]
(c)

[−.2, 2.2] by [−10.6, −9.5]
(d)

[3, 3.5] by [−.2, .2]
(e)

FIGURE 5
$y = x^3 - 3x^2 + 2x - 10$

Solution

If we begin with a decimal window (Figure 5a), we can see an x-intercept near 3, but very little else of interest. We clearly need a larger window.

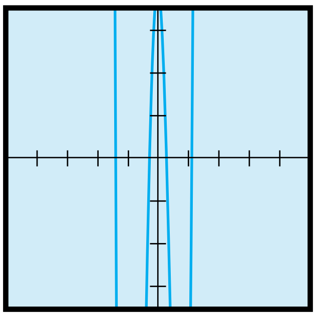
Setting a window of $[-10, 10] \times [-10, 10]$, it appears that something is happening near the y-intercept point $(0, -10)$, but the graph does not yet show enough detail to allow us even to know what we should be interested in. See Figure 5b.

To see better what is happening near $(0, -10)$, we set a window of $[-3, 4] \times [-15, 5]$ and get the graph in Figure 5c. We can see two "humps," as well as the x-intercept point, but there is too much compression in the y-direction to get much detail. Accordingly, we zoom in to look at the points of interest more closely.

When we zoom into a box like the one labeled $\boxed{1}$ in Figure 5c, we exaggerate the vertical dimensions and we can trace to get a pretty good estimate of the local maximum near $(0.42, -9.62)$ and the local minimum near $(1.57, -10.38)$. See Figure 5d.

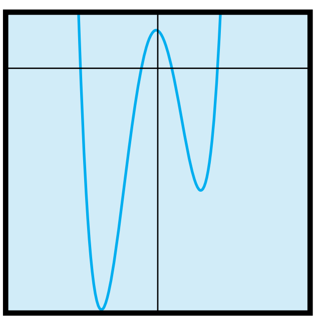
Returning to Figure 5c, if we zoom into a box like the one labeled $\boxed{2}$, we can trace to find that $y = 0$ when x is about 3.31. See Figure 5e. ◀

There are some obvious questions about what we have done in Example 4. How do we know we have located all the zeros and local extrema? At this point we have no real justification for claiming to have completed the example. Part of our task in this section is to look at enough graphs of polynomial functions to make some reasonable guesses about "typical" polynomial graphs. In the next section we get a number of theorems to justify our observations. In particular, we will learn that the graph of a cubic polynomial such as the one in Example 1 can have at most two "humps" or turning points, so that there can be no more local extrema, and the graph can never turn back to the x-axis.



[–10, 10] by [–20, 20]

(a)



[–10, 10] by [–130, 25]

(b)

FIGURE 6

$p(x) = (x^2 - 1)(x^2 + x - 20)$

▶EXAMPLE 5 Graphs, factors, and zeros

Let $p(x) = (x^2 - 1)(x^2 + x - 20) = x^4 + x^3 - 21x^2 - x + 20$.

(a) Find a window in which you can see four zeros and three turning points on the graph of $y = p(x)$.

(b) Use the factored form of $p(x)$ to find all zeros.

Solution

(a) In a decimal window, we see nothing but essentially vertical lines. Increasing our ranges to $[-10, 10] \times [-20, 20]$ is a little better. We can at least see four x-intercepts, and what appears to be a turning point near $[0, 20]$. See Figure 6a. Tracing in both directions, we can read y-coordinates below -120, so we try a y-range of $[-130, 25]$. The graph in Figure 6b shows four zeros and three turning points. There are, of course, many windows that would work as well.

(b) From the factored form, we can use the zero-product principle to assert that $p(x) = 0$ only when $x^2 - 1 = 0$ or $x^2 + x - 20 = 0$. Each of these equations is a quadratic that factors readily, so $p(x) = (x - 1)(x + 1)(x + 5)(x - 4)$. By the zero-product principle, we get one zero from each factor. The zeros are $-5, -1, 1$, and 4. ◀

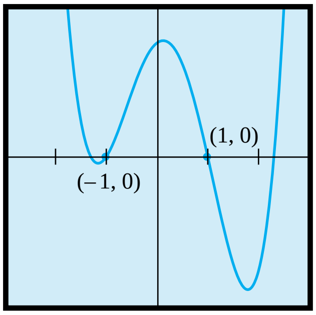
TECHNOLOGY TIP ◆ *Scaling and autoscaling*

In Example 5, the graph of p in the $[-10, 10] \times [-20, 20]$ window went "off-scale," dipping down out of our view.

Tracing displays function values as y-coordinates even for points that do not appear in the window. This allows us to estimate the y-range needed to see features that aren't visible in a particular window, as we did in the example.

Another feature available on many graphing calculators is called Autoscale or Zscale. Having set the x-range, when we use Autoscale, the calculator computes function values for the entire x-range and makes the y-range big enough to show all computed y-values.

This can be handy at times, but with many functions, including polynomials because of their steep end behavior, the resulting graph has so much vertical compression that interesting behavior is "squashed" out of sight. From the $[-10, 10] \times [-20, 20]$ window in Example 5, try autoscaling to see what happens.



[−3, 3] by [−4, 4]

FIGURE 7
$p(x) = (x^2 - 1)(x^2 - x - 3)$

►EXAMPLE 6 ***Graphs, factors, and zeros*** Repeat Example 5 for the function $p(x) = (x^2 - 1)(x^2 - x - 3) = x^4 - x^3 - 4x^2 + x + 3$.

Solution

(a) Now a decimal window is almost good enough, but one turning point is off screen. Figure 7 shows a graph in $[-3, 3] \times [-4, 4]$.

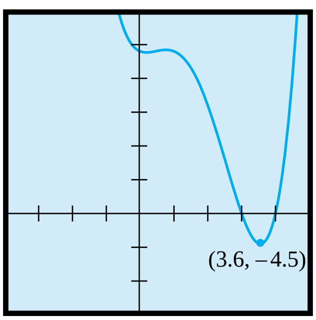
(b) From the factored form, we again have zeros at -1 and 1, from $x^2 - 1 = 0$. Solving $x^2 - x - 3 = 0$, however, requires the quadratic formula to find the two remaining zeros: $x = \frac{1 \pm \sqrt{13}}{2}$. ◄

►EXAMPLE 7 ***Finding turning points*** Let $p(x) = x^4 - 6x^3 + 7x^2 - 2x + 24$.

(a) Find a window in which you can see three turning points and two real zeros.

(b) Find the coordinates of the lowest turning point to one decimal place.

Solution



[−4, 5] by [−15, 30]

FIGURE 8
$p(x) = x^4 - 6x^3 + 7x^2 - 2x + 24$

(a) After some experimentation, we get the calculator graph of Figure 8 in the $[-4, 5] \times [-15, 30]$ window. The two turning points near the y-intercept point are not very pronounced, but there are clearly three turning points on the graph. We could set a window in which the turning points near the y-intercept are more visible, but then we would not see the lowest. The graph has only two x-intercept points.

(b) Tracing along the curve and zooming in as needed, we find that the lowest turning point is near $(3.6, -4.5)$. ◄

Not all graphs of polynomial functions have turning points. The most obvious case is the set of all polynomials of degree one or less, whose graphs are straight lines. The graph of $y = x^3$, which we have met before, levels out to run tangent to the x-axis at the origin; the function is always increasing, and so there are no turning points. The graph of the cubic function $f(x) = x^3 + 2x$ does not even level out. See Figure 9.

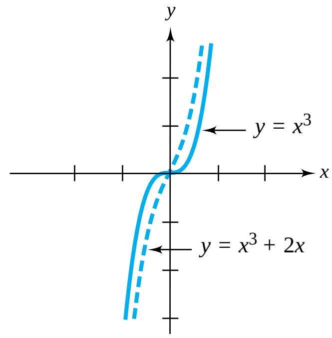


FIGURE 9

▶EXAMPLE 8 ***Finding intersections*** Let $p(x) = x^3 - 2x^2 + 5x + 3$ and $g(x) = 2\sqrt{x + 4}$. Graph both f and g in a window that shows the intersection of the curves, and locate the coordinates of the intersection to one decimal place.

Solution

After some experimentation, it appears that the graph of p has no turning points and that the intersection shown in $[-5, 5] \times [-1, 10]$ (see Figure 10a) is the only intersection of the two curves. Zooming in as needed on the point of intersection, we read the coordinates as approximately (0.2, 4.1). ◀

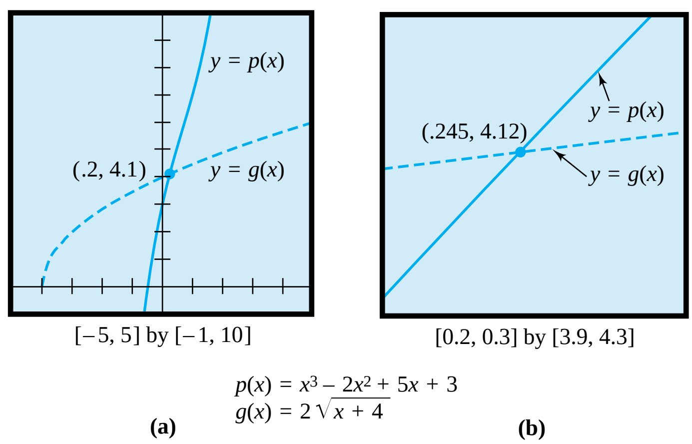


FIGURE 10

TECHNOLOGY TIP ◆ ***Trapping an intersection***

Rather than simply zooming in or drawing a box, some people prefer a process that lets us keep track of the window size and thus the accuracy. We can trace in Figure 10a and find the intersection is between $x = .2$ and $x = .3$, and between $y = 3.9$ and $y = 4.3$. Setting these numbers as range values, we get a picture something like Figure 10b, in which we can trace, knowing that the pixel increment in the new window is about $\dfrac{.3 - .2}{\#\text{ pixel cols.}}$ $(\approx .001)$.

Smoothness and End Behavior

All the graphs of polynomial functions we have looked at so far are *smooth,* with no jumps, breaks, or corners. You will learn in calculus that these properties follow from the fact that polynomial functions are *continuous* and *differentiable.* For now, we simply accept these properties about polynomial graphs. Furthermore, graphs of polynomial functions (of degree greater than 1) continue to rise or fall very steeply as we move along the graph to the right or left. We are asking what happens as x becomes large and positive or large and negative (for which we use the notation $x \to \infty$ or $x \to -\infty$). This **end behavior** depends solely on the degree of the polynomial and the sign of the leading coefficient. With a positive leading coefficient, we always have $f(x) \to \infty$ as $x \to \infty$. In most cases, we can look at the equation defining the polynomial and see what the end behavior will be.

To get a better feeling for some of the variety of graphs of polynomial functions, numbers of turning points and zeros, and so on, we have a table for several polynomial functions. We do not show graphs here. Rather, we ask you to graph each one and verify for yourself the observations we record in the table. You may select whatever window will be most helpful. Most of the pertinent information can be seen in a window such as $[-4, 5] \times [-10, 15]$, but adjust the window as needed. The arrows suggest the end behavior by indicating the direction in which the graph is heading.

Polynomial Function	*End*		*Degree*	*Real Zeros*	*Turning Points*
$y = x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12$	↙	↗	5	5	4
$y = -x^3 + 2x^2 + 3x - 1$	↖	↘	3	3	2
$y = -x^4 + 5x^2 - 4$	↙	↘	4	4	3
$y = x^6 - 2x^4 - 3x^2 - 5x + 8$	↖	↗	6	2	3
$y = -x^6 + 3x^5 + 5x^4 - 15x^3 - 3x^2 + 12x - 5$	↙	↘	6	4	5

On the basis of our discussions thus far, we can make some observations that we will substantiate in the next section.

Suppose f is a polynomial function of degree n, where $n \geq 1$.

1. The number of **real zeros** (counting multiplicities) is either n or some even number less than n (such as $n - 2$, $n - 4$, etc.).
2. The number of **turning points** or **local extrema** is $n - 1$ or some even number less than $n - 1$ (such as $n - 3$, etc.).
3. **End behavior:**

   If n is *even,* and the leading coefficient is

   positive, as $x \to \pm\infty$, then $y \to \infty$; ↖. . .↗
   negative, as $x \to \pm\infty$, then $y \to -\infty$. ↙. . .↘

   If n is *odd* and the leading coefficient is

   positive, as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$; ↙. . .↗
   negative, as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$. ↖. . .↘

EXERCISES 3.1

Check Your Understanding

True or False. Give reasons. Draw a graph whenever you think it might be helpful.

1. If k is any positive number, then the graph of $y = 1 - kx^3$ contains no points in Quadrant III.
2. The graph of $f(x) = x^3 + x^2 - 2x + 3$ has two turning points.
3. If c is a zero of f, then $(0, c)$ is a point on the graph of f.
4. Every real zero of $f(x) = (1 + x^2)(x^2 - x - 2)$ is also a zero of $g(x) = x^2 - x - 2$.
5. The graph of $f(x) = x^3 - 3x^2 - 7x - 5$ contains points in all four quadrants.
6. The degree of $f(x) = x^3 + x(1 - x^2)$ is 3.
7. The function $f(x) = x^3 - 3x^2 - 7x + 3$ has one negative zero and two positive zeros.
8. There is no fourth degree polynomial function whose graph has exactly two turning points.
9. Using the window $[-5, 5] \times [-40, 40]$ we can conclude that $f(x) = x^3 - x^2 + 5x + 4$ has a positive zero.
10. For $f(x) = x^3 - 18x^2 + 24x + 125$, using the window $[-8, 24] \times [-1300, 400]$ we can conclude that all zeros of f are between -3 and 20.

Develop Mastery

Exercises 1–4 Determine whether or not f is a polynomial function. If it is, give its degree.

1. $f(x) = 4 - 3x - 2x^2$
2. $f(x) = x^2 + \sqrt{x^2} - 3$
3. $f(x) = x(x + 1)(x + 2)$
4. $f(x) = \sqrt{x^2 + 9}$

Exercises 5–10 ***Combining Functions*** *Use the polynomial functions f, g, and h, where*

$$f(x) = 3x + 2 \quad g(x) = 5 - x \quad h(x) = 2x^2 - x.$$

(a) Determine an equation that describes the function obtained by combining f, g, and h. (b) If it is a polynomial function, give the degree, the leading coefficient, and the constant term.

5. $f + g$
6. $f - h$
7. fg
8. $h \circ f$
9. $f \circ h$
10. $\dfrac{f}{g}$

Exercises 11–12 ***Which Window?*** *In order to determine the zeros of f, which window would you use?*

11. $f(x) = 0.3x^3 + 3x^2 - 7x - 6$; three zeros.
 (i) $[-10, 10] \times [-10, 10]$
 (ii) $[-8, 10] \times [-10, 50]$
 (iii) $[-15, 10] \times [-10, 80]$
12. $f(x) = x^4 - 11x^3 - 16x^2 + 44x + 400$; two zeros.
 (i) $[-10, 10] \times [-10, 10]$
 (ii) $[-10, 10] \times [-400, 400]$
 (iii) $[-5, 15] \times [-2500, 2000]$

Exercises 13–14 ***Which Window?*** *The graph of f contains a local maximum point and a local minimum point. Which window would you use to see this feature?*

13. $f(x) = x^3 - 16x^2 - 24x + 400$
 (i) $[-10, 10] \times [-10, 10]$
 (ii) $[-5, 10] \times [-200, 200]$
 (iii) $[-5, 15] \times [-480, 450]$
14. $f(x) = -x^3 - 20x^2 + 75x + 800$
 (i) $[-10, 10] \times [-10, 10]$
 (ii) $[-20, 10] \times [-2000, 1200]$
 (iii) $[-20, 5] \times [-1000, 1000]$

Exercises 15–18 ***Zero-product Principle*** *A formula for function p is given in factored form. (a) Express $p(x)$ in standard (expanded) form, give the leading coefficient, and constant term. (b) Use the zero-product principle to find the zeros of p. (c) Use cut points to find the solution set for $p(x) < 0$.*

15. $p(x) = x(x - 1)(x + 2)$
16. $p(x) = x^2(x - 2)(x - 1)$
17. $p(x) = (x - 1)(x + 1)(2x - 1)$
18. $p(x) = (2x^2 + x - 1)(x^2 + 2x - 3)$

Exercises 19–22 (a) Factor and find all the zeros of f (including complex zeros). (b) Determine the end behavior. (c) Use a calculator graph to check your answers.

19. $f(x) = x(x - 3) + 2x(x + 2)$
20. $f(x) = (x + 2)(x - 1) - (x + 2)(2x + 3)$
21. $f(x) = x^3 - 1$
22. $f(x) = x^3 - 2x^2 - 3x + 6$

Exercises 23–26 ***Zeros and Turning Points*** *(a) Read the discussion at the end of this section and tell how many real zeros f can possibly have. Do the same for turning points. (b) Draw a calculator graph and then tell how many zeros and how many turning points there actually are. (c) In what quadrants do the turning points lie?*

23. $f(x) = x^3 - 2x^2 - 3x - 1$
24. $f(x) = -x^4 + 5x^2 - x + 1$
25. $f(x) = -x^5 - 4x^4 + 6x^3 + 24x^2 - 5x - 20$
26. $f(x) = x^4 - 3x^3 + x^2 - 3x - 8$

Exercises 27–30 ***Approximating a Zero*** *Draw a graph and use it to find an approximation (1 decimal place) for the lar- gest zero of f.*

27. $f(x) = x^3 - 2x^2 - 5x + 3$

28. $f(x) = -x^3 - 2x^2 + 5x + 4$

29. $f(x) = x^4 - 7x^2 + x + 5$

30. $f(x) = x^4 - 7x^2 - x + 5$

Exercises 31–34 ***Local Maximum***
(a) *For the functions in Exercises 27–30, determine the coordinates of any local maximum points (1 decimal place).*
(b) *Describe the end behavior for f.*

Exercises 35–38 ***Turning Points*** *Determine the coordinates of the turning point in the given quadrant (1 decimal place).*

35. $f(x) = x^3 - 2x^2 - 5x + 3$; *QII*

36. $f(x) = x^3 - 2x^2 - 5x + 3$; *QIV*

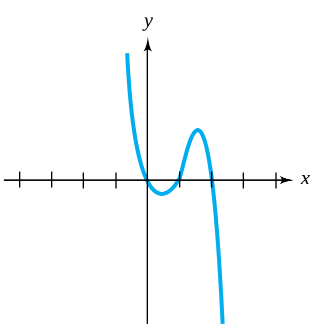
37. $f(x) = 3 + 5x - 2x^2 - x^3$; *QIII*

38. $f(x) = 3 + 5x - 2x^2 - x^3$; *QI*

Exercises 39–43 ***Graph to Formula*** *A graph of a polynomial function is given, where the vertical scale is not necessarily the same as the horizontal scale. From the following list of polynomials, select the one that most nearly corresponds to the given graph. As a check draw a calculator graph of your selection and see if it agrees with the given graph.*

(a) $f(x) = x^2(x - 1)(x - 3)$
(b) $f(x) = x^2 + 3x$
(c) $f(x) = x^2(x - 2)^2$
(d) $f(x) = x(x - 1)^2(x + 2)^2$
(e) $f(x) = 4x - x^3$
(f) $f(x) = x(x - 1)(3 - x)$
(g) $f(x) = x(1 - x)(3 - x)$
(h) $f(x) = x^4 - 5x^2 + 4$

39.



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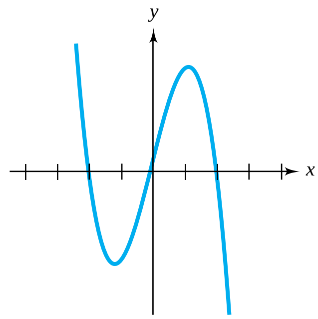

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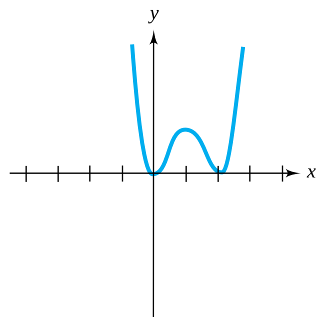

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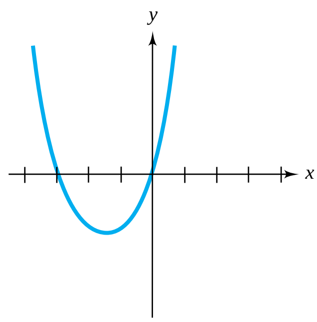

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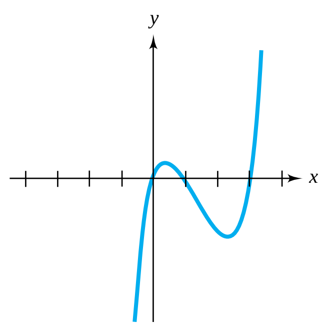


Exercises 44–47 ***Related Graphs*** *Draw graphs of f and g. From the graphs, make a guess about how the graphs are related. Prove algebraically.*

44. $f(x) = x^3 + x^2 - 6x$, $g(x) = x^3 + 7x^2 + 10x$

45. $f(x) = x^3 + 3x^2 - x - 3$,
$g(x) = x^3 + 6x^2 + 8x$

46. $f(x) = x^3 + 3x^2 - x - 3$,
$g(x) = x^3 - 3x^2 - x + 3$

47. $f(x) = x^3 - x^2 - 6x$, $\quad g(x) = x^3 - 7x^2 + 10x$

48. Determine all integer values of k for which $f(x) = x^3 - x^2 - 5x + k$ will have three real zeros. (*Hint:* Locate the local maximum and local minimum points for the graph of $g(x) = x^3 - x^2 - 5x$. Then consider vertical translations.)

49. Solve Exercise 48 for $f(x) = x^3 - 2x^2 - 5x + k$.

50. Solve Exercise 48 for $f(x) = -x^3 - x^2 + x + k$.

51. For what integer value(s) of k will f have one negative and two positive zeros where

$$f(x) = (x - k)^3 + 5(x - k)^2 + 3(x - k) - 1?$$

(*Hint:* Draw a graph of $y = x^3 + 5x^2 + 3x - 1$ and then consider horizontal translations.)

52. Solve Exercise 51 for

$$f(x) = -(x - k)^3 - 5(x - k)^2 + 5.$$

Exercises 53–54 ***Your Choice*** *Draw a rough sketch of a graph of a polynomial function satisfying the specified conditions. The answer is not unique.*

53. Function f has exactly 3 distinct zeros and $f(x) \to \infty$ as $x \to -\infty$.

54. The degree of f is 3, f has one real zero, and $f(x) \to -\infty$ as $x \to \infty$.

55. The base of a rectangle is on the x-axis and its upper two vertices are on the parabola $y = 16 - x^2$. Of all such rectangles, what are the dimensions, (1 decimal place) of the one with greatest area?

56. Solve the problem in Exercise 55 where $y = 16 - x^4$.

57. A rectangular box without a top is to be made from a rectangular piece of cardboard 12×15 inches by cutting a square from each corner and bending up the sides of the remaining piece. Of all such boxes, find the dimensions (1 decimal place) of the one having the largest volume. See illustration on p. 179.

58. ***Maximum Strength*** At a lumber mill a beam with rectangular cross section is cut from a log having cylindrical shape of diameter 12 inches. Assuming that the strength S of the beam is the product of its width w and the square of the depth d, what are the dimensions (1 decimal place) of the cross section that will give a beam of greatest strength.

59. If a polynomial function of degree 3 has no local extrema, explain why it must be one-one and therefore have an inverse.

60. **(a)** Draw a graph of $f(x) = x^3 - 3x^2 + 9x + 2$.
(b) Is it reasonable to conclude that f is one–one, and so it has an inverse given by the equation $x = y^3 - 3y^2 + 9y + 2$ giving $y = f^{-1}(x)$? Give reason.
(c) Find a decimal approximation (1 decimal place) of $f^{-1}(3)$. That is, solve the equation $y^3 - 3y^2 + 9y + 2 = 3$

61. Solve Exercise 60 for the function
$f(x) = x^3 - 3x^2 + 9x - 2$.

62. Explain why a fourth degree polynomial function cannot be one-one. Consider end behavior.

Exercises 63–64 ***Point of Intersection*** *On the same screen, draw graphs of f and g. The two graphs intersect at a single point. Find the coordinates of that point (1 decimal place).*

63. $f(x) = x^3 - 2x^2 + 5x + 4$, $\quad g(x) = x\sqrt{x + 4}$.

64. $f(x) = x^3 + 2x^2 + 3x - 5$,
$g(x) = x^2 - 8x + 15$.

Exercises 65–66 ***Intersecting Graphs*** *The graph of f and the half circle intersect at a single point. Use calculator graphs to help you find the coordinates of the point of intersection (1 decimal place).*

65. $f(x) = x^3 - 3x^2 + 5x - 8$;
upper half of circle $(x - 2)^2 + y^2 = 25$.

66. $f(x) = x^3 - 3x^2 + 5x + 8$;
lower half of circle $(x + 1)^2 + y^2 = 9$.

Exercises 67–72 Determine the end behavior of the graph of the function when $x \to \infty$ and when $x \to -\infty$.

67. $f(x) = 2x - 3x^2$

68. $g(x) = x^4 - 3x^2 + 4$

69. $h(x) = 1 + 3.4x - 5.2x^3 - 2x^5$

70. $f(x) = -\dfrac{2}{5}(2x - x^3)$

71. $g(x) = (3 - 2x)(4 - x^3)$

72. $h(x) = (1 - 2x)(3 - 4x^2)$

Exercises 73–76 ***Looking Ahead to Calculus*** *In calculus you will learn that for the given function f there is an associated function g such that the real zeros of g are the x-coordinates of the local extrema points of f. If g has no real zeros then f has no local extrema points.*
(a) *Find the zeros of g.*
(b) *Find the coordinates of the local extrema points of f, if there are any.*
(c) *Use a graph as a check.*

73. $f(x) = 2x^3 + 3x^2 - 12x + 3$,
$g(x) = x^2 + x - 2$

74. $f(x) = x^3 - x^2 - 8x + 1$,
$g(x) = 3x^2 - 2x - 8$

75. $f(x) = x^3 - 3x^2 + 12x + 1$,
$g(x) = x^2 - 2x + 4$

76. $f(x) = x^3 - 6x^2 + 9x + 4$,
$g(x) = x^2 - 4x + 3$

3.2 LOCATING ZEROS

> The man who breaks out into a new era of thought is usually himself still a prisoner of the old. Even Isaac Newton, who invented the calculus as a mathematical vehicle for his epoch-making discoveries in physics and astronomy, preferred to express himself in archaic geometrical terms.
>
> *Freeman Dyson*

I had a good teacher for freshman algebra. I think he was simultaneously the football coach. Then I took sophomore geometry. It was apparently thought that students couldn't learn geometry in one year so they had a second course in the junior year. The teacher in this second course didn't understand the subject and I did. I made a lot of trouble for her.

Saunders MacLane

In Section 3.1 we indicated that two of the important concerns for polynomials are locating zeros and local extrema. With calculator graphs we can make excellent approximations for both. On a graph, there is no obvious relation between zeros and turning points, but in calculus we learn that every turning point of a polynomial function f occurs at a zero of another polynomial function called the *derivative of* f. Thus the location of turning points also depends on locating zeros. We leave the study of derivatives to calculus, but we devote this section to understanding more about zeros of polynomial functions.

It would be nice to have something analogous to the quadratic formula for higher degree polynomials. For some polynomial functions, zeros can be expressed in exact form using radicals and the ordinary operations of algebra, but in general, we must rely on approximations. See the two Historical Notes in this section. Some of the theorems included in this section help us determine whether or not exact form solutions are available.

Locator Theorem

Graphs of all polynomial functions share some common properties; they are continuous and smooth, with no corners, breaks, or jumps. The idea of continuity (no breaks or jumps) is another topic for calculus and subsequent courses. We need some such theorem because calculator graphs are neither smooth nor continuous. Every calculator graph is the result of computing lots of discrete function values (one for each column of pixels). Dot mode shows only the isolated points. How are we to be confident that there isn't some break or jump in the graph between two adjacent pixel columns? In connected mode, a graphing calculator connects different points of the graph by *vertical strips,* which we know cannot be part of the graph of any function.

Nevertheless, our eye smooths out calculator graphs. We have come to expect graphs to be smooth and continuous, and the following theorem supports our intuition. It says, in effect, that a polynomial function cannot change from positive to negative without going through 0. The locator theorem is a special case of a theorem from analysis called the Intermediate Value Theorem.

Locator (sign-change) theorem

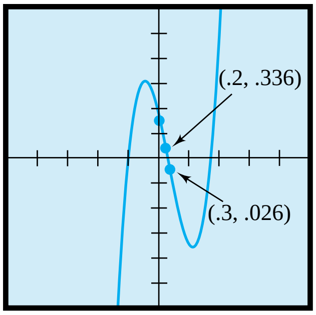
Suppose p is a polynomial function and a and b are numbers such that $p(a)$ and $p(b)$ have opposite signs. The function p has at least one zero between a and b, or equivalently, the graph of $y = p(x)$ crosses the x-axis between $(a, 0)$ and $(b, 0)$.

►EXAMPLE 1 *Using the locator theorem*

Let $p(x) = 2x^3 - 2x^2 - 3x + 1$.

(a) Use the locator theorem to verify that p has a zero between 0 and 1.

(b) Using a decimal window, graph $y = p(x)$ and trace to verify that the zero is located between 0.2 and 0.3.



[−5, 5] by [−3.5, 3.5]

$p(x) = 2x^3 - 2x^2 - 3x + 1$

FIGURE 11

Solution

(a) $p(0) = 1$ and $p(1) = -2$, so there is a sign change and hence a zero between 0 and 1.

(b) Tracing along the graph of $p(x)$ in a decimal window, we can read the coordinates shown in Figure 11. Thus $p(0.2) = 0.336$ and $p(0.3) = -0.026$. There is a sign change between 0.2 and 0.3, so there is a zero in the interval. ◀

We could, of course, use the calculator to locate the zero in Example 1 more precisely. If, for example, we simply zoom in on the point (0.2, 0) and then trace, we can locate the zero between 0.275 and 0.30. With time and patience we can locate zeros with as much accuracy as a calculator will display. A closer approximation is 0.2929.

To get zeros in exact form, to move beyond approximations, we need other tools. The most powerful technique available to us involves factoring and the zero-product principle. Unfortunately, in most cases where we need the zeros of a given polynomial function, there is no dependable procedure for finding even one zero in exact form, and with polynomial functions of higher degree, even knowing several zeros may not be enough.

The Division Algorithm

Just as we can divide one integer by another to get an integer part q and a remainder r, where the remainder must be smaller than the divisor, so we can divide one polynomial by another. The result of polynomial division is a polynomial part $q(x)$, and a remainder $r(x)$ whose degree must be smaller than the degree of the divisor. In particular, when the divisor is a linear polynomial (of the form $x - c$), then the remainder is some number r. This result is stated as a theorem known as the Division Algorithm. Properly, in the statement of the theorem, the degree of the divisor, $d(x)$, must be no greater than the degree of the polynomial we are dividing, $p(x)$. In our work, we assume a divisor that is either a linear or a quadratic polynomial.

Division algorithm

If $p(x)$ is a polynomial of degree greater than zero, and $d(x)$ is a polynomial, then dividing $p(x)$ by $d(x)$ yields unique polynomials $q(x)$ and $r(x)$, called the **polynomial part** and **remainder,** respectively, such that

$$p(x) = d(x) \cdot q(x) + r(x),$$

where the degree of $r(x)$ is smaller than the degree of $d(x)$.

If $d(x) = x - c$, then the remainder is a unique number r, such that

$$p(x) = (x - c) \cdot q(x) + r. \quad \textbf{(1)}$$

To find the polynomial part and remainder for any given pair of polynomials we use the familiar process of long division. For the special case of a linear divisor, there is a shortcut called *synthetic division.* Synthetic division is stressed in traditional courses because it is also used for several different evaluation purposes. With graphing calculators, however, the convenience of synthetic division does not justify the time required to learn the process. We outline synthetic division in the following (optional) discussion. You may divide by any method you wish, but we will do all of our polynomial division by long division.

HISTORICAL NOTE IS THERE A CUBIC FORMULA?

The Babylonians could solve some quadratics nearly four thousand years ago, as could the ancient Greeks and Egyptians although they thought only positive roots had meaning. In essence, the quadratic formula has been around for at least a thousand years.

From at least 1200 A.D. people have searched for a comparable formula for cubics. The story of who first succeeded, and when, gets muddled by conflicting claims. At least part of the credit belongs to Scipione del Ferro (ca. 1510). By about 1540, Tartaglia had learned enough to win a public contest, solving 30 cubics in 30 days. Somehow, Cardan got the method from Tartaglia and published it in 1545, much to Tartaglia's dismay. The solution is often called Cardan's even though he did credit Tartaglia.

Although Cardan is known for developing the first cubic formula, credit actually belongs to Tartaglia, pictured here.

The methods of this chapter are much easier to apply, but the formula from Cardan's book still works. Given a cubic of the form

$$x^3 + ax + b = 0,$$

first calculate

$$A = \left(\frac{a}{3}\right)^3 + \left(\frac{b}{2}\right)^2.$$

Cardan's solution is given by

$$x = \sqrt[3]{\sqrt{A} - \frac{b}{2}} - \sqrt[3]{\sqrt{A} + \frac{b}{2}}.$$

Synthetic Division Algorithm (Optional)

Synthetic division is sometimes a convenient method for factoring a polynomial function. We do not justify the steps, but the procedure is really nothing but a short-cut method of dividing, using only the coefficients. The steps are outlined in the following box.

Synthetic division algorithm (for divisors of the form $x - c$)

To divide a polynomial $p(x)$ of degree n by $x - c$:

1. On the top line write c (change sign from $x - c$), followed by all the coefficients of $p(x)$ in order of decreasing powers of x, including any zero coefficients.
2. Bring down the leading coefficient, multiply by c, and add the product to the next coefficient to get the next entry on the bottom line. Repeat, multiplying the sum by c, and adding the product to the next coefficient, and continue for all coefficients of p.
3. The first n numbers on the bottom line are the coefficients of $q(x)$, of degree $n - 1$, and the last number is the remainder r.

We illustrate the synthetic division algorithm with the problem from Example 2: Divide $p(x) = 2x^3 - 2x^2 - 3x + 1$ by $x + 1$. You may want to compare the coefficients in the long division of Example 2 with the numbers in the synthetic division. Since $x + 1 = x - (-1)$, We write -1 on the top line at the left, followed by the coefficients of $p(x)$ in order of decreasing powers of x:

$$\begin{array}{r|rrrr} -1 & 2 & -2 & -3 & 1 \\ & & -2 & 4 & -1 \\ \hline & 2 & -4 & 1 & 0 \end{array}$$

c from $x - c \rightarrow$

$\leftarrow$ Coefficients of $p(x)$

$\leftarrow$ For each entry on middle line, multiply bottom entry by -1, and add.

$q(x) = 2x^2 - 4x + 1$ $\quad r = 0$

From the last line we read the coefficients of the polynomial part $q(x)$, when $p(x)$ is divided by $x + 1$, and the remainder r. Writing $p(x)$ in the form of the division algorithm,

$$2x^3 - 2x^2 - 3x + 1 = (x + 1)(2x^2 - 4x + 1) + 0.$$

►EXAMPLE 2 ***Using the division algorithm*** Verify that -1 is a zero of the polynomial function from Example 1, $p(x) = 2x^3 - 2x^2 - 3x + 1$, and find the other two zeros in exact form.

Solution

Substituting -1 for x, $p(-1) = 0$.

Dividing $p(x)$ by $x + 1$ yields the following:

$$\begin{array}{r} 2x^2 - 4x + 1 \\ x + 1\overline{)2x^3 - 2x^2 - 3x + 1} \\ \underline{2x^3 + 2x^2} \qquad\qquad \\ -4x^2 - 3x \qquad \\ \underline{-4x^2 - 4x} \qquad \\ x + 1 \\ \underline{x + 1} \\ 0 \end{array}$$

Since the remainder is 0, $p(x)$ can be written in factored form as

$$2x^3 - 2x^2 - 3x + 1 = (x + 1)(2x^2 - 4x + 1).$$

By the zero-product principle, either $x + 1 = 0$ or $2x^2 - 4x + 1 = 0$. Using the quadratic formula for the second equation, we find that the zeros of p are -1, $\frac{2 - \sqrt{2}}{2}$, $\frac{2 + \sqrt{2}}{2}$. The second zero is about 0.29289, clearly the number we were "zeroing in on" in Example 1. ◄

►EXAMPLE 3 ***The division algorithm again***

(a) Show that $x^2 + 4$ is a factor of the polynomial $p(x) = x^4 - x^3 + 2x^2 - 4x - 8$. **(b)** Find all zeros of $p(x)$.

Solution

(a) We use long division.

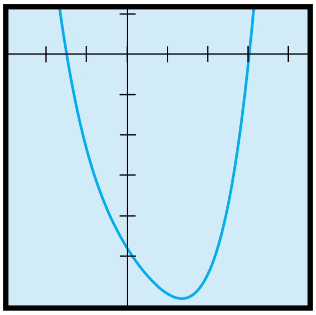
$$\begin{array}{r} x^2 - x - 2 \qquad\qquad\qquad \\ x^2 + 4\overline{)x^4 - x^3 + 2x^2 - 4x - 8} \\ \underline{x^4 \qquad + 4x^2} \qquad\qquad\quad \\ - x^3 - 2x^2 - 4x \qquad \\ \underline{- x^3 \qquad\quad - 4x} \qquad \\ - 2x^2 \qquad - 8 \\ \underline{- 2x^2 \qquad - 8} \\ 0 \end{array}$$

By the division algorithm, since the remainder is 0,

$$p(x) = (x^2 + 4)(x^2 - x - 2),$$

and we have a factorization of $p(x)$, thus reducing the problem of finding the zeros of a fourth polynomial function p to solving two quadratic equations, $x^2 + 4 = 0$, and $x^2 - x - 2 = 0$.

(b) The four zeros of $p(x)$ are $\pm 2i$, -1, and 2. With two real zeros, the graph of $y = p(x)$ should cross the x-axis just twice. See Figure 12. ◀



$[-2, 3]$ by $[-10, 2]$
$p(x) = x^4 - x^3 + 2x^2 - 4x - 8$

FIGURE 12

Factor and Remainder Theorems

Examples 2 and 3 illustrate one of the key concepts in finding exact form zeros of polynomial functions. Finding a zero is equivalent to finding a factor, and once we have factors, by the zero-product principle, the zeros of p are the zeros of the factors. Further, since the degree of q is smaller than the degree of p, $q(x)$ is called the **reduced polynomial.**

In the case where the divisor is linear, the division algorithm provides two powerful theorems. Consider again Equation (1),

$$p(x) = (x - c)q(x) + r.$$

First, suppose $r = 0$. Then $(x - c)$ is a factor, $p(x) = (x - c)q(x)$. Conversely, if $(x - c)$ is a factor of $p(x)$, then $p(x) = (x - c)q(x)$, so r must be 0.

Equation (1) is an identity, so we can replace x by any number and obtain a true statement. In particular, if we replace x by c, we obtain

$$p(c) = (c - c)q(c) + r = 0 \cdot q(c) + r = 0 + r = r.$$

Thus the remainder always equals the value of the function p at the number c.

Putting these observations together, we have the following.

Remainder and factor theorems

When $p(x)$ is divided by $x - c$, the remainder is $p(c)$.
When $p(x)$ is divided by $x - c$, then $x - c$ is a factor of $p(x)$ if and only if $p(c) = 0$.

The factor and remainder theorems give us ways to find a remainder without performing a lengthy division and can simplify many calculations.

►EXAMPLE 4 The remainder theorem

(a) Find the remainder when the polynomial $p(x) = 4x^{15} + 5x^7 + 2x^4 + 3$ is divided by $x + 1$.

(b) Find the value of k such that if the polynomial $P(x) = x^3 + x^2 + kx - 4$ is divided by $x + 2$, then the remainder is 0.

Strategy: By the remainder theorem, for **(a)** $p(-1) = r$, and for **(b)** $P(-2) = 0$, from which we can solve for k.

Solution

(a) Following the strategy, $p(-1) = 4(-1) + 5(-1) + 2(1) + 3 = -4$. Hence, $p(-1) = -4$, so $r = -4$.

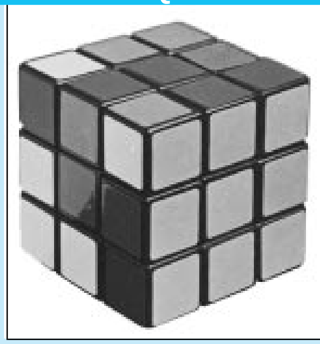
(b) Evaluating P at -2, we have $P(-2) = -8 + 4 - 2k - 4 = -8 - 2k$. Since the remainder when $P(x)$ is divided by $x + 2$ is 0, then $P(-2) = 0$. Thus,

$$-8 - 2k = 0, \quad \text{or} \quad -2k = 8, \quad \text{or} \quad k = -4.$$

The desired function is

$$P(x) = x^3 + x^2 - 4x - 4.$$ ◀

HISTORICAL NOTE **THERE IS NO "QUINTIC FORMULA"**



Very shortly after discovery of the general solution of the cubic equation (see "Is There a Cubic Formula?"), Ferrari (Italy, ca. 1545) derived a method for quartics (polynomials of degree 4). For $n = 2, 3$, or 4, solutions for equations of degree n involve nth roots. Why not for degree 5? Nearly three hundred years passed before much more was done. Then, within a few years, two brilliant young men completely resolved the question.

In 1820 Niels Henrik Abel of Norway was 18 when he thought he had the desired formula. Before it could be checked by others, however, he found his error and proved that there could be no general solution for quintic equations.

In Paris in 1829 another 18-year-old, Evariste Galois, took the final step. In papers written during 1829 and 1830, Galois found the conditions that determine just which polynomial equations of degree 5 or higher can be solved in terms of their coefficients.

Abel died of tuberculosis in 1829 at age 26. In 1831, at the age of 20, Galois was killed in a duel he himself recognized as stupid. During their brief careers, they laid the foundations for modern group theory, which has applications as diverse as solutions for Rubik's cube and the standard model of elementary particle physics at the beginning of the universe.

Clearing Fractions and Rational Zeros

Multiplying an equation by a nonzero constant to clear fractions does not change the roots of the equation. For example, multiplying both sides of the equation

$$x^3 + \frac{7}{2}x^2 + \frac{7}{3}x - \frac{2}{3} = 0,$$

by 6, we get an equation with integer coefficients having the same roots,

$$6x^3 + 21x^2 + 14x - 4 = 0.$$

If all coefficients are integers, then the following theorem provides a complete list of all the rational numbers that can possibly be zeros.

Rational zeros theorem

Let p be any polynomial function with *integer coefficients.* The only rational numbers that can possibly be zeros of p are the numbers of the form $\frac{r}{s}$, where r is a divisor of the constant term, and s is a divisor of the leading coefficient.

If none of these numbers is a zero, then p has no rational zeros.

The rational zeros theorem is useful and important because it lists all the possibilities for rational zeros. The theorem does not tell us whether a given polynomial has any rational zeros at all; many do not. Without graphing technology, the theorem is a great help in guiding the search for zeros. Used with a grapher, the theorem can tell us things about graphs that the calculator cannot. *No calculator can distinguish between rational and irrational numbers;* every decimal is truncated ("cut off") to fit the display capacity. Knowing what the rational possibilities are, we can use a calculator to verify that a particular zero is or is not a rational number.

►EXAMPLE 5 ***Rational possibilities*** Use the rational zeros theorem to list all possible zeros of the polynomial function.

(a) $P(x) = x^3 - 4x^2 + x - 6$ **(b)** $R(x) = x^4 - 4x^3 + \frac{14}{9}x^2 + \frac{44}{9}x - \frac{5}{3}$

Strategy: **(b)** To get integer coefficients, multiply through by 9 and then apply the rational zeros theorem. Zeros of $9R$ are are the same as the zeros of R.

Solution

(a) For P, begin by listing all possible numerators (factors of -6) and denominators (factors of 1):

$$\begin{array}{ll} \text{Possible numerators:} & \pm\ 1, 2, 3, 6 \\ \hline \text{Possible denominators:} & \pm\ 1 \end{array}$$

The only possibilities for rational zeros of P are the integers $-6, -3, -2, -1, 1, 2, 3,$ and 6.

(b) Follow the strategy and find all possible rational zeros of $S(x) = 9R(x)$:

$$S(x) = 9x^4 - 36x^3 + 14x^2 + 44x - 15.$$

Possible numerators are factors of 15; denominators are factors of 9.

$$\begin{array}{ll} \text{Possible numerators:} & \pm\ 1, 3, 5, 15 \\ \hline \text{Possible denominators:} & \pm\ 1, 3, 9 \end{array}$$

The rational zeros theorem tells us that S, and hence R, has only sixteen possible rational zeros (in reduced form):

$$\pm\left[1, 3, 5, 15, \tfrac{1}{3}, \tfrac{5}{3}, \tfrac{1}{9}, \tfrac{5}{9}\right]. \quad ◄$$

Having listed lots of possibilities for rational zeros of the polynomials P and R in Example 5, what do we know of the actual zeros? At this stage, we have nothing but possibilities. When we add graphing technology, we can say a great deal more, as in the next example.

►EXAMPLE 6 ***Finding rational zeros*** Find all rational zeros, if there are any, of the polynomial functions P and R in Example 5. Approximate any irrational zeros to two decimal place accuracy.

(a) $P(x) = x^3 - 4x^2 + x - 6$ **(b)** $R(x) = x^4 - 4x^3 + \frac{14}{9}x^2 + \frac{44}{9}x - \frac{5}{3}$

Solution

(a) We don't know anything about P except that as a cubic it must have at least one real zero. We begin with a graph in the $[-10, 10] \times [-10, 10]$ window. See Figure 13a. It is clear that there is exactly one real zero, very near 4. Looking at the list of rational zeros for P from Example 5, we see that the only positive rational possibilities are 1, 3, and 6. Therefore, the real zero of the polynomial P cannot be a rational number. If we zoom into a very small box around the

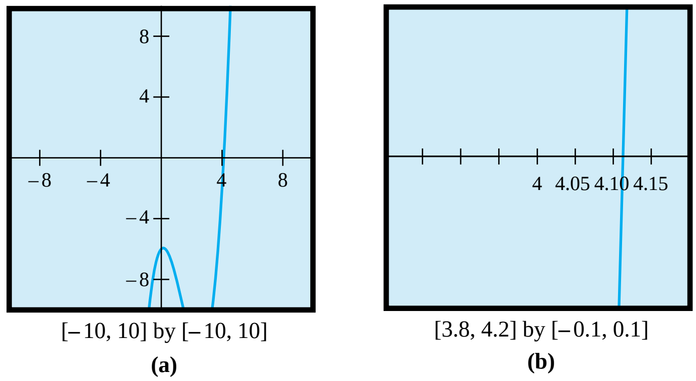


FIGURE 13
$P(x) = x^3 - 4x^2 + x - 6$

x-intercept point of the graph (Figure 13b) and then trace, we find that the zero is very near 4.11.

(b) For graphing purposes, we can choose either the polynomial $R(x)$, or the polynomial $S(x) = 9R(x)$ with integer coefficients, because R and S have the same zeros. If we were working by hand, most of us would choose S to avoid fractions; for the calculator there is no difference, except that R requires a smaller window. The graph of R is shown in the $[-10, 10] \times [-10, 10]$ window we used for part (a) in Figure 14a. When we zoom into a box just large enough to include the zeros of the graph, as in Figure 14b, we can trace along the curve to find zeros near -1, 0.3, 1.6, and 3.

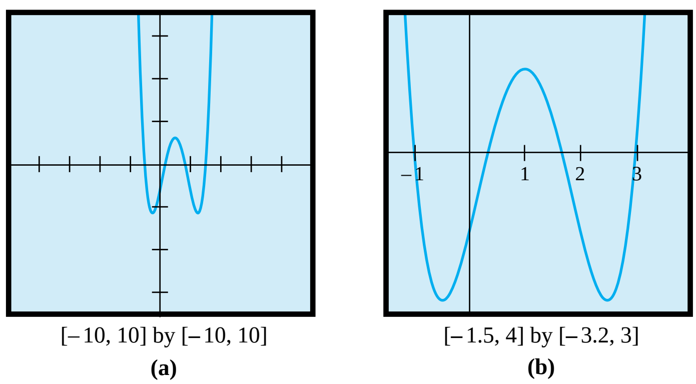


FIGURE 14

Which, if any, of these are rational zeros? The list of rational possibilities (from Example 5) includes -1, 3, $\frac{1}{3}$, and $\frac{5}{3}$, all of which are reasonable candidates, but are these actually zeros of the function? To decide, we need to evaluate R at each number (see the following Technology Tip). To calculator accuracy we find that

$$R(-1) = 0, \quad R(3) = 0, \quad R(\tfrac{1}{3}) = 0, \quad R(\tfrac{5}{3}) = 0.$$

We conclude that R has four rational zeros, -1, 3, $\frac{1}{3}$, and $\frac{5}{3}$. ◀

TECHNOLOGY TIP ◆ Function evaluation

Most calculators, when we trace along a curve, display coordinates. The y-coordinate is the calculated value corresponding to the x-coordinate of the pixel, but we have no way to specify a particular x-value unless our window happens to have it as a pixel. If our goal is to evaluate $R(\frac{1}{3})$, we don't want to settle for $R(.324076113)$. We give here some suggestions for different calculators, but there may be a more efficient way for your particular machine. The displayed value is the calculator's evaluation of $R(\frac{1}{3})$, which may involve round-off error. For example, one of our calculators displays $R(\frac{1}{3})$, = 3E-13, meaning 0.0000000000003, and which in this context we interpret as 0.

TI calculators: If you are graphing a function as Y1, return to the home screen, store the desired value in the x-register, and then enter Y1. Thus, for $R(-1)$, $-1 \rightarrow$ X Enter. Then call up Y1 from the Y-vars menu (or on the TI-85, 2nd Alpha y1, and Enter. The TI-82 will evaluate Y1(−1) directly.

Casio calculators: The function must be entered on your MEM list, so type in the function, SHIFT MEM, F1(STO) and the number, say 1 for f_1. To evaluate $f_1(3)$, EXIT and store 3 in the x-register, 3 → X EXE. Then F2(RCL) 1 EXE.

HP–38: Having entered a function as, say, F1(X), return to the home screen, type F1(1/3), and ENTER.

HP-48: The calculator will evaluate the function at any pixel-address and store the result on the stack, but direct evaluation is less convenient. One way is to store the number in, say, register A: −1 ENT 'A' STO. Then write the function as an expression in A: 'A^4 − 4*A^3 + 14/9*A^2 + 44/9*A − 5/3'. ENTER twice, so you have an extra copy on the stack. Then, purple NUM converts to a number. For another value, store it in A, Enter, and evaluate as a number.

All these calculators except the TI-81 have some sort of SOLVE routine that will approximate zeros directly, which does not make it less important for you to understand the ideas we are discussing here.

Applying the Rational Zeros Theorem

The rational zeros theorem may be applied to problems other than looking for the zeros of a particular polynomial function. For instance, if we know that some number c is a zero of a polynomial function but c is not among the possibilities for rational zeros, then we can conclude that c is not a rational number, as in the next example.

►EXAMPLE 7 Showing a number is not rational

(a) Find a polynomial equation with integer coefficients satisfied by the number $c = \sqrt[3]{5} - 1$.

(b) Use the rational zeros theorem to show that c is not a rational number.

Solution

(a) The simplest polynomial equation satisfied by c is

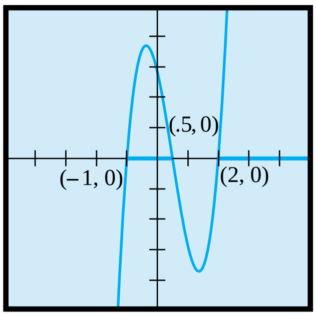
$$x = \sqrt[3]{5} - 1,$$

but of course, this does not have integer coefficients. To get integer coefficients, we add 1, cube both sides, and simplify: [See the inside front cover for the expansion of $(x + 1)^3$.]

$$
\begin{aligned}
(x + 1)^3 &= (\sqrt[3]{5})^3 \\
x^3 + 3x^2 + 3x + 1 &= 5 \\
x^3 + 3x^2 + 3x - 4 &= 0.
\end{aligned}
$$

Now we have a polynomial equation, one of whose solutions is the number c. Incidentally, a graph makes it clear that $p(x) = x^3 + 3x^2 + 3x - 4$ has only one real zero, which thus must be $\sqrt[3]{5} - 1$.

(b) Since the leading coefficient of $p(x)$ is 1, the only possible rational zeros of p are $\pm [1, 2, 4]$. The number c is a zero of p but is not one of the possible rational zeros, so $\sqrt[3]{5} - 1$ must be an irrational number. ◀



$[-5, 5]$ by $[-3.5, 3.5]$

$2x^3 - 3x^2 - 3x + 2 \geq 0$

on $[-1, 0.5] \cup [2, \infty)$

FIGURE 15

▶EXAMPLE 8 ***Solving an inequality*** Find the solution set for $2x^3 - 3x^2 - 3x + 2 \geq 0$.

Solution

A graph of $f(x) = 2x^3 - 3x^2 - 3x + 2$ is shown in Figure 15. It appears from the graph that the zeros are -1, $\frac{1}{2}$, and 2, as we may verify, either by tracing in a decimal window, or by evaluating the function. The function f is clearly positive between -1 and $\frac{1}{2}$, and whenever $x > 2$. Therefore, the solution set for the inequality is given by

$$S = [-1, 0.5] \cup [2, \infty). \quad ◀$$

EXERCISES 3.2

Check Your Understanding

Draw a graph whenever helpful.

Exercises 1–6 True or False. Give reasons.

1. The function $p(x) = 4x^3 - x$ has three real zeros.
2. The positive zero of $f(x) = x^3 - 3x$ is less than 1.73.
3. For $f(x) = x^3 - 1.6x^2 - 8.52x + 15.84$, since $f(2)$ and $f(3)$ are positive, then f contains no zeros between 2 and 3.
4. The equation $2x^3 - 5x^2 + 4x - 1 = 0$ has no rational roots.
5. The function $f(x) = (3x - 2)(x^2 - 2x - 4)$ has exactly one real zero.
6. When $x^3 - 2x^2 + 3x - 16$ is divided by $x - 3$, then the remainder is 2.

Exercises 7–10 Fill in the blank so that the resulting statement is true.

7. If $x^3 + 2x^2 + 1 = (x + 1)(x^2 + x - 1) + r$ for every value of x, then $r =$ ______.
8. The number of rational zeros of $f(x) = (x^2 - 2)(x^2 - 2x + 3)$ is ______.
9. The number of real roots of $(x^2 - 2)(x^2 - 2x + 3) = 0$ is ______.
10. If $x^{37} - 2x^{24} + 3x^2 - 5$ is divided by $x + 1$, then the remainder is ______.

Develop Mastery

Exercises 1–4 ***Locator Theorem*** *Use the locator theorem to determine which half of the interval contains a zero of the function.*

1. $p(x) = x^3 - 3x + 1$; $[-2, -1]$
2. $f(x) = 2x^3 + 3x^2 - x - 2$; $[0,1]$
3. $g(x) = x^3 - 5x^2 + 5x + 3$; $[-1, 0]$
4. $p(x) = x^3 - 5x^2 + 7x - 2$; $[2.5, 3]$

Exercises 5–8 ***Division Algorithm*** *Use division to find the polynomial part $q(x)$ and remainder r when $p(x)$ is divided by the given divisor. Write the result in the form $p(x) = (x - c)q(x) + r$, and find $p(c)$.*

5. $p(x) = 2x^3 + 3x^2 - x - 2;\ x - 1$

6. $p(x) = 2x^3 + 3x^2 - x - 2;\ x + 2$

7. $p(x) = 3x^4 + x^3 - 2x^2 + x - 1;\ x + 1$

8. $p(x) = x^3 - 3x^2 - x + 2;\ x - 4$

Exercises 9–10 ***Remainder*** *Find the remainder when the polynomial is divided by $x - c$.*

9. $4x^{12} - 3x^8 + 5x^2 - 2x + 3;\ c = -1$

10. $x^{10} - 64x^4 + 3;\ c = 2$

Exercises 11–14 ***Factor Theorem*** *Use the factor theorem to find the value of k so that the given linear expression is a factor of the polynomial.*

11. $2x^3 + 4x^2 + kx - 3;\ x + 2$

12. $x^4 + kx^2 + kx + 2;\ x - 2$

13. $kx^3 + 3x^2 - 4kx - 7;\ x - 3$

14. $x^3 - k^2x + (k + 1);\ x - k$

Exercises 15–18 ***Remainder Theorem*** *Use the remainder theorem to find the value of k so that when $p(x)$ is divided by the linear expression you get the given remainder.*

15. $p(x) = 2x^3 + 4x^2 + kx - 3;\ x + 2;\ r = 0$

16. $p(x) = 2x^3 + 4x^2 + kx + 3;\ x + 2;\ r = 3$

17. $p(x) = x^3 + kx^2 - kx + 8;\ x - 2;\ r = 1$

18. $p(x) = x^3 + kx^2 - kx + 8;\ x - 2;\ r = 0$

Exercises 19–24 ***Rational Zeros Theorem*** ***(a)*** *Apply the Rational Zeros Theorem to list all of the possible rational zeros of f. If the theorem does not apply, explain why.* ***(b)*** *Use a calculator graph to help you eliminate some to the numbers listed in part (a).*

19. $f(x) = 6x^3 + 3x^2 - 2x - 1$

20. $f(x) = 6x^3 - 2x^2 - 9x + 3$

21. $f(x) = 2x^4 - 2x^3 - 6x^2 + x + 2$

22. $f(x) = 6x^3 - x^2 - 13x + 8$

23. $f(x) = 3x^3 - 1.5x^2 + x - 0.5$

24. $f(x) = x^3 - 2x^2 + \sqrt{2}x - 2$

Exercises 25–30 ***Exact Form Zeros*** ***(a)*** *Find all zeros of f (including any complex numbers) in exact form. First look for rational zeros and express $f(x)$ in factored form (linear or quadratic factors).* ***(b)*** *Find the solution set for $p(x) < 0$.*

25. $f(x) = x^3 - 4x^2 + 2x - 8$

26. $f(x) = 4x^3 - 4x^2 - 19x + 10$

27. $f(x) = x^3 - 2.5x^2 - 7x - 1.5$

28. $f(x) = x^3 - 3.5x^2 + 0.5x + 5$

29. $f(x) = 6x^4 - 13x^3 + 2x^2 - 4x + 15$

30. $f(x) = 4x^4 - 4x^3 - 7x^2 + 4x + 3$

Exercises 31–32 ***Solving Polynomial Equations*** *Find the solution set.*

31. **(a)** $3x^2 - 12x = (x - 1)(x^2 - 4x)$

(b) $\dfrac{3x^2 - 12x}{x^2 - 4x} = x - 1$

32. **(a)** $3x^3 - 12x = (x + 2)(x^3 - 4x)$

(b) $\dfrac{3x^3 - 12x}{x + 2} = x^3 - 4x$

Exercises 33–34 ***Exact Form Roots*** ***(a)*** *Find the roots in exact form. (Hint: The equation is quadratic in x^2. Use the quadratic formula to first find x^2.)* ***(b)*** *Get approximations (2 decimal places) to the answers in part (a). Graph as a check.*

33. $x^4 - 4x^2 + 1 = 0$

34. $x^4 - 2x^2 - 1 = 0$

Exercises 35–38 ***Solution Set*** *Find the solution set for* ***(a)*** $f(x) = 0$, ***(b)*** $f(x - 1) = 0$, ***(c)*** $f(x) \le 0$. *(Hint: For part (c), first factor and get cut points.)*

35. $f(x) = 2x^3 - 3x^2 - 3x + 2$

36. $f(x) = x^4 - 2x^3 - 3x^2 + 4x + 4$

37. $f(x) = x^3 - 3x + 2$

38. $f(x) = 4x^3 - 4x^2 - 19x + 10$

Exercises 39–42 ***Nonrational Numbers*** ***(a)*** *Find a polynomial equation with integer coefficients having c as a root.* ***(b)*** *Explain why c is not a rational number. (Hint: See Example 7.)*

39. $c = \sqrt{2}$

40. $c = 2 + \sqrt{5}$

41. $c = \sqrt[3]{2} - 1$

42. $c = 2\sqrt[3]{3} + 1$

Exercises 43–46 ***Verbal to Formula*** ***(a)*** *Find a formula (in expanded form) for a polynomial function satisfying the given conditions.* ***(b)*** *How many turning points does the graph have?*

43. Degree 3; zeros are -2, 1, and 3; leading coefficient is 1.

44. Degree 3; zeros are -1, 2, and 4; leading coefficient is -2.

45. Degree 4; zeros are -1, 2, and a double zero at 1; graph of f contains the point $(0, -2)$.

46. Degree 4; zeros are 0, 2, and $(x^2 - 2x - 5)$ is a factor of $f(x)$; graph contains the point $(3, 6)$.

Exercises 47–48 ***Zeros and Turning Points*** ***(a)*** *How many real zeros does f have?* ***(b)*** *Find all rational zeros.* ***(c)*** *How many turning points does the graph of f have, if any? In what quadrants?*

47. $f(x) = (x^2 - 4)(x^2 - 8x + 15)$

48. $f(x) = (2x^2 - x - 3)(x^2 + 2x + 4)$

49. Is there a polynomial function of degree 3 that has zeros at -2, -1, and 2, whose graph passes through the points $(0, 4)$, $(3, -20)$? Explain.

Exercises 50–53 ***End Behavior and Local Minima*** *The intercept points for a polynomial function f of degree 3 are given.* ***(a)*** *Draw a rough sketch and determine the end behavior.* ***(b)*** *Determine the coordinates (one decimal place) of the local minimum point.*

50. $(-2, 0)$, $(1, 0)$, $(3, 0)$, $(0, 6)$

51. $(-2, 0)$, $(1, 0)$, $(3, 0)$, $(0, -6)$

52. $(-4, 0)$, $(-2, 0)$, $(1, 0)$, $(0, 8)$

53. $(-5, 0)$, $(-1, 0)$, $(1, 0)$, $(0, -10)$

Exercises 54–57 ***Local Maxima*** *For the function in Exercises 50–53, give the coordinates (one decimal place) of the local maximum point.*

Exercises 58–61 ***Your Choice*** *Suppose the graph of a polynomial function f of degree 3 has a local maximum point at P and a local minimum point at Q. Draw a rough sketch (or sketches) of the graph of f and use it to describe any features such as the location of x-intercept points, end behavior, and any other properties. Write an equation for your function.*

58. Both P and Q are in Quadrant I.

59. P is in Quadrant I and Q is in Quadrant IV.

60. P is in Quadrant II and Q is in Quadrant IV.

61. P is in Quadrant I and Q is in Quadrant III.

Exercises 62–65 ***Bracketing Roots*** *Find the smallest interval with integer endpoints $[b, c]$ containing all the roots of the equation; that is, b is the largest integer smaller than all roots of the equation, and similarly for c.*

62. $2x^3 - 3x^2 - 2x + 3 = 0$

63. $x^3 - 3x^2 - 2x + 4 = 0$

64. $x^4 - 2x^2 - 3 = 0$

65. $x^3 - 3x^2 - 2x + 8 = 0$

66. **(a)** For what number c is c a zero of $f(x) = 2x^3 - cx^2 + (3 - c^2)x - 6$?
(b) Using your value for c, draw a graph and verify that c is a zero of f.

Exercises 67–68 ***Bracketing Zeros*** *Find the smallest interval with integer endpoints $[b, c]$ containing all the zeros of the function. See Exercises 62–65.*

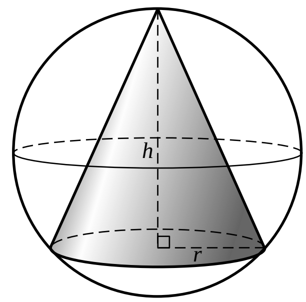
67. $f(x) = x^4 - 9x^2 + 6x - 4$

68. $f(x) = x^4 - 6x^2 + 3x + 4$

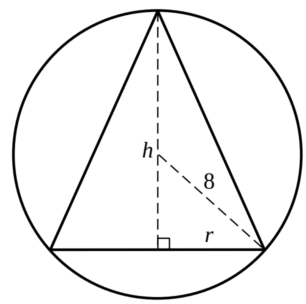
69. ***Explore*** Try integer values of c in $f(x) = x^3 - cx + 2$ until you find one for which f has a repeated zero. Then determine the other zero. Justify your conclusions algebraically.

70. Repeat Exercise 69 for $f(x) = x^3 - cx + 16$.

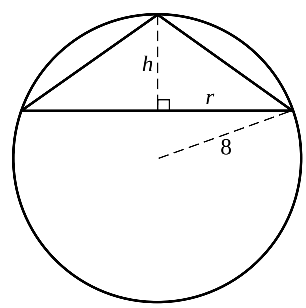
71. ***Cone in a Sphere*** A cone is inscribed in a sphere of radius 8 cm. See the diagrams showing cross sections. Let r denote the radius of the cone and h the height.



(a)



(b)



(c)

(a) Show that for both diagrams, $r^2 = 16h - h^2$. Express the volume V of the cone as a function of h.
(b) Of all such possible cones, there is one that has the greatest volume. What are the radius and height (1 decimal place) giving maximum volume? What is the maximum volume?

72. ***Cone in a Sphere (Alternative Approach)*** Solve Exercise 71 by expressing V as a function of r. Explain why it is not necessary to consider the second diagram to find the maximum volume.

3.3 MORE ABOUT ZEROS

> Why do we try to prove things anyway? I think because we want to understand them. We also want a sense of certainty. Mathematics is a very deep field. Its results are stacked very high, and they depend on each other a lot. You build a tower of blocks but if one block is a bit wobbly, you can't build the tower very high before it will fall over. So I think mathematicians are concerned about rigor, which gives us certainty. But I also think proofs are so that we can understand. I guess I like explanations better than step-by-step rigorous demonstrations.
>
> *William P. Thurston*

> *The problem that infected me with such virulence . . . concerned solving cubic equations and the answer had been known since Cardano published it in 1545. What I did not know was how to derive it. The sages who had designed the mathematics curricula . . . had stopped at solving quadratic equations. Questions by curious students about cubic and higher-order equations were deflected with answers such as "This is too advanced for you" or "You will learn this when you study higher mathematics," thereby creating a forbidden-fruit aura about the subject.*
>
> Mark Kac

We now have considerable experience with graphs of polynomial functions and a feeling for the nature of their zeros. When we have completely factored a polynomial, we obviously know its zeros, but how do we find the zeros (or equivalently, the factors) of a nonfactored polynomial?

Exact form answers are not always available, even theoretically (see the Historical Note, "There Is No Quintic Formula"). There are useful theorems about the nature of zeros, and we now have technological tools undreamed of by earlier mathematicians. This section adds to our arsenal of theorems about the nature of polynomial zeros, and then gives an informal introduction to Newton's Method, an important tool for approximating zeros with great precision.

Number of Zeros of Polynomial Functions

How many zeros does a polynomial function of degree n have? Returning again to quadratic functions (degree 2, parabola as the graph), there can be *two real zeros* (when the parabola crosses the x-axis twice), a *repeated real zero* (when the parabola is tangent to the x-axis), or *two nonreal conjugate zeros* (when the parabola does not touch the x-axis). Such behavior is typical of polynomial functions in general. We explore some cubics in the following two examples.

►EXAMPLE 1 ***Zeros of a family of cubics*** Consider the family $\mathscr{F}$ of cubic curves given by $f(x) = x^3 - cx + 2$. Graph members of $\mathscr{F}$ for $c = 0, 1, 2, 3, 4$. Decide what values of c give a cubic whose graph crosses the x-axis **(a)** just once, **(b)** exactly three times. **(c)** For what value of c does f have a repeated zero? Check by factoring.

Solution

(a) and **(b)** When $c = 0$, the graph is a vertical shift of the cubic $y = x^3$, with only one real zero. When $c = 1$ or 2, the graph has two turning points but crosses the x-axis only once, so there is only one real zero. At $c = 3$, the graph appears to just touch the x-axis at the point $(1, 0)$. See Figure 16. For values of c larger than 3, there are exactly three real zeros. **(c)** When $c = 3$, it appears that there is a repeated zero at 1, and the other crossing looks like $(-2, 0)$. To check, we want to show that a factored form of $x^3 - 3x + 2$ is $(x - 1)^2(x + 2)$.

$$(x - 1)^2(x + 2) = (x^2 - 2x + 1)(x + 2) = x^3 - 3x + 2. \quad ◄$$

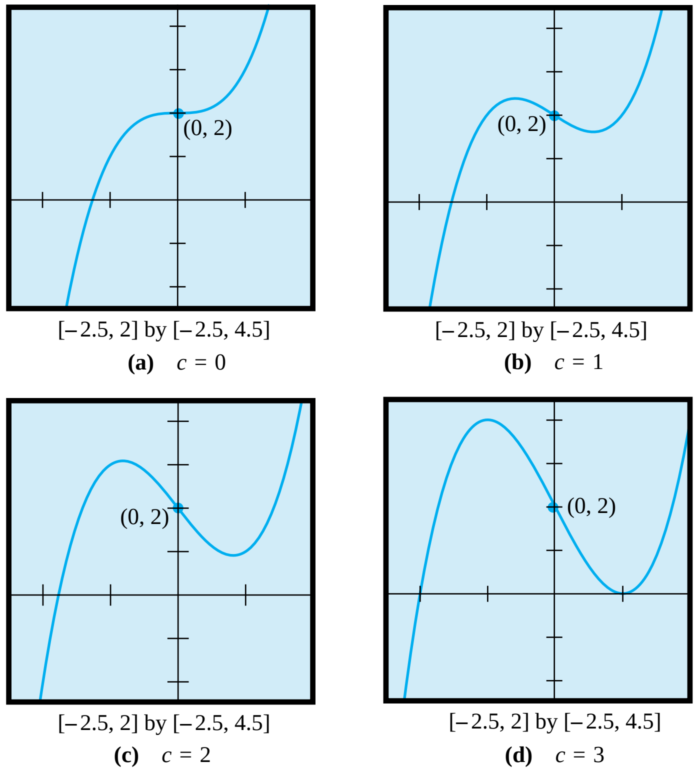


FIGURE 16
$y = x^3 - cx + 2$

▶EXAMPLE 2 ***Zeros of a family of cubics, continued*** Consider the family $\mathcal{F}$ of Example 1, $f(x) = x^3 - cx + 2$.

(a) Find a value c for which 2 is a zero of f, and find the other zeros of f in exact form. **(b)** Find all zeros in exact form when $c = -1$.

Solution

(a) To find a value of c for which $f(2) = 0$, we substitute 2 for x and solve for c:

$$2^3 - c \cdot 2 + 2 = 0$$
$$2c = 10, \quad \text{or} \quad c = 5.$$

Thus, the cubic in $\mathcal{F}$ that has 2 as a zero is $f(x) = x^3 - 5x + 2$. We know then that $(x - 2)$ is a factor, and we can use long division (or synthetic division, or simple factoring) to find the other factor:

$$x^3 - 5x + 2 = (x - 2)(x^2 + 2x - 1).$$

The remaining zeros of f are the roots of $x^2 + 2x - 1 = 0$, which the quadratic formula gives as $-1 \pm \sqrt{2}$ (*Check.*) See Figure 17.

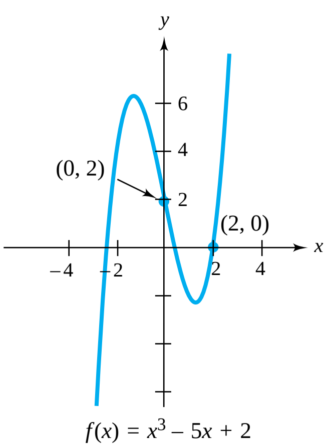


FIGURE 17

(b) For $c = -1$, $f(x) = x^3 + x + 2$ has one real zero and no turning points, and the x-intercept point appears to be $(-1, 0)$, which we can verify by substituting -1 for x: $f(-1) = (-1)^3 + (-1) + 2 = 0$. Therefore, $(x + 1)$ is a factor.

$$f(x) = x^3 + x + 2 = (x + 1)(x^2 - x + 2)$$

As in part **(a)**, the remaining zeros are the roots of $x^2 - x + 2 = 0$, which the quadratic formula gives as $\frac{1 \pm \sqrt{7}i}{2}$. ◀

Examples 1 and 2 remind us again of the utility of the factor and remainder theorems of Section 3.2, and they reinforce our ideas about the nature of zeros of cubic polynomials. A cubic always has three zeros, at least one real zero. There can be a repeated zero, and there is a possibility of two nonreal conjugate zeros.

Fundamental Theorem of Algebra

The general situation is summed up in the **fundamental theorem of algebra** and some of its consequences, called *corollaries*. These are stated in terms of the complex-number system and proofs necessarily involve complex numbers as well. The fundamental theorem was first proved by one of the greatest mathematicians of all times. See the Historical Note, "Carl Friedrich Gauss."

Fundamental theorem of algebra

Suppose p is a polynomial function of degree n, $n \geq 1$. There is at least one number c where $p(c) = 0$; that is, p has at least one zero (which may be a nonreal complex number).

Corollary 1: In the complex number system, p has exactly n zeros (counting multiplicities).

Corollary 2: If the coefficients of $p(x)$ are real numbers, then the graph of p can cross (or touch) the x-axis in at most n points.

The fundamental theorem of algebra is what mathematicians call an **existence theorem.** For any given polynomial function of positive degree, the theorem states that zeros exist, but it provides no help for finding any particular zero. For linear and quadratic functions we can find the zeros exactly; for higher-degree polynomial functions, the situation becomes more difficult.

For most of the problems that arise in applications we can only approximate zeros. Virtually all numerical techniques to do this are rooted in the locator theorem, our most fundamental tool. Mathematicians have found a number of theorems, however, that can help in the search for certain kinds of zeros.

Gauss' proof of the fundamental theorem applies to polynomial functions with both imaginary and real coefficients. We emphasize, however, that in this chapter we discuss only polynomial functions with real number coefficients.

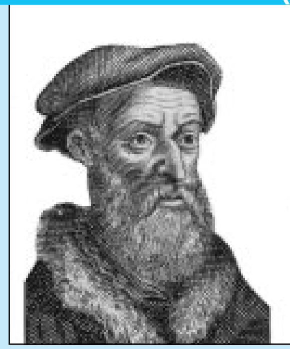
Nature of Zeros of Polynomial Functions

The next theorem generalizes what we already know about quadratic polynomials. In certain situations zeros of quadratic functions come in pairs. For example,

if $f(x) = x^2 - 4x + 1$, then the zeros of f are $2 + \sqrt{3}$ and $2 - \sqrt{3}$;
if $g(x) = x^2 - 2x + 2$, then the zeros of g are $1 + i$ and $1 - i$.

The numbers $a + bi$ and $a - bi$ are called **complex conjugates.** Certain kinds of zeros must occur in pairs in higher-degree polynomials, as well.

HISTORICAL NOTE CARL FRIEDRICH GAUSS (1777–1855)



Mathematician, astronomer, and physicist Carl Friedrich Gauss.

Called the "prince of mathematicians," Gauss is clearly among the greatest mathematicians of all time. He contributed to all areas of mathematics, as well as to astronomy and physics, and we are still building directly on foundations he laid.

Most of us could construct an equilateral triangle or a square with a compass and ruler. The Greeks also constructed a pentagon. Angle bisection allows us to double the number of sides, so it is theoretically possible to construct regular polygons of n sides if n is a power of 2 or $n = 2^k \cdot 3$ or $n = 2^k \cdot 5$, where k is any nonnegative integer. Gauss made the first significant progress in 2000 years when he discovered how to construct a regular polygon of 17 sides when he was almost 17 himself. His construction was published before he turned 19.

Gauss was the first to use i for $\sqrt{-1}$ and thoroughly understood the importance of complex numbers in the solution of equations. Although he left college before receiving his doctorate, he submitted his dissertation and attained the degree by the age of 21. His thesis established the fundamental theorem of algebra. This theorem so fascinated him that he gave three different proofs during his lifetime.

Returning to constructions, he proved in 1826 that an n-gon is constructible for an odd prime n only when n has the form $2^{2^k} + 1$ (for instance, 3, 5, and 17).

Conjugate zeros theorem

Let p be any polynomial function with *real number coefficients.*

If the nonreal complex number $a + bi$ is a zero of p, then $a - bi$ is also a zero. Here b is not zero.

If, in addition, p has *integer coefficients* and $a + \sqrt{b}$ is a zero of p, then $a - \sqrt{b}$ is also a zero. Here b is not a perfect square.

Strategy: Since the coefficients of p are real numbers, the conjugate zeros theorem applies; if i is a zero, then $-i$ must also be a zero.

The conjugate zeros theorem is illustrated in the next example.

►EXAMPLE 3 Conjugate zeros Given that $p(i) = 0$, find all zeros of the polynomial $p(x) = 2x^4 - 2x^3 + x^2 - 2x - 1$.

Solution

As the strategy suggests, both i and $-i$ are zeros, so both $x - i$ and $x + i$ are factors of $p(x)$. Thus $p(x)$ has $x^2 + 1$ as a factor since

$$(x - i)(x + i) = x^2 + 1.$$

Dividing $p(x)$ by $x^2 + 1$, we find that the other factor is $2x^2 - 2x - 1$ and so

$$p(x) = (x^2 + 1)(2x^2 - 2x - 1).$$

The remaining zeros of p are the roots of $2x^2 - 2x - 1 = 0$, which the quadratic formula gives as $\frac{1 \pm \sqrt{3}}{2}$. Therefore, p has two pairs of conjugate zeros: i and $-i$, and $\frac{1}{2} + \frac{1}{2}\sqrt{3}, \frac{1}{2} - \frac{1}{2}\sqrt{3}$. The graph in Figure 18 shows the two real zeros and suggests, as we now know, that there are no others. ◄

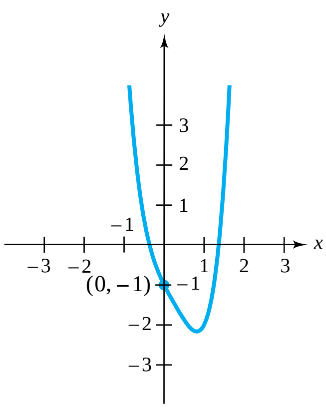


FIGURE 18
$p(x) = 2x^4 - 2x^3 + x^2 - 2x - 1$

Approximating Real Zeros: Newton's Method

Without some initial information it may be effectively impossible to find zeros of a polynomial function. In Example 3, we are given the fact that i is a zero, which, with the conjugate zeros theorem, is enough for us to find all zeros in exact form. We know how to zoom in on zeros from a graph, but the process is slow, and it is difficult to get much precision.

Many graphing calculators will immediately give approximations for all zeros of a polynomial function, to full calculator accuracy, at the press of a single key. How do they do it? A comprehensive answer is beyond the scope of this course, but most SOLVE routines essentially build on an iterative technique for approximation called Newton's (or the Newton-Raphson) method. The idea behind the process has a simple graphical interpretation that we can describe with the aid of a few pictures. For the pictures we need to speak of the "slope" or "direction" of a curve at a given point, another idea that is made precise in calculus is the *derivative* of a function. For our purposes, we will explain how to work with derivatives of polynomials of degrees 3 and 4, but we could just as well use the *numerical derivative* in the form already built-in in several graphing calculators.

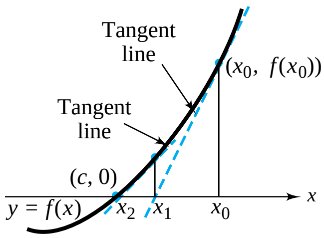


FIGURE 19

First we look at a picture of a typical function near a zero we want to approximate. See Figure 19. This could be almost any function, under any degree of magnification. If we know that the number x_0 is near the desired zero (which we have indicated as c in Figure 19), then we want a mechanical procedure for getting a better approximation, a new number nearer to c than x_0. The idea is that if we go to the point on the curve $(x_0, f(x_0))$ and essentially "take aim" along the curve in the direction of what is called the tangent line to the curve, we should hit the x-axis at a point x_1 nearer c than x_0. Repeating the process from x_1 should take us nearer still, after which we could move to a point x_2 still nearer, and so forth.

Fortunately, the general process (derived in calculus) is given in a simple formula that can be implemented easily on our calculators. We take the following information from calculus. Associated with every polynomial function f is a related function called the *derivative of* f, denoted by f'. We give formulas for the derivatives of cubic and quartic functions, leaving explanations for later courses. (It may also be possible to use a built-in program of your calculator; see the Technology Tip following Example 4.)

Function	*Derivative*
$f(x) = ax^3 + bx^2 + cx + d$	$f'(x) = 3ax^2 + 2bx + c$
Example: $f(x) = x^3 - 4x^2 + 2x - 1$	$f'(x) = 3x^2 - 8x + 2$
$f(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$	$f'(x) = 4Ax^3 + 3Bx^2 + 2Cx + D$
Example: $f(x) = 2x^4 + 5x^3 - x + 17$	$f'(x) = 8x^3 + 15x^2 - 1$

Having a way to write the derivative, we can describe the approximation process of Newton's method. If x_0 is an approximation to the zero c, then the new approximation is given by

$$x_1 = x_0 - f(x_0)/f'(x_0),$$

where $f'(x_0)$ is the *slope of the tangent line,* the number we get when the derivative function (or the numerical derivative) of f is evaluated at x_0.

The process can be programmed very simply with almost any calculator, but we outline the steps as an algorithm that can be performed on your home screen. Then we illustrate with an example. Follow Example 4 with your calculator.

Algorithm for Newton's method

1. Make an initial guess, a reasonable approximation to the desired zero. Note: the better the first guess, the more efficient the method.
2. Store your guess in memory register x. (For example, if your initial guess is 1, 1 → x, ENTER.)
3. Evaluate the next approximation, using $x_1 = x_0 - f(x_0)/f'(x_0)$, store it in memory register x and ENTER. This displays the new number and stores it for use in the next step.
4. Now press ENTER again and again, getting a new approximation with each repetition, until the display doesn't change, indicating that we have the best approximation the calculator can deliver.

►EXAMPLE 4 ***Approximating zeros*** The polynomial $p(x)=x^3 - 5x + 2$ has a real zero between 0 and 1. Use Newton's method, beginning with an initial guess of 0.5, to approximate the zero to ten decimal-place accuracy.

Solution

We would normally get our initial guess from a graph, but following directions, we will let $x_0 = 0.5$, and store: .5 → X ENT. For $p(x) = x^3 - 5x + 2$, we have $p' = 3x^2 - 5$, so we enter

X − (X³ − 5X + 2)/(3X² − 5) → X ENTER.

The display (ten decimals) reads .4117647059, and as we repeatedly ENTER, the sequence of displayed numbers is as follows.

.4142119097

.4142135624

.4142135624

This last number does not change when we continue to press ENTER. Thus, to ten decimal-place accuracy, the desired zero is .4142135624. This is the same polynomial function we considered in Example 2a, in which we found an exact form for the same zero, $-1 + \sqrt{2}$. ◄

TECHNOLOGY TIP ◆ ***Numerical derivatives***

If your calculator has a built-in function for calculating a numerical derivative, as the TI-82 (MATH 8), TI-85 (2nd CALC F2), HP–38 (▪ CHARS, Choose ∂, OK; you want ∂X(F1(X)), or Casio *fx*9700 (SHIFT d/dx), you need not write out the derivative. Enter the function in your Y1 register (or on your SHIFT MEM list as f1). Then follow the first two steps of the algorithm. In place of having to write out both the function and its derivative in Step 3, simply enter X − Y1/nDER(Y1) → X and iterate.

While this may not seem a significant advantage for polynomials, using X − Y1/nDER(Y1) → X will work with an appropriate initial guess for *any function* that has a reasonably smooth graph.

Polynomials in Application

In many applied problems, we are not interested in finding polynomial zeros in exact form; we may not care about the number of zeros. The nature of the problem may dictate that only certain zeros have physical meaning, as the next example illustrates.

►EXAMPLE 5 ***Application leading to a polynomial*** Twin radio towers are to be erected on opposite sides of a 10-foot roadway, where 50-foot guy wires reach from the top of each tower to the base of the other. The wires must cross high enough to leave a 12-foot clearance above the road. How tall can the towers be, and how close to the side of the roadway can they be built?

Solution

We sketch a diagram to help us visualize the situation. See Figure 20. Because of the symmetry, all of the pertinent information is contained in the second part of the diagram, where x is the distance from the edge of the road to the tower of height h.

From similar right triangles, $\triangle ACB$ and $\triangle AED$, we have

$$\frac{h}{2x + 10} = \frac{12}{x}, \text{ or } h = \frac{24(x + 5)}{x}.$$

Using the Pythagorean theorem with $\triangle ACB$,

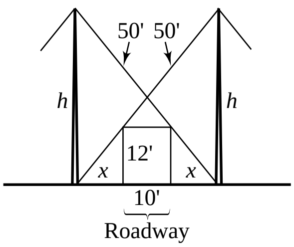
$$h^2 + (2x + 10)^2 = 50^2.$$

We substitute $24(x + 5)/x$ for h, $\dfrac{24^2(x + 5)^2}{x^2} + 4(x + 5)^2 = 2500$, then expand and simplify to get a polynomial equation,

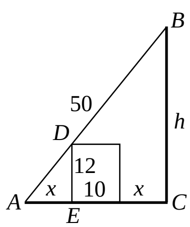
$$x^4 + 10x^3 - 456x^2 + 1440x + 3600 = 0.$$

Clearly, only positive values for x have any meaning, and from the picture, we estimate that x is probably between 5 and 15, so we look for zeros in that range. In a $[5, 15] \times [-5, 50]$ window, we see just two vertical lines, but there appear to be *two* zeros. To get a picture that looks a little more like the polynomials we are familiar with, we need a much larger y-range. With a y-range of $[-8000, 7000]$, we see the curve in Figure 21. We still see two zeros, one near 14 and one near 5.7. The corresponding heights are given by $h \approx 32.6$ and $h \approx 45.2$.

How do we interpret two different answers? When we look more closely, it turns out that we can get at least a 12-foot clearance over the road if the distance

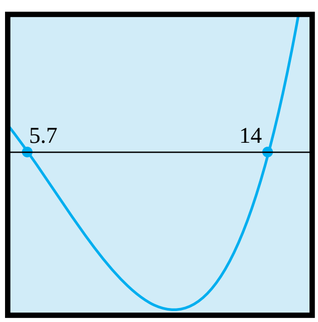


(a)



(b)

FIGURE 20



[5, 15] by [–8000, 7000]

$y = x^4 + 10x^3 - 456x^2 + 1440x + 3600$

FIGURE 21

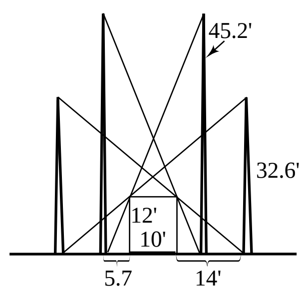


FIGURE 22

from the roadway is anything between 5.7 feet and 14 feet. The maximum tower height decreases as we move away from the road. See Figure 22. For maximum height, we can erect 45-foot towers 5.7 feet on each side of the roadway. ◀

EXERCISES 3.3

Check Your Understanding

Exercises 1–7 True or False. Draw graphs when helpful.

1. The function $f(x) = x^3 - 8x^2 + 5x + 4$ has three real zeros.

2. The graph of $f(x) = x^4 + 2x^2 + 1$ crosses the x-axis.

3. The equation $x^3 - 7x + 5 = 0$ has two negative and one positive root.

4. Since $\sqrt{3}$ is a zero of $f(x) = x^3 - 2x - \sqrt{3}$, then $-\sqrt{3}$ must also be a zero.

5. All zeros of $f(x) = x^3 - 8x^2 + 5x + 8$ lie between -1 and 8.

6. If $f(x) = -x^4 + x^3 + 7x^2 - x - 6$, then $f(x) < 15$ for every x.

7. Based on what can be seen from the graph of $f(x) = x^3 - 40x^2 - 400x + 1600$ using $[-24, 50] \times [-5100, 17{,}000]$, we can conclude that f has one negative and two positive zeros.

Exercises 8–10 Fill in the blank so that the resulting statement is true.

8. For the family of functions $f(x) = x^3 + cx - 5$, the value of c for which -1 is a zero is ______.

9. The number of real zeros of $f(x) = x^4 + 2x^3 - 2x^2 - 4x$ is ______.

10. In applying Newton's method, if $f(x) = x^3 - 2x + 3$, then $f'(x) =$ ______.

Develop Mastery

Exercises 1–4 ***Zeros from Graph*** *(**a**) Graph $y = p(x)$ and locate each zero between two consecutive integers. (**b**) Find an approximation (one decimal place) for the largest zero.*

1. $p(x) = x^3 - 3x^2 - x + 2$

2. $p(x) = x^3 - 3x^2 - 3$

3. $p(x) = x^3 - 2x^2 - x + 3$

4. $p(x) = x^3 + 2x^2 - 3x - 2$

Exercises 5–8 ***Locate Intersection*** *Graph the two equations on the same screen and find the coordinates of the point of intersection (one decimal place).*

5. $y = x^3 - 3x$; $y = -3$

6. $y = x - x^3$; $y = 1$

7. $y = x^3$; $y = x + 1$

8. $y = 1 - x^3$; $y = x^2 - 2$

Exercises 9–12 ***Zeros to Function*** *Find a polynomial function of lowest degree with integer coefficients, a leading coefficient of 1, and the given numbers as zeros. Give the result in standard (expanded) form. Use the conjugate zeros theorem, if needed.*

9. $1 + i, 1 - \sqrt{2}$

10. $1, \sqrt{2}$

11. $2, -1 + \sqrt{3}$

12. $\sqrt{2}, 1 - i$

Exercises 13–16 ***Conjugate Zeros Theorem*** *The given number is a zero of f. Find the remaining zeros. (Hint: Use the conjugate zeros theorem and long division.)*

13. $f(x) = x^3 - 4x^2 + 3x + 2$; $1 - \sqrt{2}$

14. $f(x) = x^3 - 2x^2 - 9x - 2$; $2 + \sqrt{5}$

15. $f(x) = 2x^3 - 9x^2 + 2x + 1$; $2 - \sqrt{5}$

16. $f(x) = 2x^3 - 3x^2 - 4x - 1$; $1 + \sqrt{2}$

Exercises 17–24 ***Find All Zeros*** *Assume that the domain of the variable is the set of complex numbers. Find all zeros in exact form.*

17. $f(x) = x^3 - 4x^2 + 2x - 8$

18. $f(x) = 4x^3 - 4x^2 - 19x + 10$

19. $f(x) = x^3 - 2.5x^2 - 7x - 1.5$

20. $f(x) = 3x^3 - 1.5x^2 + x - 0.5$

21. $f(x) = 6x^4 - 13x^3 + 2x^2 - 4x + 15$

22. $f(x) = 2x^4 + 3x^3 + 2x^2 - 1$

23. $f(x) = 4x^4 - 4x^3 - 7x^2 + 4x + 3$

24. $f(x) = 4x^4 + 8x^3 + 9x^2 + 5x + 1$

Exercises 25–32 ***Exact Form Roots*** *Find all roots in exact form.*

25. $6x^3 - 2x^2 - 9x + 3 = 0$

26. $6x^3 - x^2 - 13x + 8 = 0$

27. $x^4 - x^3 - 3x^2 + x + 2 = 0$

28. $x^4 - x^3 - 8x + 8 = 0$

29. $x^3 - 3x + 2 = 0$

30. $18x^3 + 27x^2 + 13x + 2 = 0$

31. $x^3 + 2 = -\dfrac{(14x^2 + 17x)}{3}$

32. $x^4 + 4x^3 - 5x^2 = 36x + 36$

Exercises 33–34 ***Solution Set, Exact Form*** *Find the solution set for* ***(a)*** $f(x) \geq 0$ ***(b)*** $f(x - 1) \geq 0$. *Give answers in exact form.*

33. $f(x) = x^3 + x^2 - 11x - 15$

34. $f(x) = x^3 - 6x^2 + 4x + 16$

Exercises 35–38 ***Evaluating Inverse*** ***(a)*** *Graph the function to support the claim that f is either an increasing or decreasing function (tell which), so that f has an inverse,* f^{-1}. ***(b)*** *Find* $f^{-1}(3)$ *(1 decimal place). (Hint: In* $x = f(y)$ *replace x by 3 and solve for y.)*

35. $f(x) = 2x^3 + 3x - 4$

36. $f(x) = x^3 - 3x^2 + 4x + 5$

37. $f(x) = 2 - x + x^2 - 2x^3$

38. $f(x) = 4 - 3x - 2x^3$

39. For $f(x) = x^3 - 3x^2 - x + 3$ and $g(x) = |x|$, how many zeros does the function $f \circ g$ have? Draw a graph. Does your answer contradict the corollaries to the fundamental theorem of algebra? Explain.

40. Repeat Exercise 39 for $f(x) = x^3 - 3x^2 - 4x - 4$.

41. ***Explore*** For $f(x) = x^3 + cx + 4$, choose several integer values for c (positive and negative) and draw a graph of the corresponding function. What values of c give graphs with **(a)** turning points, **(b)** more than one x-intercept point?

42. Repeat Exercise 41 for $f(x) = x^3 + cx - 4$.

Exercises 43–46 ***Explore*** *The equation defines a family of polynomial functions of degree 3. Experiment with several integer values of c and for each draw a graph. Describe the role that c appears to play. Include information about local extrema, number of real zeros, and any other graphical features you observe. Use complete sentences.*

43. $f(x) = x^3 + cx^2 - 4x + 3$

44. $f(x) = x^3 + 2x^2 + cx + 3$

45. $f(x) = x^3 + 2x^2 - 4x + c$

46. $f(x) = cx^3 + 2x^2 - 4x + 3$

Exercises 47–50 ***Explore*** *For each real number k, f is a polynomial function of degree 3. Experiment with several integer values of k and determine the values of k for which f will have* ***(a)*** *three real zeros,* ***(b)*** *exactly one real zero,* ***(c)*** *one negative and no positive zeros.*

47. $f(x) = x^3 + kx - 2$

48. $f(x) = x^3 + kx + 3$

49. $f(x) = x^3 + kx + 8$

50. $f(x) = x^3 + kx - 5$

Exercises 51–53 ***Explore Composition with Absolute Value*** *(See Section 2.6)*

51. Try several polynomial functions f of degree 3 and in each case draw a graph of f and then a graph of $g(x) = f(|x|)$. For what functions f will the composition function g have
(a) no zeros **(b)** two zeros **(c)** four zeros
(d) six zeros

52. Draw a graph of $f(x) = |x^3 + x^2 - 2x|$. Explain why the graph of f cannot be the graph of a polynomial function. (*Hint:* Read "Smoothness and End Behavior" in Section 3.1.)

53. **(a)** Take any polynomial function f of degree 3 and draw the graph of $g(x) = |f(x)|$. Use the graph to explain why g cannot be a polynomial function.
(b) Is there a polynomial function f of degree 4 for which $g(x) = |f(x)|$ is also a polynomial function? Explain.

54. For each real number c, the graph of $f(x) = 0.1(x + c)^3 + 0.3(x + c)^2 - (x + c) - 8$ is a horizontal translation of the graph of $g(x) = 0.1x^3 + 0.3x^2 - x - 8$. For what *integer* values of c will f have a negative zero?

55. ***Explore*** The graph of $f(x) = x^3 - 6x^2 + 9x - 6$ has one real zero and local extrema at $(1, -2)$ and $(3, -6)$. Draw a graph in a shifted decimal window to support this claim. Consider the family of functions $g(x) = f(x) + c$.
(a) For what integer values of c will g have three distinct real zeros?
(b) For what integer values of c will g have a repeated zero? Justify algebraically.
(c) For what integer values of c will g have a negative zero?

56. ***Explore*** Consider the family of functions $f(x) = 0.3x^3 - 4x + c$. Let P be the local maximum point and Q be the local minimum point. For what integer values of c will **(a)** P and Q be below the x-axis? **(b)** P be above and Q be below the x-axis? **(c)** P and Q be above the x-axis?

Exercises 57–60 ***Newton's method*** *Use Newton's method to find the largest zero of f (nine decimal places). See Example 4.*

57. $f(x) = x^3 - 4x + 2$

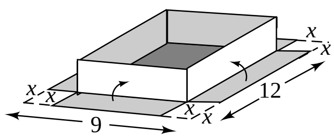
58. $f(x) = x^3 + 4x^2 - 8$

59. $f(x) = x^4 - 5x^2 + 3x - 1$

60. $f(x) = x^4 - 9x^2 + 8x$

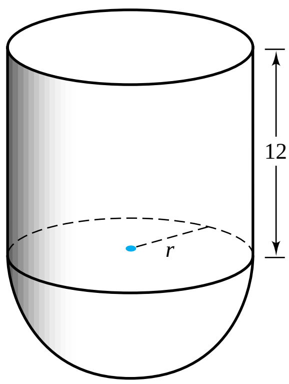
61. A box with an open top is constructed from a rectangular piece of tin that measures 9 inches by 12 inches by cutting out from each corner a square of side x, and then folding up the sides as shown in the diagram.

(a) If V denotes the volume of the box, find a formula for V as a function of x. What is the domain of the function?

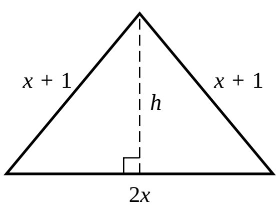


(b) What size corners should be cut out so that the box will have a capacity of 81 cubic inches? (*Hint:* There are two answers, both between 1 and 2.)

62. A storage tank consists of a right circular cylinder mounted on top of a hemisphere, as shown in the diagram. If the height of the cylindrical portion is 12 feet and the tank is to have a capacity of 1250π cubic feet, find the radius r of the cylinder to 3 significant digits.



63. An isosceles triangle has the dimensions shown in the diagram. If the area is equal to 10 square units, find the length of the altitude h (to the nearest tenth).



64. What are the dimensions of a rectangle of area 7 square inches that has a diagonal 1 inch longer than the length of one of its sides? Give the result rounded off to 3 significant digits.

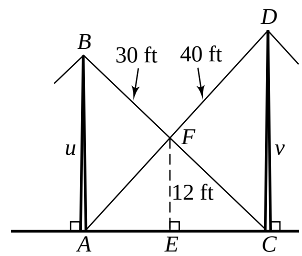
65. A storage tank has the shape of a cube. If one of the dimensions is increased by 2 and another by 3, while the third is decreased by 4, then the resulting rectangular tank will have a volume of 600 cubic units. What is the length of an edge of the original cube to 3 significant digits?

66. A rectangular storage container is 3 by 4 by 5 feet.

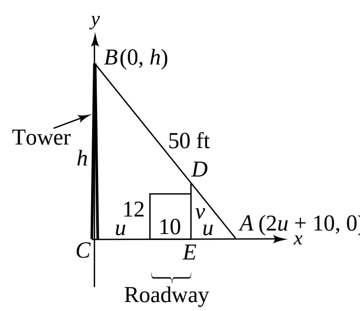
(a) What is its capacity (volume)?

(b) If we increase each of the dimensions by x feet in order to get a container with a capacity five times as large as the original, how large must x be, rounded off to 3 significant digits?

67. Two vertical poles, AB and CD, are connected by guy wires of lengths 30 ft and 40 ft, intersecting at point F, 12 feet from the ground. See the diagram. Find the heights u and v, of the poles and the distance d between the poles.



68. In Example 5 we found that the two towers can be placed anywhere from 5.7 ft to 14 ft from the road. How far from the road should they be located if we want maximum clearance above the road? (*Hint:* Let u be the desired distance and v be the distance, $|\overline{DE}|$, from the right edge of the road to the guy wire AB.) Write v as a function of u. See diagram.



69. Solve the problem in Example 5 if the guy wires are 48 feet long.

70. In Exercise 69, how far from the road should the two towers be located so that there is maximum clearance above the road? See Exercise 68.

3.4 RATIONAL FUNCTIONS

> What is mathematics about? I think it's really summed up in what I frequently tell my classes. That is that proofs really aren't there to convince you that something is true—they're there to show you why it is true. That's what it's all about—it's to try to figure out how it's all tied together.
>
> *Freeman Dyson*

In this section we consider **rational functions,** which are defined as **quotients of polynomial functions.**

Definition: rational function

Suppose p and q are polynomial functions, where q is not the zero function. Then the function f given by

$$f(x) = \frac{p(x)}{q(x)}$$

is called a **rational function,** sometimes written $f = \frac{p}{q}$. The domain of f consists of all real numbers for which $q(x) \neq 0$.

If $p(x)$ and $q(x)$ have no common factors, then we say that $\frac{p}{q}$ is in **reduced form.**

> *The only idea of real mathematics that I had came from* Men of Mathematics. *In it, I got my first glimpse of a mathematician per se. I cannot overemphasize the importance of such books about mathematics in the intellectual life of a student like myself completely out of contact with research mathematicians.*
>
> Julia Robinson

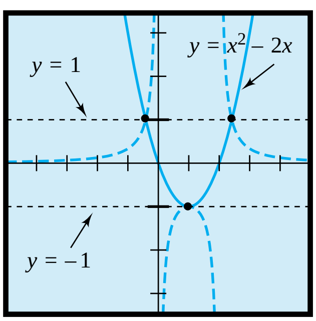
For our discussion in this section, we assume that all rational functions are reduced unless we specify to the contrary. The most significant information for analyzing the behavior of a rational function is the set of zeros of the polynomial functions in the numerator and the denominator.

To get a feeling for the meaning of zeros in the denominator, we look at a variety of graphs. In Example 1 we consider pairs of graphs, each consisting of a polynomial and its reciprocal.

►EXAMPLE 1 ***Functions and reciprocals*** Sketch the graphs of both functions, together with the horizontal lines $y = 1$ and $y = -1$, in the specified window. Then describe where the graphs of f and g meet, and where the graph of g goes off-scale (out of sight in the window).

(a) $f(x) = x^2 - 2x, \quad g(x) = \dfrac{1}{x^2 - 2x}$. Decimal window

(b) $f(x) = x^3 - 3x^2, \; g(x) = \dfrac{1}{x^3 - 3x^2}$. Change y-range to $[-5, 5]$



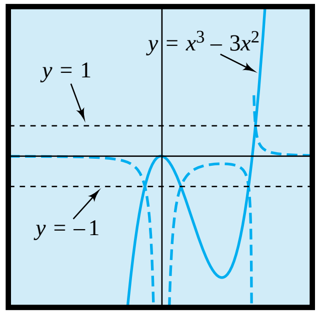
[–5, 5] by [–3.5, 3.5]

FIGURE 23

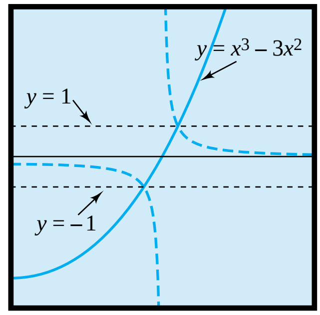
Solution

(a) The graph of f is the solid parabola in Figure 23, and $y = g(x)$ is the dotted curve. It appears that the graphs of f and g meet where $f(x) = \pm 1$. The graph of g appears to go off-scale at $x = 0$ and $x = 2$; that is, where $f(x) = 0$.

(b) In Figure 24a, again the graphs of f and g meet where $f(x) = \pm 1$. The graph of g appears to go off-scale at $x = 0$. It is less clear what happens to g near $x = 3$ (the other point where $f(x) = 0$), so we look closer by reducing the x-range to $[2, 4]$ (Figure 24b). Now we still see the intersections where



[–5, 5] by [–5, 5]
(a)



[2, 4] by [–5, 5]
(b)

FIGURE 24

$f(x) = \pm 1$, but the graph of g clearly goes off-scale as x nears 3. Your calculator may show a nearly vertical column of pixels near 3, this is a "*false (calculator) asymptote.*" It occurs because the calculator connects widely separated y-values in adjacent columns, but it is *not* part of the graph of g. ◀

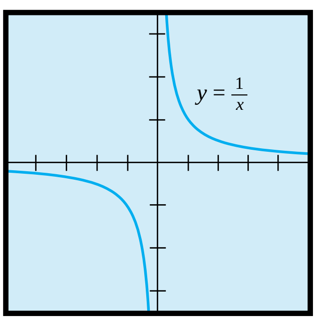
TECHNOLOGY TIP ◆ ***Rational functions and parentheses***

In graphing rational functions on a calculator, one of the most common errors involves the use of parentheses. We are so accustomed to using the fraction bar as a separator that we can get careless with our calculator. In Example 1(a), $g(x) = \dfrac{1}{x^2 - 2x} = \dfrac{1}{x(x+2)}$. On the calculator we could enter

Y = 1/(X² − 2X) or Y = 1/(X(X − 2)) or Y = 1/(X*(X − 2))
but NOT Y = 1/X² − 2X or Y = 1/X*(X − 2).

Every calculator is programmed to handle operations differently. Try each of the above on your calculator and make sure you know how to get a graph that looks like Figure 23.

Vertical Asymptotes



[–5, 5] by [–3.5, 3.5]

FIGURE 25

The kind of "off-scale" behavior we observed in the calculator graphs in Example 1 deserves closer examination. When we plot $y = 1/x$ in a decimal window (Figure 25), the graph goes off-scale in both directions, with the graph seeming to get closer and closer to the y-axis. If, however, we increase the y-range to $[-15, 15]$, the calculator graph climbs up the y-axis and then *stops*; the highest point is $(\frac{1}{10}, 10)$.

Does the graph really stop at $(\frac{1}{10}, 10)$, or does it keep climbing? We could see more by zooming in near (0, 10), but the calculator is not the best tool for answering the question. Looking at the equation $y = 1/x$, it is easy to see that we can get a y-value of 100 (at $x = 1/100$), or 1000, or ten million, by taking x-values small enough. *There is no highest point*. Taking positive x-values closer and closer to 0, the y-values keep growing without bound.

We use arrows to describe such behavior and write: as $x \to 0^+$, $y \to \infty$ (or $1/x \to \infty$), or more compactly, $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

Arrow notation

$x \to a^+$ means that x approaches a from above; that is, x takes on values near a, but greater than a (such as $a + 0.01, a + 0.001, \ldots$).
$x \to a^-$ means that x approaches a from below; that is, x takes on values near a, but less than a (such as $a - 0.01, a - 0.001, \ldots$).

Similarly, $x \to \infty$ or $x \to -\infty$ means that x assumes larger and larger positive or negative values, respectively. The same notation is used to indicate functional behavior.

In calculus the concept of limit has a very important, precise meaning. Here we use the notation only for the intuitive notion embodied in our arrows. Looking back at Figure 23, we see the following:

$$\lim_{x\to 0^-} g(x) = \infty, \qquad \lim_{x\to 0^+} g(x) = -\infty,$$
$$\lim_{x\to 2^-} g(x) = -\infty, \qquad \lim_{x\to 2^+} g(x) = \infty.$$

The vertical lines $x = 0$ (the y-axis) and $x = 2$ are called **vertical asymptotes** for the curve $y = g(x)$. Without attempting a more precise definition, we say that a line is an **asymptote** for a curve if *the distance between the curve and the line goes to zero as we move out along the line.*

From the graphs in Example 1, it is clear that each reciprocal function $1/f(x)$ has a vertical asymptote at each zero of $f(x)$, and furthermore, that the x-axis is a **horizontal asymptote** for each reciprocal function since $g(x) \to 0$ as $x \to \infty$ and $g(x) \to 0$ as $x \to -\infty$.

Asymptotes for rational functions can be vertical, horizontal, or oblique lines, as illustrated in Figure 26.

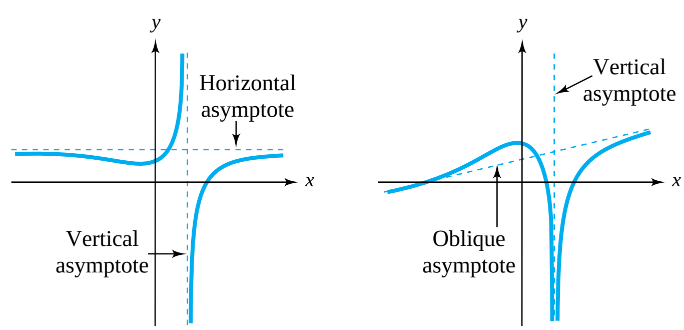


FIGURE 26

Shifts and reflections are also useful in graphing rational functions.

►EXAMPLE 2 ***Shifts and reflections*** Use the graph of $f(x) = \frac{1}{x}$ to graph

(a) $y = \dfrac{1}{x - 1}$ **(b)** $y = \dfrac{2}{2 - x}$ **(c)** $y = \dfrac{1 + x}{x}$

Strategy: Try to relate each function to $f(x) = \frac{1}{x}$. In **(a)** $f(x - 1)$ gives a horizontal translation. In **(b)** factor out -2 to get $-2 \cdot f(x - 2)$. In **(c)** $y = \frac{1}{x} + 1$, for a vertical translation.

Solution

(a) Since $f(x - 1) = \frac{1}{x-1}$, graph $y = \frac{1}{x-1}$ by translating the graph of f one unit to the right. As a useful check, observe that $y = \frac{1}{x-1}$ has a vertical asymptote where the denominator is 0, at $x = 1$. The result of the translation is shown in Figure 27a.

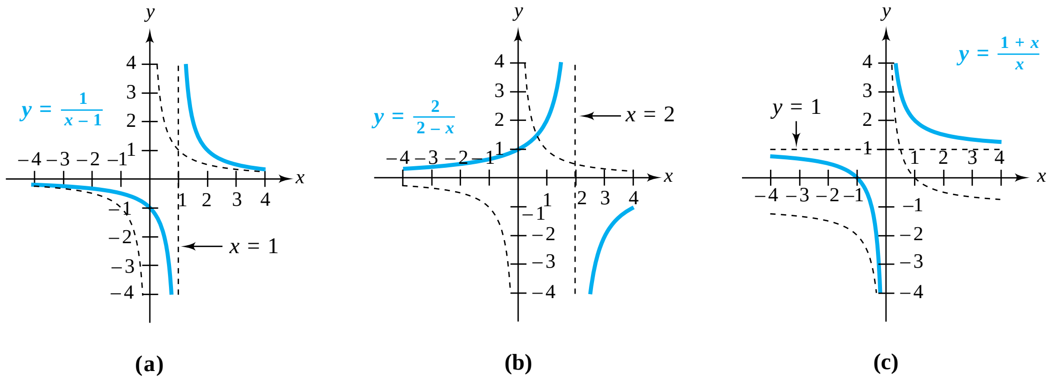


FIGURE 27

Shifts and reflections of $y = \dfrac{1}{x}$.

(b) If we factor out -1 from the denominator, then $\frac{2}{2-x} = \frac{-2}{x-2}$, so $\frac{2}{2-x} = -2f(x-2)$. Translate the graph of f two units to the right, reflect it through the x-axis and stretch it vertically by a factor of 2. Plotting a few points gives the graph shown in Figure 27b.

(c) To relate $\dfrac{1+x}{x}$ to $f(x)$, rewrite $\dfrac{1+x}{x}$ as

$$\frac{1+x}{x} = \frac{1}{x} + \frac{x}{x} = \frac{1}{x} + 1 = f(x) + 1.$$

Translate the graph of f one unit up, as shown in Figure 27c. ◀

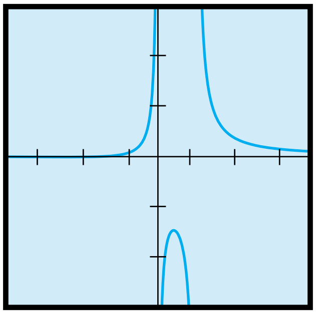
Graphing Other Rational Functions

All of the rational functions we have graphed thus far are either reciprocals or shifts of reciprocals of polynomial functions, where we have observed that *there is a vertical asymptote at every zero of the denominator*. What is the significance of the zeros of the numerator? Suppose $f(x) = \dfrac{p(x)}{q(x)}$ and f is in reduced form (so that p and q have no common zeros). Then if c is a zero of the numerator, we have $f(c) = \dfrac{p(c)}{q(c)} = \dfrac{0}{q(c)} = 0$. That is, *the zeros of a rational function are the zeros of the numerator*. Collectively, we call the zeros of the numerator and the zeros of the denominator the set of **cut points** for a rational function. As in Chapter 1, cut points identify the location of possible sign changes. In addition, for rational functions, cut points in the numerator identify x-intercept points, cut points in the denominator correspond to actual breaks in the graph, at each vertical asymptote. Summing up, we have the following.

Vertical asymptotes and intercepts

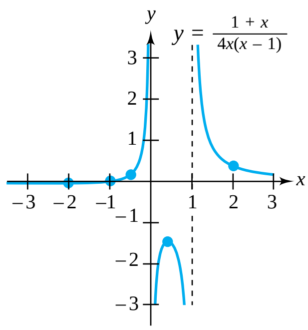
Suppose $f(x) = \dfrac{p(x)}{q(x)}$ and that p and q have no common zeros.

Then there is a **vertical asymptote** at every zero of the denominator, and there is an ***x*-intercept point** at every zero of the numerator.



[−4, 4] by [−3, 3]

(a)



(b)

FIGURE 28
Graph of $y = \dfrac{x + 1}{4x(x - 1)}$

►EXAMPLE 3 ***Graphing a rational function*** Find the intercepts, cut points and asymptotes and sketch the graph of the rational function

$$f(x) = \frac{x + 1}{4x(x - 1)}.$$

Solution

The zeros of the denominator are 0 and 1, so the vertical asymptotes are the lines $x = 0$ (the y-axis) and $x = 1$. The other cut point is the single zero of the numerator, -1, and so we have only one x-intercept point, $(-1, 0)$. Since the y-axis is a vertical asymptote, there is no y-intercept. A calculator graph in a decimal window (Figure 28a) makes it appear that the graph ends near $(-1, 0)$, but of course we know that the domain of f consists of all real numbers except 0 and 1. We also know that the numerator, and hence f, changes sign at $x = -1$. Tracing along the curve near $(-1, 0)$ or zooming in near the same point verifies that the graph crosses the x-axis there, and the x-axis is a horizontal asymptote, as in Figure 28b. ◄

Horizontal and Slant Asymptotes

In all of the examples we have graphed thus far, the x-axis has been a horizontal asymptote, with the exception of one translated graph. The general principle that we list below is illustrated in all of these examples. What happens as $x \to \infty$ or $x \to -\infty$ depends on the degrees of the numerator and denominator.

Horizontal and slant asymptotes

$$f(x) = \frac{ax^m + \cdots}{bx^n + \cdots}$$

If the degree of the denominator is larger $(m < n)$,

then the **x-axis is a horizontal asymptote.**

If the degrees are equal $(m = n)$,

then the line $\boldsymbol{y = \frac{a}{b}}$ **is a horizontal asymptote.**

Use long division to get additional information.

If the degrees differ by 1 $(m = n + 1)$,

then there is a **slant (oblique) asymptote.**

Use long division to find an equation for the asymptote.

►EXAMPLE 4 ***Equal degrees*** Graph the rational function

$$f(x) = \frac{x^2 - 4}{x^2 + 2}.$$

Solution

We note first that $f(-x) = f(x)$, so f is an even function; the graph is symmetric about the y-axis. Setting the numerator equal to 0, we find that its zeros are ± 2, so the x-intercept points are $(2, 0)$ and $(-2, 0)$. The denominator has no real zeros, so the graph has no vertical asymptotes. Since $f(0) = -2$, the y-intercept point is $(0, -2)$.

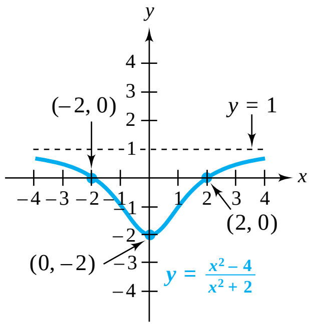
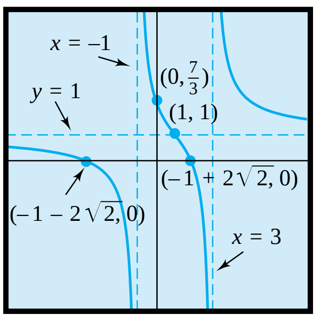


FIGURE 29



[−8, 8] by [−6, 6]

FIGURE 30
$g(x) = \dfrac{x^2 + 2x - 7}{x^2 - 2x - 3}$

The degree of both numerator and denominator is 2, so $m = n$. Thus the line $y = \frac{a}{b} = \frac{1}{1} = 1$ is a horizontal asymptote. The graph is shown in Figure 29. Note that without the boxed information above, it would be difficult to tell from a calculator graph that the graph really does flatten out along the line $y = 1$, although increasing the x-range makes that conclusion very plausible. ◀

▶EXAMPLE 5 ***Equal degrees*** Graph the function, identifying all asymptotes and intercept points in exact form.

$$g(x) = \frac{x^2 + 2x - 7}{x^2 - 2x - 3}.$$

Solution

The zeros of the denominator are easy to find because the denominator factors: $(x - 3)(x + 1)$. There are zeros, and hence vertical asymptotes at $x = -1$ and $x = 3$. The numerator has zeros at $x = -1 \pm 2\sqrt{2}$, so we have two x-intercept points. $g(0) = \frac{7}{3}$, so the y-intercept point is $(0, \frac{7}{3})$.

The degree of both numerator and denominator is 2, so $m = n$. Thus the line $y = \frac{a}{b} = \frac{1}{1} = 1$ is a horizontal asymptote. If we use long division, as suggested in the box above, we have

$$g(x) = 1 + \frac{4x - 4}{x^2 - 2x - 3} = 1 + \frac{4(x - 1)}{(x - 3)(x + 1)}.$$

In this form, it is easy to see that $g(x) = 1$ when $x = 1$, so the graph crosses the horizontal asymptote at the point $(1, 1)$. Graphing in a decimal window doesn't show much of the asymptotic behavior. To see a little more, we zoom out by a factor of 2 to get the graph shown in Figure 30. ◀

Intersections of Graphs and Asymptotes

Because graphs get so close to asymptotes it is sometimes difficult to look at a graph and tell whether or not it crosses an asymptote. Calculator graphs are particularly difficult to read in some areas where we most need detail. In Example 3 the graph crosses the horizontal asymptote at the x-intercept point $(-1, 0)$ and in Example 5 the graph crosses the horizontal asymptote as well. When numerator and denominator have equal degrees, long division gives us a form from which we can use algebraic techniques to find intersections. On the other hand, *no graph can cross a vertical asymptote.* If $x = c$ is a vertical asymptote for a rational function f, then there would have to be a point $(c, f(c))$ on the graph, but vertical asymptotes occur at zeros of the denominator, where by definition, f is undefined.

Slant Asymptotes

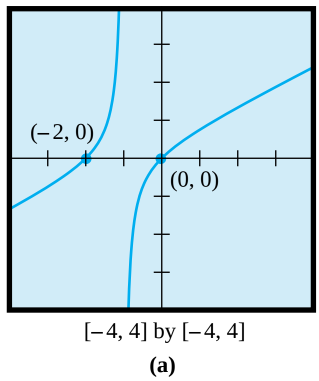
From the box above, slant (oblique) asymptotes occur when the degree of the numerator is 1 greater than the degree of the denominator, as in the next example.

▶EXAMPLE 6 ***Degrees differ by 1*** Graph the rational function

$$h(x) = \frac{x^2 + 2x}{2x + 2}.$$

Solution

Both numerator and denominator can be factored: $\dfrac{x(x+2)}{2(x+1)}$. Thus the cut points are 0, -2, and -1. The first two give us x-intercept points (0, 0) and $(-2, 0)$; the zero in the denominator indicates a vertical asymptote at $x = -1$. A calculator graph in $[-4, 4] \times [-4, 4]$ shows the graph near the vertical asymptote (Figure 31a), but it isn't clear what happens to the graph as we move further to the right or the left. If we zoom out, essentially looking at the graph from further away, say in the window $[-10, 10] \times [-6, 6]$, the graph looks very much like a line, except near the vertical asymptote. What line does it approach? The slant asymptote.



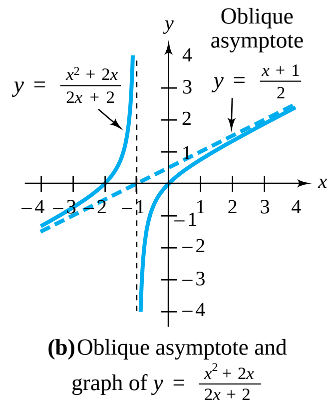


FIGURE 31

The numerator has degree 2 and denominator degree 1, so there is a slant asymptote. Dividing $x + 1$ into $x^2 + 2x$ using long division, we get

$$h(x) = \frac{1}{2}(x+1) - \frac{1}{2(x+1)}.$$

The slant asymptote is the line $y = (x+1)/2$. Adding that line to the graph in the $[-10, 10] \times [-6, 6]$ window shows that the two graphs are indistinguishable except in the region near the vertical asymptote.

As with horizontal asymptotes, we want to know whether the graph of $y = h(x)$ intersects the oblique asymptote. Such an intersection would come from a solution to the equation $h(x) = (x+1)/2$, which clearly has no solution. The graph is shown in Figure 31b. ◀

Rational Functions Not Reduced

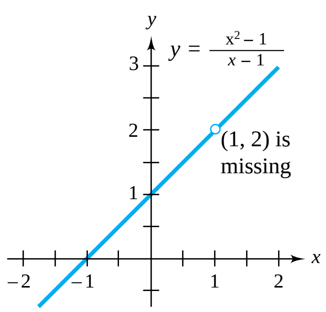


FIGURE 32

A rational function $f(x) = \dfrac{p(x)}{q(x)}$ is not in reduced form if there are common factors in the numerator and denominator. To handle common zeros, remember that if $p(x)$ and $q(x)$ have the same zero, say $x = c$, then since $\frac{0}{0}$ is not defined, c is not in the domain of the function. In such a case the graph has a single point removed. Consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

Factor the numerator.

$$f(x) = \frac{(x-1)(x+1)}{x-1}.$$

This is identical to the function $g(x) = x + 1$ *except* when x is 1. When x is 1, $g(1) = 2$, but $f(1)$ is not defined. Therefore, the graph of $y = f(x)$ is the same as the graph of $y = x + 1$ with the point $(1, 2)$ removed (see Figure 32).

TECHNOLOGY TIP ◆ ***Missing points***

In graphing nonreduced rational functions, we may or may not be able to see that a point is missing. Such a gap is visible on a calculator graph if and only if there is a pixel with x-coordinate corresponding to value where the function has the form $\frac{0}{0}$.

Thus for the example considered above, $f(x) = (x^2 - 1)/(x - 1)$, in any window where there is a pixel for $x = 1$, such as a decimal window, the graph makes the missing point quite apparent. If you then change the x-range by almost any small amount, you will no longer be able to see where the point is missing; the graph looks just like the line $y = x + 1$.

EXERCISES 3.4

Check Your Understanding

Use a graph whenever you think it will be helpful. True or False. Give reasons.

1. If the graph of $y = \frac{1}{x}$ is translated two units left, then the resulting graph will be that of $y = \frac{1}{x-2}$.

2. If the graph of $y = \frac{1}{x+2}$ is translated down one unit, then the resulting graph will be that of $y = \frac{-x-1}{x+2}$.

3. If the graph of $y = \frac{1}{x+2}$ is translated up one unit, then the resulting graph will be that of $y = \frac{x+1}{x+2}$.

4. The line $y = \frac{x}{2}$ is an asymptote to the graph of $y = \frac{x+1}{2x+1}$.

5. The horizontal line $y = -2$ is an asymptote to the graph of $y = \frac{1-2x^2}{5+2x+x^2}$.

6. The graph of $y = \frac{x-2}{x^2-x+2}$ has no vertical asymptotes.

Exercises 7–8 Suppose $f(x) = \frac{3x^2+1}{x^2+1}$.

7. There is no value of x for which $f(x) = 3$.

8. For every real number x, $f(x)$ is in the interval $[1, 3)$.

9. If $f(x) = \frac{x^2+100}{x}$, then the graph of f has a local minimum point in the third quadrant.

10. The graph of $f(x) = \frac{2x^3 - 3x^2 + 500}{x^4 + 8x + 50}$, has one zero and no vertical asymptotes.

Develop Mastery

1. The line $x = 1$ is a vertical asymptote for the function

$$f(x) = \frac{x^3}{x^2 - 2x + 1}.$$

(a) To see what happens to the graph of f as $x \to 1^-$, evaluate f at $x = 0.8, 0.9, 0.99$, etc. For what values of x is $f(x)$ greater than 100? Greater than 1000?

(b) Repeat part **(a)** as $x \to 1^+$.

2. The line $y = 1$ is a horizontal asymptote for the function

$$f(x) = \frac{x^2 + 2x}{x^2 + 1}.$$

(a) Draw a graph of $y = f(x)$, Does the graph approach the line $y = 1$ from above or below as $x \to \infty$?
(b) Use TRACE and appropriate windows to find some values of x for which $f(x)$ is within 0.01, 0.001, of $y = 1$.
(c) Do the same for $x \to -\infty$.
(d) Does the graph intersect the line $y = 1$? If it does, find the coordinates of the point of intersection algebraically.

3. Repeat Exercise 2 for the function

$$f(x) = \frac{x^2 + 2x}{x^2 + 2}.$$

Exercises 4–9 Use translations, reflection, or stretching of the graph of $f(x) = \frac{1}{x}$ to sketch a graph. See Example 2.

4. $y = \dfrac{1}{x + 2}$ **5.** $y = \dfrac{1}{1 - x}$ **6.** $y = \dfrac{2}{x - 3}$

7. $y = \dfrac{2}{3 - x}$ **8.** $y = \dfrac{2x + 1}{x}$ **9.** $y = \dfrac{x - 1}{x}$

Exercises 10–13 ***Related Graphs*** *Graph $f(x) = \dfrac{2x^2}{x^2 + 1}$ and the given function g simultaneously.* ***(a)*** *Determine how the graphs are related and verify algebraically.* ***(b)*** *Find the coordinates of any intersection points (1 decimal place).*

10. $g(x) = \dfrac{4x^2 + 2}{x^2 + 1}$ **11.** $g(x) = \dfrac{-2}{x^2 + 1}$

12. $g(x) = \dfrac{2x^2 + 4x + 2}{x^2 + 2x + 2}$

13. $g(x) = \dfrac{2x^2 - 8x + 8}{x^2 - 4x + 5}$.

Exercises 14–21 ***Graph Rational Functions*** *Sketch a graph of f, identifying asymptotes and intercepts.*

14. $f(x) = \dfrac{1}{x - 1}$ **15.** $f(x) = \dfrac{2}{x + 2}$

16. $f(x) = \dfrac{x}{x - 1}$ **17.** $f(x) = \dfrac{2x - 3}{x + 2}$

18. $f(x) = \dfrac{2x^2}{(x - 1)^2}$ **19.** $f(x) = \dfrac{x^2}{x^2 - 4}$

20. $f(x) = \dfrac{x + 2}{x^2 - 3x - 4}$ **21.** $f(x) = \dfrac{x}{x^2 - 2x + 1}$

Exercises 22–28 ***Intercepts, Domain, Range*** *Draw a graph and use it to find* ***(a)*** *the x-intercept points,* ***(b)*** *the domain of f,* ***(c)*** *the range of f.*

22. $f(x) = \dfrac{2}{(x + 2)^2}$ **23.** $f(x) = \dfrac{2}{x^2 + 1}$

24. $f(x) = \dfrac{2x}{x^2 + 1}$ **25.** $f(x) = \dfrac{x^2 + 2}{x^2 + 1}$

26. $f(x) = \dfrac{4}{x^2 - 3x - 4}$

27. $f(x) = \dfrac{x - 2}{x^3 - 3x^2 - 4x}$

28. $f(x) = \dfrac{2x - 4}{x^3 - 3x^2 - 4x}$

Exercises 29–32 ***Increasing, Decreasing*** *Determine the intervals on which f is* ***(a)*** *increasing,* ***(b)*** *decreasing.*

29. $f(x) = \dfrac{x^2 + 4x + 3}{2x + 4}$ **30.** $f(x) = \dfrac{x^2 - 2x - 3}{x^2 - 2x + 3}$

31. $f(x) = \dfrac{x - 1}{x^2 - x - 2}$ **32.** $f(x) = \dfrac{4}{x^2 + 2x - 3}$

Exercises 33–36 ***Solution Set*** *Find the solution set algebraically.*

33. $\dfrac{2}{x} - \dfrac{4}{x - 2} - 6 = 0$ **34.** $\dfrac{x^2 - 6x + 5}{x^2 + 3} = 3$

35. $\dfrac{x^3 - 4x}{x^2 + 1} = \dfrac{3}{2}$

36. $\dfrac{2x}{x + 1} + \dfrac{4}{2x + 1} = \dfrac{14}{2x^2 + 3x + 1}$

Exercises 37–40 ***Solution Set*** ***(a)*** *Find the solution set algebraically.* ***(b)*** *Use a graph to support your answer.*

37. $\dfrac{1}{x + 1} > 2$ **38.** $\dfrac{x - 3}{x + 2} < 0$

39. $\dfrac{2x^2 + x - 3}{x^2 + 1} \le 0$ **40.** $\dfrac{x^2 - 3x - 4}{x^2 + 2x + 1} \ge 0$

Exercises 41–44 ***Not Reduced*** *Sketch a graph of f. (Hint: First express the function in reduced form; keep in mind the domain.)*

41. $f(x) = \dfrac{x}{x^2 + 2x}$ **42.** $f(x) = \dfrac{x^3}{x^2 - 2x}$

43. $f(x) = \dfrac{x^2 - 2x + 1}{x^2 + 2x - 3}$ **44.** $f(x) = \dfrac{x^2 + 2x - 3}{x^2 + x - 2}$

Exercises 45–48 Find an equation for the horizontal asymptote and find the coordinates of the points (if any) where the graph of $y = f(x)$ intersects the horizontal asymptote.

45. $f(x) = \dfrac{x^2 - 6}{x^2 - 2x}$ **46.** $f(x) = \dfrac{2x^2 - 2}{x^2 - 3x + 2}$

47. $f(x) = \dfrac{x^2 - x}{x^2 + x + 2}$ **48.** $f(x) = \dfrac{x^2 - x}{x^2 - x - 2}$

Exercises 49–52 ***Oblique Asymptotes*** *The graph of f has an oblique asymptote. Find an equation for the asymptote and find the coordinates of the points (if any) where the graph of* $y = f(x)$ *intersects the asymptote. Graph the function.*

49. $f(x) = \dfrac{x^2 - 3}{x - 1}$ **50.** $f(x) = \dfrac{x^2 + x - 2}{x + 1}$

51. $f(x) = \dfrac{2x^2 + 3x + 1}{x}$ **52.** $f(x) = \dfrac{x^2 + 3x - 2}{x + 1}$

Exercises 53–54 ***Minima*** *Find the minimum value of f algebraically and check graphically. What value of x gives the minimum value of f?*

53. $f(x) = \dfrac{x^4 + x^2 + 1}{x^2}$

Hint: $f(x) = x^2 + 1 + \dfrac{1}{x^2} = \left(x - \dfrac{1}{x}\right)^2 + 3$

54. $f(x) = \dfrac{x^4 + 2x^2 + 4}{x^2}$

Hint: See Exercise 53.

Exercises 55–56 ***Your Choice*** *Give a formula for a rational function whose graph satisfies the given conditions. Check with a graph.*

55. x-intercept point $(2, 0)$, vertical asymptote $x = -1$, horizontal asymptote $y = 2$.

56. x-intercept points $(-2, 0)$, vertical asymptote $x = 1$, horizontal asymptote $y = 2$.

Exercises 57–58 ***Local Maxima*** *Find the coordinates of the local maximum point(s) on the graph of f.*

57. $f(x) = \dfrac{x^2 - 7x + 16}{x - 3}$ **58.** $f(x) = \dfrac{x^2 + x + 1}{x}$

Exercises 59–60 ***Local Minima*** *Find the coordinates of the local minimum point(s) on the graph of f.*

59. $f(x) = \dfrac{x^2 + 5x + 7}{x + 2}$ **60.** $f(x) = \dfrac{x^2 - x + 4}{x}$

Exercises 61–68 ***Match Functions*** *Match the graph with the appropriate function from the following list. Check by graphing.*

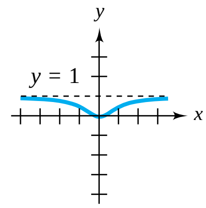
(a) $f(x) = \dfrac{x + 1}{x - 1}$ **(b)** $f(x) = \dfrac{1}{x - 1}$

(c) $f(x) = \dfrac{x + 1}{(x - 1)^2}$ **(d)** $f(x) = \dfrac{x^2}{x^2 + 1}$

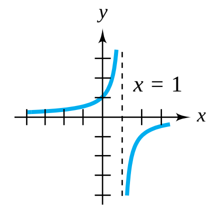
(e) $f(x) = \dfrac{1}{1 - x}$ **(f)** $f(x) = \dfrac{x^2}{x^2 - x - 2}$

(g) $f(x) = \dfrac{x}{x^2 - x - 2}$ **(h)** $f(x) = \dfrac{x^3}{x^2 - x - 2}$

61.



62.



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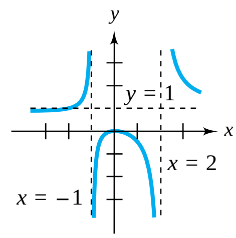

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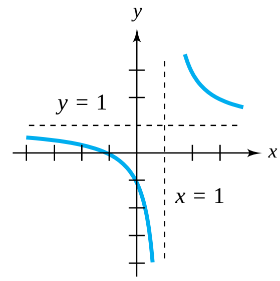
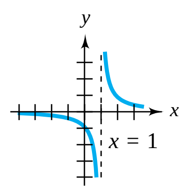


65.



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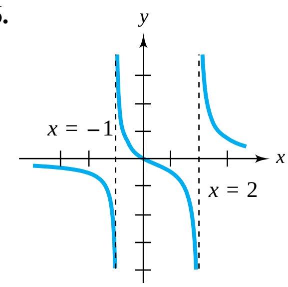
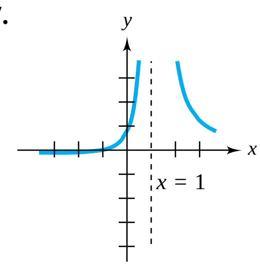
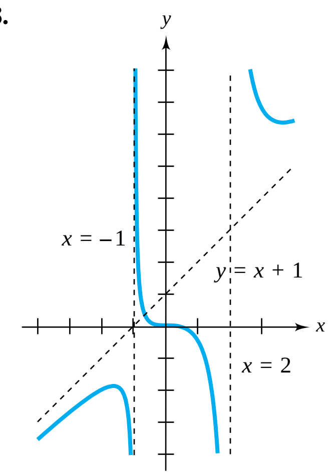


67.



68.



Exercises 69–72 ***Solution Set*** *Functions g and h are given and function f is defined by* $f(x) = \dfrac{g(x)}{h(x)}$. *Find the solution set for* $f(x) < 0$ *algebraically. Draw a graph to support your answer. (Hint: First show that* $h(x) > 0$ *for every value of x (draw a graph).) Why is the solution set for* $f(x) < 0$ *the same as the solution set for* $g(x) < 0$?

69. $g(x) = x^2 - 2x - 3, \quad h(x) = x^2 - 2x + 3$

70. $g(x) = x^3 - 2x^2 - 8x, \quad h(x) = x^4 - 3x^2 + 4$

71. $g(x) = x^3 + 2x^2 - x - 2, \quad h(x) = x^2 + 2x + 2$

72. $g(x) = 1 - x^3, \quad h(x) = 1 + x^2$

73. For $f(x) = \dfrac{2x^3 + 3x^2 + x - 2}{x^2}$,

(a) show that $y = 2x + 3$ is an oblique asymptote.

(b) Draw graphs of $y = f(x)$ and $y = 2x + 3$ simultaneously and see that the graph of $y = f(x)$ is approaching the asymptote as $x \to \infty$. Is it approaching from above or from below?

(c) Does the graph of $y = f(x)$ intersect the asymptote? If it does, find the point of intersection algebraically.

(d) Find some values of x for which the difference of the two y values is less than 0.01.

74. Solve the same problem as in Exercise 72 except use $f(x) = \dfrac{x^3 - 3x^2 + 2x - 1}{x^2 + 1}$. First find an equation for the oblique asymptote.

75. Show that if g and h are polynomial functions with no common zeros and the degree of h is 3, then the function $f(x) = \dfrac{g(x)}{h(x)}$ (lowest terms) must have at least one vertical asymptote.

76. **(a)** Function $f(x) = \dfrac{x - 3}{|x| + 2}$ is not a rational function, Why?

(b) Draw a graph and see that the graph f has two horizontal asymptotes.

(c) By considering two cases, $x \geq 0$, and $x < 0$, find equations for the two horizontal asymptotes.

Exercises 77–78 ***Intercepts and Asymptotes*** *Does the graph of f have **(a)** x-intercept points? **(b)** Any vertical asymptotes? (Hint: Draw graphs of the numerator and denominator separately.)*

77. $f(x) = \dfrac{x^2 - 2x + 5}{x^4 + 3x - 4}$ **78.** $f(x) = \dfrac{x^3 + x - 1}{x^2 - 3x + 4}$

79. Of all rectangles with an area of 160 square inches, what are the dimensions of the one having the smallest perimeter?

80. Solve the problem in Exercise 79 for a rectangle of area 240 square inches.

81. A cylindrical can is to contain 48 ounces (87 cubic inches) of apple juice. If the can is to use the least amount of tin, what should the radius and the height be?

82. Suppose x ounces of pure acid are added to 50 ounces of 40% solution of acid. Let u denote the concentration (percent) of the resulting solution.

(a) Express u as a function of x.

(b) Why cannot u be 100 or greater?

(c) What is the domain of this function?

(d) How many ounces of acid must be added to get a 65% solution? Support your answer by drawing a graph and finding the value of x that gives 65 for u.

83. Find the positive number (2 decimal places) such that the sum of its square and its reciprocal is a minimum.

84. A rectangular box is to have a base whose length is twice its width, and whose volume is 2460 cubic inches. Of all such boxes, what are the dimensions (1 decimal place) of the one that will require the least amount of material if the box has **(a)** a top? **(b)** no top?

85. Solve the problem in Exercise 84 if the length of the base is three times its width.

Exercises 86–89 ***Applied Minima***

86. A rectangular printed page is to have margins 2 inches wide at the top and bottom and 1 inch wide on each of the two sides. If the page is to have 60 square inches of printed material,

(a) What is the minimum possible area of the page?

(b) What are the dimensions of the page?

87. A factory has a fixed daily overhead cost of $600. If it produces x units daily, then the cost for labor and materials is $3x$ dollars. The daily cost of equipment maintenance is $\dfrac{x^2}{240000}$ dollars.

(a) Find a function giving the total daily cost, $c(x)$, when x units are produced.

(b) How many units should be produced each day to minimize the cost per unit $\left(\text{minimize } \dfrac{c(x)}{x}\right)$? (*Hint:* $x > 10000$.)

88. The x-axis, y-axis and any line with negative slope passing through the point $P(3, 5)$ determine a triangle. Of all such triangles, determine the line for which the area of the triangle is a minimum.

89. Solve the problem in Exercise 88 if P is the point $(5, 4)$.

CHAPTER 3 REVIEW

Test Your Understanding

True or False. Give reasons.

1. $F(x) = x^{-2} + x^{-1} + 1$ is a polynomial function of degree -2.

2. The equation $x^3 + x + 1 = 0$ has no positive roots.

3. The equation $x^3 + x - 1 = 0$ has no negative roots.

4. The equation $x^3 + x^2 - 1 = 0$ has no positive roots.

5. Every polynomial function of degree 3 has at least one real zero.

6. Every polynomial function of degree 4 has at least one real zero.

7. The graph of $y = x^3 + x^2 + 1$ is the same as that of $y = x^3 + x^2$ translated upward by 1 unit.

8. The equation $x^3 + 2x^2 - 1 = 0$ has no rational roots.

9. The equation $x^4 + 3x^3 - x + 1 = 0$ has no rational roots.

10. The function $f(x) = x(x - \sqrt{3})(x + \sqrt{3})$ is a polynomial function with integer coefficients.

11. The function $f(x) = \sqrt{x^3 + x - 1}$ is a polynomial function of degree 3.

12. The graph of $y = x^3 + x^2 - 2x - 1$ crosses the x-axis at exactly two points.

13. An x-intercept point on the graph of $y = x^4 + x^2 - 2x$ is $(1, 0)$.

14. The graph of $y = x^4 + x^2 + 1$ crosses the x-axis at four points.

15. All real roots of $x^4 + 3x^3 - 3x - 1 = 0$ are irrational.

16. The graphs of $f(x) = x^3 + 5x - 4$ and $g(x) = (x^2 + 1)(x^3 + 5x - 4)$ cross the x-axis at precisely the same points.

17. Given that $\sqrt{3}$ is a root of $x^3 + \sqrt{3}x^2 - 6x = 0$, then $-\sqrt{3}$ is also a root.

18. The graph of $y = x^4 + 1$ does not cross the y-axis.

19. The graph of every polynomial function crosses the y-axis at exactly one point.

20. $f(x) = \dfrac{x - 1}{x^2 + 1}$ is a rational function.

21. $f(x) = x^{-2} - x$ is a rational function.

22. An irrational root of $x^4 - 2x^2 - 3 = 0$ is $\sqrt{3}$.

23. A factor of $3x^4 - 2x^3 + x - 4$ is $x + 1$.

24. When $x^{15} - 2x^{10} + x^8 - 3x^2 + 1$ is divided by $x + 1$, the remainder is -4.

25. A factor of $x^{12} - 2x^8 + x^5 - 4x - 2$ is $x + 1$.

26. If $f(x) = x^3 + 2$ and $g(x) = x^2 - 1$, then $f \circ g$ is a polynomial function of degree 5.

27. The function $F(x) = x^3 - 2x^2 + x - 1$ has an irrational zero between -1 and -2.

28. The graph of every rational function has at least one vertical asymptote.

29. Every polynomial function is also a rational function.

30. The graph of $y = \dfrac{3x^2 + 1}{x^2 + 1}$ has no horizontal asymptotes.

31. The graph of $y = \dfrac{x^2 - 4x}{x^2 - 1}$ crosses the x-axis at exactly two points.

32. If c is a root of the polynomial equation $f(x) = 0$, then $c + 1$ is a root of $f(x - 1) = 0$.

33. If c is a root of the polynomial equation $f(x) = 0$, then $c - 3$ is a root of $f(x + 3) = 0$.

34. If $f(x) = (x + 3)(x + 1)(x - 2)$, then $f(x)$ is negative for every x in the interval $(-1, 2)$.

35. The graph of $y = x^4 - x^3 + 2x^2 + 1$ has no points in the third quadrant.

36. The graph of $y = x^3 + 2x + 1$ has no points in the fourth quadrant.

37. Every horizontal line must intersect the graph of any polynomial function of degree 3 in at least one point.

38. Every vertical line will intersect the graph of any polynomial function of degree 4 in exactly one point.

39. The graph of every rational function must have a horizontal asymptote.

40. The graph of every polynomial function of degree 4 must have a y-intercept point.

41. Every vertical line will intersect the graph of any polynomial function.

42. If f is a polynomial function and both $f(1)$ and $f(2)$ are positive, then f cannot have a zero between 1 and 2.

43. The graph of $y = 2x^3 - 3x^2 - 12x - 8$ has no local maximum point in the fourth quadrant.

44. The function $f(x) = 2x^3 - 3x^2 - 4x - 4$ has three real zeros.

45. The function $f(x) = 2x^3 + 9x^2 + 24x + 5$ is an increasing function.

46. The function $f(x) = x^3 - x^2 + 3x - 4$ has an inverse.

47. The function $f(x) = x^4 - 2x^2 + x - 1$ has a minimum value.

48. The function $g(x) = 3 - 4x - x^4$ has a minimum value.

49. The graph of $y = \dfrac{x + 1}{x^2 + 1}$ has no horizontal asymptote.

50. The graph of $y = \dfrac{x^2 + 1}{x^2 - 2x + 3}$ has no vertical asymptotes and no x-intercept points.

51. The graph of $y = \dfrac{2x^2 + 1}{x - 1}$ has a slant asymptote.

52. If f is a polynomial function of degree 4, there is at least one horizontal line that will not intersect its graph.

53. The graph of $f(x) = x^3 + 3x - 4$ has a local minimum point.

54. The graph of every polynomial function of degree 4 must have at least one local maximum or local minimum point.

55. The graph of a polynomial function of degree 3 cannot have more than one local maximum point.

56. The graph of $f(x) = \dfrac{2x^2 - 3x}{x^2 + 4}$ approaches the line $y = 2$ from above as $x \to \infty$.

57. The line $y = 2x - 1$ intersects the graph of $y = x^3 + 4x - 5$ at exactly one point.

58. If $f(x) = 5 - 4x - x^3$, then f is a decreasing function.

59. The solution set for $x^3 + 4x^2 + x - 6 \geq 0$ is $\{x \mid -3 \leq x \leq -2 \quad \text{or} \quad x \geq 1\}$.

60. The solution set for $\dfrac{x^2 - 5x + 6}{x^2 + 3} \geq 0$ is the same as the solution set for $x^2 - 5x + 6 \geq 0$.

61. Suppose f is a polynomial function and $f(x) \to \infty$ as $x \to \infty$. If $g(x) = f(-x)$, then $g(x) \to -\infty$ as $x \to -\infty$.

Review for Mastery

Use a graph whenever you think it might be helpful.

1. Find the quotient and remainder when $3x^3 - 4x^2 - x + 1$ is divided by $x + 1$.

2. For $f(x) = x^4 - 3x^2 + 2x - 5$, find $q(x)$ and r such that $f(x) = (x - 3)q(x) + r$.

3. Determine the remainder when $3x^{16} + 2x^{10} - 5x^3 + 3x^2 - 1$ is divided by $x + 1$.

4. Sketch a graph of $y = (x - 1)(x^2 - 4)$. Label on your graph the x- and y-intercept points.

5. Sketch a graph of $y = (x + 2)(x - 1)^2$. Label the x- and y-intercept points on your graph.

6. For $f(x) = 2x^3 - 3x^2 + x - 4$ locate a zero between successive integers.

7. Suppose $f(x) = 2x^3 - x^2 - 27x - 30$.
(a) List all *possible* rational zeros given by the rational zeros theorem.
(b) Draw a graph. Which of the numbers listed in part **(a)** can be eliminated?
(c) Find all of the zeros in exact form.

8. Find all rational zeros of $f(x) = 2x^3 - 3x^2 - 12x - 5$.

9. Find all zeros of $f(x) = 2x^3 + 9x^2 + 7x - 6$ in exact form.

10. Locate each of the irrational roots of $x^4 + 2x^3 - 4x^2 - 6x + 3 = 0$ between two consecutive integers.

11. Locate each of the irrational roots of $x^3 - 5x + 3 = 0$ between **(a)** two consecutive integers and **(b)** two consecutive tenths. **(c)** Determine the largest root rounded off to two decimal places.

12. **(a)** Locate each of the zeros of $f(x) = 3x^3 - 2x^2 - x + 1$ between two consecutive integers.
(b) Determine the largest zero rounded off to one decimal place.

13. **(a)** Find all roots of $x^4 - 3x^2 = 0$ in exact form.
(b) Sketch a graph of $y = x^4 - 3x^2$.

14. Find a polynomial function of lowest degree with leading coefficient 1 that has -1, 1, and 3 as zeros. Give your answer in expanded form.

15. Find a polynomial function of degree 4 that has each of -2 and 2 as double zeros. Give your answer in expanded form.

16. Find a polynomial function of lowest degree that has integer coefficients, a leading coefficient of 2, and $\frac{1}{2}$ and $\sqrt{3}$ as zeros. Give your answer in expanded form.

17. Sketch the graph of $y = x^3 - 3x^2 - x + 3$. Label the x- and y-intercept points on your graph.

18. **(a)** Find all roots of $x^3 + x^2 - 6x - 6 = 0$ in exact form.
(b) Draw a graph of $y = x^3 + x^2 - 6x - 6$ and label the intercept points.

19. Draw a graph of each function.
(a) $y = x^3 - 4x$ **(b)** $y = x^3 - 4x - 1$
(c) $y = (x - 1)^3 - 4(x - 1)$
(d) How are the graphs related?

20. Find the zeros in exact form for the function f and g.
(a) $f(x) = x^3 + x^2 - 5x - 5$
(b) $g(x) = (x - 1)^3 + (x - 1)^2 - 5(x - 1) - 5$

Exercises 21–23 ***Intercepts and Asymptotes*** *For the graph of the function,* ***(a)*** *give the x- and y-intercept points,* ***(b)*** *determine the equations of any vertical or horizontal asymptotes, and* ***(c)*** *sketch.*

21. $f(x) = \dfrac{x + 3}{2 - x}$

22. $f(x) = \dfrac{x^2 - 9}{x^2 - x - 2}$

23. $f(x) = \dfrac{x^2 - 2x + 1}{x^2 - 4x}$

Exercises 24–28 ***Solution Set*** *Find the solution set. (Hint: Draw a graph or use cut points.)*

24. $x^3 - 4x^2 + x + 6 < 0$

25. $x^3 - 3x^2 + x - 3 \geq 0$

26. $x^4 + 4x^3 + 2x^2 - 4x - 3 > 0$

27. $x^2 + \dfrac{4}{x - 3} \geq 0$

28. $x^2 - \dfrac{3x}{x - 2} \geq 0$

Exercises 29–30 ***Evaluate Inverse*** *Find the indicated values of f^{-1}.*

29. $f(x) = x^3 + x + 8$
(a) $f^{-1}(18)$ **(b)** $f^{-1}(-2)$

30. $f(x) = 2x^3 - 4x^2 + 3x - 5$
 (a) $f^{-1}(22)$ **(b)** $f^{-1}(-14)$

Exercises 31–32 ***Related Functions*** *Find the solution set for* **(a)** $f(x) = 0$ **(b)** $f(x-1) = 0$ **(c)** $f(x+2) = 0$.

31. $f(x) = 2x^3 - 5x^2 - 4x + 3$

32. $f(x) = (2x - 3)(12 - x - x^2)$

Exercises 33–34 ***Solution Set*** *Find the solution set for* **(a)** $f(x) \geq 0$ **(b)** $f(x-1) \geq 0$.

33. $f(x) = 2x^3 + 5x^2 - 4x - 3$

34. $f(x) = (x + 3)(x^2 - 16)$

Exercises 35–36 ***Finding Intersections*** *Find the points of intersection of the graphs of f and g.*

35. $f(x) = x^3 - 2x + 3$ and $g(x) = x^2 - 3x - 11$.

36. $f(x) = 2x^3 - 4x^2 + 5$ and $g(x) = -2x + \frac{11}{4}$.

37. For what values of x is $2x^3 - 5x^2 + 9x - 9$ positive?

38. For what values of x is $2x^3 + 5x^2 + 9x + 9$ negative?

39. For what values of x is $x^3 + 2x + 3$ greater than $x^2 + 3x - 7$?

40. The length of each of the two equal sides of an isosceles triangle is $x + 2$, and the base is $2x$. The area is 10. Find the dimensions (2 decimal places) of the triangle and its height.

41. Find the maximum value of $f(x)$ where $f(x) = 4 - (x-1)^4$.

Exercises 42–43 ***Local Maxima*** *Determine the coordinates of the local maximum point for the graph of f,*

42. $f(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 6x - 4$

43. $f(x) = -x^3 + 4x^2 - x - 6$

44. Find the solution set for $x^3 + 4x^2 + x \geq 6$.

45. Find the point(s) of intersection of the graphs of $y = x^2 - 6$ and $y = x^3 - 3x$. Support your answer with a calculator graph.

46. Find the point(s) of intersection of the graphs of $y = x^3 + x^2 - 1$ and $y = x + 9$.

47. ***Local Extrema***

(a) For $f(x) = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 2x + 3$, draw a graph. Does the graph have any local maximum or minimum points?

(b) If it does, determine the coordinates of the local maximum.

48. Repeat Exercise 47 for $f(x) = 3 + 2x + \dfrac{x^2}{2} - \dfrac{x^3}{3}$.

49. ***Oblique Asymptotes***

For $f(x) = \dfrac{2x^3 - x^2 + 3x - 3}{x^2 + 1}$,

(a) find an equation for the oblique asymptote for the graph of f.

(b) Draw a calculator graph of $y = f(x)$ and the asymptote simultaneously. Check to see that the graph of $y = f(x)$ is approaching the asymptote as $x \to \infty$.

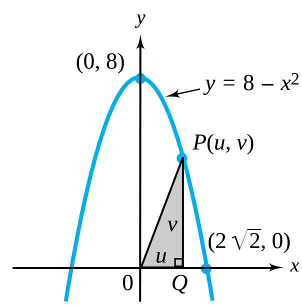
(c) Does the graph intersect the asymptote? If it does, find the point of intersection algebraically.

50. Do the problem in Exercise 49 for

$$f(x) = \frac{-2x^3 + 3x^2 - x + 5}{x^2 + 1}$$

Exercises 51–56 ***Applied Extrema Problems*** *Round off results to one decimal place.*

51. Point $P(u, v)$ is any point in the first quadrant and on the graph of $y = 8 - x^2$. A right triangle POQ is drawn as shown in the diagram. Of all such possible triangles find the dimensions of the one that has a maximum area.



52. Use the diagram in Exercise 51 and suppose the triangle is revolved about the leg $\overline{PQ}$ generating a cone. What are the coordinates of P that will give a cone of maximum volume?

53. A rectangle with an area of 40 square inches has a diagonal that is 4 inches longer than one of its sides. What are the dimensions of the rectangle?

54. The base of a rectangle is on the x-axis and its upper two vertices are on the graph of $y = 4x - x^2$. Of all such possible rectangles, find the dimensions of the one with greatest area.

55. Of all rectangles with an area of 128 square inches, what are the dimensions of the one with smallest perimeter?

56. A manufacturer wants to make a cylindrical can that contains 24 ounces (43 cubic inches) of tomato juice. Of all such possible cans, find the radius and height of the one that uses the least amount of tin.

4 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

We had a thing at high school called the algebra team, which consisted of five kids, and we would travel to different schools as a team and have competitions. A teacher, who was running the contest, would take out an envelope, and on the envelope it says, "forty-five seconds." She . . . writes the problem on the blackboard, and says, "Go!" One thing was for sure: It was practically impossible to do the problem in any conventional, straightforward way . . . so you had to think, "Is there a way to see it?"

Richard P. Feynman

THE PEOPLE IN THE TOWN of Calburn have a long tradition of summer leisure time swimming and boating in the Hatchee reservoir outside town. On May 1, an industrial accident in the county upstream released a toxic chemical that left a dangerous level of pollutants in the reservoir. The leak was stopped almost immediately, but water testing showed that the pollution level had only dropped to half of the initial level by May 16.

Assuming that the water continues to clear at the same rate, with the pollution level dropping by half every fifteen days, and that the water won't be safe until 95% of the chemical is gone, how soon can people use the dam again? Can the city safely have its Fourth of July Centennial celebration at the dam as planned?

Given these assumptions, we know that in another 15 days (May 31) the pollution level will be down to 25% (half of half), by June 15 to 12.5%, by June 30 to 6.25%, and by July 15 the water will have become safe, at just over 3% of the original level of contamination, but how much will be gone by July 4? On what day will enough of the chemical flow out to drop the level below 5%? The mathematical model that allows us to predict such information (and many, many other phenomena) is called an **exponential function.** We will return to the question of Hatchee Reservoir in Section 4.5, after we have developed the tools we need.

In Chapter 3 we discussed polynomial functions. This chapter looks at two closely related families of functions, exponential and logarithmic functions. Section 4.1 reviews properties of exponents and uses those properties to introduce exponential functions. Exponential functions are one–one functions, so they have inverses. Sections 4.2 and 4.3 explore these inverses, called logarithmic functions. The last two sections of the chapter show how to evaluate and apply both exponential and logarithmic functions.

4.1 EXPONENTS AND EXPONENTIAL FUNCTIONS

> What I really am is a mathematician. Rather than being remembered as the first woman this or that, I would prefer to be remembered, as a mathematician should, simply for the theorems I have proved and the problems I have solved.
>
> *Julia Robinson*

Elementary algebra courses define expressions of the form b^x for integer exponents (and a few rational-number exponents). We need to expand this to allow more kinds of numbers as exponents. This requires extending definitions to ***n*th roots** and then to **rational exponents.** The extension to irrational exponents is properly left to calculus, but we can at least get a feeling for what a calculator does when we evaluate an expression such as $3^{\sqrt{2}}$ or 2^{π}. In the following definitions, n and m denote positive integers.

Definition: exponents, roots, and radicals

Integer Exponents	$b^n = b \cdot b \cdot \ldots \cdot b$, *product of* n *factors*, if $n > 0$ $b^0 = 1$ if $b \neq 0$; $b^{-n} = 1/b^n$, $n > 0$ and $b \neq 0$.
Principle n^{th} Root	$b^{1/n} = \sqrt[n]{b}$ is the *real number root of* $x^n = b$ when there is only one root; when there are two, $b^{1/n}$ is the *positive root*. When $n = 2$ we write $b^{1/2} = \sqrt{b}$.
Rational Exponents	If m/n is in *lowest terms,* then $b^{m/n} = (b^{1/n})^m$. When $b > 0$, $b^{m/n}$ is also equal to $\sqrt[n]{b^m}$, which is called *radical form.*

Irrational Exponents

Certain theoretical considerations require care in defining a number like $2^{\sqrt{2}}$ but properties of the real number system guarantee its existence. We use calculators to evaluate exponential expressions. Since

$$\sqrt{2} \approx 1.41421356\ldots,$$

we would expect the numbers $2^{1.4}, 2^{1.41}, 2^{1.414}, \ldots$ (where all of the exponents are rational) to approach $2^{\sqrt{2}}$. The calculator makes the conclusion plausible:

$$2^{1.4} = 2^{7/5} \approx 2.639 \qquad 2^{1.41} \approx 2.6574 \qquad 2^{1.414} \approx 2.66475$$
$$2^{\sqrt{2}} \approx 2.6651441.$$

Properties of Exponents

In the expression b^x we call b the **base** and x the **exponent.** If b is a positive number, then b^x is a real number for every value of x. If, however, b is negative, then b^x is a real number for some values of x, but it is nonreal for other values of x. For instance, $(-4)^{5/3}$ is a real number (see Example 2b), but $(-4)^{3/2}$ is a nonreal complex number. Our primary interest in this chapter is the exponential function, which requires a positive base. Therefore, the following properties of exponents assume b and c are positive.

Properties of exponents

If b and c are positive numbers and x and y are any real numbers, then

$$\textbf{E1.}\ b^x b^y = b^{x+y} \qquad \textbf{E2.}\ \frac{b^x}{b^y} = b^{x-y} \qquad \textbf{E3.}\ (b^x)^y = b^{xy}$$

$$\textbf{E4.}\ (bc)^x = b^x c^x \qquad \textbf{E5.}\ \left(\frac{b}{c}\right)^x = \frac{b^x}{c^x}.$$

TECHNOLOGY TIP ◆ **Roots of negative numbers**

Different calculators handle roots of negative numbers differently. Check to see how your calculator evaluates $(-1)^{1/3}$. We know that -1 is the only real root of $x^3 = -1$, so $(-1)^{1/3} = -1$. Your calculator may use $\boxed{y^x}$ or $\boxed{\wedge}$. Remember parentheses for both -1 and the $\frac{1}{3}$. If the display returned is -1, then your calculator evaluates $(-1)^{1/3}$ as you expect. Your calculator may display an ERROR message (which means that your machine does not evaluate roots of negative numbers), or you may get something like (.5, .866 . . .), which means your calculator is giving you a *complex number root*. When b is negative and the exponent is irrational, do not expect a real number result.

If your calculator doesn't return what you expect, *you have to be more clever than your calculator*. Remember that cube roots of negative numbers are defined and that

$$(-b)^{1/3} = -(b^{1/3}).$$

To make certain that you know how to use the above definitions and to get your calculator to evaluate exponential expressions, make certain that you can do everything suggested in the first example.

▶**EXAMPLE 1** ***Exponential expressions*** Simplify and evaluate (in exact form if possible, five-decimal place approximation otherwise):

(a) $\sqrt[3]{-64}$ **(b)** $4^{2/3}$ **(c)** $(-8)^{5/3}$ **(d)** $4^{\sqrt{2}}$

Solution

(a) $\sqrt[3]{-64} = (-64)^{1/3} = -(64^{1/3}) = -(2^6)^{1/3} = -(2^2) = -4.$
We use the calculator to check by evaluating -64^(1/3).

(b) $4^{2/3}$ can be rewritten in other forms, as, for example, $(4^2)^{1/3} = \sqrt[3]{16}$, but other equivalent forms are no easier to evaluate. In decimal form, 4^(2/3) ≈ 2.51984 (*be careful about parentheses*).

(c) $(-8)^{5/3} = -((2^3)^{5/3}) = -2^5 = -32.$

(d) $4^{\sqrt{2}} \approx 7.10299.$ ◀

▶**EXAMPLE 2** ***Calculator evaluation*** Give a four-decimal place approximation to illustrate E2 and E3.

(a) $5^{\pi-\sqrt{2}}$ and $5^{\pi}/5^{\sqrt{2}}$ **(b)** $5^{\pi\cdot\sqrt{2}}$ and $(5^{\pi})^{\sqrt{2}}$

Solution

(a) Evaluating $5^{\pi-\sqrt{2}}$ and rounding off to four decimal places gives 16.1208. Evaluating 5^{π} and $5^{\sqrt{2}}$ and then dividing, also returns 16.1208.

(b) Rounding to four decimal places, both $5^{\pi\cdot\sqrt{2}}$ and $(5^{\pi})^{\sqrt{2}}$ are given by the calculator as 1274.7996. ◀

Strategy: **(a)** Use E4 first, followed by E3, and simplify. **(b)** First replace x^{-2} by $\frac{1}{x^2}$ and $4x^{-1}$ by $\frac{4}{x}$, then simplify.

►EXAMPLE 3 ***Getting rid of negative exponents*** Simplify. Express the result without negative exponents.

(a) $(x^{-2}y^3)^{-2}$ **(b)** $\dfrac{x^{-2} - 4x^{-1} - 5}{5x - 1}$

Solution

(a) $(x^{-2}y^3)^{-2} = (x^{-2})^{-2}(y^3)^{-2} = x^4y^{-6} = x^4\left(\dfrac{1}{y^6}\right) = \dfrac{x^4}{y^6}.$

(b) $$\frac{x^{-2} - 4x^{-1} - 5}{5x - 1} = \frac{\dfrac{1}{x^2} - \dfrac{4}{x} - 5}{5x - 1} = \frac{\dfrac{1 - 4x - 5x^2}{x^2}}{5x - 1} = \frac{(1 - 5x)(1 + x)}{x^2(5x - 1)}$$
$$= -\frac{x + 1}{x^2}.$$ ◄

►EXAMPLE 4 ***Rationalize denominator*** Rationalize the denominator of $\frac{x - 1}{\sqrt{x} + 1}$.

Solution

Strategy: Multiply numerator and denominator by $\sqrt{x} - 1$ and simplify to get rid of the radical in the denominator.

Follow the strategy.

$$\frac{x - 1}{\sqrt{x} + 1} = \frac{(x - 1)(\sqrt{x} - 1)}{(\sqrt{x} + 1)(\sqrt{x} - 1)} = \frac{(x - 1)(\sqrt{x} - 1)}{x - 1} = \sqrt{x} - 1.$$ ◄

►EXAMPLE 5 ***Disguised quadratic equation*** Solve the equation $2x^{-2} + 7x^{-1} - 4 = 0$.

Solution

Strategy: First get rid of the negative exponents by multiplying both sides by x^2, then solve the resulting quadratic equation.

Follow the strategy.

$$x^2(2x^{-2} + 7x^{-1} - 4) = x^2 \cdot 0 \qquad 2 + 7x - 4x^2 = 0$$

Factoring, $(2 - x)(1 + 4x) = 0$. By the zero-product principle, the solutions are 2 and $-\frac{1}{4}$. ◄

►EXAMPLE 6 ***Equating powers of 3*** Solve the equation $3^{2x+1} - \dfrac{27}{\sqrt[3]{9}} = 0$.

Solution

Strategy: First express $\frac{27}{\sqrt[3]{9}}$ as a power of 3, then use properties of exponents.

Follow the strategy.

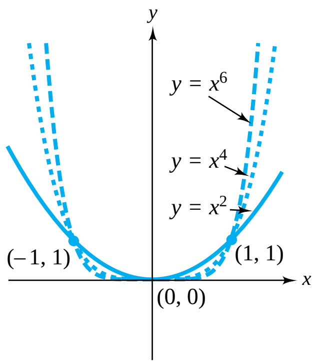
$$\frac{27}{\sqrt[3]{9}} = \frac{3^3}{\sqrt[3]{3^2}} = \frac{3^3}{3^{2/3}} = 3^{3-(2/3)} = 3^{7/3}$$

Therefore, the given equation is equivalent to

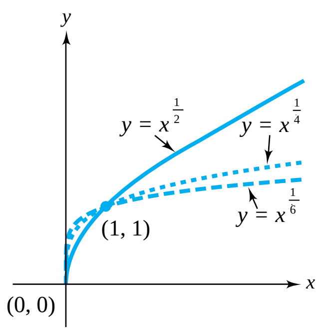
$$3^{2x+1} = 3^{7/3}.$$

In this form it is intuitively clear that the two exponents must be equal: $2x + 1 = \frac{7}{3}$. Thus the solution is $\frac{2}{3}$.

If we had been unable to express $\frac{27}{\sqrt[3]{9}}$ as a simple power of 3, then the solution of this problem would have had to await the techniques of Section 4.4. ◄



(a) Even powers



(b) Even roots

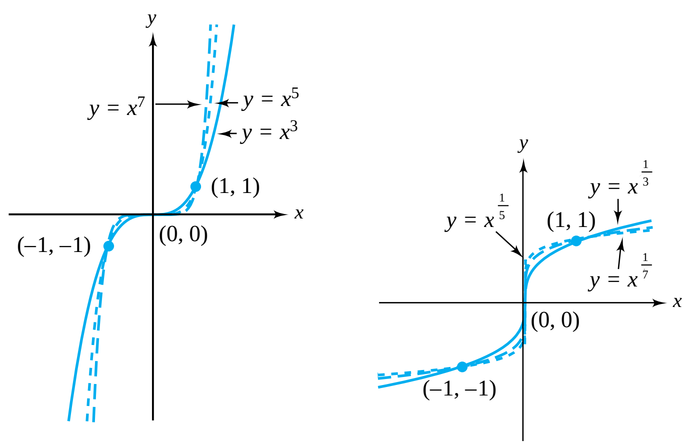
FIGURE 1

Rational Power Functions, $x^{m/n}$

In Chapter 3, our focus was on polynomial functions, which can all be expressed as sums of **power functions,** $x^0, x^1, x^2, x^3, \ldots$. With the definition of rational exponents, it makes sense to consider graphs of **rational powers** of x, functions of the form $f(x) = x^{m/n}$, where m and n are positive integers (for a negative exponent, we would take the reciprocal). We have already looked at the graph of $y = \sqrt{x} = x^{1/2}$, which we recognize as the inverse of the function $f(x) = x^2$, $x \geq 0$.

There is a basic difference between graphs of even and odd powers of x. The even powers form a family, all of whose graphs contain the points $(-1, 1)$, $(0, 0)$, and $(1, 1)$. See Figure 1a. As the power increases, the graphs become progressively flatter around the origin and then increase more and more steeply, as if a slightly flexible parabola had been "jammed nose first" into the x-axis. None of these even powers is one-one, but each is increasing if we restrict the domain to $x \geq 0$. Thus for even numbers n, restricting the domain to the nonnegative real numbers gives a function $y = x^n$, $x \geq 0$ with an inverse function $y = x^{1/n}$, $x \geq 0$. See Figure 1b.

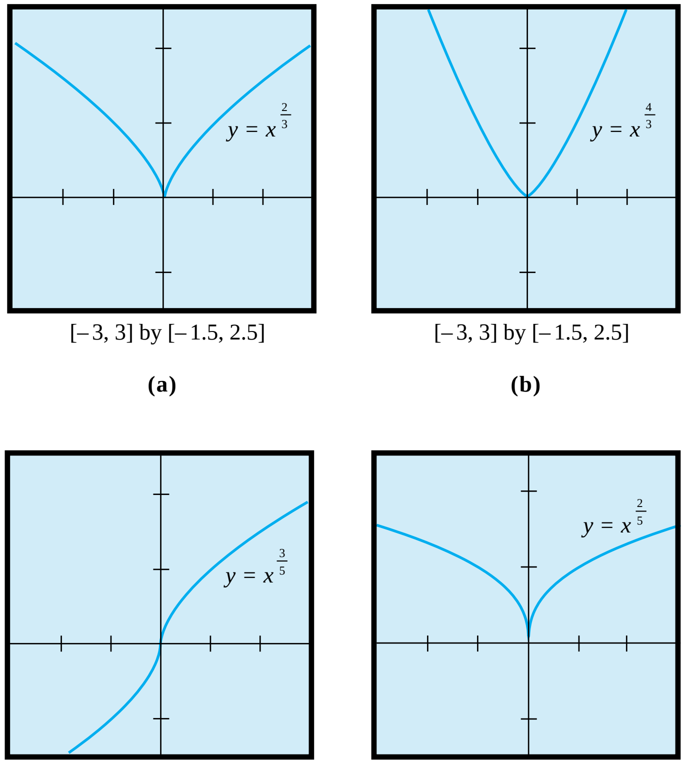
The odd powers of x also form a family. All graphs contain the points $(-1, -1)$, $(0, 0)$, $(1, 1)$. All odd power functions are *increasing* and hence one-one. Therefore every odd power function $y = x^n$ has an inverse function $y = x^{1/n}$ that is also increasing, and the domain (and range) for every member of the family, including inverses, consists of all real numbers. See Figure 2.



(a) Odd powers **(b)** Odd roots

FIGURE 2

From the definition, $x^{m/n} = (x^{1/n})^m$. Thus to graph $y = x^{2/3}$, we enter `Y = (X^(1/3))²`. The parentheses are critical to make sure that the calculator is graphing what we intend. The graphs of several rational power functions are shown in Figure 3 and are typical of such functions in general. The variations in shape depend on the parity (odd or even) of m and n. Rather than trying to describe all possible combinations, we suggest that you experiment and observe the patterns, being careful with parentheses. See the following Technology Tip.



[−3, 3] by [−1.5, 2.5]

(a)

[−3, 3] by [−1.5, 2.5]

(b)

[−3, 3] by [−1.5, 2.5]

(c)

[−3, 3] by [−1.5, 1.5]

(d)

FIGURE 3

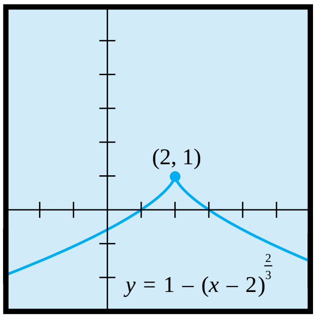
TECHNOLOGY TIP ◆ ***What your calculator may not show you***

To graph rational power functions correctly, you need to be sure that you know how your calculator handles such expressions. The graph of $y = x^{2/3}$ is shown in Figure 3a. The function is defined for all real numbers x and has what is called a "cusp" at the origin, a sharp corner at a local minimum. Without care, your calculator will almost surely not duplicate the graph in Figure 3a. If you enter `Y = X^2/3`; you will probably get the parabola $y = \frac{1}{3}x^2$. Graphing `Y = X^(2/3)`, you will get a function whose domain is the set of nonnegative numbers. To get the graph in Figure 3a, you will probably have to enter `Y = (X^(1/3))²`. On the HP-38 and HP-48, even that function will produce only the right half of the graph, the points where $x \geq 0$. You simply must recognize that the graph of the function contains more than the calculator shows in that case.

►EXAMPLE 7 ***A shifted rational power function*** Describe the graph of $f(x) = 1 - (x - 2)^{2/3}$ in terms of basic transformations of a rational power function. For what values of x is f increasing? Find all local extrema.

Solution

If $g(x) = x^{2/3}$, then the graph of $y = (x - 2)^{2/3}$ is a horizontal shift of the graph of g, 2 units right, and $y = -(x - 2)^{2/3}$ is a reflection of the shifted graph through the



[−3, 6] by [−3, 6]

FIGURE 4

x-axis (tipping it upside down). Finally, the graph of f is obtained by shifting up 1 unit. Graphing Y = 1 − ((X − 2)^(1/3))² gives a picture something like Figure 4. It is clear that f is increasing on $(-\infty, 2)$ and that there is a local maximum at $(2, 1)$. Because $g(x) = x^{2/3}$ has a minimum at the origin, f has only the one local extremum. ◀

Beyond Calculator Precision

There are times when we need more precision than a calculator can display. If we understand some basic principles, we may be able to do more than the calculator alone can provide. The idea of one–one functions has some unexpected applications that are used a number of times in this chapter. For example, suppose $a^2 = b^2$. What can we say about a and b? Because two numbers can have the same square (as $2^2 = (-2)^2$), without more information, all we can say is that $a = \pm b$. If, however, $a^2 = b^2$ *and we know that both a and b are positive, then we can conclude that* $a = b$. We are using the fact that the function $y = x^2$ is a one–one function *on the limited domain* where $x \geq 0$. We use this idea in the next example.

▶EXAMPLE 8 ***Do equal decimals imply equality?*** Which, if any, of the following are equal?

$$a = \sqrt{5} + 1 \qquad b = \sqrt{5 + \sqrt{21 + 4\sqrt{5}}} \qquad c = \frac{5702887}{1762289}$$

Solution

When we evaluate the three numbers by calculator, each shows the same display, 3.2360679775, so relying on the calculator alone, we would have to conclude that the numbers are equal. Their appearance is so different, though, that we want more confirmation.

For the first pair, a and b, we can get rid of some of the radicals by squaring.

$$a^2 = 6 + 2\sqrt{5} \qquad b^2 = 5 + \sqrt{21 + 4\sqrt{5}}$$

These numbers still appear very different, but rather than squaring again immediately, we observe that it would be much easier to square $b^2 - 5$, so we subtract 5 from each and then square again.

$$(a^2 - 5)^2 = (1 + 2\sqrt{5})^2 = 1 + 4\sqrt{5} + 20 = 21 + 4\sqrt{5}.$$
$$(b^2 - 5)^2 = 21 + 4\sqrt{5}.$$

Since $(a^2 - 5)^2 = (b^2 - 5)^2$ and $a^2 - 5$, $b^2 - 5$ are both positive, we have $a^2 - 5 = b^2 - 5$, so $a^2 = b^2$, and finally, since a and b are positive, $a = b$.

Now, how about a and c? Since c is clearly a rational number, we might be able to use the technique of Example 7 from Section 3.3 to show that a is an irrational number. As an alternative, we use an approach that shows how to go beyond the number of digits a calculator can display. We begin with the idea of squaring. Before squaring, though, we subtract 1 from both a and c, and then we can clear fractions. That is, we want to know if $a - 1 = c - 1$, or if $\sqrt{5} = \frac{3940598}{1762289}$, and then if $(1762289\sqrt{5})^2 = (3940598)^2$. What the calculator shows for both is 1.55283125976E13. That is, the display tells us only that each number equals 155283125976??; the last two digits are not displayed. Here we use what we know about properties of multiplication. While we cannot display the entire number, we can use the calculator for either the first digits or the last.

$$3940598^2 \text{ ends } (\ldots 598)^2 = \ldots 7604, \text{ and}$$
$$1762289^2 \cdot 5 \text{ ends } (\ldots 289)^2 5 = \ldots 7605.$$

Putting the information together, we have

$$(1762289\sqrt{5})^2 = 15528312597605, \text{ and}$$
$$(3940598)^2 = 15528312597604.$$

We conclude that $a \neq c$, so that a and b are equal to each other, but c is different from either a or b. ◀

Exponential Functions

We assume the properties of real numbers that assure us that for any real number x, the expression 5^x is a positive real number, so the equation $f(x) = 5^x$ defines a positive-valued function whose domain is R, called an **exponential function.** The number 5 is the **base** of this exponential function, but any other positive number (except 1) can be used as a base for an exponential function as well.

Definition: exponential function

An **exponential function, base *b*,** is any function that can be expressed in the form

$$f(x) = b^x$$

where b is a fixed positive number ($\neq 1$).

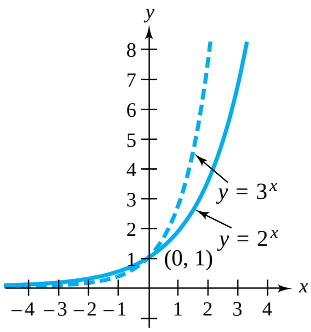


FIGURE 5
Exponential functions with bases greater than 1

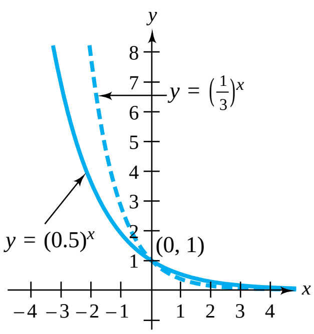


FIGURE 6
Exponential functions with bases less than 1

Graphs of Exponential Functions

Graphs of all exponential functions have one of essentially two different shapes, depending on whether $b > 1$ or $b < 1$.

To get a feeling for the graphs when $b > 1$, we use the graphing calculator to graph exponential functions for two different bases. See Figure 5. The graphs in Figure 5 are drawn on the same axes, but we suggest that you graph each of the same functions, preferably on different screens, or at least sequentially, to see how similar they are. If the base b is a number near 1, then the curve is relatively flat; as b increases, the curve $y = b^x$ rises more and more steeply to the right of the y-axis.

The graphs of exponential functions when the base is a number less than 1 are reflections through the y-axis of the kinds of curves in Figure 5. For example, if $f(x) = 3^x$, then for $g(x) = (\frac{1}{3})^x$, we have

$$g(x) = (3^{-1})^x = 3^{-x} = f(-x),$$

so the graph of g is the reflection through the y-axis of the graph of f. See Figure 6 and graph a variety of such functions yourself. Again, when the base b is a number near 1, the exponential curve is flatter, becoming steeper to the left of the y-axis as b decreases toward 0.

Properties of Exponential Functions

The graphs in Figures 5 and 6 suggest some general properties of exponential functions.

Properties of exponential functions

Suppose b is a positive number different from 1 and $f(x) = b^x$.

Domain: $(-\infty, \infty)$ *Range:* $(0, \infty)$
Intercepts: x-intercept points, none; y-intercept point $(0, 1)$.
Asymptotes: the x-axis is always a *horizontal asymptote.*
If $b > 1$, then f is an *increasing function;*
if $b < 1$, then f is a *decreasing function.*

Every exponential function f is *one-one* and thus has an *inverse function.*

The Euler Number e and the Natural Exponential Function

It turns out, as we shall see, that in one very important sense, all exponential functions can be considered as transformations of a single exponential function. That being the case, we should be able to choose any particular exponential function and use it as *the* exponential function, from which we can obtain all others. As a matter of fact, however, nature has made a selection for us. There is an important number just a little less than 3, denoted by e, which is the base of what is almost universally called the **natural exponential function,** and denoted by

$$f(x) = e^x \quad \text{or} \quad f(x) = \exp(x).$$

Justification for the name "natural" usually comes in a calculus course; for our purposes, we simply state that all sorts of natural growth and decay phenomena are most easily described in terms of e^x.

The number e, sometimes called the **Euler number,** can be defined in many different ways (see the Historical Note, "π and e," Part I) and appears in as many unexpected mathematical contexts as the number π. Your calculator is programmed to evaluate e^x for real numbers x. The number itself is an irrational number that has been calculated to many decimal places, the first twenty-five of which are given by

$$e \approx 2.71828\ 18284\ 59045\ 23536\ 02875.$$

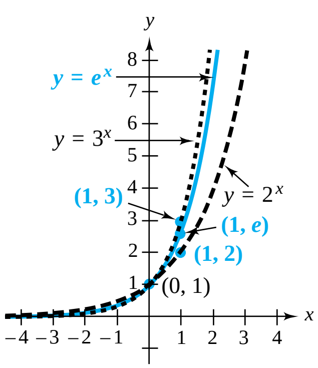


FIGURE 7
Natural exponential function, $y = e^x$

You should see what your calculator displays by evaluating e^1 or EXP (1).

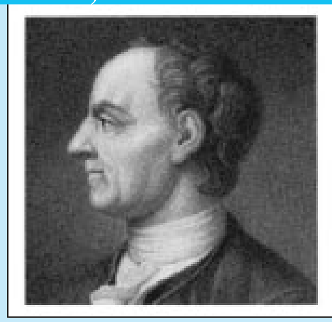
Since e is a number between 2 and 3, and much nearer 3, we would expect the graph of natural exponential function to lie between the graphs of $y = 2^x$ and $y = 3^x$, closer to the latter, as can be seen in Figure 7. You should be able to draw a calculator graph similar to Figure 7, using the built-in function key for e^x, which is paired on almost all calculators with the $\boxed{\text{LN}}$ key. Use a decimal window and trace along the curves to the point where $x = 1$. Compare the y-coordinates at that point. On the natural exponential function, you should see part of what your calculator displays for e.

One of the ways to define the number e is as the limit of the function $f(x) = (1 + \frac{1}{x})^x$ as x increases without bound. That is, $\lim_{x\to\infty}(1 + \frac{1}{x})^x = e$. The expression $(1 + \frac{1}{x})^x$ appears when we compute compound interest on investments. See "Compound Interest" formula in Section 4.5. We show how the graph of $y = (1 + \frac{1}{x})^x$ is related to the number e in the next example.

HISTORICAL NOTE π AND e, PART I

Leonhard Euler (Switzerland, 1707–1783) first used the letter e for the base of natural logarithms; e is often called *Euler's number*. Euler proved in 1737 that e is irrational, 24 years before π was shown to be irrational. Euler discovered a relationship between π and e that some thought to have mystical significance:

$$e^{\pi i} + 1 = 0, \text{ where } i = \sqrt{-1}.$$



Leonhard Euler

The numbers π and e share another property. Any number that is a root of a polynomial equation with integer coefficients is called an **algebraic number.** The set of algebraic numbers includes all of the rational numbers and some of the irrational numbers. For example, $\sqrt{2} + \sqrt{6}$ is an irrational number and it is also algebraic since it is a root of $x^4 - 16x^2 + 16 = 0$. Real numbers that are not algebraic are called **transcendental numbers.** It was long suspected that π and e might be transcendental, but not until 1873 did Hermite (France) prove the transcendence of e. Nine years later, Lindemann (Germany) extended Hermite's result to include π (as well as many numbers involving trigonometric and logarithmic functions).

►EXAMPLE 9 ***The number e*** Draw calculator graphs of $g(x) = e$ and $f(x) = (1 + \frac{1}{x})^x$ and describe what happens as x becomes large and positive.

Solution

We are not told how large x must be, so we may need to experiment with various windows. Note first that g is a constant function, *not* an exponential function; g has the same value, e, for every x, and the graph is a horizontal line. We enter Y1 = e^1 and Y2 = (1 + 1/X)^X and then choose a window. Since e is just a little less than 3, our y-range should include 3, but we have little experience to guide us with the function f. Fortunately, one of the advantages of a graphing calculator is that we can look at a function in a particular window and explore different possibilities until we have a view that gives us the information we need.

If we try $[0, 10] \times [0, 6]$ we get a graph like Figure 8a. It appears that the graph of f is approaching the graph of g, that the line $y = e$ is a horizontal

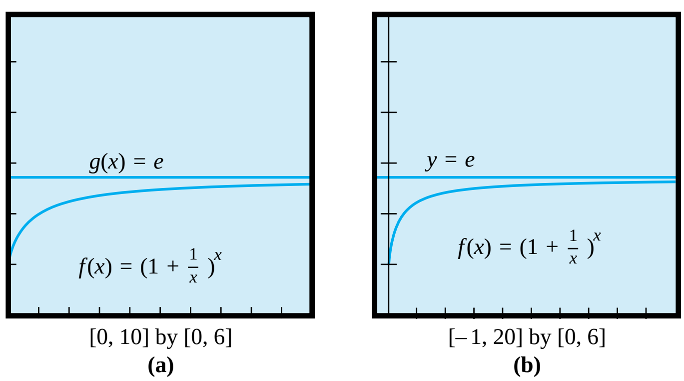


FIGURE 8

asymptote for the graph of f. Tracing to the right along f shows that the values of f are getting closer to e. To see more of the graphs, we can expand the window. Figure 8b shows the view in $[-1, 20] \times [0, 6]$. To support our feeling that the graph of f has a horizontal asymptote, we can return to the home screen and eval- uate f at larger values of x. For example, $f(100) \approx 2.705$, and $f(2000) \approx 2.7176$. It seems clear that $f(x)$ is approaching e as x gets large; it seems equally clear that evaluating f is not a good way to approximate e with much accuracy. ◀

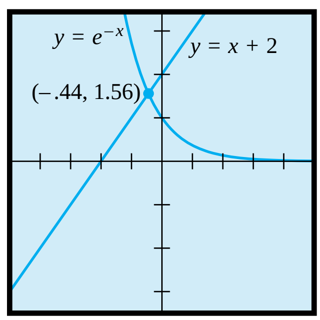
▶EXAMPLE 10 ***Solve an exponential equation*** Draw calculator graphs of $f(x) = e^{-x}$ and $g(x) = x + 2$ on the same screen. To two decimal places,

(a) find the coordinates of the point of intersection, and
(b) approximate the solution to the equation $e^{-x} - x - 2 = 0$.

Solution

(a) If we graph both functions on the same screen, we see something like Figure 9. (Note that the graph of f has the same shape as the other exponential functions in Figure 6.) When we zoom in on the point of intersection, and trace, we read the coordinates as about $(-.44, 1.56)$.
(b) The given equation is the same as $f(x) - g(x) = 0$, or $f(x) = g(x)$. Thus the one solution is the x-coordinate of the point of intersection, which we just estimated as -0.44.

Alternative Solution Many graphing calculators have a solve routine that will find the desired intersection for the first two functions, or the zero of the second function. We invite you to investigate your calculator and to become familiar with any such built-in routines. In this case, the desired x-coordinate is approximately -0.44285440100. ◀



[−5, 5] by [−3.5, 3.5]

FIGURE 9

Basic Transformations and Exponential Functions

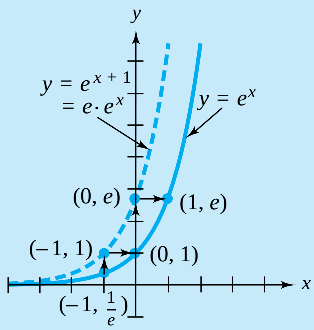
Basic transformations apply to the family of exponential functions just as they do all functions, but some of the properties of exponents and the way we write exponential functions have unexpected consequences when we explore transformations. We list below some of the relations to keep in mind as you graph and work with exponential functions.

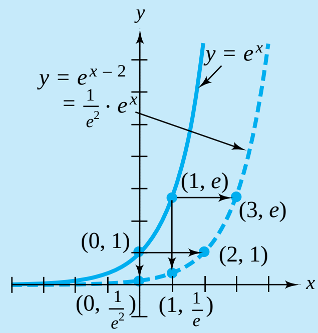
Basic transformations and the family of exponential functions

$$f(x) = e^x,\ c > 0$$

Vertical Shifts $f(x) \pm c = e^x \pm c$

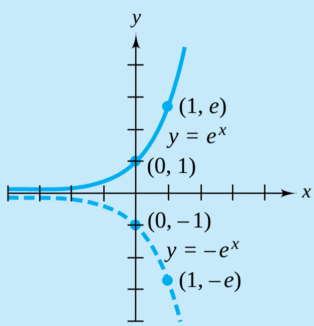
Horizontal Shifts/Dilations $f(x \pm c) = e^{x \pm c} = e^x \cdot e^{\pm c} = ke^x$ (because e^c and e^{-c} are constants). Thus *every horizontal shift is the same as some vertical dilation.*

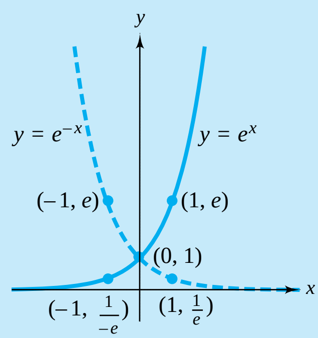




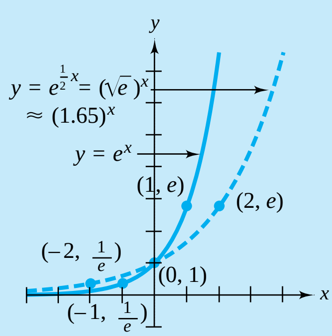
Reflections x-axis: $-f(x) = -e^x$ y-axis: $f(-x) = e^{-x}$

This is why the graphs in Figure 6 all have the same basic shape; each is the reflection of one in Figure 5.





Horizontal Dilation/Change of Base $f(cx) = e^{cx} = (e^c)^x$, an exponential function with base e^c. This is why *every exponential function is really a transformed natural exponential function.*



Although every horizontally-dilated natural exponential function can be considered as an exponential function with another base, the last statement in the preceding box (that any exponential function is a transformed natural exponential function) is less obvious. We get a better feeling for such relationships in the next couple of examples.

▶EXAMPLE 11 ***Recognizing transformations*** Simplify the equations that describe functions f and g. Describe the relation of each to the natural exponential function and draw graphs of both f and g on the same screen as $y = e^x$.

$$f(x) = \frac{e^{2x} - e^x}{e^x}, \qquad g(x) = \frac{e^x}{e^2}$$

Solution

$$f(x) = \frac{e^{2x} - e^x}{e^x} = \frac{e^{2x}}{e^x} - \frac{e^x}{e^x} = e^x - 1, \quad \text{and} \quad g(x) = e^x \cdot e^{-2} = e^{x-2}.$$

In simplified form, we can recognize that the graph of $y = f(x)$ is a vertical shift of $y = e^x$. The graph of $y = g(x)$ may be considered either as a horizontal shift (2 units right) or as a vertical dilation (by a factor of e^{-2}). Both graphs are shown in Figure 10. ◀

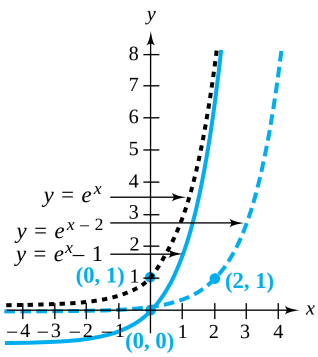


FIGURE 10
Translations of $y = e^x$

Changing Base for Exponential Functions

Every exponential function may be considered as a horizontal dilation of the natural exponential function. Thus, by an appropriate choice of c in $y = e^{cx}$ we can get an equivalent formula for $y = b^x$. To have $e^{cx} = b^x$, we need c such that $e^c = b$. In the next section we introduce the natural logarithmic function, $\ln x$, and see that $c = \ln b$. For now, we illustrate the change in an example.

▶EXAMPLE 12 ***Changing base***

(a) Show that $e^{0.69315} \approx 2$. Is $e^{0.69315}$ larger or smaller than 2?
(b) Draw graphs of $f(x) = e^{0.69315x}$ and $g(x) = 2^x$ on the same screen. Which graph is higher to the right of the y-axis?

Solution

(a) Evaluating `e^0.69315`, the calculator displays 2.00000563889, so $e^{0.69315} > 2$, but not by very much. We would certainly expect the graphs of $y = 2^x$ and $y = e^{0.69315x}$ to be *very* close to each other. The graphs are so close together that we cannot see two graphs at all; both share the same y-pixels in most reasonable windows. In fact, when $x = 10$, $f(x) \approx 1024.03$, and $g(x) = 1024$. By the time $x = 20$, $f(x) \approx 1{,}048{,}635.13$, and $g(x) = 1{,}048{,}576$. To the right of the y-axis the graph of f is always higher than the graph of g. ◀

EXERCISES 4.1

Check Your Understanding

Draw a graph whenever it may be helpful.
Exercises 1–5 True or False. Give reasons.

1. For every real number x, $\sqrt[3]{\sqrt{64x^6}} = 2x$.
2. For every real number x, $(x - 2)^0 = 1$.
3. For every negative number x, $\sqrt{(-x)^2} = -x$.
4. $(-2)^{248} + (-1)^{215} > (-2)^{248}$.
5. The function $f(x) = 1 + 2^{-x}$ is increasing.

Exercises 6–10 *Fill in the blank so that the resulting statement is true.*

6. The number of zeros for $f(x) = e^x - x - 2$ is ______.
7. The number of roots of $2^{-x} - x = 2$ is ______.
8. The graphs of $y = e^{0.25x}$ and $x + y = 2$ intersect at a point in Quadrant ______.
9. The graphs of $y = e^{-x} - 2$ and $x + y + 4 = 0$ intersect at a point in Quadrant ______.
10. The graphs of $y = e^x + 2$ and $y = 4 - x^2$ intersect in Quadrant(s) ______.

Develop Mastery

Exercises 1–3 *Simplify. Give answer in exact form.*

1. **(a)** $\dfrac{3^{-1} + 2^{-2}}{6^{-1}}$ **(b)** $(2^{-3} + 4^{-1})^{-3}$
2. **(a)** $\left(\dfrac{3^{-1} \cdot 2^2}{6^{-1}}\right)^{-1}$ **(b)** $(\sqrt{8} - \sqrt{2})^{-4}$
3. **(a)** $\dfrac{7^{5/2} - 63^{3/2}}{\sqrt{7}}$ **(b)** $\dfrac{\sqrt{105}(\sqrt{35})^{-1}}{3}$

Exercises 4–6 ***Decimal Approximations*** *Give a calculator approximation, rounding off to four significant digits.*

4. **(a)** $(-3)^{5/3}$ **(b)** $(\sqrt{5} - 1)^{\pi}$
5. **(a)** $(1 + \pi)^{-2/5}$ **(b)** $(\sqrt{2} + \sqrt{5})^{-1/2}$
6. **(a)** $5^{\sqrt{5}}$ **(b)** $(-1.47)^{2/3}$

Exercises 7–8 ***Rationalize*** *Rationalize the denominator and simplify.*

7. **(a)** $\dfrac{8}{\sqrt{5} + 1}$ **(b)** $\dfrac{x - 4}{\sqrt{x} - 2}$
8. **(a)** $\dfrac{6}{\sqrt{3} + 1}$ **(b)** $\dfrac{x^2 - 9x}{\sqrt{x} + 3}$

Exercises 9–14 *Simplify.*

9. $\sqrt{1 - (1 + x)(1 - x)}$
10. $\dfrac{x^2(x^{-2} - 2x^{-1} + 1)}{x - 1}$
11. $x^{5/2}x^{-3/2}$
12. $4^{2-3x}\,8^{x-2}$
13. $\dfrac{27^{x-1}}{9^{1-3x}}$
14. $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2 - \dfrac{1}{x}$

Exercises 15–22 ***Solution Set*** *Find the solution set.*

15. $x^{-2} - 3x^{-1} + 2 = 0$
16. $x^{-2} - 2x^{-1} + 1 = 0$
17. $2x^{-2} - 5x^{-1} = 0$
18. $4^{x-1} - 4\sqrt{2} = 0$
19. $(x^2 - 1)^0 = 1$
20. $(x^2 - 2x - 3)^0 = 1$
21. $2^4 \cdot 3^{2x+1} = 6^4$
22. $2^3 \cdot 5^{2x-1} = 10^3$

Exercises 23–26 *Does the equation define an exponential function? Give reasons.*

23. $y = \pi^x$
24. $y = (\sqrt{3} - 1)^x$
25. $y = (1 - \sqrt{2})^x$
26. $y = \dfrac{e^{2x}}{e^x}$

Exercises 27–28 ***Function Evaluations*** *Give results rounded off to two decimal places.*

27. $F(x) = xe^x$; find
 (a) $F(-1)$ **(b)** $F(\sqrt{2})$
28. $F(x) = e^x + e^{-x}$; find
 (a) $F(-1)$ **(b)** $F(\sqrt{5})$

Exercises 29–30 *Determine which of the two numbers is greater.*

29. $3^{\sqrt{3}}$ or $(\sqrt{3})^3$
30. e^3 or 3^e

Exercises 31–36 ***Graphs*** ***(a)*** *Draw a graph and use it to determine the range of the function. Note that -3^x means $-(3^x)$, not $(-3)^x$.* ***(b)*** *Is the function increasing, decreasing, or neither?*

31. $f(x) = -3^x$
32. $g(x) = -3^{-x}$
33. $f(x) = e^{x+1}$
34. $g(x) = e^x + 1$
35. $f(x) = 1 - e^x$
36. $f(x) = (\sqrt{2} - 1)^x$

Exercises 37–40 ***Graphs*** ***(a)*** *From the graph of $f(x) = e^x$, describe the translations and/or reflections required to get the graph of g.* ***(b)*** *Draw graphs of f and g on the same screen. Does the graph of g support your description in part (a)?*

37. $g(x) = -e^x$
38. $g(x) = e^{x-2}$
39. $g(x) = e^x + 2$
40. $g(x) = e^{-x} + 1$

Exercises 41–46 ***Solution Set*** *Find the solution set. Solve algebraically and then check graphically.*

41. $x^2 \cdot 2^x - 4 \cdot 2^x = 0$
42. $2^x = 6 - 8 \cdot 2^{-x}$
43. $2^{2x-1}\,4^{2x+5} = 8^{2x+3}$
44. $(2^x)^2 - 2(2^x) - 8 = 0$
45. $x^2e^{-x} - 5e^{-x} > 0$
46. $2^x \geq 6 - 8 \cdot 2^x$

Exercises 47–48 *Simplify the equation describing f and then draw a graph. Use translations where appropriate.*

47. $f(x) = e^2e^x$
48. $f(x) = 1 + \dfrac{2^x}{8}$

Exercises 49–51 ***End Behavior*** *In example 9 we considered the end behavior of $y = (1 + \frac{1}{x})^x$ and concluded that as $x \to \infty$, $y \to e$. Draw calculator graphs of f and g on the same screen and observe what happens to $f(x)$ as $x \to \infty$.*

49. $f(x) = (1 + \frac{2}{x})^x$, $g(x) = e^2$;

50. $f(x) = (1 - \frac{1}{x})^x$, $g(x) = e^{-1}$;

51. $f(x) = (1 + \frac{1}{x})^{2x}$, $g(x) = e^2$;

Exercises 52–53 Is b equal to c, less than c, or greater than c? Calculator evaluation is not sufficient to establish equality. (Hint: See Example 8.)

52. $b = \sqrt{3} + \sqrt{5}$ $\quad c = \sqrt{8 + 2\sqrt{15}}$

53. $b = 1 + \sqrt{3}$ $\quad c = \sqrt{3 + \sqrt{13 + 4\sqrt{3}}}$

54. ***Large Numbers, Exact Form***

(a) Evaluate $\sqrt{3}$ and $\frac{3650401}{2107560}$ and see that the decimal approximations agree to the capacity of your calculator.

(b) To show that these two numbers are not equal, let $b = \sqrt{3}\,(2107560)$ and $c = 3650401$, and evaluate b^2 and c^2 in exact form. You should get two 14 digit numbers that differ by 1. See Example 8.

55. What is the smallest integer that is greater than $(1 + \sqrt{2})^4$? than $(\sqrt{2} + \sqrt{3})^5$?

56. If $f(x) = (x^x)^x$ and $g(x) = (x)^{(x^x)}$, which number is larger

(a) $f(2)$ or $g(2)$? **(b)** $f(3)$ or $g(3)$?
(c) $f(0.5)$ or $g(0.5)$?

57. If $f(x) = 1 + 2^x$, then evaluate $\dfrac{1}{f(x)} + \dfrac{1}{f(-x)}$ and simplify your answer.

58. If $f(x) = 3^x$, show that

(a) $f(u + v) = f(u) \cdot f(v)$
(b) $f(2x) = [f(x)]^2$ **(c)** $f(3x) = [f(x)]^3$.

59. ***Rational Exponents, Domain, Graphs*** For $f(x) = x^{4/5}$,

(a) Determine the domain. To support your answer, try drawing graphs separately by entering

(i) `Y = (X^(1 ÷ 5))^4`
(ii) `Y = X^(4 ÷ 5)` **(iii)** `Y = (X^4)^(1 ÷ 5)`.

(b) Which graphs support your answer in part **(a)**?

60. Repeat Exercise 59 for $f(x) = x^{3/5}$.

61. ***Composition*** For $f(x) = x^4$ and $g(x) = x^{1/2}$, determine the domain of **(a)** $f \circ g$ and **(b)** $g \circ f$. As a check, draw graphs of `Y = (X^(1 ÷ 2))^4` for **(a)** and `Y = (X^4)^(1 ÷ 2)` for **(b)**. Do the graphs agree with your answers in **(a)** and **(b)**?

62. Repeat Exercise 61 for $f(x) = x^5$ and $g(x) = x^{1/3}$.

Exercises 63–64 ***Transformations of Rational Power Function*** ***(a)*** *Describe the graph of f in terms of basic transformations of the graph of a rational power function.* ***(b)*** *What are the zeros of f?* ***(c)*** *Find the coordinates of any local extrema. Check graphically. See Example 7.*

63. $f(x) = 1 - (x + 2)^{2/3}$ **64.** $f(x) = (x - 2)^{4/3} - 1$

65. ***Explore*** Consider the family of functions $f(x) = b^{0.2x}$ where $b > 0$ and $b \neq 1$.

(a) Experiment with several values of b and draw graphs. Describe what you observe and the role that b appears to play.

(b) For what *integer* values of b do the graphs pass between the points $P(4, 3)$ and $Q(4, 6)$?

66. Repeat Exercise 65 for $f(x) = b^{-0.2x}$, and $P(-4, 3)$, $Q(-4, 6)$.

67. ***Explore*** Consider the family of functions $f(x) = (0.5)^{bx}$ where b is a positive constant. Repeat the instructions for Exercise 65, with $P(2, 2)$ and $Q(2, 8)$.

68. Repeat Exercise 67 for $f(x) = (0.5)^{-bx}$ and $P(-2, 2)$, $Q(-2, 8)$.

69. **(a)** Draw graphs of $y_1 = 2^x$, $y_2 = 3^x$, $y_3 = 4^x$ on the same screen. Use $[-2, 3] \times [-1, 10]$.

(b) For what values of x is $2^x > 3^x > 4^x$?

(c) Describe how the graph of $y = 2.5^x$ would fit into the picture. Draw it.

70. Functions f and g are given by $f(x) = 2^x + 2^{-x}$, $g(x) = x^2 + 2$.

(a) Are f and g even functions?

(b) Draw a graph of f using $[-5, 5] \times [0, 5]$. Is the graph a parabola?

(c) Include the graph of g on the same screen. Are the graphs identical? At how many points do the graphs intersect?

(d) Change the range to $[-6, 6] \times [0, 40]$. Now how many points of intersection can you see? Find these points (1 decimal place).

71. Function f is given by $f(x) = 2^x - 2^{-x}$.

(a) Is f an odd function? What symmetry does the graph have?

(b) Draw a graph to see if it supports your answer in part **(a)**.

(c) Does f appear to be a 1–1 function?

Exercises 72–73 ***Intersections*** *Determine the coordinates of the point(s) of intersection of the graphs of f and g (2 decimal places). See Example 10.*

72. $f(x) = 2^x$, $g(x) = x + 2$

73. $f(x) = 3^x$, $g(x) = x + 2$

74. In what quadrants do the graphs of $y = e^{0.5x}$ and $y = 3 + 4x - x^2$ intersect?

75. Solve the equation $e^{0.5x} + x^2 - 4x - 3 = 0$, (2 decimal places). See Exercise 74.

76. The volume V of a sphere of radius r is given by $V = \frac{4\pi r^3}{3}$.

(a) Solve for r and get an equation that gives r as a function of V.

(b) Use the results in part **(a)** to find the radius of a sphere whose volume is 148.4 cubic centimeters. Give the result rounded off to four significant digits.

77. A container in the form of a right circular cone with its vertex at the bottom has a height of 16 cm. There is a control valve at the vertex through which the container can be emptied. When the height of the water in the container is h cm $(0 \le h \le 16)$ and the valve is opened, it will take T seconds to empty the container, where T is given by

$$T = 0.04[16^{5/2} - (16 - h)^{5/2}].$$

How long will it take to empty the container, when h is equal to **(a)** 12 cm **(b)** 6 cm **(c)** 3 cm?

Exercises 78–79 ***Revenue*** *A demand function p that determines the unit price (in dollars) of a certain product is given, where x is the number of units sold.* ***(a)*** *Calculate p when* 1200 *units are sold.* ***(b)*** *Find the corresponding revenue R (where $R = x \cdot p$) when* 2000 *units are sold.*

78. $p = 400 - 0.4(2^{0.003x})$

79. $p = 300\left(1 - \dfrac{5}{5 + 2^{-0.003x}}\right)$

80. ***Population Model*** It is predicted that the population P of Brouwer's Ferry is given by $P = 2000(2^{0.03t})$, where t is the number of years after 1990. What does this model predict for the population at the end of
(a) The year 1996? **(b)** The year 2000?
(c) The year 2020?

Exercises 81–84 ***Huge Numbers and Estimation*** *It is often difficult to get a good feeling for the size of numbers that appear in the daily news. For instance, the size of the federal debt ($5 trillion) is so huge that we have little basis for comparison, but working with more familiar numbers may help.*

81. At the end of 1995 the federal debt reached $5.3 trillion and was increasing at the rate of $13,000 per second. Assume that the debt continues to increase at this rate.
(a) What will the debt be at the end of 1996?
(b) During what year will the debt reach $8 trillion?

82. If you can stack 250 dollar bills per inch, how many miles high would a stack eight trillion (8×10^{12}) dollars be? (*Hint:* The distance from the earth to the moon is approximately 240,000 miles.)

83. **(a)** Suppose you are rich, *extremely rich,* and would like to give away 5 trillion dollars by giving one thousand dollars every minute. How long would it take? Express your answer in reasonable units such as days, months, or years. First make a guess.
(b) The federal debt is more than 5 trillion dollars. Write a paragraph describing your feelings about a debt of that magnitude.

84. In 1994 the journal *Science* reported the discovery of the large prime number, $2^{859433} - 1$. This number is a side benefit of a program developed by David Slowinsk and Paul Gage to debug supercomputers. If this number were written out in our usual base 10 notation, it would have 258,716 digits compared to the previous record holder of 227,832 digits. Estimate the number of pages it would take to print out P. (Assume as many characters on a page as on a typical page of this book.)

85. ***Looking Ahead to Calculus*** Using tools of calculus we can show that the natural exponential function $f(x) = e^x$ can be approximated by certain polynomial functions such as

$$g(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040}.$$

Draw graphs of f and g on the same screen using $[0, 5] \times [-1, 100]$. Trace and use the arrow keys to move from one graph to the other and find the values of x for which the two y values agree in the first two decimal places.

4.2 LOGARITHMIC FUNCTIONS

> A piece of advice: do examples. Do a million examples. I think there are shameful cases of people making (I'll even say) silly and reckless conjectures just because they didn't take the trouble to look at the first few examples. A well-chosen example can teach you so much.
>
> *Irving Kaplansky*

In the preceding section we observed that every exponential function has an inverse. The inverse of an exponential function is called a **logarithmic function.** In this section and the next we study properties of such functions.

When I was a junior, Nowlan was giving a course in the theory of equations. He watched for the capable students and encouraged them. He would give out special problems, and if you solved those he would give you more special problems. . . . He got me to explain, without notes, the nature of that long proof. Well, it was rather fun. . . .
Ivan Niven

In Section 2.7 we developed a useful algorithm to find equations for inverse functions. Basically, we write $y = g(x)$ and interchange the x and y values. If we can solve the resulting equation for y, the result gives the inverse of the function g. Unfortunately, this algorithm depends on solving an equation for y, which is not always an easy task.

The Inverse of an Exponential Function

Consider the exponential function $f(x)=3^x$. Since f is one–one, we know that it has an inverse. Applying the algorithm, we write $y = 3^x$, interchange variables, $x = 3^y$, and we stop; we have no way to solve for y. To describe the value of y verbally:

y is the power to which 3 must be raised to get x. **(1)**

Such a rule describes a function, but it is not easy to apply. Without something more, we have no way to find the power of 3 that gives 2, for example, even though the graph of $y = 3^x$ indicates that there is exactly one such number.

We introduce a new name and notation for the function described in (1):

$$y = \log_3 x$$

That is, $\log_3$ (read "log base 3") is the name of a function, the inverse of the exponential function $f(x) = 3^x$. We usually write $\log_3 x$ without parentheses around x unless needed for clarity.

If $f(x) = 3^x$, then $f^{-1}(x) = \log_3 x$, where $\log_3 x$ is the power to which 3 must be raised to get x.

►EXAMPLE 1 ***Logarithmic evaluation*** Evaluate **(a)** $\log_3 1$, **(b)** $\log_3 \sqrt{3}$, and **(c)** $\log_3 9$.

Solution

(a) Recall that $\log_3 1$ is the power to which we must raise 3 to get 1. Since $3^0 = 1$, $\log_3 1$ is 0. We write $\log_3 1 = 0$.

(b) In the same way $\log_3 \sqrt{3}$ is the power of 3 that gives $\sqrt{3}$. Since $\sqrt{3} = 3^{1/2}$, then $\log_3 \sqrt{3} = \frac{1}{2}$.

(c) Similarly, since $3^2 = 9$, $\log_3 9 = 2$. ◄

We used our knowledge of some of the powers of 3 to find the values in Example 1, but we need a calculator to evaluate numbers such as $\log_3 2$. In Section 4.4, we will learn how to find that $\log_3 2 \approx 0.6309297536$. Check this by using your calculator to evaluate $3^{0.6309297536}$.

Graph, Domain, and Range of $\log_3 x$

In Section 2.7 we discussed several key ideas about the graph, domain, and range of an inverse function. Since the number pairs that define a function are interchanged in defining the inverse, the graph of f^{-1} is the reflection of the graph of f through the line $y = x$; the domain and range are also interchanged accordingly. We graphed $y = 3^x$ in the preceding section (Figure 5). We can use that graph to draw the graph of $y = \log_3 x$ (see Figure 11), and we can easily read the domain and range of both functions from the graphs. Every point pair (u, v) from $y = 3^x$ gives a corresponding point pair (v, u) for the inverse function, $y = \log_3 x$.

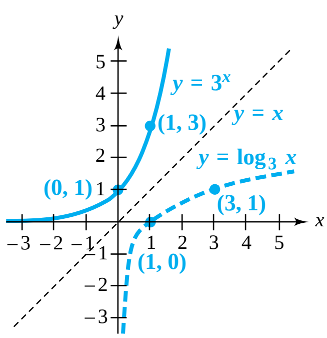


FIGURE 11
The graph of
$y = \log_3 x$
is reflection of the graph of $y = 3^x$ in the line $y = x$.

Logarithmic Functions

There is nothing special about the base 3 in the discussion above. The exponential function $f(x) = 3^x$ has an inverse function, namely the logarithmic function for base 3, denoted by $\log_3$. Just as there is an exponential function for every positive base b other than 1, there is a corresponding logarithmic function for every such base.

Definition: logarithmic functions

Suppose b is any positive number other than 1. The exponential function $f(x) = b^x$ has an inverse function called the **logarithmic function**, $f^{-1}(x) = \log_b x$, where $\log_b x$ is the power to which b must be raised to get x, that is, $b^{\log_b x} = x$.

Since each logarithmic function is the inverse of an exponential function, knowledge of exponential functions and their graphs implies much about the domains and ranges of logarithmic functions. Since $b^0 = 1$ and $b^1 = b$, the definition gives two values that are common to all logarithmic functions, as well as a very helpful equivalence.

Domain and range: special values and equivalence

The domain of $\log_b$ is $\{x \mid x > 0\}$. The range of $\log_b$ is R.

$$\log_b 1 = 0 \quad \text{and} \quad \log_b b = 1. \tag{2}$$

$$y = \log_b x \text{ is equivalent to } b^y = x. \tag{3}$$

►EXAMPLE 2 ***Converting logarithms to powers*** Evaluate
(a) $\log_5 25$, **(b)** $\log_{10} 0.01$, **(c)** $\log_{0.5} 2\sqrt{2}$.

Solution

Strategy: Use equivalence (3) and knowledge of powers.

(a) If $y = \log_5 25$, then by (3), $5^y = 25 = 5^2$, so y is 2. Thus $\log_2 25 = 2$.

(b) Let $y = \log_{10} 0.01$. By (3), $10^y = 0.01 = 10^{-2}$. Hence y is -2, or $\log_{10} 0.01 = -2$.

(c) Let $y = \log_{0.5} 2\sqrt{2}$. By (3), $(0.5)^y = 2\sqrt{2}$. Since $2\sqrt{2} = 2^{3/2}$ and $0.5 = \frac{1}{2} = 2^{-1}$,

$$(2^{-1})^y = 2^{3/2} \qquad \text{or} \qquad 2^{-y} = 2^{3/2}$$

so y is $-\frac{3}{2}$. Thus, $\log_{0.5} 2\sqrt{2} = -\frac{3}{2}$. ◄

Inverse Function Identities

In Section 2.7 we observed that, if a function f has an inverse, then

$f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}, and
$f^{-1}(f(x)) = x$ for every x in the domain of f.

For the exponential function $f(x) = b^x$ and its inverse $f^{-1}(x) = \log_b x$, this gives the following inverse function identities.

Inverse function identities

$$b^{\log_b x} = x \text{ for every } x > 0 \tag{4}$$

$$\log_b b^x = x \text{ for every real number } x \tag{5}$$

▶EXAMPLE 3 ***Exact form evaluation*** Evaluate
(a) $7^{\log_7 5}$ **(b)** $\log_3 (\log_5 5)$

Solution

(a) From Equation (4), where b is 7 and x is 5, $7^{(\log_7 5)} = 5$.
(b) From Equation (2), $\log_5 5 = 1$. Therefore, $\log_3 (\log_5 5) = \log_3 1$. Also by Equation (2), $\log_3 1 = 0$, so $\log_3 (\log_5 5) = 0$. ◀

▶EXAMPLE 4 ***Inverse function identities*** Simplify. Give the values of x for which the result is valid.

(a) $3^{\log_3(x-2)}$ **(b)** $\log_5 5^{\sqrt{x}}$

Solution

(a) From Equation (4),

$$3^{\log_3(x-2)} = x - 2 \text{ for } x - 2 > 0.$$

Thus, $3^{\log_3(x-2)}$ is identically equal to $x - 2$ when x is greater than 2, but it is undefined if x is less than or equal to 2.
(b) From Equation (5),

$$\log_5 5^{\sqrt{x}} = \sqrt{x} \text{ for every } x \geq 0.$$ ◀

▶EXAMPLE 5 ***Logarithmic to exponential form*** Solve the equation $\log_3(x^2 - 3x + 5) = 2$.

Solution

Strategy: Use Equation (3) to write the given equation in exponential form, then solve the resulting quadratic equation.

Follow the strategy.

$$x^2 - 3x + 5 = 3^2, \qquad x^2 - 3x - 4 = 0, \qquad (x - 4)(x + 1) = 0$$

The solutions are -1 and 4. Exercise 21 asks you to verify that 4 and -1 are solutions of the equation $\log_3(x^2 - 3x + 5) = 2$. ◀

▶EXAMPLE 6 ***Domain of logarithmic functions*** Determine the domains of f and g.

(a) $f(x) = \log_3(x^2 - 2x - 3)$ and **(b)** $g(x) = \log_3 5^x$

Solution

Strategy: For each part, find where the argument is positive; that is, for f, find the solution set for $x^2 - 2x - 3 > 0$. For g, solve $5^x > 0$.

It is important to understand that any logarithmic function can be evaluated only at positive numbers. Follow the strategy.

$$x^2 - 2x - 3 > 0, \quad (x + 1)(x - 3) > 0.$$

(a) The solution set is $\{x \mid x < -1 \text{ or } x > 3\}$. Consequently, the domain of f is $\{x \mid x < -1 \text{ or } x > 3\}$, or in interval notation, $(-\infty, -1) \cup (3, \infty)$.
(b) Since 5^x is positive for every real number x, $\log_3 5^x$ is defined for any real number x; the domain of g is R. ◀

Graphs of Logarithmic Functions

For any given base b, the graph of $y = \log_b x$ is the reflection about the line $y = x$ of the graph of $y = b^x$. It is helpful to become very familiar with the shapes of the graphs of the exponential and logarithmic functions. For $b > 1$, the general shapes of $y = b^x$ and $y = \log_b x$ are shown in Figure 12. For $0 < b < 1$, the graphs of $y = b^x$ and $y = \log_b x$ look like those shown in Figure 13.

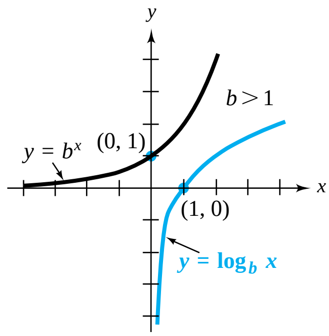


FIGURE 12

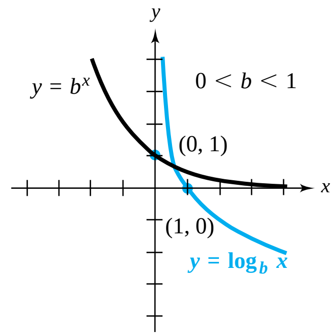


FIGURE 13

►EXAMPLE 7 ***Graphs of logarithmic functions*** Draw a graph of **(a)** $y = \log_2 x$ and **(b)** $y = \log_{0.5} x$.

Solution

(a) The graph of $y = \log_2 x$ is a reflection about the line $y = x$ of the graph of $y = 2^x$. See Figure 14a.

(b) Reflect the graph of $y = (0.5)^x$ about the line $y = x$ to get the graph of $y = \log_{0.5} x$. See Figure 14b. ◄

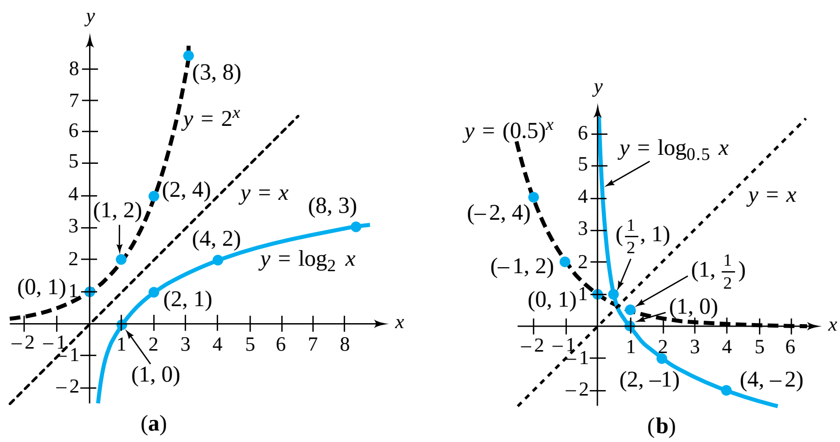


FIGURE 14

►EXAMPLE 8 ***Shifting logarithm graphs*** Use translations to draw a graph of **(a)** $y = \log_2 x - 1$ and **(b)** $y = \log_2(x - 1)$.

Strategy: If $f(x) = \log_2 x$, then $\log_2 x - 1 = f(x) - 1$ and $\log_2(x - 1) = f(x - 1)$.

Solution

(a) Following the strategy, translate the graph of $y = \log_2 x$ shown in Figure 14a one unit down to get the graph of $y = \log_2 x - 1$.

(b) The graph of $y = \log_2(x - 1)$ is a horizontal translation of the graph of $y = \log_2 x$ one unit to the right. Both graphs are shown in Figure 15. ◄

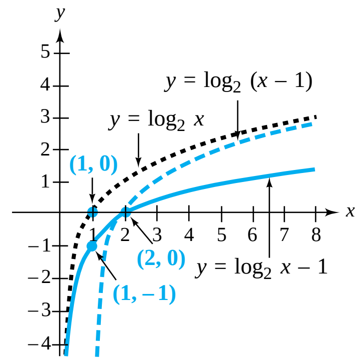


FIGURE 15
Translations of $y = \log_2 x$

►EXAMPLE 9 ***Solving an inequality graphically*** Find the solution set for the inequality $\log_2 x - 1 < 0$.

Solution

Let $y = \log_2 x - 1$. Find all values of x for which y is negative. From the graph in Figure 15, when x is between 0 and 2, y is negative. Therefore, the solution set is $\{x \mid 0 < x < 2\}$, or in interval notation, (0, 2). ◄

Calculator Graphs of Logarithmic Functions

Up to this point we have not used the calculator to draw graphs of logarithmic functions. There is an exponentiation key on all calculators, usually labeled $\boxed{\wedge}$ or $\boxed{y^x}$, so that we can enter any desired base b and graph Y = b^X. We have no corresponding generic logarithm key with which we can specify the base and graph Y = LOG b X.

There are two ways around this limitation. One way, using parametric equations as we learned in Chapter 2, can be used to graph any function and its inverse. More directly, every graphing calculator has a key labeled $\boxed{\text{LN}}$, almost always the same key as the one activating the natural exponential function, $\boxed{e^x}$.

The inverse of the natural exponential function is called the **natural logarithm** function. For consistency, we should denote this inverse in the same way as the inverses of the other exponential functions:

natural exponential function: $y = e^x$;

natural logarithm function: $y = \log_e x$.

History, however, is seldom consistent. Logarithms were invented for computational purposes (see the Historical Note in Section 4.4). It took a long time before the functional relationships were recognized and we learned how extremely important they are for purposes having nothing to do with computations. By historical accident, the notation $\log x$ (with no base indicated) is usually reserved for the **common logarithm** function, with base 10, the inverse of the exponential function $y = 10^x$. The natural logarithm function is denoted **ln.**

Calculator logarithmic function notation

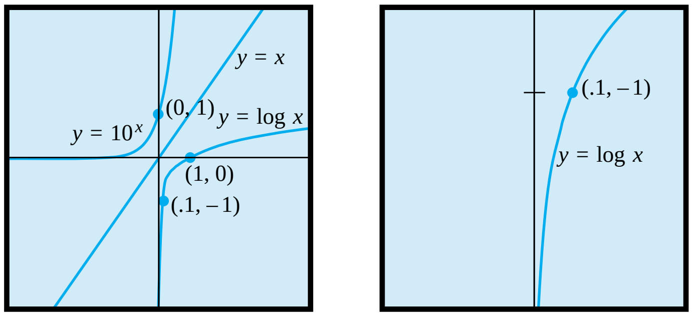
Natural logarithm function $\ln x = \log_e x$ (inverse of e^x)
Common logarithm function $\log x = \log_{10} x$ (inverse of 10^x)

▶EXAMPLE 10 ***Inverse pairs*** In the decimal window plot the graph of $y = x$ together with **(a)** $y = \log x$ and $y = 10^x$ **(b)** $y = \ln x$ and $y = e^x$.

Solution

(a) The calculator graphs are shown in Figure 16. It looks on the screen as if the graph begins at the point $(0, -1)$. If we trace, however, the calculator shows that the function $f(x) = \log x$ is undefined at $x = 0$ because, as we know, the domain of any logarithm function is the set of *positive* numbers. The first point on the graph in the decimal window is $(.1, -1)$:

$$f(.1) = f\left(\frac{1}{10}\right) = f(10^{-1}) = \log_{10} 10^{-1} = -1.$$



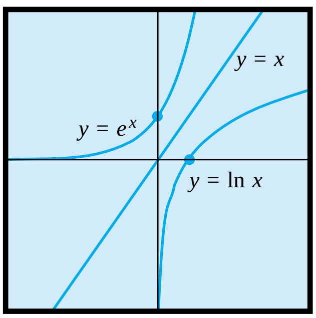
[−5, 5] by [−3.5, 3.5]
(a)

[−.4, .4] by [−2, −.6]
(b)

FIGURE 16

Since the x-axis is a horizontal asymptote for the graph of $y = 10^x$, when we interchange variables for the inverse function, the y-axis must be a vertical asymptote for the graph of $y = \log x$, and if we zoom in near $(0, -1)$, we can see some of the asymptotic behavior. See Figure 16b.

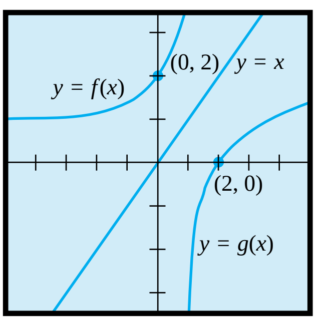
(b) The calculator graphs of $y = \ln x$, $y = x$, and $y = e^x$ show clearly that $\ln x$ and e^x are inverse functions. See Figure 17. Since $e < 10$, the graph of $y = e^x$ rises less steeply than the graph of $y = 10^x$, and the inverses are similarly related; the graph of $y = \ln x$ is not nearly as flat as the graph of $y = \log x$. ◀



[−5, 5] by [−3.5, 3.5]

FIGURE 17

Each basic transformation of exponential functions has a corresponding transformation for logarithmic functions. A horizontal shift of the natural exponential function corresponds to a vertical shift of the natural logarithm, as shown in the next example.



[−5, 5] by [−3.5, 3.5]

FIGURE 18

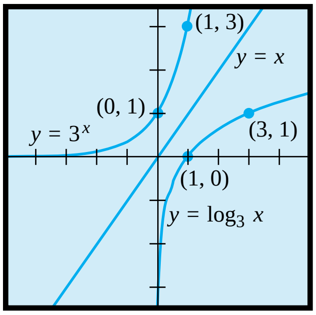
▶EXAMPLE 11 ***Transformations and inverses*** Graph $y = x$, $f(x) = e^x + 1$, and $g(x) = \ln(x - 1)$ on the same screen. Describe the relationship between the graphs of f and g.

Solution

Enter Y1 = X, Y2 = e^X + 1, and Y3 = LN(X − 1). The calculator graphs are shown in Figure 18. It appears that the graph of $y = g(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$. Since the two graphs are symmetric about the line $y = x$, the functions must be inverses of each other, which we can verify by the inverse function identities:

$$f(g(x)) = f(\ln(x-1)) = e^{\ln(x-1)} + 1 = (x-1) + 1 = x$$
$$g(f(x)) = g(e^x + 1) = \ln((e^x + 1) - 1) = \ln e^x = x.$$

The graph of $y = f(x)$ is a vertical shift of the graph of $y = e^x$, 1 unit up. When we interchange the roles of the variables, a 1 unit shift upward should become a horizontal shift of the graph of $y = \ln x$, 1 unit right, which is what we see as the graph of $y = \ln(x-1)$. ◀



[−5, 5] by [−3.5, 3.5]

FIGURE 19

$y = 3^x;\ \begin{cases} x = t \\ y = 3^t \end{cases}$

$y = \log_3 x;\ \begin{cases} x = 3^t \\ y = t \end{cases}$

Just as each exponential function can be viewed as a horizontal dilation of the natural exponential function, we would expect each logarithmic function to be obtainable as a vertical dilation of the natural logarithm function. We discuss that in Section 4.4. For now, we can use parametric equations to graph logarithmic functions with different bases.

▶EXAMPLE 12 ***Graphing logarithm functions parametrically*** Graph $f(x) = 3^x$, $y = x$, and $g(x) = \log_3 x$ on the same screen, using the parametric mode of graphing.

Solution

With the calculator in parametric mode, we enter X1 = T,Y1 = 3^T for f, X2 = T, Y2 = T for the line $y = x$, and X3 = 3^T,Y3 = T for the inverse of f, which we know to be the function g. Graphs are shown in Figure 19. ◀

EXERCISES 4.2

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. $\log_3 7 < \log_2 7$ (*Hint:* Think exponents.)
2. $\log_5 \pi < \log_2 \pi$
3. If $f(x) = \log_2 x$, then $f^{-1}(x) = 2^x$.
4. The domain of $g(x) = \log_2 x^2$ is the set of all real numbers.
5. The domain of $g(x) = \log_2(x^2 + 1)$ is the set of all real numbers.

Exercises 6–10 Fill in the blank so that the resulting statement will be true.

6. The graphs of $y = 2^{-x}$ and $x + y = 3$ intersect in Quadrant(s) ______.
7. The graphs of $y = 3^{-x}$ and $x - y + 3 = 0$ intersect in Quadrant(s) ______.
8. The number of points of intersection of the graph of $y = \log_2 x$ and $y = 4 - x^2$ is ______.
9. A point on both of the graphs of $y = 2^x$ and $y = 3^x$ is ______.
10. The graphs of $y = -\ln x$ and $y = e^{x-2}$ intersect in Quadrant ______.

Develop Mastery

Exercises 1–2 ***Exponents to Logarithms*** *Write the equation in equivalent logarithmic form with appropriate base.*

1. **(a)** $5^3 = 125$ **(b)** $4^{-2} = \dfrac{1}{16}$ **(c)** $3^{x-1} = 5$
2. **(a)** $(0.5)^2 = 0.25$ **(b)** $7^{-1} = \dfrac{1}{7}$ **(c)** $5^{x+3} = 7$

Exercises 3–6 ***Logarithms to Exponents*** *Express the equation in exponential form and then solve for y.*

3. **(a)** $y = \log_4 16$ **(b)** $y = \log_4(\frac{1}{16})$
4. **(a)** $y = \log_8 512$ **(b)** $y = \log_8(\frac{1}{64})$
5. **(a)** $y = \log_{\sqrt{3}}(\frac{1}{3})$ **(b)** $y = \log_{\sqrt{3}}(9\sqrt{3})$
6. **(a)** $y = \log_{\sqrt{5}} \sqrt{5}$ **(b)** $y = \log_{\sqrt{5}}(\frac{1}{25})$

Exercises 7–11 ***Exact Form Evaluation*** *Evaluate. Express the result in exact form.*

7. **(a)** $\log_3 1$ **(b)** $\log_e \sqrt[3]{e}$
8. **(a)** $\log_5 5$ **(b)** $\log_4 16$
9. **(a)** $e^{\log_5 \sqrt{5}}$ **(b)** $7^{\log_3 1}$
10. **(a)** $5^{\log_5 \sqrt{3}}$ **(b)** $3^{\log_{10} 10}$
11. **(a)** $4^{\log_2 4}$ **(b)** $7^{\log_7 17}$

Exercises 12–15 ***Simplify*** *Simplify and state the values of x for which the result is valid.*

12. **(a)** $\log_5 5^x$ **(b)** $5^{\log_5(x+1)}$

13. **(a)** $\log_3 3^{x-2}$ **(b)** $3^{\log_3(x-2)}$

14. **(a)** $\log_3(\sqrt{3})^{4x}$ **(b)** $5^{\log_5(5x)}$

15. **(a)** $\log_4 2^{2x}$ **(b)** $\log_5(\sqrt{5})^{2x}$

Exercises 16–20 ***Solve*** *Treat the equations in Exercises 19 and 20 as quadratic equations.*

16. **(a)** $\log_7(2x - 3) = 1$ **(b)** $\log_5(4 - 3x) = 2$

17. **(a)** $\log_2(x^2 - 2x - 1) = 1$
(b) $\log_3(x^2 - 4x) = 2$

18. **(a)** $3^{\log_3(x^2-1)} = 4$ **(b)** $5^{\log_5(x^2-2x-2)} = 1$

19. $(\log_3 x)^2 = 3 + 2\log_3 x$

20. $(\log_2 x)^2 = 8 + 2\log_2 x$

21. Verify that 4 and -1 are solutions of the equation $\log_3(x^2 - 3x + 5) = 2$. (See Example 5.)

Exercises 22–23 *Find b in terms of c.*

22. $b = \log_4 49$, $c = \log_8 7$

23. $b = \log_8 289$, $c = \log_2 17$

Exercises 24–25 *Determine b.*

24. **(a)** $\log_b \pi = 1$ **(b)** $\log_b 0.49 = 2$

25. **(a)** $\log_b 2 = 2$ **(b)** $\log_b 7 = -\dfrac{1}{2}$

Exercises 26–27 ***Bracketing Logarithms*** *The given number is between which two consecutive integers? (Hint: Think in terms of exponents.)*

26. **(a)** $\log_3 31$ **(b)** $\log_6 0.16$

27. **(a)** $\log_2(1 + \sqrt{35})$ **(b)** $\log_3 47$

Exercises 28–29 ***Ordering Logarithms*** *Which of the pair of numbers is larger?*

28. **(a)** $\log_3 4$, $\log_5 120$ **(b)** $\log_2 6$, $\log_3 6$

29. **(a)** $\log_5 36$, $\log_6 32$ **(b)** $\log_2 0.4$, $\log_2 0.2$

Exercises 30–31 *Find the smallest even integer that is greater than the number.*

30. **(a)** $\log_2 16$ **(b)** $\log_3 17$

31. **(a)** $\log_3 9$ **(b)** $\log_5 120$

Exercises 32–33 *Determine how many integers lie between the number pair.*

32. **(a)** $\log_2 8$, $\log_2 64$ **(b)** $\log_3 7$, $\log_3 250$

33. **(a)** $\log_3 2$, $\log_3 96$ **(b)** $\log_2 3$, $\log_3 47$

Exercises 34–35 *Without using a calculator, graph the function using appropriate translations or reflections of core graphs. See Example 8.*

34. **(a)** $y = \log_3 x$ **(b)** $y = \log_3(x - 1)$

35. **(a)** $y = 2 + \log_2 x$ **(b)** $y = \log_2(-x)$

Exercises 36–39 ***Domain, Graph*** ***(a)*** *Determine the domain,* ***(b)*** *simplify the equation, and* ***(c)*** *graph the function.*

36. $y = \log_3 3^{2x}$ **37.** $y = x\log_3 3^{-x}$

38. $y = 3^{\log_3 x}$ **39.** $y = 2^{\log_2(2-x)}$

Exercises 40–41 ***Domain*** *Determine the domain.*

40. **(a)** $f(x) = \log_3(x - 4)$ **(b)** $f(x) = \log_5(5^x - 1)$

41. **(a)** $f(x) = \log_3(-x)$ **(b)** $f(x) = \log_3(x^2 - 2x)$

Exercises 42–45 ***Graph*** ***(a)*** *Draw a graph of f. Does the graph suggest that f is 1-1?* ***(b)*** *Find a formula for f^{-1}.* ***(c)*** *Use calculator evaluations to find $f(2)$ and $f^{-1}(4)$ (2 decimal places). As a check use graphs.*

42. $f(x) = 2e^x$ **43.** $f(x) = e^{-x}$

44. $f(x) = 3 + e^{-x}$ **45.** $f(x) = 6 - e^x$

Exercises 46–49 ***Inverse Functions*** ***(a)*** *Find a formula for f^{-1}.* ***(b)*** *Draw graphs of f, f^{-1}, and $y = x$ on the same screen. See Example 11.* ***(c)*** *Do the graphs of f, f^{-1} and $y = x$ intersect at a common point? If they do, state the quadrant(s) in which they intersect.*

46. $f(x) = \ln x$ **47.** $f(x) = -\ln(x - 1)$

48. $f(x) = -\ln x$ **49.** $f(x) = 1 + \ln(x + 2)$

Exercises 50–53 ***Graph Intersections*** ***(a)*** *Draw a calculator graph of f and g on the same screen.* ***(b)*** *Find the coordinates of the point of intersection of the graphs (2 decimal places).*

50. $f(x) = \ln(x - 1)$, $g(x) = 3 - x$

51. $f(x) = 2 - \ln x$, $g(x) = \dfrac{x - 4}{2}$

52. $f(x) = 1 + \ln 0.5x$, $g(x) = 4 - x^2$

53. $f(x) = 1 + \ln(-x)$, $g(x) = 5 - x^2$

Exercises 54–55 ***Graphical Transformations*** ***(a)*** *Give a verbal description of how translations and/or reflections can be used to draw a graph of f from the graph of $y = \ln x$.* ***(b)*** *Draw graphs of $y = \ln x$ and $y = f(x)$ on the same screen. Do the graphs support your description in part (a)?* ***(c)*** *Find the solution set for $f(x) < 0$ (2 decimal places).*

54. $f(x) = 3 + \ln(x - 2)$ **55.** $f(x) = 2 - \ln x$

Exercises 56–57 ***Explore*** *Consider the family of functions $f(x) = c^{0.2x} + c^{-0.2x}$ where $c > 0$.* ***(a)*** *Experiment* *with several values of c and draw graphs. Describe how the graphs change as c changes.* ***(b)*** *For what integer values of c do the graphs pass between P and Q?*

56. $P(5, 4), Q(5, 7)$ **57.** $P(-5, 4), Q(-5, 7)$

Exercises 58–59 ***Your Choice*** *From the family of functions $f(x) = ce^{kx}$, where c and k are nonzero constants, choose c and k so that f satisfies the specified conditions.*

58. The graphs of f and f^{-1} intersect in **(a)** QI **(b)** QIII. **(c)** Is there an f such that the graphs intersect in QIV? Explain.

59. **(a)** The graphs of f and $y = x - 2$ intersect in QIII and QIV.
(b) The graphs of f and $y = x$ intersect in QIII.

Exercises 60–61 ***(a)*** *Find a formula for f.* ***(b)*** *If the graphs of f and f^{-1} intersect, find the point of intersection. Use parametric mode.*

60. $f^{-1}(x) = \log_3(4 - x)$ **61.** $f^{-1}(x) = \dfrac{2^{4-x}}{8}$

Exercises 62–64 ***Domains, Ranges, Graphs*** *Function f has an inverse.* ***(a)*** *Find the domain and range of f.* ***(b)*** *Find a formula for f^{-1} and give its domain and range.*

62. $f(x) = \dfrac{1}{1 + 3^x}$ **63.** $f(x) = \dfrac{3^x}{1 + 3^x}$

64. $f(x) = \dfrac{3^{-x}}{1 + 3^{-x}}$

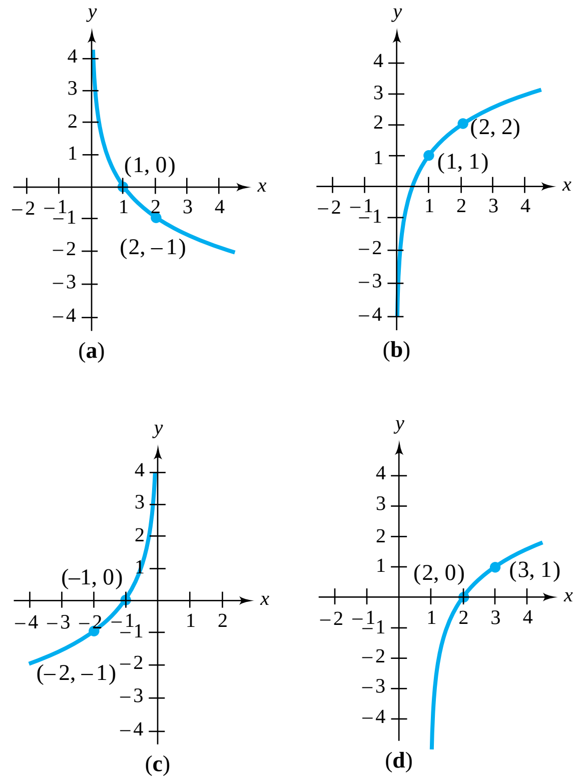
65. If $f(x) = 2^{0.5x}$ and $g(x) = Int(x)$ solve $(f \circ g)(x) = 8$. *(Hint:* Use dot mode in a decimal window.)

66. Repeat Exercise 65 for $(f \circ g)(x) = 4$.

Exercises 67–70 ***Graph to Formula*** *Use the graph of $y = \log_2 x$ to help match the function with one of the graphs a, b, c, or d. Think in terms of translations and reflections.*

67. $f(x) = 1 + \log_2 x$ **68.** $f(x) = -\log_2 x$

69. $f(x) = \log_2(x - 1)$ **70.** $f(x) = -\log_2(-x)$



4.3 PROPERTIES OF LOGARITHMIC FUNCTIONS

> If one remembers . . . the useful concepts . . . [as well as] the countless misconceptions and errors that rigorous mathematical development avoids without touching, then mathematics begins to resemble not as much a nerve as the thread that Ariadne used to guide her lover Theseus out of the Labryinth in which he slew the dreaded Minotaur.
>
> *Hans C. von Baeyer*

As defined in the preceding section, logarithms are exponents, so we would expect logarithms to have properties analogous to those of exponents. The list of some of the most important properties of both logarithms and exponents emphasizes the parallels between them.

Properties of logarithms and exponents

Logarithms	*Exponents*
L1. $\log_b(uv) = \log_b u + \log_b v$	E1. $b^u b^v = b^{u+v}$
L2. $\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$	E2. $\frac{b^u}{b^v} = b^{u-v}$
L3. $\log_b(u^p) = p(\log_b u)$	E3. $(b^u)^p = b^{up}$
L4. $\log_b 1 = 0$ and $\log_b b = 1$	E4. $b^0 = 1$ and $b^1 = b$

Because logarithmic functions are defined only for positive numbers, L1, L2, and L3 are valid **only when both *u* and *v* are positive.**

Use the following equivalence statement to change a logarithmic equation to an exponential equation, and vice versa:

$$\boldsymbol{y = \log_b x} \quad \textbf{is equivalent to} \quad \boldsymbol{b^y = x}. \tag{1}$$

I think it was during that semester in Berkeley, when I was not quite fifteen, that I really switched into being serious about mathematics. As soon as I saw what geometry was about, it was immediately clear to me how the whole thing worked—I mean absolutely clear. I could visualize the figures rather well, and I didn't have any problem with understanding what proofs were supposed to be.
Andrew M. Gleason

We outline a proof of logarithm property L1; proofs for properties L2 and L3 are similar and are left as exercises (see Exercises 43 and 44). In words, property L1 states that the logarithm of a product is the sum of the logarithms.

$$\log_b u = s \qquad \text{and} \qquad \log_b v = t.$$

In terms of exponents

$$u = b^s \qquad \text{and} \qquad v = b^t.$$

Since the equation in property L1 involves uv, multiply the two exponential equations and apply exponent property E1 to get

$$uv = b^s b^t = b^{s+t}.$$

Returning to logarithmic form,

$$\log_b(uv) = s + t.$$

Replacing s and t by $\log_b u$ and $\log_b v$,

$$\log_b(uv) = \log_b u + \log_b v$$

The first three logarithm properties involve logarithms of products, quotients, and powers. We do not give similar formulas for sums and differences because there are no simple ways to express $\log_b(u + v)$ and $\log_b(u - v)$ in terms of $\log_b u$ and $\log_b v$. Similarly, for exponents, we have for instance, $b^2 \cdot b^3 = b^5$, but there is no simpler expression for $b^2 + b^3$.

►EXAMPLE 1 ***Using logarithm properties*** Use properties L1 through L4 to evaluate

(a) $\log_3 81$ **(b)** $\log_{10} 0.001$ **(c)** $\log_3(\frac{1}{\sqrt{3}})$.

Strategy: Rewrite each argument as a power of the base: $81 = 3^4$, $0.001 = 10^{-3}$, $\frac{1}{\sqrt{3}} = 3^{-1/2}$, then use logarithm properties as appropriate.

Solution

We indicate under the equals sign the property that gives the equality. Follow the strategy.

(a) $\log_3 81 = \log_3 3^4 \underset{L_3}{=} 4(\log_3 3) \underset{L_4}{=} 4(1) = 4$, hence $\log_3 81 = 4$.

(b) $\log_{10} 0.001 = \log_{10} 10^{-3} \underset{L_3}{=} (-3)\log_{10} 10 \underset{L_4}{=} (-3)(1) = -3$,
hence $\log_{10} 0.001 = -3$.

(c) $\log_3\left(\frac{1}{\sqrt{3}}\right) \underset{L_2}{=} \log_3 1 - \log_3 \sqrt{3} \underset{L_4}{=} 0 - \log_3 3^{1/2} \underset{L_3}{=} -\left(\frac{1}{2}\right)\log_3 3 \underset{L_4}{=} -\frac{1}{2}$,
hence $\log_3\left(\frac{1}{\sqrt{3}}\right) = -\frac{1}{2}$ ◀

▶EXAMPLE 2 ***Using logarithm properties*** Simplify.

(a) $\log_5 10 - \log_5 2$ **(b)** $\log_7 3 + 4(\log_7 2)$

Solution

(a) $\log_5 10 - \log_5 2 \underset{L_2}{=} \log_5\left(\frac{10}{2}\right) = \log_5 5 \underset{L_4}{=} 1$

(b) $\log_7 3 + 4(\log_7 2) \underset{L_3}{=} \log_7 3 + \log_7 2^4 = \log_7 3 + \log_7 16$
$\underset{L_1}{=} \log_7(3 \cdot 16) = \log_7 48$ ◀

It is important to learn to use logarithm properties L1 through L3 going from right to left, as well as left to right. For instance, in Example 2b we used property L3 to write

$$4(\log_7 2) \underset{L_3}{=} \log_7 2^4,$$

and then we used property L1 to combine logarithms and get

$$\log_7 3 + \log_7 16 \underset{L_1}{=} \log_7(3 \cdot 16) = \log_7 48.$$

We call $\log_7 48$ a simplified form of $\log_7 3 + 4(\log_7 2)$. In a similar manner, using L1 and L3 gives

$$\log_7 48 = \log_7(3 \cdot 2^4) \underset{L_1}{=} \log_7 3 + \log_7 2^4 \underset{L_3}{=} \log_7 3 + 4(\log_7 2).$$

Thus, $\log_7 48$ can be written as a sum of logarithms, $\log_7 3 + 4(\log_7 2)$.

▶EXAMPLE 3 ***Combining logarithms*** Simplify.

$$\log_b x + 4\log_b(x-1) - \log_b 5.$$

Solution

$$\begin{aligned}\log_b x + 4\log_b(x-1) - \log_b 5 &\underset{L_3}{=} \log_b x + \log_b(x-1)^4 - \log_b 5\\ &\underset{L_1}{=} \log_b[x(x-1)^4] - \log_b 5\\ &\underset{L_2}{=} \log_b \frac{x(x-1)^4}{5}\end{aligned}$$

Hence, $\log_b x + 4\log_b(x-1) - \log_b 5 = \log_b \dfrac{x(x-1)^4}{5}$ ◀

▶EXAMPLE 4 ***Numerical approximations*** In the next section we will show that four-place decimal approximations to $\log_5 3$ and $\log_5 6$ are:

$$\log_5 3 \approx 0.6826 \qquad \log_5 6 \approx 1.1133.$$

Use these values along with logarithm properties L1 through L4 to get three-decimal place approximations for **(a)** $\log_5 2$ and **(b)** $\log_5 18$.

Solution

Follow the strategy.

Strategy Write each of 2 and 18 as a product, quotient, power, etc. in terms of the numbers 3 and 6, whose logarithms we have: $2 = \frac{6}{3}$, $18 = 3 \cdot 6$, then use the properties of logarithms.

(a) $\log_5 2 = \log_5(\frac{6}{3}) = \log_5 6 - \log_5 3$
$$\approx 1.1133 - 0.6826$$
$$= 0.4307.$$

Therefore, $\log_5 2 \approx 0.431$ to three decimal places.

(b) $\log_5 18 = \log_5(3 \cdot 6) = \log_5 3 + \log_5 6$
$$\approx 0.6826 + 1.1133$$
$$= 1.7959.$$

Hence, $\log_5 18 \approx 1.796$. ◀

▶EXAMPLE 5 *Using properties to solve equations*

Solve **(a)** $\log_4 x - \log_4(x - 1) = \frac{1}{2}$ **(b)** $\log_4 x - \log_4(x + 1) = \frac{1}{2}$.

Solution

Strategy Combine the terms on the left by using property L2 to get a single logarithm equal to a constant, then express the result in exponential form. Since property L2 applies only to positive numbers, check all results in the original equation.

(a) Follow the strategy,

$$\log_4 x - \log_4(x - 1) \underset{L_2}{=} \log_4\left(\frac{x}{x-1}\right)$$

so the given equation can be written as

$$\log_4\left(\frac{x}{x-1}\right) = \frac{1}{2}, \qquad \frac{x}{x-1} = 4^{1/2}, \qquad \text{or} \qquad \frac{x}{x-1} = 2.$$

Solving for x, we find that $x = 2$. Since $\log_4(2)$ and $\log_4(2 - 1)$ are both defined, we know that 2 belongs to the replacement set for the original equation and therefore 2 is the desired solution.

(b) As in part **(a),** the given equation can be written as

$$\log_4\left(\frac{x}{x+1}\right) = \frac{1}{2}, \qquad \frac{x}{x+1} = 4^{1/2}, \qquad \text{or} \qquad \frac{x}{x+1} = 2.$$

In this case, when we solve for x we find $x = -2$. However, if we replace x with -2, the left side involves $\log_4(-2)$ and $\log_4(-2 + 1)$, neither of which is defined. Since -2 is not in the replacement set for the original equation, it cannot be the solution. The given equation has no solution. ◀

Example 5 illustrates an important point. Properties L1, L2, and L3 are valid for only positive values of all arguments; logarithmic functions are defined for only positive arguments. We could check the domains at each step, but it is good enough to check the final result in the original equation.

▶EXAMPLE 6 *A logarithmic equation*

Solve the equation

$$2 \log_9 x + 2 \log_9(x + 2) = 1.$$

Solution

Divide through by 2 and write the left side in simpler form:

$$\log_9 x + \log_9 (x + 2) = \frac{1}{2}, \qquad \log_9[x(x + 2)] = \frac{1}{2}.$$

In exponential form,

$$x(x + 2) = 9^{1/2}, \qquad x^2 + 2x = 3, \qquad \text{or} \qquad x^2 + 2x - 3 = 0.$$

Solutions to the quadratic equation are 1 and -3. Since the domain of the original equation is the set of positive numbers, 1 is a solution but -3 is not. ◀

Strategy: There is no formula to simplify the logarithm of a sum, rewrite $8^x + 8^x$ as a power of 2 (the base) and then simplify.

▶EXAMPLE 7 ***Logarithms and sums*** Find the solution set for

(a) $\log_2(8^x + 8^x) = x - 1$ **(b)** $\log_2(8^x + 8^x) = 3x + 1$
(c) $\log_2(8^x + 8^x) = 3x$.

Solution

To simplify all three equations, begin with the expression $8^x + 8^x$

$$8^x + 8^x = 2 \cdot 8^x = 2 \cdot (2^3)^x = 2 \cdot 2^{3x} = 2^{3x+1}$$

from which $\log_2(8^x + 8^x) = \log_2(2^{3x+1}) = 3x + 1$. Since $8^x + 8^x$ is positive for every x in R, $\log_2(8^x + 8^x) = 3x + 1$ for every real number. In each case, replace $\log_2(8^x + 8^x)$ by $3x + 1$ and solve the resulting equation.

(a) $3x + 1 = x - 1$, so $x = 1$; the solution set is $\{1\}$.
(b) $3x + 1 = 3x + 1$, which is an identity, so the solution set is R.
(c) $3x + 1 = 3x$, or $0 \cdot x = 1$. The solution set is the empty set. ◀

Strategy: The domain of the $\log_3$ function is the set of positive real numbers. For f this requires $x^2 - 5x + 6 > 0$, and for g, both $x - 2 > 0$ *and* $x - 3 > 0$.

▶EXAMPLE 8 ***Domains of logarithmic functions*** If $f(x) = \log_3(x^2 - 5x + 6)$ and $g(x) = \log_3(x - 2) + \log_3(x - 3)$, then find the domain of each function. Are functions f and g equal? Explain.

Solution

Follow the strategy. To find the domain of function f, solve the inequality

$$x^2 - 5x + 6 > 0, \qquad \text{or} \qquad (x - 2)(x - 3) > 0.$$

The solution set is $\{x \mid x < 2 \text{ or } x > 3\}$, so the domain of f is $(-\infty, 2) \cup (3, \infty)$.

For the function g, the strategy emphasizes that both $x > 2$ *and* $x > 3$. The solution set is $\{x \mid x > 3\}$, so the domain of g is $(3, \infty)$.

Finally, since functions f and g have different domains, they cannot be equal. However, $f(x) = g(x)$ for all $x > 3$. ◀

▶EXAMPLE 9 ***Finding an inverse function*** If $f(x) = \ln(x - 1) + 2$,

(a) Find a formula for f^{-1}. Graph f and f^{-1} on the same screen.
(b) Describe the graphs in terms of transformations of the natural exponential and logarithmic functions.

Solution

(a) Using the algorithm from Section 2.7, we write $y = f(x)$, interchange variables, and solve for y.

$$y = \ln(x - 1) + 2$$
$$x = \ln(y - 1) + 2$$
$$\ln(y - 1) = x - 2$$

By Equation (1), the last equation is equivalent to

$$e^{x-2} = y - 1, \quad \text{or} \quad y = e^{x-2} + 1.$$

Thus the inverse function is given by

$$f^{-1}(x) = e^{x-2} + 1.$$

The graphs of f and g are shown in Figure 20.

(b) The graph of f is the graph of the natural logarithmic function shifted 1 unit right and 2 units up. The inverse of the natural logarithmic function is the natural exponential function, so the graph of f^{-1} is the graph of the natural exponential function shifted 2 units right and 1 unit up. ◀

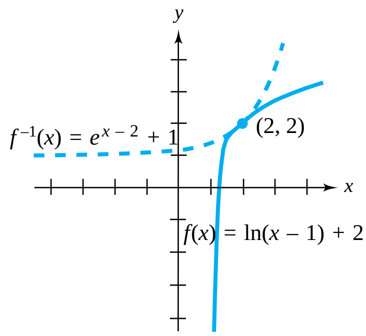
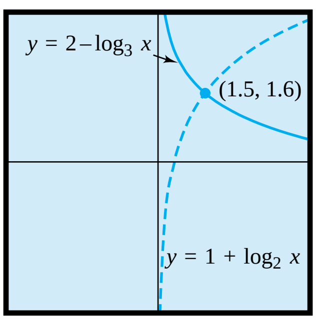


FIGURE 20

Strategy: At this point, we cannot graph $y = \log_b x$ directly, but we can graph its inverse. Find formulas for f^{-1} and g^{-1} and then graph their inverses parametrically, because f is the inverse of f^{-1}.



[–5, 5] by [–3.5, 3.5]

FIGURE 21

▶EXAMPLE 10 ***Finding an intersection of calculator graphs*** Find the coordinates (one decimal place) of the point of intersection of the graphs of $f(x) = 1 + \log_2 x$ and $g(x) = 2 - \log_3 x$.

Solution

Follow the strategy. We use the algorithm to find formulas for f^{-1} and g^{-1}, interchanging x and y and then solving for y:

f: $y = 1 + \log_2 x$, $\log_2 y = x - 1$, $y = 2^{x-1}$, and so $f^{-1}(x) = 2^{x-1}$.

g: $y = 2 - \log_3 x$, $\log_3 y = 2 - x$, $y = 3^{2-x}$, and so $g^{-1}(x) = 3^{2-x}$.

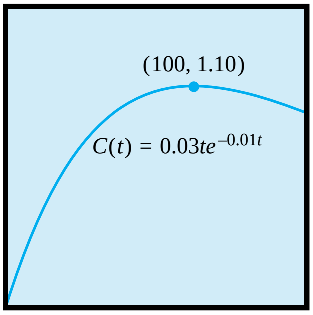
We want to use parametric equations to graph the *inverses* of f^{-1} and of g^{-1}, thus giving us the graphs of f and g. In parametric mode, enter X1 = 2^(T − 1), Y1 = T for f, X2 = 3^(2 − T), Y2 = T for g.

A calculator graph is shown in Figure 21. Tracing and zooming as necessary, we find that the intersection point is approximately (1.5, 1.6). ◀

▶EXAMPLE 11 ***Answering medical questions*** The concentration $C(t)$ of a drug in the bloodstream (in mg/cm^3) is given by $C(t) = 0.03te^{-0.01t}$, where t is the number of minutes after injection. **(a)** In how many minutes after injection will the concentration reach 0.5 mg/cm^3? **(b)** At what time will the concentration be the greatest?

Solution

(a) We want a graph of Y = .03Xe^(−.01X), but we must find a reasonable window first. If we evaluate the function at several values of x, we can get a feeling for how large the y-values are and when the concentration seems to be increasing and decreasing. Some typical values are listed in the table:



[0, 160] by [0, 1.5]

FIGURE 22

x	5	10	20	40	80	120	160
y	0.14	0.27	0.49	0.80	1.08	1.08	0.97

From the table, it looks as if a $[0, 160] \times [0, 1.5]$ window should show us the information we need. A calculator graph is shown in Figure 22. We trace to find when y is 0.5 and find that x is between 20 and 21, which means that the concentration will reach 0.5 mg/cm^3 in just over 20 minutes.

(b) We could zoom in to locate the highest point on the curve more precisely, but the coordinates we read as we trace near the high point indicate that the concentration is not changing very rapidly there. The high point appears to be near (100, 1.10), so we conclude that after 100 minutes, the maximum concentration will be about 1.10 mg/cm^3. ◀

EXERCISES 4.3

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. $\log_5(\sqrt{2} + \sqrt{3}) = \frac{1}{2}(\log_5 2 + \log_5 3)$.
2. The graph of $y = -2^x$ and $y = -\log_2(x + 2)$ intersect at $(0, -1)$.
3. The graph of $y = \log_4(4x)$ is the same as the graph of $y = 1 + \log_4 x$.
4. The graph of $y = \log_2(4x)$ can be drawn by translating the graph of $y = \log_2 x$ up 2 units.
5. For every real number x, $\log_2(2x) = 1 + \log_2 x$.

Exercises 6–10 Fill in the blank so that the resulting statement will be true.

6. The domain of $f(x) = \log_2(x + 2) + \log_2(1 - x)$ is ______.
7. The sum of all the prime numbers between $\log_2 0.5$ and $\log_2 256$ is ______.
8. If $f(x) = \log_4 x$ then $f^{-1}(x) =$ ______.
9. The graphs of $y = -3^x$ and $y = \log_3 x$ intersect in Quadrant ______.
10. The integers between $\log_2 1$ and $\log_2 128$ are ______.

Develop Mastery

Exercises 1–2 ***Logarithm to Exponent*** *(a) Express the equation in exponential form.* ***(b)*** *Use the properties of logarithms to find a simpler expression for k.*

1. **(a)** $k = \log_3(3\sqrt{3})$ **(b)** $k = \log_5(5\sqrt[3]{25})$
2. **(a)** $k = \log_e(e^2\sqrt{e})$ **(b)** $k = \log_e(\frac{\sqrt{e}}{e^3})$

Exercises 3–5 ***Combining Logarithms*** *Simplify. See Example 2.*

3. **(a)** $\log_3 6 - \log_3 2$ **(b)** $\log_7 2 + 3 \log_7 3$
4. **(a)** $2 \log_3 2 + \frac{1}{2} \log_3 4$ **(b)** $3 \log_5 2 - \frac{1}{4} \log_5 16$
5. **(a)** $\log_{10} 50 - 2 \log_{10} 5$ **(b)** $\frac{2}{3} \log_2 27 - 3 \log_2 4$

Exercises 6–8 ***Using Logarithm Properties*** *Use properties of logarithms to write the expression as a sum or difference.*

6. **(a)** $\log_3(2x^3)$ **(b)** $\log_4\left(\frac{16}{x^4}\right)$
7. **(a)** $\log_5(x\sqrt{x^2 + 4})$ **(b)** $\log_5(25x\sqrt{x^2 + 1})$
8. **(a)** $\log_2\left(\frac{8x^2}{\sqrt{x^2 + 1}}\right)$ **(b)** $\log_3[9x(x + 1)]$

Exercises 9–10 ***Using Logarithm Properties*** *Simplify.*

9. **(a)** $2 \log_3 x - \log_3(x + 2)$
   (b) $\log_5 3 + \log_5 x - \log_5 \sqrt{x}$
10. **(a)** $\frac{1}{2}\log_3 x^2 - 2 \log_3 x + \log_3 4$
    (b) $\frac{3}{2}\log_5 x^2 + \log_5 3 - 2 \log_5 \sqrt{x}$

Exercises 11–14 If $\log_{10} 2 = u$ and $\log_{10} 3 = v$, express in terms of u and v.

11. **(a)** $\log_{10} 5$, **(b)** $\log_{10}(\frac{1}{5})$
12. **(a)** $\log_{10} 30$, **(b)** $\log_{10} 1.5$
13. **(a)** $\log_{10} \sqrt[3]{18}$, **(b)** $\log_{10} \sqrt{24}$
14. **(a)** $\log_{10} \frac{16}{27}$, **(b)** $\log_{10} 80$

Exercises 15–18 *If $\log_b 5 = u$ and $\log_b 45 = v$, express in terms of u and v.*

15. **(a)** $\log_b 9$, **(b)** $\log_b 3$

16. **(a)** $\log_b 15$, **(b)** $\log_b 1.8$

17. **(a)** $\log_b 25$, **(b)** $\log_b 135$

18. **(a)** $\log_b \frac{1}{3}$, **(b)** $\log_b \sqrt{1.8}$

Exercises 19–26 ***Roots of Logarithmic Equations*** *Solve. Check to see that your solutions are in the domain of the original equation. See Examples 5 and 6.*

19. $\log_3(x + 1) - \log_3 x = 1$

20. $2 \log_2 x - \log_2 32 = 1$

21. $\log_4(2x + 3) - \log_4 x = 2$

22. $\log_3(2x + 3) - \log_3 x = \log_3 5$

23. $\log_2 x + \log_2(x + 2) = 3$

24. $\log_4(3x - 2) - \log_4(2x) = \log_4 3$

25. $\log_3(x + 8) + \log_3 x = 2$

26. $\log_5(4x) - \log_5 (2x - 1) = 0$

Exercises 27–28 ***Domain*** *Determine the domain of the function.*

27. **(a)** $f(x) = \log_3(x - 3) + 5$
(b) $f(x) = \log_5(x^2 - 2x)$

28. **(a)** $g(x) = \log_7(4x - x^2)$
(b) $g(x) = \log_4(x + 3) + \log_4(2 - x)$

29. For what values of x is $\log_5(x^2 - 3x - 4)$ equal to $\log_5(x - 4) + \log_5(x + 1)$?

30. **(a)** For what values of x is $\log_3 x^2$ equal to $2 \log_3 x$?
(b) For what values of x is $\log_3 x^2$ equal to $2 \log_3(-x)$?
(c) For what values of x is $\log_3 x^2$ equal to $2 \log_3 |x|$?

Exercises 31–34 ***Points of Intersection*** *Draw graphs of f and g on the same screen. Find the coordinates (2 decimal places) of the point of intersection.*

31. $f(x) = 2 \ln x$, $g(x) = 2 - e^{-x}$

32. $f(x) = 2 - \ln x$, $g(x) = e^{0.5x}$

33. $f(x) = 2 + \ln x$, $g(x) = 3 - \ln x$

34. $f(x) = 2 - \ln x$, $g(x) = 3 + \ln x$

Exercises 35–36 ***Intercept Points*** *Find the coordinates (2 decimal places) of the x-intercept points of the graph of f algebraically. Check graphically.*

35. $f(x) = 0.5 - \ln(x - 1)$

36. $f(x) = 2 \ln x - \ln(4 - x)$

Exercises 37–40 ***Graphs and Zeros*** *Draw graphs of f and g separately. Use graphs to find the zero(s) of **(a)** f the zero(s) of **(a)** f **(b)** g. Explain why f and g do not have the same zeros.*

37. $f(x) = \ln(x - 2) + \ln(x - 4)$,
$g(x) = \ln((x - 2)(x - 4))$

38. $f(x) = \ln(x - 3) + \ln(x - 1)$,
$g(x) = \ln((x - 3)(x - 1))$

39. $f(x) = \ln(x^2 - 3) - \ln(2x - 1)$,
$g(x) = \ln \dfrac{x^2 - 3}{2x - 1}$

40. $f(x) = \ln(x^2 - 5) - \ln(2x - 3)$,
$g(x) = \ln \dfrac{x^2 - 5}{2x - 3}$

Exercises 41–42 ***Compare Graphs*** *Draw graphs of f and g separately. **(a)** Explain why the graphs are not identical. **(b)** For what values of x do the graphs coincide?*

41. $f(x) = \ln x^2$, $g(x) = 2 \ln x$

42. $f(x) = \ln((2x - 3)(x - 3))$,
$g(x) = \ln(2x - 3) + \ln(x - 3)$

43. Prove the validity of logarithm property L2.

44. Prove the validity of logarithm property L3.

45. If $a = 8$ and $b = 16$, show that $\log_2(ab)$ is not equal to $(\log_2 a)(\log_2 b)$.

46. If $a = 16$ and $b = 8$, show that $\log_2(\frac{a}{b})$ is not equal to $\dfrac{\log_2 a}{\log_2 b}$.

47. If $c = 4$ and $n = 3$, show that $\log_2(c^n)$ is not equal to $(\log_2 c)^n$.

Exercises 48–50 *Find the solution set. See Example 7.*

48. **(a)** $\log_2(2^x + 2^x) = x + 1$
(b) $\log_3(3^x + 3^x) = 1$

49. **(a)** $\log_3(3^x + 3^x) = x$
(b) $\log_3(3^x + 3^x + 3^x) = x + 1$

50. **(a)** $\log_4(4^x + 4^x) = 2x$
(b) $\log_2(4^x + 4^x) = x$

51. Show that $\log_3(\sqrt{3} + \sqrt{2}) = -\log_3(\sqrt{3} - \sqrt{2})$.

52. Show that $\log_5(\sqrt{6} + \sqrt{5}) = -\log_5(\sqrt{6} - \sqrt{5})$.

53. Show that for any positive number k,
$\log_b(\sqrt{k + 1} + \sqrt{k}) = -\log_b(\sqrt{k + 1} - \sqrt{k})$.

Exercises 54–55 *Given that function f has an inverse, find an equation that describes f^{-1}. What is the domain of f? (Hint: Use the algorithm in Section 2.7.)*

54. $f(x) = \log_2(\sqrt{x^2 + 1} + x)$

55. $f(x) = \log_3(\sqrt{x^2 + 1} - x)$

Exercises 56–57 ***Maximum Value*** ***(a)*** *For what value(s) of x is y equal to 36 (one decimal place)?* ***(b)*** *What value of x will give a maximum value of y? What is the maximum value? (Hint: The window* $[0, 300] \times [0, 60]$ *should give you a start.)*

56. $y = 4 + xe^{-0.01x}$

57. $y = 6 + x \cdot 3^{-0.01x}$

58. ***Maximum Concentration*** The concentration C of a drug in the bloodstream at t minutes after injection is given by

$$C = 0.036te^{-0.015t} \text{ mg/cm}^3.$$

(a) In how many minutes will the concentration reach 0.6 mg/cm^3?

(b) How many minutes after injection will the concentration be the greatest? What is the maximum concentration? See Example 11.

59. ***True or False*** Draw graphs to support your answer. Assume that L is a line.

(a) If L and the graph of $y = \ln x$ intersect at two points, then the slope of L must be positive.

(b) If L and the graph of $y = e^{-x}$ intersect at two points, then the slope of L must be positive.

60. ***Explore*** For what *integer* values (positive and negative) of c will the graphs of $y = 1 + \frac{x}{c}$ and $y = \ln x$ intersect at **(a)** exactly one point? **(b)** two points?

61. ***Explore*** For what *integer* values (positive and negative) of c will the graphs of $y = cx - 3$ and $y = 2^x + 6$ intersect at **(a)** exactly one point? **(b)** two points?

62. ***Explore*** What is the smallest prime number c for which the graph of $y = cx - 5$ will intersect the graph of $y = 3^x + 5$ at exactly two points?

63. ***Your Choice*** Give a formula for a linear function f (with nonzero slope) that satisfies the specified conditions.

(a) The graphs of f and $y = \ln x$ intersect in Quadrant I and Quadrant IV.

(b) The graphs of f and $y = e^x$ intersect in Quadrant I and Quadrant II.

64. ***Your Choice*** From the family of functions $f(x) = c \ln(kx)$, where c and k are nonzero constants, select c and k so that f satisfies the specified condition.

(a) The graph of f intersects the graph of $y = 2x - 4$ at two points.

(b) The graphs of f and of $y = x + 4$ intersect in QII.

Exercises 65–66 ***Is It a Function?*** *Explain what you observe when you graph the equation.*

65. $y = \ln(-x^2 + 2x - 3)$

66. $y = \ln(x - 3) + \ln(2 - x)$

4.4 COMPUTATIONS WITH LOGARITHMIC AND EXPONENTIAL FUNCTIONS

> Galileo's observation that all bodies accelerate equally in the Earth's gravity is counterintuitive precisely because it is usually wrong. Everybody knows that a lump of coal falls faster than a feather. Galileo's genius was in spotting that the differences which occur in reality are an incidental complication caused by air resistance, and are irrelevant to the properties of gravity as such.
>
> *P. W. C. Davies*

Because logarithms are exponents, evaluation in exact form is possible only in special cases. We can, for example, evaluate $\log_3(9\sqrt{3})$ in exact form because $9\sqrt{3}$ is a power of the base 3:

$$9\sqrt{3} = 3^{5/2}, \text{ so } \log_3 9\sqrt{3} = \frac{5}{2}.$$

More generally, we need assistance to approximate logarithms. This section covers the use of calculators to evaluate logarithmic and exponential functions to any base. All scientific calculators are programmed to evaluate the natural exponential function, $f(x) = e^x$, and its inverse, the natural logarithm function $f^{-}(x) = \ln x$. As we will see, these functions are sufficient to handle calculator evaluation for exponential and logarithmic functions with any base.

> *I was in the ninth grade of Powell Junior High School in Washington, D.C. I was doing very poorly in my first course in algebra. To be precise, I was flunking. Later on, after recovering from my poor start in algebra, I began to get top marks. I was good in math and science in high school.*
>
> George B. Dantzig

Change of Base and Evaluating Logarithms in Other Bases

As observed in the previous section graphing calculators have an exponentiation key, $\boxed{\wedge}$ or $\boxed{y^x}$, that allows us to evaluate exponential expressions or to graph exponential functions for any given base. In contrast, there is no built-in logarithm key that directly evaluates logarithms for any bases except e ($\boxed{\text{LN}}$) and 10 ($\boxed{\log}$). Fortunately, there is a simple change-of-base formula that allows us to evaluate any logarithmic function by means of the natural logarithm function, $\ln x$. To evaluate $\log_3 4$, we can express the relationship $y = \log_3 4$ in exponential form, (by equivalence Equation (1)), then apply the natural logarithm function to both sides and solve for y. The same steps work for any base b, as follows:

$$
\begin{array}{lll}
y = \log_3 4 & y = \log_b c & \\
3^y = 4 & b^y = c & \text{By EQ1} \\
\ln 3^y = \ln 4 & \ln b^y = \ln c & \text{Applying ln function} \\
y \ln 3 = \ln 4 & y \ln b = \ln c & \text{By L3} \\
y = \dfrac{\ln 4}{\ln 3} & y = \dfrac{\ln c}{\ln b} & \text{Solving for } y
\end{array}
$$

Thus we have $\log_3 4 = \dfrac{\ln 4}{\ln 3} \approx 1.2619$, and we have a general formula for evaluating any logarithmic function.

Change-of-base formula

For any positive real numbers c and b where b is not 1, $\log_b c = \dfrac{\ln c}{\ln b}$.

The change-of-base formula allows us to evaluate logarithmic functions for any base, including base 10, so that $\boxed{\log}$ is not really necessary.

►EXAMPLE 1 ***Evaluating logarithms*** Find an approximation rounded off to four decimal places.

(a) $\log_5 0.43$ **(b)** $\log_8(1 + \sqrt{3})$ **(c)** $\log 79.442$

Solution

Use the change-of-base formula.

(a) $\log_5 0.43 = \dfrac{\ln 0.43}{\ln 5} \approx -0.5244.$

(b) $\log_8(1 + \sqrt{3}) = \dfrac{\ln(1 + \sqrt{3})}{\ln 8} \approx 0.4833.$

(c) With no base shown, $\log 79.442$ refers to the common logarithm (base 10). Use $\boxed{\log}$ directly if your calculator has such a key, or use the change-of-base formula.

$$\log 79.442 = \frac{\ln 79.442}{\ln 10} \approx 1.9001.$$

Check each of the above computations using your calculator. ◄

There are many occasions when we have functions given by two different formulas and we want to determine whether the functions are identical. A graphing calculator can be very helpful in this regard, and there are at least three convenient methods.

TECHNOLOGY TIP ◆ *Graphing identical functions*

We want to determine whether two functions, f and g, are identical.

- *Method 1* Plot the graphs of $y = f(x)$ and $y = g(x)$ on the same screen. The advantage and disadvantage of this method is that you see only one graph. Differences in domain may not be apparent. To check, trace along the curve, using the up or down arrows to jump from one curve to the other, and watch the y-coordinates.
- *Method 2* Translate one graph up or down by some constant, say 1 or 0.5. That is, plot the graphs of $y = f(x)$ and $y = g(x) + .5$ on the same screen. If the functions are identical, the graphs will differ by the same amount all the way across the screen.
- *Method 3* Shift the graph of $f - g$ so that the *difference* is visible on the screen. Plotting $y = f(x) - g(x) + 1$ will yield the horizontal line $y = 1$, which can also be checked by tracing (or replace 1 by any other constant).

►EXAMPLE 2 ***Verifying the change of base formula*** Use graphs to support the claim that the functions $f(x) = \log x$ and $g(x) = \ln x/\ln 10$ are identical.

Solution

Following the suggestions in the Technology Tip above, we enter `Y1 = LOG X` and `Y2 = LN X/LN 10`. We can graph both `Y1` and `Y2` on the same screen and see a single logarithm function, or we can graph `Y1` and `Y2 + 1` (for *Method 2*), or `Y3 = Y1 − Y2 + 1` (for *Method 3*). By whichever method we choose, the calculator shows that, at least to calculator accuracy, the functions are identical. ◄

Using Inverse Function Identities

Restating the inverse function identities in terms of the natural exponential function and the natural logarithmic function is useful as a reminder of relations that can simplify much of our work.

If f is the natural exponential function, $f(x) = e^x$, then $f^{-1}(x) = \ln x$. Since $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f, we have two identities.

Inverse function identities

$e^{\ln x} = x$	for all *positive* numbers x.	**(1)**
$\ln e^x = x$	for *all* real numbers x.	**(2)**

HISTORICAL NOTE INVENTION OF LOGARITHMS

As the need for more accuracy in trigonometric computations grew (see the Historical Note, Trigonometric Tables in Section 5.3), so did the need for better ways to do the arithmetic. Logarithms have been called "the most universally useful mathematical discovery of the seventeenth century." They significantly reduced the time required to perform computations and may have been as important for the exploration of the globe as any improvement in marine technology in two hundred years.

One basic idea motivated the development of logarithms: to multiply powers of the same base, simply add exponents. For example, to multiply 16 by 64, use tables to identify equivalent numbers 2^4 and 2^6, from which

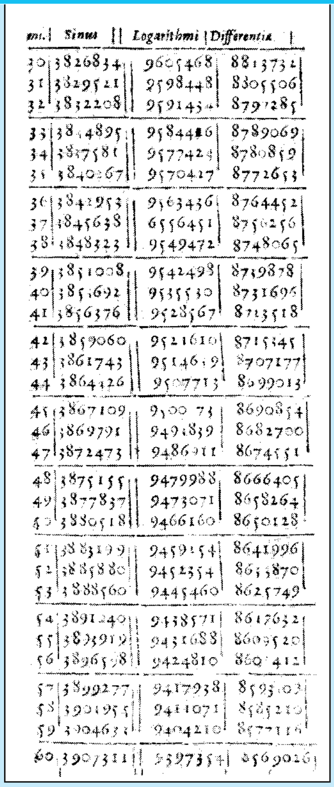
$$\begin{aligned} 16 \cdot 64 &= 2^4 \cdot 2^6 = 2^{4+6} \\ &= 2^{10} = 1024. \end{aligned}$$

To be useful, of course, tables must identify the exponents of all the numbers we might need to multiply.

John Napier (1550–1617) spent twenty years compiling tables of exponents (called *logarithms* or *ratio numbers*). He started with a large number for accuracy ($N = 10{,}000{,}000$) and calculated a hundred terms in a geometric sequence, successively subtracting $\frac{1}{10{,}000{,}000}$ of each number from the one before, and rounding each to 14 digits.

This produced one table of exponents. If he had simply continued with this sequence, it would have required years of calculation just to get from 10 million to 5 million, producing an unusable table with nearly 7 million entries. Napier's genius lay in his construction of other tables to allow interpolation between numbers. Rather than millions of entries, his second table had only 50 entries, and the third had fewer than 1500. A user would locate a pair of exponents from the first two tables and then use the third table to compute the logarithm.

After his logarithms of numbers, Napier produced a table to give seven-place logarithms of sines of angles for every minute from 0° to 90°. Kepler credited Napier's tables for making possible the incredible calculations required to analyze the motion of the planets about the sun.



mi.	Sinus	Logarithmi	Differentia
30	3826834	9605468	8813732
31	3829521	9598448	8805506
32	3832208	9591434	8797285
33	3834895	9584426	8789069
34	3837581	9577424	8780859
35	3840267	9570417	8772653
36	3842953	9563436	8764452
37	3845638	6556451	8756256
38	3848323	9549472	8748065
39	3851008	9542498	8739878
40	3853692	9535530	8731695
41	3856376	9528567	8723518
42	3859060	9521610	8715345
43	3861743	9514659	8707177
44	3864426	9507713	8699013
45	3867109	9500773	8690854
46	3869791	9493839	8682700
47	3872473	9486911	8674551
48	3875155	9479988	8666405
49	3877837	9473071	8658264
50	3880518	9466160	8650128
51	3883199	9459254	8641996
52	3885880	9452354	8633870
53	3888560	9445460	8625749
54	3891240	9438571	8617632
55	3893919	9431688	8609520
56	3896598	9424810	8601412
57	3899277	9417938	8593303
58	3901955	9411071	8585210
59	3904633	9404210	8577116
60	3907311	9397354	8569026

Part of a page from Napier's Logarithmic Tables.

►EXAMPLE 3 ***Using inverse function identities*** Use inverse function identities to simplify. Express the result in exact form and then give a five-decimal-place approximation.

(a) $e^{\ln\sqrt{3}}$ **(b)** $e^{-2\ln 7}$ **(c)** $\ln e^{-\sqrt{5}}$

Strategy: Rewrite each part as needed to use inverse function identities.

Solution

(a) By identity (1), $e^{\ln\sqrt{3}} = \sqrt{3} \approx 1.73205$. The exact form is $\sqrt{3}$ and 1.73205 is the desired approximation.

(b) For the exact form, first use logarithm property L3 to rewrite $-2\ln 7$ as $\ln 7^{-2}$, or $\ln(\frac{1}{49})$, then use identity (1).

$$e^{-2\ln 7} = e^{\ln(1/49)} = \frac{1}{49} \approx 0.02041$$

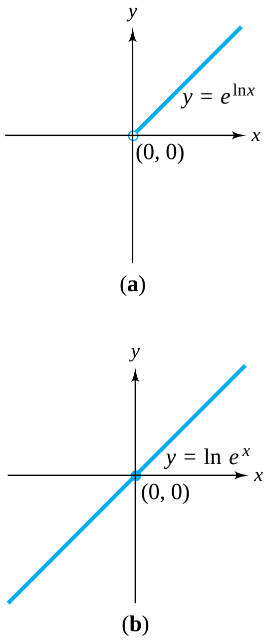


FIGURE 23

Thus $e^{-2\ln 7}$ is exactly equal to $\frac{1}{49}$ and 0.02041 is the five-decimal-place approximation.

(c) Identity (2) gives $\ln e^{-\sqrt{5}} = -\sqrt{5} \approx -2.23607$. An exact form for $\ln e^{-\sqrt{5}}$ is $-\sqrt{5}$ and -2.23607 is the desired approximation. ◀

▶EXAMPLE 4 ***Identical functions?*** Graph the functions $f(x) = e^{\ln x}$ and $g(x) = \ln e^x$ separately. Describe and explain the differences between the graphs of f, g, and the line $y = x$.

Solution

The graphs of $y = e^{\ln x}$ and $y = \ln e^x$ are shown in Figures 23a and 23b. The graph of f is the same as the first quadrant portion of the line $y = x$, but the domain of f is limited to $x > 0$. We cannot tell visually whether the origin is included, but in a decimal window, tracing verifies that f is undefined at $x = 0$.

Since e^x is always positive, $\ln e^x = x$ is defined for all real numbers x. The graph of g is identical with the graph of $y = x$.

The graphs of both f and g give graphical confirmation of the inverse function identities. ◀

Using Inverse Function Identities to Solve Equations

In Section 4.1 we solved the equation $3^{2x+1} = 3^{7/3}$ by using our intuitive understanding of exponents. To justify equating exponents, we now know that exponential and logarithmic functions are one–one; if two numbers are equal, their logarithms are equal, or in mathematical notation, if $u = v$, then $\log_b u = \log_b v$. Applying the log function to both sides, if $3^{2x+1} = 3^{7/3}$, then $\log_3(3^{2x+1}) = \log_3(3^{7/3})$, from which $2x + 1 = \frac{7}{3}$, and so $x = \frac{2}{3}$.

▶EXAMPLE 5 ***Solving exponential equations*** Solve. Express your solution in exact form and give a four-decimal-place approximation.

Strategy: Apply the natural logarithm function to both sides and simplify, using properties of logarithms.

(a) $e^{2x-1} = 4$ **(b)** $5^x = 3 \cdot 4^{1-x}$

Solution

(a) From the strategy,

$$\ln e^{2x-1} = \ln 4 \qquad \text{or} \qquad 2x - 1 = \ln 4.$$

Therefore $x = \frac{1 + \ln 4}{2} \approx 1.1931$, so $\frac{1 + \ln 4}{2}$ is the exact solution and 1.1931 is the desired approximation.

(b) In a similar fashion, $\ln 5^x = \ln(3 \cdot 4^{1-x})$. By logarithm property L3, $\ln 5^x = x \ln 5$, and by properties L1 and L3, $\ln(3 \cdot 4^{1-x}) = \ln 3 + (1 - x)\ln 4$. Therefore, the given equation is equivalent to

$$x \ln 5 = \ln 3 + (1 - x)\ln 4.$$

We now have a linear equation in x. Solve it as follows:

$$x \ln 5 = \ln 3 + \ln 4 - x \ln 4$$
$$x(\ln 5 + \ln 4) = \ln 3 + \ln 4$$
$$x = \frac{\ln 3 + \ln 4}{\ln 5 + \ln 4} \underset{\text{L1}}{=} \frac{\ln 12}{\ln 20} \approx 0.8295.$$

Therefore, the exact solution is $\frac{\ln 12}{\ln 20}$ and 0.8295 is the approximation. ◀

Notice that $\frac{\ln 12}{\ln 20}$ cannot be simplified further in the exact form solution of Example 5. In particular, $\frac{\ln 12}{\ln 20}$ is not equal to $\ln \frac{12}{20}$, since $\frac{\ln 12}{\ln 20} \approx 0.8295$ and $\ln \frac{12}{20} \approx -0.5108$.

▶EXAMPLE 6 ***Another exponential equation*** Solve the equation $e^x + e^{-x} = 4$.

Strategy: Note that the strategy of Example 5 is not helpful, since $\ln(e^x + e^{-x})$ does not simplify. Multiply through by e^x to get a quadratic equation in e^x. Use the quadratic formula to solve for e^x, and then take logarithms to solve for x.

Solution

Follow the strategy and multiply both sides by e^x.

$$e^{2x} + e^x e^{-x} = 4e^x \quad \text{or} \quad (e^x)^2 - 4e^x + 1 = 0.$$

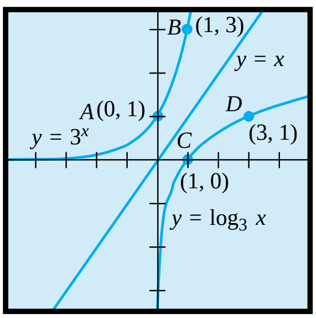
Use the quadratic formula to solve for e^x,

$$e^x = 2 + \sqrt{3} \quad \text{and} \quad e^x = 2 - \sqrt{3}.$$

Apply the ln function to both sides of each and use identity (2) to get

$$\ln e^x = \ln(2 + \sqrt{3}) \quad \text{or} \quad x = \ln(2 + \sqrt{3}) \approx 1.317$$
$$\ln e^x = \ln(2 - \sqrt{3}) \quad \text{or} \quad x = \ln(2 - \sqrt{3}) \approx -1.317.$$

The exact solutions are $\ln(2 + \sqrt{3})$ and $\ln(2 - \sqrt{3})$. Decimal approximations are 1.317 and -1.317, respectively. ◀



[−5, 5] by [−3.5, 3.5]

FIGURE 24

▶EXAMPLE 7 ***Inverse functions*** Graph the functions $f(x) = 3^x$, $g(x) = \log_3 x$, and $y = x$ in the same decimal window. Find at least two pairs of points on the graphs of f and g that are reflections of each other in the line $y = x$. What do the graphs suggest about the domain and range of f and g?

Solution

To graph $y = \log_3 x$, we use the change of base formula and enter Y = LN X/LN 3. The graphs of all three are shown in Figure 24.

From the figure, it looks as if the graph of g is the reflection of the graph of f in the line $y = x$. For partial verification, we trace along the graph of $y = 3^x$ and find points $A(0, 1)$ and $B(1, 3)$. On the graph of $y = \log_3 x$ are the points $C(1, 0)$, the reflection of point A, and $D(3, 1)$, the image of B. ◀

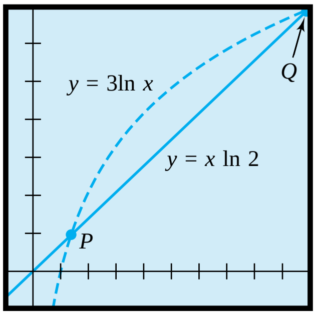
Exponential functions are said to "grow faster" than any polynomial function. We are not prepared to prove such a general statement, but it is illustrative to see how unexpected the intersections of polynomial and exponential functions may be, as suggested in the next example.

▶EXAMPLE 8 ***Hidden intersections*** Let $f(x) = 2^x$ and $g(x) = x^3$.

(a) Graph f and g in the $[-1, 6] \times [-1, 10]$ window. To the right of the visible intersection, which graph appears to be growing faster? Use an x-range of $[-1, 12]$ and keep increasing the y-range until you find another intersection.

(b) Find the "hidden intersection," (one decimal place) by setting $2^x = x^3$ and taking the natural logarithm of both sides.

(c) Discuss alternative ways to use technology to find the intersection in part **(b).**



$[-1, 10]$ by $[-1, 7]$

FIGURE 25

Solution

(a) In the specified window, there is an intersection near (1.4, 2.7), and then the cubic function rises much faster than the exponential. Trying larger and larger y-ranges, it isn't until we get to something near 1000 that the exponential function "catches up." Tracing, the intersection is close to (9.9, 980). From this point on, the exponential graph grows faster.

(b) When we take the natural logarithm of both sides of the equation $2^x = x^3$ and apply properties of logarithms, we get the equation, $x \ln 2 = 3 \ln x$, for which we have no direct way of solving. Nevertheless, graphical tools are available. Graphing `Y1 = X LN 2` and `Y2 = 3 LN X` in $[-1, 10] \times [-1, 7]$ gives a picture something like Figure 25. The intersection Q is near (9.94, 6.89). The "hidden intersection" of the original graphs is given by $x \approx 9.94$, for which $y \approx 2^{9.94} \approx 982$ and $(9.94)^3 \approx 982$.

(c) Among the many alternative approaches using a graphing calculator, we could locate graphically the root of either the equation $2^x - x^3 = 0$ or of $x \ln 2 - 3 \ln x = 0$, or, if our calculator has one, we could use a solve routine (mentioned in Example 10 of Section 4.1) for any of the above equations. Any of the solve routines require a starting guess. In this case, we must indicate that we want the solution near 9.9, which we will find is approximately 9.9395351414. In summary, we conclude that the equation $2^x = x^3$ has two roots, $x_1 \approx 1.4$ and $x_2 \approx 9.9$. ◀

Applications

Exponential and logarithmic functions are used to model many natural phenomena. The following section is devoted entirely to such applications. Here we discuss just one example.

The sounds we hear Logarithmic functions are used in modeling the sounds we hear. Loudness of sound is a sensation in the brain. We cannot measure it directly, but there is a related physically measurable quantity: the *intensity* of the sound wave. Sound waves travel through the air, and these wave vibrations force the eardrums to vibrate, producing a sound sensation. The intensity I of a sound wave is measured in watts per square meter $\left(\frac{w}{m^2}\right)$.

The intensity of a barely audible sound wave, about $10^{-12}\,\frac{w}{m^2}$, corresponds to pressure vibrations less than a billionth of the atmospheric pressure at sea level. The human ear is very sensitive. A sound wave of intensity of $1\,\frac{w}{m^2}$ would damage the eardrum.

The human ear does not respond to sound intensity in a linear fashion. If the intensity doubles, we do not hear the sound as twice as loud. The sound level β is logarithmically related to the intensity I.

$$\beta(I) = 10 \log\left(\frac{I}{I_0}\right) = 10(\log I - \log I_0) \tag{3}$$

where I is the measured intensity and I_0 is the intensity of sound we can just barely hear, $10^{-12}\,\frac{w}{m^2}$. The sound level β is measured in decibels (dB), a unit named for Alexander Graham Bell.

For a sound just at the hearing threshold, I is I_0, so

$$\beta(I) = 10 \log\left(\frac{I_0}{I_0}\right) = 10 \log 1 = 10 \cdot 0 = 0.$$

Thus 0 dB measures the threshold hearing level. At an intensity of $10\ I_0$, $\beta(10\ I_0) = 10 \log 10 = 10$. Similarly, if I is $100\ I_0$, then the sound level is given by $\beta(100\ I_0) = 10 \log 100 = 10 \cdot 2 = 20$. Multiplying the intensity by a factor of 10 only doubles the loudness of the sound we hear.

►EXAMPLE 9 ***Adding trumpets*** Four trumpets are playing at the same time, each at an average loudness of 75 dB. What is the resulting sound level?

Solution

If $\beta(I_1)$ denotes the loudness level of one trumpet, then Equation (3) can give the corresponding intensity I_1.

$$\beta(I_1) = 10 \log\left(\frac{I_1}{I_0}\right) = 10 \log I_1 - 10 \log I_0.$$

Since $I_0 = 10^{-12}$, $\log I_0 = -12$. Since $\beta(I_1) = 75$, we have

$$75 = 10 \log I_1 + 120 \qquad \log I_1 = -4.5 \qquad \text{and} \qquad I_1 = 10^{-4.5}.$$

The intensity of sound for one trumpet is $10^{-4.5}$ so four trumpets have a sound intensity of $4 \cdot I_1$, or $4 \cdot 10^{-4.5}$. Thus

$$\beta(4 \cdot I_1) = 10 \log\left[\frac{4\ I_1}{I_0}\right] = 10 \log\left[4\left(\frac{I_1}{I_0}\right)\right]$$

$$= 10 \log 4 + 10 \log\left(\frac{I_1}{I_0}\right) = 10 \log 4 + 75 \approx 81.02$$

Therefore, the loudness of the four trumpets is about 81 dB. A fourfold increase in sound wave intensity increases the loudness level by less than 10 percent. This is why a solo instrument can be heard in a symphony concert even when the full orchestra is playing at the same time. ◄

EXERCISES 4.4

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. $\log 16 < \ln 5$

2. $\ln(\sqrt{2} + \sqrt{5}) = \frac{1}{2}(\ln 2 + \ln 5)$

3. For all positive numbers c and d, $\ln(c + d) = \ln c + \ln d$.

4. The graph of $y = \log x$ is above the graph of $y = \ln x$ for all $x > 1$.

5. The graph of $y = \ln x$ is above the graph of $y = \log_3 x$ for every $x > 1$.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. The number of integers between ln 4 and 5ln 25 is ______.

7. The sum of the integers between ln 4 and 2ln 25 is ______.

8. If $S = \{x \mid \ln 0.5 \le x \le 5\ln 25\}$, then the smallest positive integer that is not in S is ______.

9. The graph of $y = \ln(x^3 + x^2 - 4x + 6)$ has turning points in Quadrants ______.

10. The local minimum point (2 decimal places) for the graph of $y = 2 + \ln(x^3 + x^2 - 4x + 4)$ is ______.

Develop Mastery

Exercises 1–8 ***Logarithmic Evaluations*** *Evaluate. Give the result rounded off to four decimal places. If your calculator indicates an error, explain why.*

1. (a) ln 5 **(b)** log 15.6

2. (a) $\ln \sqrt{3}$ **(b)** $\log(1 + \sqrt{3})$

3. **(a)** $\dfrac{\ln 3}{\ln 5}$ **(b)** $\ln\left(\dfrac{3}{5}\right)$ **(c)** $\dfrac{\ln 3 + \ln 4}{\ln 5}$

4. **(a)** $\ln\sqrt{0.5}$ **(b)** $\sqrt{\ln 0.5}$ **(c)** $(0.5)\ln 0.5$

5. **(a)** $\ln\sqrt{2}$ **(b)** $\sqrt{\ln 2}$ **(c)** $2(\ln\sqrt{2})$

6. **(a)** $\log_3 7$ **(b)** $\log_8 0.8$ **(c)** $\log_4(3-\sqrt{10})$

7. **(a)** $\log_5(\ln 7)$ **(b)** $\log_3(\ln 0.3)$ **(c)** $\log_8 2^{\sqrt{3}}$

8. **(a)** $\log_2(\ln 4)$ **(b)** $\log_5(\ln 0.4)$ **(c)** $\log_3 3^{\sqrt{3}}$

Exercises 9–12 Simplify. Express your result in exact form and give a two-decimal-place approximation.

9. **(a)** $e^{\ln\sqrt{5}}$ **(b)** $e^{-\ln\sqrt{5}}$

10. **(a)** $e^{-\ln\sqrt{6}}$ **(b)** $e^{-2(\ln 13)}$

11. **(a)** $\ln e^{\sqrt{3}}$ **(b)** $\ln\sqrt[6]{e}$

12. **(a)** $\ln e^{-\sqrt{7}}$ **(b)** $\ln\sqrt[7]{e}$

Exercises 13–18 ***Compare Logarithm Values*** *Enter =, <, or > in the blank to get a true statement.*

13. $\log_2 3$ ______ $\log_3 2$ **14.** $\log_5 15$ ______ $\log_2 5$

15. $\log_5 25$ ______ $\log_3 9$

16. $\log_2 12$ ______ $\log_{12} 60$

17. $\log_{0.5} 5$ ______ $\log_3 0.04$

18. $\log_5 0.3$ ______ $\log_3 0.5$

Exercises 19–20 ***Comparing Large Numbers*** *Of the three numbers a, b, and c, which one is the* ***(a)*** *largest?* ***(b)*** *smallest? (Hint: If your calculator cannot handle such large numbers, then compare* ln *a,* ln *b, and* ln *c. Explain why you can conclude that if* $\ln u < \ln v$ *then* $u < v$.)

19. $a = 2^{333}$, $b = 3^{210}$, $c = 5^{144}$,

20. $a = 2^{485}$, $b = 4^{243}$, $c = 7^{172}$

Exercises 21–24 ***Comparing Graphs*** *Draw graphs of f and g. See Technology Tip. Are the graphs identical? Explain.*

21. $f(x) = \ln x + \ln(x+2)$, $g(x) = \ln[x(x+2)]$

22. $f(x) = \ln x - \ln(x-2)$, $g(x) = \ln\dfrac{x}{x-2}$

23. $f(x) = 2\ln x$, $g(x) = \ln x^2$

24. $f(x) = \ln(2x)$, $g(x) = \ln x + \ln 2$

Exercises 25–28 ***Composition Graphs*** ***(a)*** *Draw graphs of* $f \circ g$ *and* $g \circ f$. *Are the graphs identical?* ***(b)*** *What is the domain of* $f \circ g$? *of* $g \circ f$?

25. $f(x) = \log x$, $g(x) = 10^x$

26. $f(x) = \ln x$, $g(x) = e^x$

27. $f(x) = 1 + \ln x$, $g(x) = e^{x-1}$

28. $f(x) = -\ln x$, $g(x) = e^{-x}$

Exercises 29–32 ***Exponential Function Inverses*** ***(a)*** *Find a formula for* f^{-1}. ***(b)*** *Draw graphs of* f, f^{-1}, *and* $y = x$ *on the same screen. If the graphs intersect, find the coordinates of the point(s) of intersection (1 decimal place).*

29. $f(x) = e^{-x} + 2$ **30.** $f(x) = 3 + 10^{-0.2x}$

31. $f(x) = 4.10^{1-x}$ **32.** $f(x) = 2e^{1-x}$

Exercises 33–36 ***Logarithmic Function Inverses*** ***(a)*** *Draw a graph of f. Does it suggest that f is one-one?* ***(b)*** *Find a formula for* f^{-1}. ***(c)*** *Draw a graph of* f^{-1} *and give its domain and range.*

33. $f(x) = \ln(x-1)$ **34.** $f(x) = \ln x - 1$

35. $f(x) = \ln\dfrac{x}{x-1}$, $x > 1$

36. $f(x) = \ln(x^2 - 2x)$, $x > 2$

Exercises 37–49 ***Exponential and Logarithmic Equations*** *Solve. Express the result in exact form and also give a three- decimal-place approximation.*

37. $3\ln x - 1 = 0$ **38.** $2\ln x - 1 = 0$

39. $\ln(3x-2) + \ln 5 = 1$

40. $\log(3x-1) - \log x = -1$

41. $2\log x - 2\log(x-1) = 1$

42. $2\ln x - \ln(2x+1) = 1$

43. $3^x = 4$

44. $e^{-x} = 0.56$ **45.** $3^x - \ln 4 = 0$

46. $e^x = 3 \cdot 4^x$ **47.** $e^{-x} + 1 = \ln 8$

48. $4^{-x} - \ln 5 = 0$ **49.** $5^{-x} = 6 \cdot 7^x$

Exercises 50–53 ***Intercept Points*** *For the graph of the equation, find the x- and y-intercept points algebraically. Round off to two decimal places as needed. Use a graph as a check.*

50. $y = \ln(x+1) - 1$ **51.** $y = 2 \cdot 4^x - 5$

52. $y = \ln(x+2) - \ln(x+1) - 1$

53. $y = 3 \cdot 2^x - 5^{-x}$

Exercises 54–57 ***Graphs*** ***(a)*** *Use appropriate translations of a core graph to sketch the graph of* $y = f(x)$. *Label the x-intercept points.* ***(b)*** *Use the graph to help find the solution set for* $f(x) \geq 0$.

54. $f(x) = \ln(x-1) - 1$ **55.** $f(x) = \ln(x+2)$

56. $f(x) = e^x - 2$ **57.** $f(x) = e^{-x} - 2$

Exercises 58–59 ***Disguised Quadratic Equations*** *First express the equation in quadratic form and then solve for x.*

58. $e^x - 2e^{-x} - 1 = 0$ **59.** $5^x + 10 \cdot 5^{-x} = 7$

Exercises 60–63 ***Disguised Quadratic Equations*** *Solve. Express the result in exact form and give a three-decimal-place approximation. (Hint: Consider quadratic equations.) Check graphically.*

60. $(\ln x)^2 - 2\ln x - 3 = 0$ **61.** $(\ln x)^2 = \ln x$

62. $e^{2x} + 2e^x - 3 = 0$ **63.** $e^{2x} + 4e^x + 4 = 0$

Exercises 64–66 Find the solution set (1 decimal place). See Example 8.

64. $2^x = x^2$ **65.** $2^x = x^4$

66. $2^x = x^5$

67. ***Hidden Root*** Find the largest root (1 decimal place) of $2^x = x^7$ See Example 8.

68. **(a)** ***Hidden Root*** Solve $2^x = x^{10}$ using techniques similar to that used in Example 8 (1 decimal place). You can find two roots by drawing graphs of $y = 2^x$ and $y = x^{10}$ on the same screen.
(b) Find the third root by applying ln to both sides and then solving the resulting equation.

69. See Exercise 68. One of the roots of $2^x = x^{10}$ is approximately 58.8. If you were to draw a graph of $y = 2^x$ (or $y = 10^x$), what size graph paper would be needed to find the point for $x = 58.8$ if the scale on your graph paper is one tenth of an inch on both the x and y axes? Compare your answer with the distance from the earth to the sun, 93 million miles.

Exercises 70–79 ***Solution Set*** *Find the solution set. Solve algebraically and then use a graph to support your answer.*

70. **(a)** $e^{-x} = -3$ **(b)** $\ln(-x) = -3$

71. $2\log x = \log 2x$

72. $\ln(e^x + 1) = \ln(e^{-x} + 1) + x$

73. $x^{\log x} = \dfrac{x^4}{1000}$ (*Hint:* Take the log of each side.)

74. $x^{\ln x} = x^2e^3$ (*Hint:* Take the ln of each side.)

75. **(a)** $x^{\log 3} = 3$ **(b)** $x^{\ln x} = e^4$

76. $\log(x^2 + 3) - 2\log x = 1$

77. $(\log_5 x)(\log_x 7) = \log_5 7$ (*Hint:* Use the change-of-base formula.)

78. $(\log_2 x)(\log_x 5) = \log_2 5$ (See Exercise 77.)

79. $\dfrac{1}{\log_2 x} + \dfrac{1}{\log_3 x} = \dfrac{1}{\log_6 x}$ (See Exercise 77.)

Exercises 80–83 ***The Sounds We Hear***

80. How many times more intense is a 70 dB sound than **(a)** a 60 dB sound? **(b)** a 40 dB sound?

81. The loudness level near a lawn mower is 90 dB. What is the corresponding intensity in $\frac{w}{m^2}$?

82. The average loudness level of one trombone is about 70 dB.
(a) What is the loudness level when 76 trombones are playing at the same time?
(b) What is the percentage increase in the loudness level from one trombone to 76 trombones?

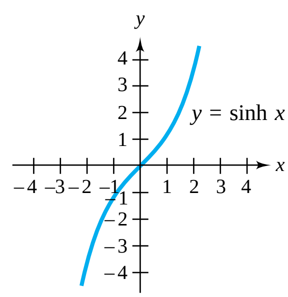
83. What is the loudness level of 110 cornets playing simultaneously if the average loudness level of each is 75 dB? What is the percentage increase in loudness level over that of one cornet?

84. ***Fruit Flies*** The number N of fruit flies in a colony after t days of breeding is given by

$$N = \frac{320}{1 + 7e^{-0.17t}}$$

(a) How many fruit flies are there initially?
(b) Draw a calculator graph using $[0, 60] \times [40, 350]$. Find the number of days it will take for N to be 200.
(c) Describe the end behavior of this function. That is, what happens to N as $t \to \infty$?

Exercises 85–86 ***Looking Ahead to Calculus*** *In calculus we define a function, called the hyperbolic sine, by* $\sinh x = \frac{1}{2}(e^x - e^{-x})$. *The graph of the hyperbolic sine is shown in the diagram. The graph shows that the sinh function is one–one and hence has an inverse.*



85. If $f(x) = \sinh x$, find an equation that describes f^{-1}. (*Hint:* Use the algorithm in Section 2.7.)

86. The function $g(x) = \ln(\sqrt{x^2 + 1} + x)$ has an inverse. Use the algorithm in Section 2.7 to find an equation that describes g^{-1}. Compare with Exercise 85.

4.5 MODELS FOR GROWTH, DECAY, AND CHANGE

> I find that I may have emphasized the need to escape from the devils of mathematics to embark on the pleasures of the real world. But it works both ways, and sometimes the devils of the real world drive one into the pleasures of studying mathematics.
>
> *Cathleen S. Morawetz*

Exponential and logarithmic functions are used to model many real-world processes, some of which we mentioned in earlier sections. In this section we look at additional applications.

> *I developed a proficiency [in junior high school] in simple algebra that lasted for a long time and has been very useful. My mother gave me her college algebra book. I learned from it how to solve word problems, although I remember distinctly that I never really understood them. I could do only the problems that followed the pattern of the examples in the book. My view is that if you have a firm grasp of technique, you can then concentrate on theory without having to think about the technical details.*
>
> Ralph P. Boas, Jr.

Exponential Growth

When scientists measure population size, they see regular changes. Whether they study fish, bacteria, or mammals, they observe that the rate of change is proportional to the number of organisms present; with more bacteria in a culture, colonies grow faster (as long as there is adequate food). A similar kind of growth occurs in a financial setting with compound interest. The amount of interest depends on the amount of money invested, and a larger investment grows faster.

We learn in calculus that exponential functions can model any kind of growth for which the rate of change is proportional to the amount present. Hence this kind of growth is called **exponential growth.**

To express exponential growth mathematically, suppose $A(t)$ denotes the amount of substance or the number of organisms present at time t. Then $A(t)$ is given by

$$A(t) = Ce^{kt},$$

where C and k are constants. When t is 0, the formula gives $A(0) = Ce^{k \cdot 0}$, or $A(0) = C$. Hence, for any exponential growth, C is the amount present at the time measurement begins, when t is 0; we replace C by A_0.

Exponential growth formula

Suppose the rate of change of some substance or quantity is proportional to the amount present, then the amount or number $A(t)$ at time t is given by

$$A(t) = A_0 e^{kt} \qquad \textbf{(1)}$$

where A_0 is the initial amount (the amount present when t is 0), and k is a positive constant determined by the particular substance.

In many problems the constant k is determined experimentally. For instance, a scientist may find that the number of bacteria in a culture doubles every 72 minutes. This information is enough to determine the value of k, as shown in the following example.

Strategy: Use Equation (1) with $A_0 = 500$ and $A(1.2) = 1000$, since 72 minutes is 1.2 hours. Find k.

►EXAMPLE 1 ***Exponential growth*** A sample culture medium contains approximately 500 bacteria when first measured, and 72 minutes later the number has doubled to 1000.

(a) Determine a formula for the number $A(t)$ at any time t hours after the initial measurement.
(b) What is the number of bacteria at the end of 3 hours?
(c) How long does it take for the number to increase tenfold to 5000?

Solution

(a) Follow the strategy. When t is 1.2, Equation (1) becomes

$$1000 = 500e^{k(1.2)}.$$

Divide by 500, take the natural logarithm of both sides, and solve for k.

$$2 = e^{1.2k}, \qquad \ln 2 = \ln e^{1.2k} = 1.2k, \qquad k = \frac{\ln 2}{1.2} \approx 0.578.$$

Replacing k by 0.578 and A_0 by 500 gives the desired equation.

$$A(t) = 500e^{0.578t} \tag{2}$$

(b) When t is 3, $A(3) = 500e^{0.578(3)} = 500e^{1.734} \approx 2832$, so at the end of 3 hours there are approximately 2800 bacteria in the culture.

(c) To find t when $A(t)$ is 5000, substitute 5000 for $A(t)$ in Equation (2) and solve for t.

$$5000 = 500e^{0.578t} \qquad 10 = e^{0.578t}$$
$$\ln 10 = \ln e^{0.578t} = 0.578t$$
$$t = \frac{\ln 10}{0.578} \approx 3.98$$

It takes about 4 hours for the number of bacteria to increase tenfold.

Graphical Draw a graph of $A(x) = 500e^{0.578x}$ in $[0, 5] \times [500, 5100]$. Then trace and zoom as needed to see that when $x = 3$, $A = 2832$, and when $A = 5000$, $x = 3.98$. ◀

Compound and Continuous Interest

If money is invested in an account that pays interest at a rate r compounded n times a year, the growth is not described by Equation (1). We need another formula. When the annual interest rate is given as a percentage, we express r as a decimal; for a rate of 6 percent we write $r = 0.06$.

Compound interest formula

Suppose A_0 dollars are invested in an account that pays interest at rate r compounded n times a year. The number of dollars $A(t)$ in the account t years later is given by

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{nt}. \tag{3}$$

Compound interest is paid only at the end of each compounding period. If interest is compounded quarterly, then the interest is credited at the end of each three-month period. To apply Equation (3) for other values of t, we should replace the exponent nt by the greatest integer $[nt]$.

As the number of times a year that interest is compounded increases, we approach what is called **continuous compounding.** To see what happens to $A(t)$ as n becomes large ($n \to \infty$), replace $\frac{r}{n}$ by x and rewrite the exponent nt as $(\frac{n}{r})(rt)$, or $(\frac{1}{x})(rt)$, so Equation (3) becomes

$$A(t) = A_0[(1 + x)^{1/x}]^{rt}. \tag{4}$$

Now, as $n \to \infty$, $\frac{r}{n} \to 0$, so $x \to 0$. We are interested in what happens to the expression $(1 + x)^{1/x}$ as $x \to 0$. This is equivalent to the problem we considered in Section 4.1. See Exercise 38, where you are asked to show that $(1 + x)^{1/x} \to e$ as $x \to 0$. Thus, when interest is compounded continuously at rate r for t years, *compound interest becomes exponential growth.* Equation (4) becomes $A(t) = A_0 e^{rt}$.

Continuous interest formula

Suppose A_0 dollars are invested in an account that pays interest at rate r compounded continuously. Then the number of dollars $A(t)$ in the account t years later is given by

$$A(t) = A_0 e^{rt}. \tag{5}$$

▶EXAMPLE 2 Compound interest Suppose \$2400 is invested in an account in which interest is compounded twice a year at the rate of 8 percent.

(a) How much is in the account at the end of ten years?
(b) How long does it take to double the initial investment?
(c) Answer the same questions if the money is compounded continuously.
(d) Draw graphs of Y1 = 2400 (1.04) ^ (2X) (interest compounded twice a year) and Y2 = 2400 e ^ (.08X) (continuous compounding) on the same screen $[0, 12] \times [2400, 5500]$. How soon does the continuous interest curve become visibly higher? Trace and zoom as needed to answer questions **(a)** and **(b).**

Strategy: For **(a)** and **(b)**, replace A_0 by 2400, r by 0.08, and n by 2 in Equation (3), then use the resulting equation. For **(c)**, replace A_0 by 2400 and r by 0.08 in Equation (5), then use the resulting equation.

Solution

Follow the strategy.

$$A(t) = 2400(1 + 0.04)^{2t} = 2400(1.04)^{2t}, \text{ so } A(t) = 2400(1.04)^{2t}.$$

(a) In ten years, t is 10, so

$$A(10) = 2400(1.04)^{20} \approx 5258.70$$

At the end of ten years the account will be worth \$5258.70.

(b) Solve the following for t:

$$4800 = 2400(1.04)^{2t}, \qquad 2 = (1.04)^{2t},$$

$$\ln 2 = \ln(1.04)^{2t} = 2t \ln 1.04, \qquad \text{or} \qquad t = \frac{\ln 2}{2 \ln 1.04} \approx 8.8.$$

The \$2400 investment doubles in about 8 years and 10 months, but the account will not be credited with the last interest until the end of the year.

(c) Using Equation (5) instead of Equation (3),

$$A(t) = 2400e^{0.08t}.$$

In ten years, $A(10) = 2400e^{0.8} \approx 5341.30$, so continuous interest returns nearly \$83 more on a \$2400 investment than semiannual compounding over that time. To see how long it takes to double the investment, solve for t:

$$4800 = 2400e^{0.08t} \qquad t = \frac{\ln 2}{0.08} \approx 8.66.$$

The investment doubles in 8 years and 8 months.

(d) The graphs of the two functions are indistinguishable for the first half of the time interval (until x is about 6) even though when we trace, we can see that continuously compounded interest yields about \$36 more when $x = 6$. We can get the same answers from tracing along the graphs that we obtained above. It may help to adjust your window as suggested in the following Technology Tip, and in addition, to see how the interest is actually added to the account—at the end of each accounting period—graph `Y1 = 2400 (1.04) ^ Int (2X)` in dot mode rather than connected mode. ◀

TECHNOLOGY TIP ◆ ***"Nice-pixel" windows***

If you feel that nice pixel coordinates are helpful in reading information from a graph, use an x-range that is a multiple of your decimal window range. In the example above, we want something that includes [0, 12]. On the TI-82 and Casio fx–7700, the decimal window goes from -4.7 to 4.7, a total of 9.4 units or 94 tenths. If we multiply the number of pixel columns by 1.5, we have $9.4 \times 1.5 = 14.1$, so $[-2, 12.1]$, works well on the TI-82. Similarly, $[-2, 12.25]$ is good on the TI-81, and [0, 12.6] for the TI-85 or Casio fx-9700, $[-1, 12]$ on the HP-38 and HP-48. The y-range must include [2400, 6200].

Exponential Decay

Certain materials, such as radioactive substances, decrease with time, rather than increase, with the rate of decrease proportional to the amount. Such negative growth is described by exponential functions, very much like exponential growth except for a negative sign in the exponent.

Exponential decay formula

Suppose the rate of decrease of some substance is proportional to the amount present. The amount $A(t)$ at time t is given by

$$A(t) = A_0 e^{-kt} \qquad \textbf{(6)}$$

where A_0 is the initial amount (the amount present when t is 0), and k is a positive constant determined by the particular substance.

▶***EXAMPLE 3*** ***Radioactive decay*** Strontium-90 has a half-life of 29 years. Beginning with a 10 mg sample, **(a)** determine an equation for the amount $A(t)$ after t years and **(b)** find how long it takes for the sample to decay to 1 mg. **(c)** Check your result in part **(b)** by drawing a calculator graph of $A(t)$ in $[0, 100] \times [0, 8]$ and zooming in as needed.

Strategy: First, replace A_0 by 10 in Equation (6), then in the resulting equation use $A(29) = 5$ and solve for k. Using this value of k in Equation (6) gives the decay equation for strontium-90.

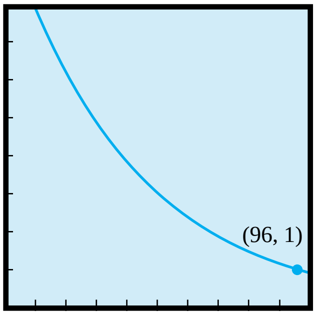
Solution

(a) Follow the strategy. $A(t) = 10e^{-kt}$, and in 29 years, half the sample will remain, so $A(29) = 5$. Substitute 29 for t and 5 for A, so

$$5 = 10e^{-29k} \quad \text{or} \quad e^{-29k} = \frac{1}{2}, \text{ so } -29k = \ln\left(\frac{1}{2}\right).$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{-29} = \frac{-\ln 2}{-29} \approx 0.0239.$$

Therefore, the decay equation for strontium-90 is $A(t) = 10e^{-0.0239t}$.



[0, 100] by [0, 8]
$A(t) = 10e^{-0.0239t}$

FIGURE 26

(b) To find when $A(t)$ is 1, replace $A(t)$ by 1 and solve the resulting equation for t.

$$1 = 10e^{-0.0239t}, \qquad \ln\left(\frac{1}{10}\right) = \ln e^{-0.0239t} = -0.0239t,$$

$$t = \frac{\ln\left(\frac{1}{10}\right)}{-0.0239} \approx 96.$$

It takes 96 years for 90 percent of the original amount of strontium-90 to decay.

(c) The graph of $A(t) = 10e^{-0.0239t}$ is shown in Figure 26. Observe that the function is decreasing. If we zoom in near the point where $A(t) = 1$, we find that $t \approx 96$. ◀

Hatchee Reservoir Revisited

The contamination of Hatchee Reservoir described in the chapter introduction is a classic case of exponential decay, where the water flowing through the reservoir flushes out half of the pollutants every fifteen days. According to Equation (6), the amount $A(t)$ of the toxic chemical left after t days is given by $A(t) = A_0e^{-kt}$, where we must determine the constant k.

▶EXAMPLE 4 ***Will Hatchee be clean by July 4?*** Find the constant k in the equation $A(t) = A_0e^{-kt}$ and determine how much of the toxic chemical will be left on July 4.

Solution

In fifteen days, $t = 15$, and so $A(15) = 0.5A_0$. Substituting these values into the equation for A, we have $0.5A_0 = A_0e^{-15k}$. Dividing by A_0 and taking the natural logarithm of both sides, we can solve for k:

$$\begin{aligned} 0.5 &= e^{-15k} \\ \ln 0.5 &= -15k \\ k &= (\ln 0.5)/(-15) \approx 0.0462. \end{aligned}$$

We store the entire display for computing, but we have $A(t) \approx A_0e^{-0.0462t}$.

As a check, when $t = 60$ (June 30), the formula gives $A(60) \approx 0.0625A_0$, confirming the simple analysis we gave in the introduction that 6.25% of the original contamination would remain on June 30. We can now determine the pollution level on July 4, when $t = 64$: $A(64) \approx 0.0520A_0$. Thus just *over* 5% will remain on July 4. City officials will have to decide whether the risk outweighs town tradition. Since the pollution level is predicted to be so near the declared safe level, it might pay the city to invest in more testing to see how accurately the mathematical model predicts the measured pollution level. ◀

Carbon Dating

Radioactive decay is used to date fossils. The method involves the element carbon. Carbon-12 is a stable isotope, while carbon-14 is a radioactive isotope with a half-life of approximately 5700 years. Fortunately for us, the concentration of C^{14} in the air we breathe and the food we eat is extremely small (about 10^{-6} percent).

Although C^{14} disintegrates as time passes, the amount of C^{14} in the atmosphere remains in equilibrium because it is constantly being formed by cosmic rays. All living things regularly take in carbon, and the proportion of C^{14} in living organisms reflects the proportion in the atmosphere. When an organism dies, however, the C^{14} is not replenished and the decay process decreases the ratio of C^{14} to C^{12}. By measuring this ratio in organic material, it is possible to determine the number of years since the time of death. The technique is known as *carbon dating* (see the Historical Note, "Exponential Functions, Dating, and Fraud Detection").

▶EXAMPLE 5 ***Dating Crater Lake*** A tree felled by the eruption that created Crater Lake in Oregon was found to contain 44 percent of its original amount of carbon-14. Use 5700 years as the half-life of carbon-14 and determine the age of Crater Lake.

Strategy: Crater Lake was formed when the tree died, so find how long the tree has been dead. Use Equation (6) and when t is 5700, $A(t)$ is $\frac{A_0}{2}$. Solve for k. Substitute this value in Equation (6) to get the decay equation for carbon-14. Now replace $A(t)$ by $0.44A_0$ and solve the resulting equation for t.

Solution

Follow the strategy.

$$A(t) = A_0 e^{-kt}$$

$$\frac{1}{2}A_0 = A_0 e^{-5700k} \qquad \text{or} \qquad \frac{1}{2} = e^{-5700k}$$

Take natural logarithms and use the fact that $\ln(\frac{1}{2}) = -\ln 2$:

$$-\ln 2 = \ln e^{-5700k} \qquad \text{or} \qquad -\ln 2 = -5700k$$

$$\text{or } k = \frac{\ln 2}{5700} \approx 0.0001216.$$

Therefore, the decay equation for carbon-14 is

$$A(t) = A_0 e^{-0.0001216t} \tag{7}$$

Since 44 percent of the original amount of carbon-14 still remained when the tree was discovered, find the value of t for which $A(t)$ is $(0.44)A_0$. Substitute $(0.44)A_0$ for $A(t)$ in Equation (7):

$$(0.44)A_0 = A_0 e^{-0.0001216t} \qquad \text{or} \qquad 0.44 = e^{-0.0001216t}$$

$$\ln 0.44 = \ln e^{-0.0001216t} \qquad \text{or} \qquad \ln 0.44 = -0.0001216t$$

$$t = -\frac{\ln 0.44}{0.0001216} \approx 6751.$$

Crater Lake was formed approximately 7000 years ago. ◀

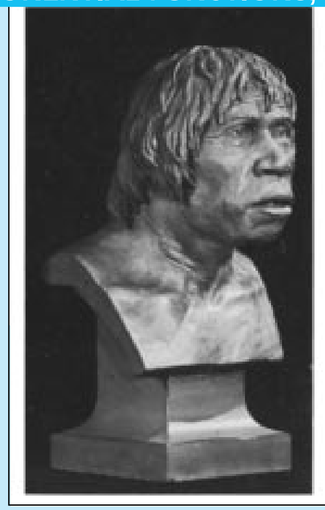
TECHNOLOGY TIP ◆ ***What about A_0?***

The formula for $A(t)$ in Example 5 involves the initial amount, A_0. If we want to use graphical methods to solve the problem, once we have the decay equation, we can take any constant for the initial amount, say $A_0 = 1$. Then to find the value of t when 44% of the initial amount remains, trace along the curve $A(t) = e^{-0.0001216t}$ and find the t-value for which $A(t) \approx .44$.

HISTORICAL NOTE EXPONENTIAL FUNCTIONS, DATING, AND FRAUD DETECTION

The discovery of radiocarbon dating in 1949 by Willard F. Libby opened new ways to learn about the past. The half-life of carbon-14 allows dependable dating of organic material up to a range of 10,000 or 20,000 years.

Potassium allows dating on a much longer scale, albeit less precisely. Each of our bodies contains about a pound of potassium, including a miniscule fraction of radioactive potassium-40, which is changing (into argon gas) at a rate of about 500 atoms per second. Potassium-argon dating established the age of the fossil hominid Lucy at over 3 million years.



Bust of Piltdown Man

In 1908 bits of bone that comprised part of a human skull were found in a gravel pit in Piltdown, Sussex, England. Four years later part of an apelike jawbone showed up in the same location. Thus was born Piltdown Man, one of the strangest puzzles in human paleontology.

Joining a human cranium with an apelike jaw raised problems for students of human evolution and fueled a vigorous controversy that raged for years. Not until 1953 did fluorine dating (based on the fact that bones and teeth absorb fluorine from soil and groundwater at a constant rate) finally show that the cranium and jawbone did not belong together. The newly discovered radiocarbon dating showed that the skull dated from near Chaucer's time (about 600 years earlier—hardly prehistoric), and the jaw was even younger. It had belonged to an orangutan from the East Indies.

The whole Piltdown affair was perhaps the greatest hoax in the history of science. Professionals and amateurs alike (including Sir Arthur Conan Doyle, creator of Sherlock Holmes) became embroiled in the disputes. The identity of the perpetrators remains unresolved, but progress in the method of science, including mathematical dating analyses, helped to uncover the fraud.

The next example presents another illustration of exponential decay.

▶EXAMPLE 6 Atmospheric pressure Standard atmospheric pressure at sea level is 1035 g/cm^2. Experimentation shows that up to about 80 km ($\approx$ 50 mi), the pressure decreases exponentially. The atmospheric pressure (in g/cm^2) at an altitude of h kilometers is given by

$$P(h) = 1035e^{-0.12h} \tag{8}$$

Find **(a)** the atmospheric pressure at 40 km, and **(b)** the altitude where the atmospheric pressure drops to 20 percent of that at sea level.

Solution

(a) From Equation (8), $P(40) = 1035e^{(-0.12)(40)} \approx 8.5$. Hence the atmospheric pressure at 40 km ($\approx$25 mi) is only 8.5 g/cm^2, less than 1 percent of the pressure at sea level.

(b) Find the value of h for which $P(h)$ is 20 percent of the pressure at sea level. Replace $P(h)$ in Equation (8) by (0.2)(1035) and solve for h.

$$(0.2)(1035) = 1035e^{-0.12h}, \qquad e^{-0.12h} = 0.2,$$

$$-0.12h = \ln 0.2, \qquad h = \frac{\ln 0.2}{-0.12} \approx 13.4.$$

Since $h \approx 13.4$, at an altitude of 13.4 km ($\approx$8.3 mi, not quite 44,000 ft), the atmospheric pressure drops to 20 percent of the atmospheric pressure at sea level. ◀

In the previous section we saw an example of an application of logarithms to measure sound levels. The next two examples also illustrate models that apply logarithms.

Measuring Earthquakes

An earthquake produces seismic waves whose amplitude is measured on a seismograph. Charles Richter, an American geologist, recognized the great variation in amplitudes of earthquakes and proposed a logarithmic scale to measure their severity. The magnitude $M(A)$ of an earthquake with amplitude A is a number on the Richter scale given by

$$M(A) = \log\left(\frac{A}{A_0}\right), \tag{9}$$

where A_0 is a standard amplitude.

▶EXAMPLE 7 ***Comparing earthquakes*** How many times larger was the amplitude of the Alaskan earthquake on March 28, 1964, which measured 8.6 on the Richter scale, than the amplitude of a relatively minor aftershock that measured 4.3?

Strategy: Use Equation (9). Let A_1 and A_2 be the two amplitudes, and replace $M(A_1)$, $M(A_2)$, by 8.6 and 4.3, respectively. Find A_1 and A_2 in terms of A_0.

Solution

Follow the strategy.

$$8.6 = \log\left(\frac{A_1}{A_0}\right) \qquad \text{and} \qquad 4.3 = \log\left(\frac{A_2}{A_0}\right).$$

Write each equation in exponential form and solve for A_1 and A_2.

$$A_1 = A_0 10^{8.6} \qquad \text{and} \qquad A_2 = A_0 10^{4.3}.$$

Solve the second equation for A_0 and substitute into the first equation,

$$A_0 = A_2 10^{-4.3}, \text{ so } A_1 = (A_2 10^{-4.3})10^{8.6} = 10^{4.3}A_2 \approx 19{,}953A_2.$$

The amplitude A_1 of the 8.3 magnitude earthquake is nearly 20,000 times larger than the amplitude of the 4.3 aftershock, which explains the enormous amount of damage done by the original earthquake. ◀

Acidity Measurement

Chemists determine the acidity of a solution by measuring the hydrogen ion concentration (denoted by $[H^+]$, in moles per liter). Such concentrations are very small numbers. To deal with numbers in a more familiar range, the quantity denoted by pH essentially puts hydrogen ion concentration on a logarithmic scale.

Formula for determining acidity of a solution

For a solution with hydrogen ion concentration of $[H^+]$ moles per liter, the corresponding pH value is given by

$$\text{pH} = -\log[H^+]. \quad \textbf{(10)}$$

If the pH number for a solution is less than 7, then the solution is called *acidic;* if the pH is greater than 7, then the solution is called *basic*. Solutions with pH equal to 7 are called *neutral*. For a solution with 10^{-1} moles of hydrogen ions per liter ($[H^+] = 10^{-1}$), the pH is $-\log 10^{-1} = -(-)1 = 1$; such a solution is very strongly acidic (even one-tenth of a mole of hydrogen ions indicates *lots* of freely reacting ions in the solution). At the other end of the scale, if $[H^+] = 10^{-13}$, then $\text{pH} = -\log(10^{-13}) = 13$, indicating a strongly basic solution.

►EXAMPLE 8 ***Fruit juice acidity*** A certain fruit juice has a hydrogen ion concentration of 3.2×10^{-4} moles per liter. Find the pH value for the juice and decide whether it is acidic or basic.

Solution

Given that $[H^+] = 3.2 \times 10^{-4}$, substitute into Equation (10):

$$\text{pH} = -\log(3.2 \times 10^{-4}) = -\log(0.00032) \approx 3.5.$$

A pH of less than 7 indicates that the juice would be classified as acidic. ◄

EXERCISES 4.5

Check Your Understanding

Exercises 1–6 If \$1000 is invested in an account that earns interest compounded continuously at an interest rate that doubles the investment in value every 12 years, then select from the choices below the amount that is closest to the total value of the investment after the indicated period of time. As in the text $A(t)$ denotes the amount of money in the account t years after the investment is made.

(a) \$1400 **(b)** \$1500 **(c)** \$2000
(d) \$2800 **(e)** \$3000 **(f)** \$4000
(g) \$6000 **(h)** \$7000 **(i)** \$8000

1. $A(24) =$ ______. **2.** $A(36) =$ ______.

3. $A(6) =$ ______. **4.** $A(18) =$ ______.

5. The interest earned during the first 18 years is ______.

6. The interest earned during the years from $t = 12$ to $t = 24$ is ______.

Exercises 7–10 A radioactive substance has a half-life of 30 days. Select from the list below the choice that is closest to the amount of the substance that remains after the indicated period of time. A_0 denotes the number of grams of the substance when t is 0, and $A(t)$ denotes the number of grams t days later.

(a) $0.25A_0$ **(b)** $0.35A_0$ **(c)** $0.50A_0$
(d) $0.70A_0$ **(e)** $0.75A_0$ **(f)** $0.80A_0$

7. $A(60) =$ ______. **8.** $A(15) =$ ______.

9. $A(45) =$ ______.

10. The amount of the substance that decays during the first 60 days is ______.

Develop Mastery

Exercises 1–2 ***Number of Bacteria***

1. The number of bacteria in a culture doubles every 1.5 hours. If 4000 are present initially,
(a) How many will there be three hours later?
(b) Four hours later?
(c) How long does it take for the number to increase to 40,000?

2. If the number of bacteria in a sample increases from 1000 to 1500 in two hours, how long does it take for the number of bacteria
(a) to double? **(b)** to triple?

Exercises 3–5 ***Population Growth***

3. The world population in 1968 was 3.5 billion; in 1992 it was 5.5 billion. Assume exponential growth.
(a) Predict the population in the year 2000.
(b) When will the population reach 7 billion?

4. Assuming an annual population increase of 1.5 percent since 1968 (when the world population was 3.5 billion)
- **(a)** Show that n years after 1968, the population $P(n)$ is $(3.5)(1.015)^n$ billion.
- **(b)** Determine the population for 1992. How does your calculation agree with the information in Exercise 3?

5. In 1960 the population of the United States was 180 million; in 1970 it was 200 million. Assume an exponential rate of growth and predict the population for the year 2000.

Exercises 6–12 ***Compound Interest*** *Assume interest is compounded continuously and that all interest rates are annual.*

6. Suppose \$1000 is invested in an account that earns 8 percent interest.
- **(a)** How much interest is in the account 10 years later?
- **(b)** How long does it take the money to double?
- **(c)** To triple?

7. Suppose \$1000 invested in a savings account increases over three years to \$1200. What rate of interest is being paid?

8. An investment of \$800 in a savings certificate that pays 10 percent interest has grown to \$2000. How many years ago was the certificate purchased?

9. An investment doubles in 8 years. What is the rate of interest?

10. How long does it take for an investment to double if the rate of interest is
(a) 8 percent? **(b)** 12 percent? **(c)** r percent?

11. Suppose you invest \$1000 in a savings account at 5 percent interest, and at the end of 6 years you use the accumulated total to purchase a savings certificate that earns 6 percent interest. What is the value of the savings certificate 6 years later?

12. An annuity pays 12 percent interest. What amount of money deposited today will yield \$3000 in 8 years?

Exercises 13–18 ***Radioactive Decay***

13. A radioactive isotope, radium-226, has a half-life of 1620 years. A sample contained 10 grams in 1900. How many grams will remain in the year
(a) 2000? **(b)** 3000?

14. Radioactive lead, lead-212, has a half-life of 11 days. How long will it take for 20 pounds of lead-212 to decay to 8 pounds?

15. Another isotope of lead, lead-210, has a half-life of 22 years. How much of a 10-pound sample would remain after 10 years?

16. Radioactive iodine-131 is a component of nuclear fallout.
- **(a)** If 10 mg of iodine-131 decays to 8.4 mg in 2 days, what is the half-life of the isotope?
- **(b)** In how many days does the 10 mg sample decay to 2 mg?

17. After two years, a sample of a radioactive isotope has decayed to 70 percent of the original amount. What is the half-life of the isotope?

18. A 12 mg sample of radioactive polonium decays to 7.26 mg in 100 days.
- **(a)** What is polonium's half-life?
- **(b)** How much of the 12 mg sample remains after six months (180 days)?

Exercises 19–22 ***Carbon Dating*** *Use the carbon dating information discussed in this section.*

19. A piece of petrified wood contains 40 percent of its original amount of C^{14}. How old is it?

20. If the Dead Sea Scrolls contain about 80 percent of their original C^{14}, how old are they?

21. How old is a fossil skeleton that contains 85 percent as much C^{14} as a living person?

22. If the Piltdown cranium (the Historical Note, "Exponential Functions, Dating, and Fraud Detection") was found to contain 93 percent of the C^{14} found in a modern skeleton, what is the approximate age of the cranium?

Exercises 23–25 ***Earthquake Comparisons***

23. A 1933 earthquake in Japan registered 8.9 on the Richter scale, the highest reading ever recorded. Compare its amplitude to that of the 1971 earthquake in San Fernando, California, which measured 6.5.

24. The famous San Francisco earthquake of 1906 registered 8.4 on the Richter scale. Compare its amplitude with that of the 1976 earthquake in Guatemala, which measured 7.9.

25. If an earthquake in Ethiopia had an amplitude 100 times larger than an earthquake that measured 5.7 on the Richter scale, what would the Ethiopian earthquake measure?

Exercises 26–27 ***Atmospheric Pressure***

26. Example 6 contains a formula for the atmospheric pressure $P(h)$ (in g/cm^2) at an altitude of h km. If h is measured in miles and pressure is measured in lb/in^2, then the corresponding equation is $P(h) = 14.7e^{-0.19h}$.
- **(a)** Find the atmospheric pressure at an altitude of 25 miles.
- **(b)** At what altitude is the atmospheric pressure one-tenth of that at sea level?

27. Use the equation for atmospheric pressure in Example 6 to find the altitude at which the atmospheric pressure is 100 g/cm^2.

28. A satellite is powered by a radioactive isotope. The power output $P(t)$ (measured in watts) generated in t days is given by $P(t) = 50e^{-t/250}$.
(a) How much power is available at the end of a year (365 days)?
(b) What is the half-life of the power supply?
(c) If the equipment aboard the satellite requires 10 watts of power to operate, what is the operational life of the satellite?

Exercises 29–30 ***Acidity***

29. The hydrogen ion concentration for a sample of human blood is found to be 4.5×10^{-8} moles per liter. Find the pH value of the sample. Is it acidic or basic?

30. Find the pH value for
(a) vinegar, $[H^+] = 6.3 \times 10^{-4}$
(b) milk, $[H^+] = 4 \times 10^{-7}$
(c) water, $[H^+] = 5.0 \times 10^{-8}$
(d) sulphuric acid, $[H^+] = 1$.

31. Oil is being pumped from a well. If we assume that production is proportional to the amount of oil left in the well, then it can be shown that the number of barrels of oil, $A(t)$, left in the well t years after pumping starts, is given by $A(t) = Ce^{-kt}$, where C and k are constants. When t is 0 it is estimated that the well holds 1 million barrels of oil, and after six years of pumping, 0.5 million barrels remain. It is not profitable to keep pumping when fewer than 50,000 barrels remain in the well. What is the total number of years during which pumping remains profitable?

32. The population of Taunton is growing exponentially at an annual rate of 5 percent.
(a) Show that after t years the population increases from 13,000 to N (in thousands) given by $N = 13(1.05)^t$.
(b) In how many years will the population double?
(c) In how many years will the population triple?

33. ***Looking Ahead to Calculus*** A 500 gallon tank of brine starts the day with 150 pounds of salt. Fresh water runs into the tank at the rate of 5 gallons per minute and the well-stirred mixture drains at the same rate. In calculus it can be shown that the number of pounds, $A(t)$, of salt still in the tank t minutes later is given by $A(t) = 150e^{-0.01t}$.
(a) How many pounds of salt remain in the tank after 30 minutes?
(b) How many minutes does it take to reduce the amount of salt in the tank to 50 pounds?

34. In Example 3 we developed an equation for the amount $A(t)$ of strontium-90 (half-life 29 years) left after t years, starting with an initial amount A_0: $A(t) = A_0e^{-0.0239t}$. Show that $A(t)$ is also given by $A(t) = A_0(2^{-t/29})$. (*Hint:* In Example 3 $k = \frac{\ln 2}{29}$, so $A(t) = A_0e^{-(t\ln 2)/29}$.)

35. ***Spreading a Rumor*** A rumor is spreading about the safety of county drinking water. Suppose P people live in the county and $N(t)$ is the number of people who have not yet heard the rumor after t days. If the rate at which $N(t)$ decreases is proportional to the number of people who have not yet heard the rumor, then $N(t)$ is given by $N(t) = Pe^{-kt}$, where k is a constant to be determined from observed information. In Calaveras County, population 50,000, suppose 2000 people have heard the rumor after the first day (when t is 1).
(a) How many people will have heard the rumor after 10 days?
(b) After how many days will half of the population have heard the rumor?

36. Use the equation in Exercise 35. If 10 percent of a county population of 20,000 have heard the rumor after the first two days, then how many people will have heard the rumor after three additional days?

37. ***Hatchee Reservoir Contamination*** In Example 4 suppose the pollutant level decreases by one half every 10 days. What percentage of the toxic chemical will be present on
(a) May 21? **(b)** July 4?

38. Draw a calculator graph of $y = (1 + x)^{1/x}$. Use the graph to see what happens to y when $x \to 0$.

CHAPTER 4 REVIEW

Test Your Understanding

True or False. Give reasons.

1. $\ln x$ is positive for every positive x.

2. $\ln x^2$ is defined for every real number x.

3. $\ln 1 = e$.

4. $\ln e = 1$.

5. $(\ln x)^2 = 2(\ln x)$ for every $x > 0$.

6. If x and y are positive numbers, then $\ln(\frac{x}{y}) = \frac{\ln x}{\ln y}$.

7. $\frac{1}{2}(\ln x^2) = \ln x$ for every positive x.

8. For any real numbers x and y, $3^x + 3^y = 3^{x+y}$.

9. For any real number x, $e^{x+1} = e^x + e$.

10. For every real number x, $4^x = 2^{2x}$.

11. If $f(x) = 3^{x/2}$ and $g(x) = (\sqrt{3})^x$, then $f(x) = g(x)$ for every real number x.

12. For every real number x, e^{-x} is positive.

13. The domain of $\ln(x - 1)$ is $\{x \mid x > 0\}$.

14. The solution set for $e^{x-1} < 1$ is $\{x \mid x < 1\}$.

15. There is only one real number x that satisfies $\ln(x + 2)^2 = \ln(x^2 + 4)$.

16. If x is any number between 0 and 1, then $\ln x$ is negative.

17. The graph of $f(x) = e^x$ crosses the x-axis at $(0, 1)$.

18. $e^{-\ln x} = \dfrac{1}{x}$ for every positive number x.

19. There is no number x for which $e^{-x} = e^x$.

20. $\log(5 + 2) = \log 5 + \log 2$.

21. $\log 5 + \log 2 = 1$.

22. The solution set for $(\ln x)^2 = \ln x$ is the set $\{1, e\}$.

23. $e^0 = 0$.

24. $\log(\frac{1}{10}) = -1$.

25. For every $x \geq 0$, $2^{-x} \geq 3^{-x}$.

26. The formula $f(x) = (-3)^x$ does not define an exponential function.

27. For every real number x, $2^x \leq 3^x$.

28. $e^{\ln(-5)} = -5$.

29. The graph of $y = 1 + \ln x$ crosses the x-axis at $\left(\dfrac{1}{e}, 0\right)$.

30. The equation $\ln x + \ln(x + 1) = 0$ has no solutions.

31. $\ln x + \ln(x - 1) = \ln[x(x - 1)]$ when $x > 1$.

32. For every real number x, $2^x + 2^x = 2^{x+1}$.

33. If $\log_2 x = 3$, then $x = 8$.

34. **(a)** $2^0 = 1$ **(b)** $(-2)^0 = 1$
(c) $0^{-2} = 0$ **(d)** $0^0 = 1$

35. There is no real number x for which $\ln x = -1$.

36. The formula $g(x) = (\sqrt{5} - 2)^x$ defines g as an exponential function.

37. The formula $g(x) = (2 - \sqrt{5})^x$ defines g as an exponential function.

38. The graph of $y = \ln(x + 2)$ crosses the x-axis at $(-1, 0)$.

39. $(x^2 - 1)^0 = 1$ for every real number x.

40. $\ln(-x)$ is undefined for any real number x.

41. The equation $e^{-x} + 1 = 0$ has no solution.

42. A root of $e^{-x} - 1 = 0$ is 0.

43. The function $f(x) = 2^{-x}$ is a decreasing function.

44. The graph of the function $f(x) = e^x + 1$ lies above the line $y = 1$.

45. The range of the natural exponential function is R.

46. The smallest prime number greater that 3^3 is 23.

47. The sum of all of the integers between $\ln 3$ and $\ln 500$ is 20.

48. The graph of every exponential function contains the point $(0, 1)$.

49. If $b > 1$, then $\log_b x$ is an increasing function.

50. **(a)** $\log_2 4 > \log_4 2$ **(b)** $(\log_2 4)(\log_4 2) = 1$

51. For every positive x, $\ln(-x) = -\ln x$.

52. For every positive x, $\ln\left(\dfrac{1}{x}\right) = \dfrac{1}{\ln x}$.

53. $\dfrac{\ln 5}{\ln 3} = \ln 5 - \ln 3$.

54. $\sqrt[3]{x}$ is a real number only for $x \geq 0$.

55. $\sqrt[4]{x}$ is a real number only for $x \geq 0$.

56. $-(3^x) = (-3)^x$ for every real number x.

57. To draw a graph of $y = e^{x+2}$, translate the graph of $y = e^x$ horizontally 2 units to the left.

58. If $f(x) = \dfrac{1}{1 + 2^x}$, then f is a decreasing function.

59. For every real number x, $\dfrac{3^x}{1 + 3^x}$ is a number between 0 and 1.

60. The graph of $y = \dfrac{3^{-x}}{1 + 3^{-x}}$ is the same as the graph of $y = \dfrac{1}{1 + 3^x}$.

61. The graph of $y = e^x + e^{-x}$ is symmetric about the y-axis.

62. If $f(x) = e^x + e^{-x}$, then $(f(x))^2 = 2 + f(2x)$.

63. If $f(x) = 1 + e^{-x}$, then f is an increasing function.

64. The graph of $y = 1 + e^{0.5x}$ and $x + y = 3$ intersect in the second quadrant.

65. The domain of $f(x) = \ln(3 - x) + \ln(x + 1)$ is $\{x \mid x > 0\}$.

66. If $f(x) = 1 - \ln x$, then f has an inverse.

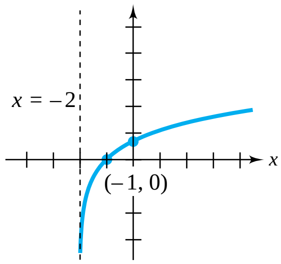
67. The graph of $f(x) = \ln(x^3 + x^2 - 3x + 4)$ has no turning points.

68. If $f(x) = e^x - 4$ and $g(x) = \ln(x + 4)$, then f and g are inverses of each other.

Exercises 69–74 A graph is shown. Select from the following list the function corresponding to the graph.

(a) $f(x) = \ln x$ **(b)** $f(x) = \ln(x - 2)$
(c) $f(x) = \ln(x + 2)$ **(d)** $f(x) = \ln x + 1$
(e) $f(x) = e^{x+1}$ **(f)** $f(x) = 2^{-x} + 1$
(g) $f(x) = e^{\ln x}$ **(h)** $f(x) = \ln e^x$
(i) none of the above

69.



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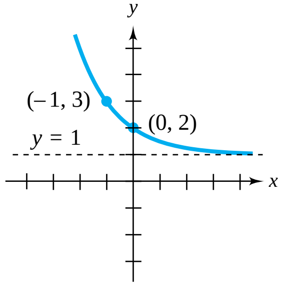

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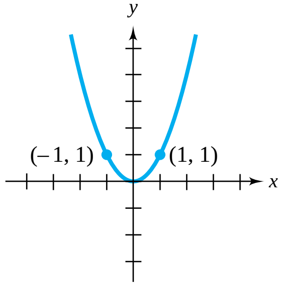

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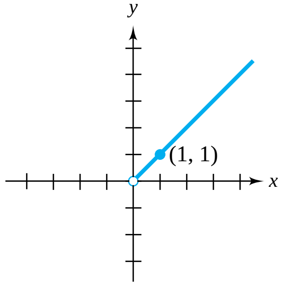

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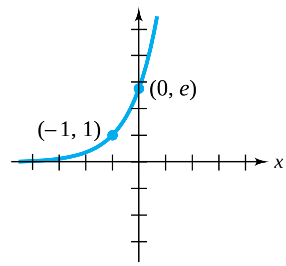

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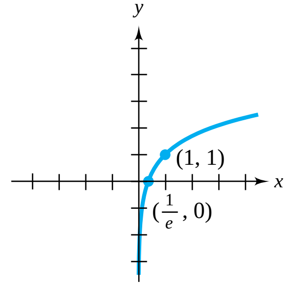


Review for Mastery

Exercises 1–9 ***Exact Form Evaluations*** *Evaluate and give the result in exact form.*

1. $5(5^{-1} + 5^{-2})$ **2.** $(49 \cdot 7^{-1})^{-1}$ **3.** $\log_3 \sqrt{27}$
4. $\ln \sqrt{e}$ **5.** $\ln(\log 10)$ **6.** $e^{-\ln 7}$
7. $10^{-2(\log 7)}$ **8.** $\log_3\left(\dfrac{3}{\sqrt{27}}\right)$
9. $\log \sqrt{40} - \log 2$

Exercises 10–15 ***Approximations*** *Give an approximation rounded off to three decimal places.*

10. $\log 6$ **11.** $\ln 47$ **12.** $\log(\ln 5)$
13. $\log(e - 1)$ **14.** $e + e^{-1}$ **15.** e^{π}

16. What is the product of the smallest and the largest integers between e^2 and e^3?

17. How many prime numbers are in the set $\{x \mid e < x < e^2\}$?

Exercises 18–19 ***Manipulating Radicals***

18. (a) If $b = \sqrt{2}$ and
$$c = \frac{\sqrt{\sqrt{5} + 2} - \sqrt{\sqrt{5} - 2}}{\sqrt{\sqrt{5} - 1}},$$
use your calculator to get approximations for b and c.

(b) Is $b = c$? Justify your conclusion.

19. If $c = \dfrac{\sqrt{3}(\sqrt{2} + \sqrt{6})}{3 + \sqrt{3}}$ find a simpler expression for c. (*Hint:* First find c^2.)

Exercises 20–23 ***Function Evaluations*** *Evaluate $f(x)$ at the given values of x. Give the result rounded off to two decimal places.*

20. $f(x) = xe^x$; $-1, \sqrt{2}$

21. $f(x) = x\ln(x-1)$; $3, \sqrt{3}$

22. $f(x) = \log_3 x$; $5, \sqrt{5}$

23. $f(x) = e^x - e^{-x}$; $-1, \pi$

Exercises 24–32 ***Exact Form Solutions*** *Solve algebraically. Give the result in exact form.*

24. $5^{2x+1} = \sqrt{5}$

25. $\log(2x+1) = 1$

26. $\log_5(5^{x-1}) = 2$

27. $\log_7(2x-1) = 1$

28. $9^x = 27^{1-2x}$

29. $\ln x + \ln(x+1) = \ln 2$

30. $1 + \log x = 0$

31. $\ln x - \ln(x-1) = 1$

32. $x + \ln e^x = 2$

Exercises 33–41 ***Decimal Approximations*** *Solve algebraically. Give the result rounded off to two decimal places.*

33. $\ln(2x+1) = 1$

34. $e^x = 10^{1-x}$

35. $5^x = 3(2^x)$

36. $3^{x-3} = 4$

37. $3e^x - 4 = 0$

38. $\log x + \log(x-1) = 1$

39. $\ln(x-e) + \ln(x-1) = 1$

40. $e^x = \ln 2$

41. $e^{2x} + e^x - 2 = 0.$

Exercises 42–47 ***Domain*** *Determine the domain of f. Check with a calculator graph.*

42. $f(x) = \ln(x-2)$

43. $f(x) = \ln(x^2-2x)$

44. $f(x) = \ln e^{x-1}$

45. $f(x) = \ln(e^x-1)$

46. $f(x) = \ln[x(x-1)]$

47. $f(x) = \ln x + \ln(x-1)$

Exercises 48–53 ***Domain, Range*** *Use a graph to help determine the domain and range.*

48. $y = \ln(x-1)$

49. $y = 1 + \ln x$

50. $y = e^{-x}$

51. $y = 1 + e^{-x}$

52. $y = e^{\ln x}$

53. $y = x^2e^{-\ln x}$

Exercises 54–57 ***Intercept Points*** *Use a graph to find the x-intercept point for the graph of f.*

54. $f(x) = \ln x + \ln(x-1)$

55. $f(x) = e^x - 2$

56. $f(x) = e^x - e^{-x} - 1$

57. $f(x) = \ln(2x-4)$

58. Is there a real number x such that
(a) $3^{-x} = 1$? **(b)** $3^{-x} = -1$?
(c) $e^x + 1 = 0$? **(d)** $\ln(-e^{-x}) = 1$?
Give reasons.

59. Is the graph of $y = \ln x + \ln(x-1)$ the same as the graph of $y = \ln(x^2 - x)$?

60. From a graph of $y = 1 + \ln x$, find the x-intercept point, and the solution set for the inequality $1 + \ln x < 0$.

61. Show that $\dfrac{1}{\log_2 x} + \dfrac{1}{\log_3 x} + \dfrac{1}{\log_4 x} = \log_x 24$ for every positive number x not equal to 1.

62. Draw a graph of $f(x) = \ln(x^2 - 6x + 9.5)$
(a) How many zeros does f have?
(b) Find the zeros (1 decimal place).

Exercises 63–64 ***Local Minima*** *Find the exact coordinates of the local minimum point for the graph of f. Check graphically.*

63. $f(x) = \log_2(x^2 - 4x + 12)$

64. $f(x) = \log_3(x^2 - 2x + 10)$

Exercises 65–68 ***Find Inverse*** ***(a)*** *Find a formula for f^{-1}.* ***(b)*** *Draw graphs of f and f^{-1} on the same screen. If the graphs intersect, find the coordinates of the point of intersection (1 decimal place).*

65. $f(x) = 4 - \ln x$

66. $f(x) = 4 - \log_3 x$

67. $f(x) = 1 + 2^{x+1}$

68. $f(x) = e^{1-x} - 3$

Exercises 69–71 *Solve the equation (1 decimal place).*

69. $3 + \ln x = x$

70. $4 + \log_3 x = x$

71. $e^x - 2 = x$

72. The equation $2^x = x^6$ has three roots, one is negative and the other two are positive. Find the largest root (1 decimal place).

73. Describe a strategy for finding the largest root of $3^x = x^6$.

74. Solve the equation algebraically in exact form. Use a graph to support your solution. Explain why the solutions in **(a)** and **(b)** are identical.
(a) $\log_2(x^2 - 4x + 5) = 1$
(b) $\log_2|x^2 - 4x + 5| = 1$

75. A sum of \$800 is invested in a savings account that earns 5 percent interest compounded continuously.
(a) How much is in the account at the end of 10 years?
(b) How long will it take for the account to reach \$5000?

76. How much money should be invested in an annuity that earns 10 percent interest compounded continuously to have an investment worth \$5000 in 8 years?

77. Radioactive iodine-131 has a half-life of 8 days. What percentage of a sample will remain after 3 days? After 20 days?

78. A fossil tree has 75 percent as much carbon-14 as a living tree. How old is the fossil tree?

79. The population of a city increases at the rate of 8 percent yearly. Assuming exponential growth, in how many years will the population double?

80. Two firecrackers produce a sound of 90 dB. What would be the loudness level of one alone?

81. The following table lists measured intensity values for some commonly heard sound sources. Complete the table by entering the corresponding loudness levels. Recall Equation (3) of Section 4.4, $I_0 = 10^{-12}$. Extended exposure to sounds of loudness levels exceeding 90 dB usually results in permanent ear damage and hearing loss.

Source of Sound	*Intensity* (w/m^2)	*Loudness Level (dB)*
Whisper	1×10^{-10}	______
Busy downtown traffic	1×10^{-5}	______
Siren (at 30 meters)	1×10^{-2}	______
Indoor rock concert	1	______
Jet plane (at 30 meters)	100	______

5 TRIGONOMETRIC AND CIRCULAR FUNCTIONS

IN CHAPTER 2 WE INTRODUCED the concept of functions. Chapters 3 and 4 explored important special functions, namely polynomial, exponential, and logarithmic functions. Equally important in both applications and theoretical mathematics are the trigonometric functions we introduce in this chapter.

Trigonometry (meaning "triangle measurement") has a long and remarkable history. Some of its roots and applications go back to antiquity, but it continues to find new applications through the space age and beyond. Trigonometry has provided tools for surveying and navigation for thousands of years. Today it is built into sophisticated devices that, for example, help satellites navigate among the planets or determine how fast the spreading ocean floor is pushing continents apart.

Partly because it has served so many different uses, trigonometry may appear somewhat schizophrenic in its presentation. Triangle and circle measurement commonly use degree measure, while all modern applications of trigonometry that describe periodic phenomena—from tides to orbiting satellites to the wave nature of quantum physics—require functions of real numbers, not degrees. In Section 5.1 we introduce both modes of angle measure because it is important to become familiar with both. In Section 5.2 the trigonometric functions are also defined in both modes.

5.1 ANGLES AND UNITS OF MEASURE

> . . . mathematics, just as all other scientific branches, is developed in the process of examining, verifying, and modifying itself.
>
> *Yi Lin*

> *I intended to take either physics or mathematics . . . and intended to become a high school teacher. I found myself very excited by a course called Physical Measurements. We kept measuring things to more and more decimal places by more and more ingenious methods.*
>
> Frederick Mosteller

The study of plane geometry considers all geometric figures as sets of points in a plane. An angle, for instance, is the union of two rays with a common endpoint. In trigonometry we talk about angles of a triangle as the union of two line segments that have a common endpoint. More critically, however, the *measure* of an angle involves the notion of rotation. For most purposes, we consider an angle as being generated by rotating a ray in the plane about its endpoint, from an initial position to a final position. The initial position is called the **initial side** and the final position is called the **terminal side** of the angle. The point about which the ray rotates is called the **vertex** of the angle. An angle is the union of two rays together with a rotation.

The measure of an angle is described by the amount of rotation. An angle has **positive measure** if the rotation is counterclockwise, and **negative measure** if the rotation is clockwise. For brevity, we say the angle is positive if its measure is positive. Figure 1 illustrates the labeling of angles and rotation. The curved arrow indicates the direction and amount of rotation. Angles A and B are positive while angle is negative. The rotation in angle B is greater than one revolution.

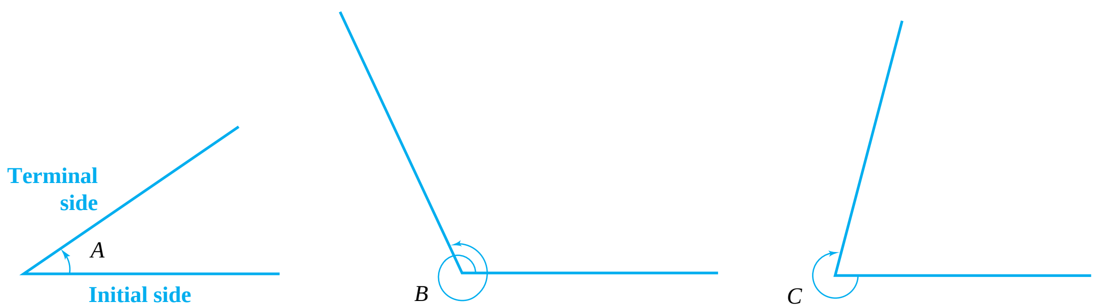


FIGURE 1

Units of Angular Measure

Your calculator operates both in *degree mode* and in *radian mode,* reflecting two different ways of measuring angles. The modes are related by their measures of one complete revolution. The measure of one revolution is 360 degrees, or 2π radians.

Degree measure. In geometry, angles are measured most often in degrees, minutes, and seconds, or decimal fractions of degrees. Degree measure is part of our legacy from Babylonian mathematics, with numeration based on multiples and fractions of 60. Units of time (hours, minutes, seconds) have the same historical basis.

Figure 2 illustrates several angles and their degree measures. For brevity, we write, for example, $A = 90°$ to denote "the measure of angle A is 90°."

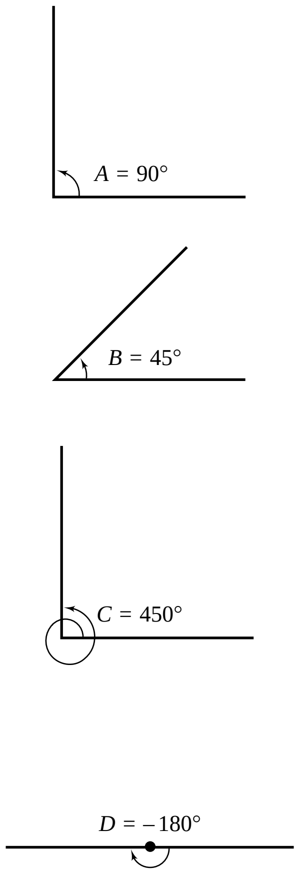


FIGURE 2

Degrees, minutes, and seconds are related by the following:

1 **degree,** written 1°, is $\dfrac{1}{360}$ of a complete rotation.

1 **minute,** written $1'$, is $\dfrac{1}{60}$ of a degree.

1 **second,** written $1''$, is $\dfrac{1}{60}$ of a minute, or $\dfrac{1}{3600}$ of a degree.

Your calculator may allow you to enter angles in degree–minute–second (DMS) form (see your instruction manual), but you can also use the relations above to change between DMS and decimal forms, as in the next example.

►EXAMPLE 1 ***DMS to decimal form*** Express $36°16'23''$ in decimal form and round the result to three decimal places.

Solution

$16'$ is $\frac{16}{60}$ of a degree, and $23''$ is $\frac{23}{3600}$ of a degree, so we have

$$36°16'23'' = \left(36 + \frac{16}{60} + \frac{23}{3600}\right)^{\circ} \approx 36.273°. \quad ◄$$

►EXAMPLE 2 ***Decimal form to DMS*** Express 64.24° in degrees, minutes, and seconds.

Solution

First convert the decimal part, 0.24°, into minutes. Since 1° is $60'$,

$$0.24° = (0.24)(60') = 14.4'.$$

Next convert the $0.4'$ into seconds.

$$0.4' = (0.4)(60'') = 24''.$$

Therefore, 64.24° is $64°14'24''$. ◄

Radian measure. The radian measure of an angle is determined as a ratio of arc length to radius. That is, if we have a segment of length equal to the radius and lay it out along the circle, the central angle is **1 radian.** An arc length of two radii (or a diameter of the circle) measures a central angle of **2 radians.** See Figure 3. Since the total arc length of a circle (its circumference) is $2\pi r$, the *radian measure of one revolution is* 2π *radians.*

Greek letters such as θ (theta) and ϕ (phi) are often used to refer to angles. For an arbitrary angle θ, take a circle of a radius r with center at the vertex, with the initial side meeting the circle at A and the terminal side at B. Think of a point P moving around the circle from A to B. The directed distance s that P travels is the *directed arc length associated with* θ (see Figure 4.) For counterclockwise rotation, s is positive; for clockwise rotation, s is negative. If the rotation is greater than one revolution, the arc length is greater than $2\pi r$ (positive or negative), so s can be any real number. The radian measure of θ is defined as the *ratio of* s *to* r. Note that this definition is independent of any particular circle.

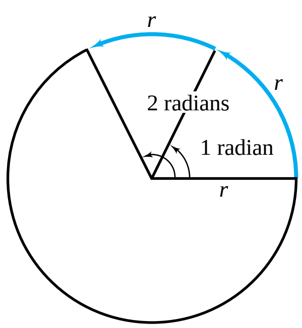


FIGURE 3
An arc length of one radius measures 1 radian

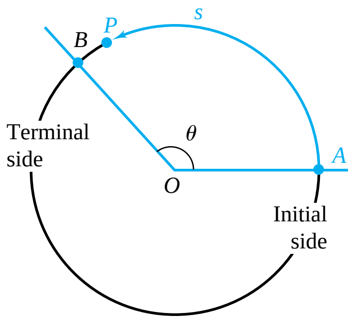


FIGURE 4
Directed arc length s is the distance P travels along the circle from A to B.

Definition: radian measure of an angle

Suppose θ is any angle and C is a circle of radius r with its center at the vertex of θ. If s is the directed arc length associated with θ, then the radian measure of θ is $\frac{s}{r}$; that is,

$$\theta = \frac{s}{r}.$$

In taking the ratio of two lengths, units cancel; thus *radian measure is simply a real number* (no units). If, for example, θ is an angle such that in a circle of radius 2 centimeters, the associated arc length is 2.6 centimeters (see Figure 5). Then the measure of θ in radians is given by

$$\theta = \frac{s}{r} = \frac{2.6 \text{ cm}}{2 \text{ cm}} = 1.3.$$

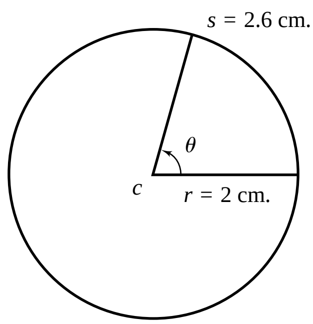


FIGURE 5

The units cancel, and we write simply $\theta = 1.3$. We could emphasize that the radian measure of θ is 1.3 by writing $\theta = 1.3$ radians, but our normal convention is that radians need not be written. *When the measure of an angle is given as a real number, it is understood that the measure is radians.*

▶EXAMPLE 3 ***Radian measure of central angles*** In Figure 6, α and β are central angles of a circle of radius 2. The lengths of the subtended arcs are $s_\alpha = 3.6$ for α and $s_\beta = 13.6$ for β. Determine the measures of α and β in radians.

Solution

$$\alpha = \frac{s_\alpha}{r} = \frac{3.6}{2} = 1.8$$

The measure of α is 1.8 radians.

$$\beta = \frac{s_\beta}{r} = \frac{13.6}{2} = 6.8$$

The measure of β is 6.8 radians. The measure of β is greater than one revolution ($2\pi \approx 6.28$), as the arrow in the figure indicates. ◀

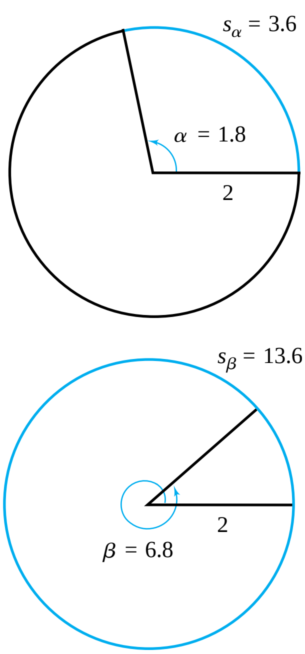


FIGURE 6

Degree–Radian Relationships

In many cases we may have the measure of an angle in degrees when we need the radian measure, or vice versa. This requires a technique for conversion. Since one complete rotation is measured by either 360° or 2π radians, we have the necessary equivalence. The basic relationship $360° = 2\pi$ radians connects degree and radian measures.

Degree–radian conversions

$$180° = \pi \text{ radians.} \qquad \textbf{(1)}$$

$$1° = \frac{\pi}{180} \text{ radians} \quad \text{or} \quad 1° \approx 0.017453 \text{ radians,}$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^{\circ} \quad \text{or} \quad 1 \text{ radian} \approx 57.296°.$$

See Figure 7 for equivalent measures of selected angles.

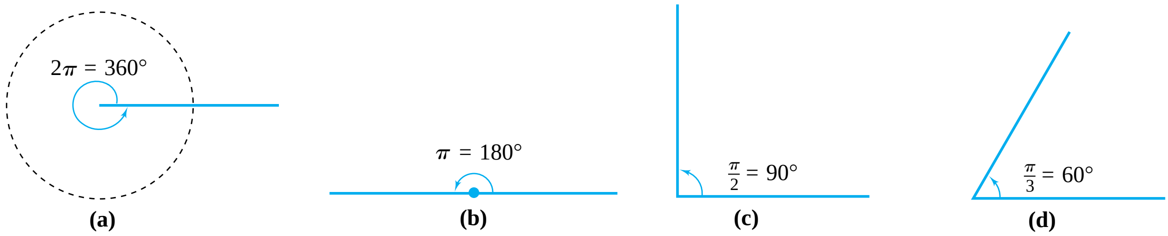


FIGURE 7
Degree–radian measure for some familiar angles.

Most of us are familiar with degree measure, and because radian measure is so important in calculus, we need to learn some convenient equivalences. There are a few angles that are used so often, we need to recognize them immediately in both degree and radian measure:

$$\pi = 180°, \frac{\pi}{2} = 90°, \frac{\pi}{3} = 60°, \frac{\pi}{4} = 45°, \frac{\pi}{6} = 30°.$$

For most of us, a visual reminder of these relations is helpful, as in Figure 8.

In addition to thinking in terms of fractions of π radians, you need to develop a feeling for radian measure expressed simply as numbers. For example, 1 radian is about 57.3°, almost 60°. Similarly, an angle of 3 radians is very nearly a straight angle (remember that $3.14 \approx \pi = 180°$). See Figure 9. Right angles are so common that the decimal approximation of $\pi/2$ as 1.57 will become very familiar.

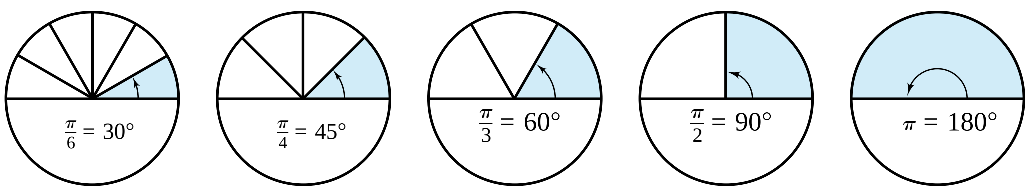


FIGURE 8
Degree and radian measure for some familiar angles.

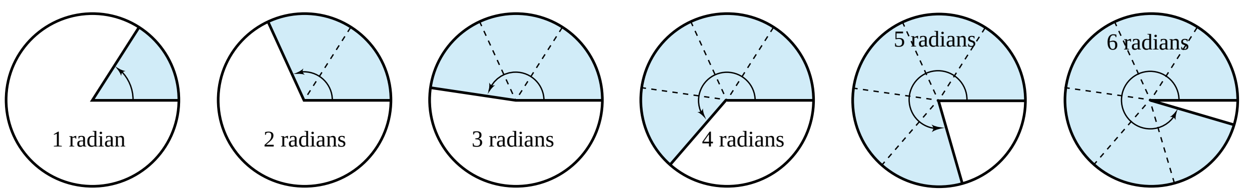


FIGURE 9
Integer multiples of 1 radian.

▶*EXAMPLE 4* ***Degree–radian conversion*** Draw a diagram that shows the angle and then find the corresponding radian measure. Give the result both in exact form and as a decimal approximation rounded off to two decimal places.

(a) $\alpha = 210°$ **(b)** $\beta = 585°$ **(c)** $\gamma = -150°$

Solution

Diagrams in Figure 10 show the angles. To convert from degree measure to radian measure, multiply by $\frac{\pi}{180}$.

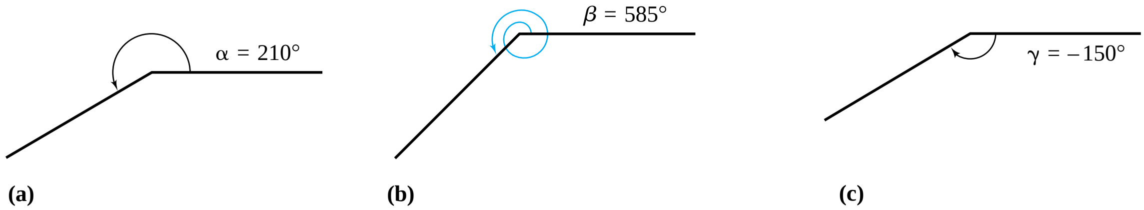


FIGURE 10

(a) $\alpha = 210° = 210\left(\frac{\pi}{180}\right) = \frac{7\pi}{6} \approx 3.67$

(b) $\beta = 585° = 585\left(\frac{\pi}{180}\right) = \left(\frac{13\pi}{4}\right) \approx 10.21$

(c) $\gamma = -150° = -150\left(\frac{\pi}{180}\right) = -\frac{5\pi}{6} \approx -2.62$ ◀

▶***EXAMPLE 5*** ***Radian–degree conversion*** If the radian measure of θ is 2.47, find its degree measure rounded off to one decimal place and then to the nearest minute.

Strategy: To get the degree measure, multiply by $\frac{180}{\pi}$. Convert the decimal part of a degree to minutes by multiplying by 60.

Solution

Follow the strategy.

$$2.47 \text{ radians} = 2.47\left(\frac{180}{\pi}\right)^{\circ} \approx 141.5205754°.$$

Rounded off to one decimal place, 2.47 radians $\approx$ 141.5°; to the nearest minute, 2.47 radians $\approx$ 141°31′. Hence, θ is approximately 141°31′. ◀

Applications of Radian Measure: Arc Length and Area

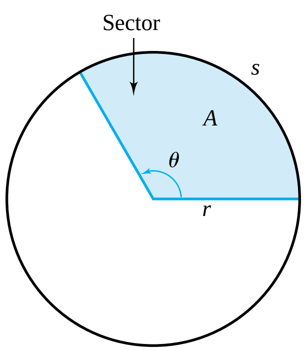


FIGURE 11

In a circle a given central angle between 0 and 2π determines a portion of the circle called a **sector,** as indicated by the shaded region shown in Figure 11. For the sector shown in the figure with central angle θ, suppose the length of the subtended arc is s and the area of the sector is A. The ratios of θ, s, and A to the respective measures 2π, $2\pi r$, and πr^2 for the entire circle are equal, that is,

$$\frac{\theta}{2\pi} = \frac{s}{2\pi r}, \qquad \frac{\theta}{2\pi} = \frac{A}{\pi r^2}, \qquad \frac{s}{2\pi r} = \frac{A}{\pi r^2}.$$

Solving for s and A ,

$$s = r\theta, \qquad A = \frac{1}{2}r^2\theta, \qquad A = \frac{1}{2}rs.$$

Arc length and the area of a circular sector

Suppose θ is a central angle of a circle of radius r. Let s denote the length of the subtended arc and let A denote the area of the sector. If θ is measured in radians, then s and A are given by

$$s = r\theta \tag{2}$$

$$A = \frac{1}{2}r^2\theta \qquad A = \frac{1}{2}rs \tag{3}$$

▶EXAMPLE 6 ***Arc length/area of circular sector*** The radius of a circle is 3.6 centimeters and the central angle of a circular sector is 150°. Draw a diagram to show the sector and find the arc length and the area of the sector.

Strategy: Equations (2) and (3) require θ to be in radians. First convert 150° to radians and then use Equations (2) and (3).

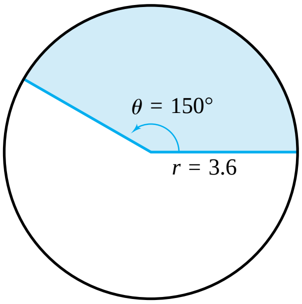


FIGURE 12

Solution

The sector is the shaded region in Figure 12. Following the strategy,

$$\theta = 150° = 150\left(\frac{\pi}{180}\right) = \frac{5\pi}{6}.$$

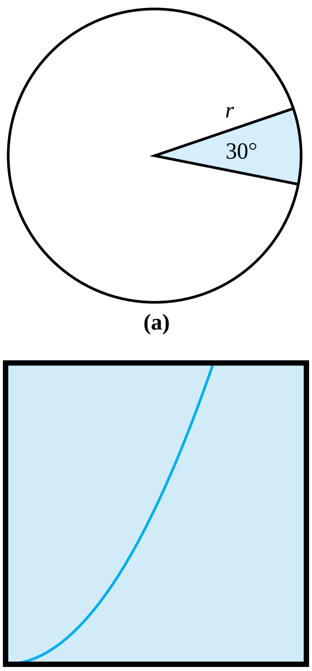
Substitute 3.6 for r and $\frac{5\pi}{6}$ for θ in Equations (2) and (3)

$$s = 3.6\left(\frac{5\pi}{6}\right) = 3\pi \approx 9.42,$$

$$A = \frac{1}{2}(3.6)^2\left(\frac{5\pi}{6}\right) = 5.4\pi \approx 16.96.$$

Round off to two significant digits to get an arc length of 9.4 cm and an area of 17 cm^2. ◀

▶EXAMPLE 7 ***Area as a function*** Given a circular sector with central angle of 30°. **(a)** Give a formula for the area A of the sector as a function of the radius r. Draw a graph of $A(r)$ in the window $[0, 20] \times [0, 50]$. **(b)** Evaluate A when $r = 12.7$ (one decimal place). **(c)** Find r when $A = 25.6$ (one decimal place).



[0, 20] by [0, 50]
(b)
$A = \dfrac{\pi x^2}{12}$

FIGURE 13

Solution

(a) Since the formula for area (Equation (3)) requires radian measure for the central angle, we use $30° = \pi/6$, and the area (see Figure 13a) is given by

$$A(r) = \frac{\pi}{12}r^2$$

Graphing Y = πX^2/12 in the specified window gives us a calculator graph as shown in Figure 13b, clearly part of a parabola.

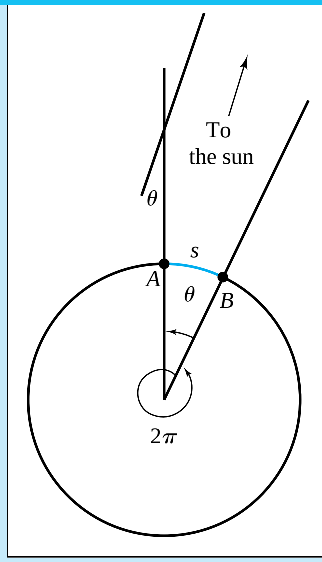
(b) Tracing on the calculator graph does not allow us to read with sufficient accuracy the value of A when $r = 12.7$. We can zoom in for more accuracy, or we can return to the home screen and evaluate $\pi(12.7)^2/12$. Either way, we get an area of approximately 42.2.

(c) If $\pi r^2/12 = 25.6$, then $r = \sqrt{(12)(25.6)/\pi} \approx 9.9$. ◀

▶EXAMPLE 8 ***Circular motion*** Assume that the moon travels around the earth in a circular path of radius 239,000 miles and that it makes one complete revolution every 28 days.

(a) Find a formula for the distance $D(x)$ (in thousands of miles) that the moon travels in x days. Draw a calculator graph. How far does the moon travel **(b)** in 10 days? in 21 days and 6 hours? **(c)** How many days does it take for the moon to travel a million miles? A billion miles?

HISTORICAL NOTE MEASUREMENT OF THE CIRCUMFERENCE OF THE EARTH



One of the earliest and most dramatic applications of trigonometry was made by Eratosthenes a little before 200 B.C. As his name suggests, Eratosthenes was of Greek descent, but he spent his life in Egypt during the reign of Ptolemy II and later became the head of the greatest scientific library in the ancient world at Alexandria.

Travelers reported that at Syene (the modern city Aswan), the sun cast no shadow at noon on the summer solstice (the longest day of the year). Eratosthenes reasoned, then, that at Syene on that date, the sun's rays were coming directly toward the center of the Earth. Alexandria was supposed to be directly north of Syene. By measuring the angle of the sun's rays at Alexandria at noon on the same day, Eratosthenes realized that he could use geometric relationships to find the circumference of the Earth.

In the diagram, A represents Alexandria and B, Syene. The ratio of the distance from A to B, the arc length s, to the entire circumference C must equal the ratio of angle θ to the entire central angle (a complete revolution).

Symbolically,

$$\frac{s}{C} = \frac{\theta}{\text{One revolution}}$$

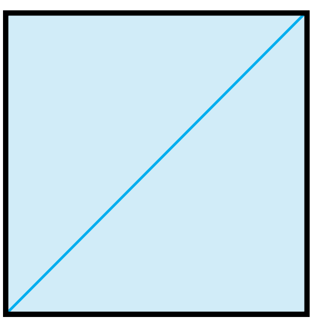
By the best estimates of the day, the distance from Alexandria to Syene was 5000 stadia, a distance estimated from travel by camel caravans by surveyors trained to count paces of constant length. Eratosthenes measured angle θ at Alexandria to be $\frac{1}{50}$ of a complete revolution, leading to the equation,

$$\frac{5000}{C} = \frac{1}{50},$$

from which he calculated the circum-circumference of the earth to be

$$C = 50 \cdot 5000 = 250{,}000 \text{ stadia.}$$

Comparison with modern measurements is difficult because the measuring unit, the stadium, varied in size, but Eratosthenes' estimate would compare to something near 24,000 miles. Several compensating errors probably contributed to the truly remarkable accuracy of his figure, but the real genius Eratosthenes lay in his analysis of the problem and his recognition that geometric figures can tell us something about the nature of the world that we can learn in no other way.



[0, 30] by [0, 1600]

$D(x) = \frac{239\pi x}{14}$

FIGURE 14

Solution

(a) The distance traveled in one revolution is $2\pi r$, or $478{,}000\pi$ miles (≈ 1.50 million), every 28 days. One day's distance is $2\pi r/28$, where the radius is 239 thousand miles, from which the number of thousands of miles is given by

$$D(x) = \frac{239\pi}{14}x.$$

A calculator graph in $[0, 30] \times [0, 1600]$ is shown in Figure 14.

(b) Either from the graph or from the formula, $D(10) \approx 536{,}000$ miles. Six hours is a quarter of a day; in 21 days, 6 hours, $D(21.25) \approx 1{,}140{,}000$ miles.

(c) A million miles is a thousand thousand ($10^6 = 10^3 10^3$), so we want to solve for x when $D(x) = 1000$: $x = 14 \cdot 1000/(239\pi) \approx 18.6$ days, less than three weeks. A billion miles is $10^9/10^3 = 10^6$ thousands. When $D(x) = 10^6$, $x = 14{,}000{,}000/(239\pi) \approx 18{,}600$ days. Assuming 365 days in a year, it will take more than 51 *years* for the moon to travel a billion miles in circling the earth. ◀

▶EXAMPLE 9 ***Making a paper cup*** Draw a circle of radius 4 on a piece of paper. Cut from a point A on the circle to the center O. You can make a conical cup by sliding OA to any other radius OB, effectively cutting out a sector with central angle $x = \angle AOB$. See Figure 15.

(a) Using the circumference of the cone given by $C(x) = 4(2\pi - x)$, as in the diagram, find a formula for the radius $r(x)$ at the top of the cone and the height $h(x)$.
(b) Express the volume $V(x)$ as a function of x and draw a calculator graph.
(c) Find the approximate value of x giving the maximum volume. What is the maximum volume?

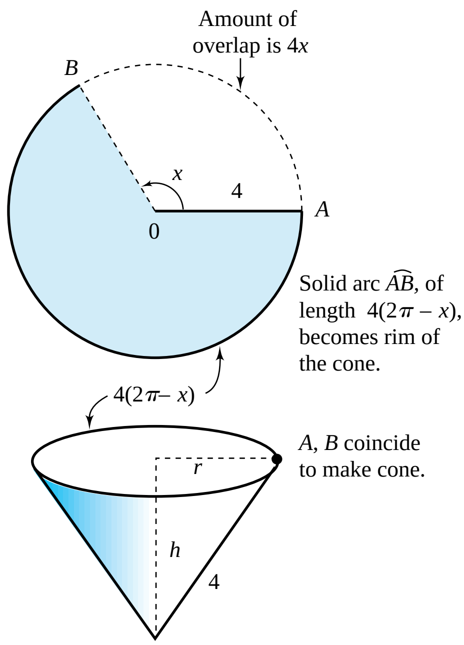


FIGURE 15

Solution

(a) For the top of the cup, $C = 2\pi r$, so

$$r(x) = C(x)/2\pi = \frac{4(2\pi - x)}{2\pi} = \frac{2}{\pi}(2\pi - x).$$

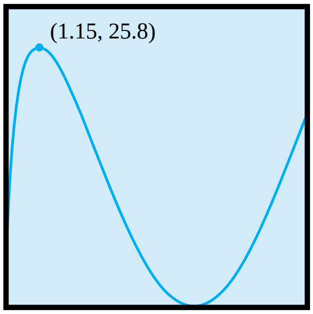
Looking at a cross-section of the cup, r and h are legs of a right triangle with hypotenuse 4 (the radius of the original circle), so $r^2 + h^2 = 4^2$.

$$h^2 = 16 - \frac{4}{\pi^2}(2\pi - x)^2 = 16 - \frac{4}{\pi^2}(4\pi^2 - 4\pi x + x^2),$$

or

$$h^2 = \frac{4(4\pi x - x^2)}{\pi^2}.$$

Thus, $h(x) = \dfrac{2}{\pi}\sqrt{4\pi x - x^2}$.



[0, 10] by [0, 30]

FIGURE 16

(b) The volume of a cone is a third of the volume of the cylinder with the same base, or $V = \frac{1}{3}\pi r^2 h$, so we have

$$V(x) = \frac{\pi}{3}(r(x))^2 h(x) = \frac{\pi}{3}\frac{4}{\pi^2}(2\pi - x)^2 \frac{2}{\pi}\sqrt{4\pi x - x^2}.$$

Thus,

$$V(x) = \frac{8}{3\pi^2}(2\pi - x)^2\sqrt{4\pi x - x^2},\ 0 < x < 2\pi.$$

(c) Graphing V in $[0, 10] \times [0, 30]$ gives the graph shown in Figure 16. Tracing and zooming to find the high point, we find that the maximum volume is about 25.8 cubic inches when x is about 1.15 radians, or about 66°. A volume of 25.8 cubic inches is just over 14 ounces. We suggest that you make a cone with $x \approx 66°$ and observe that your cone is not standard. Since that is the shape with the maximum volume and the least waste (overlap), why do you suppose paper cups are not made to hold the maximum volume? ◀

Linear and Angular Speed

There are two kinds of speeds associated with rotational motion. To introduce the basic ideas, consider an example. Suppose a bicycle wheel is rotating at a constant rate of 40 revolutions per minute (40 rpm). One measure of speed, **angular speed,** gives the rate of rotation, frequently denoted by the Greek letter ω. The bicycle wheel's angular speed is 40 rpm by one measure. Since one revolution is equivalent to 2π radians, the angular speed can also be expressed as $40(2\pi)$ or 80π radians per minute, or 4800π radians per hour.

Suppose the diameter of the bicycle wheel is 26.4 inches, so that its radius r is 13.2 inches or 1.1 feet. In 1 minute a point on the circumference turns through an angle θ equal to 80π radians, and the distance s traveled by point on the circumference in 1 minute is

$$s = r\theta = (1.1)(80\pi) = 88\pi.$$

Hence, the point travels 88π feet in 1 minute. The **linear speed** of a point on the circumference is $88\pi \frac{\text{ft}}{\text{min}} \approx 276$ ft/min.

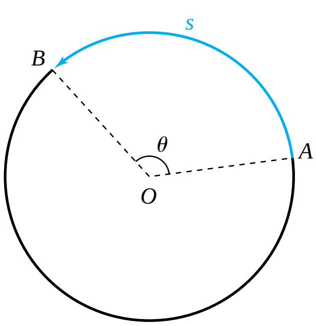


FIGURE 17

Relationship between linear and angular speeds. For a particle moving in a circular path at a uniform rate, the linear and angular speeds are obviously related. Suppose that such a particle P moves from point A to point B along a circle of radius r, as indicated in Figure 17. If the central angle AOB is θ, the distance (arc length) from A to B is s, and P moves from A to B in time t, then the linear and angular speeds are

$$v = \frac{s}{t} \quad \text{and} \quad \omega = \frac{\theta}{t}.$$

Since $s = r\theta$ (where θ is measured in radians),

$$v = \frac{s}{t} = \frac{r\theta}{t} = r\left(\frac{\theta}{t}\right) = r\omega$$

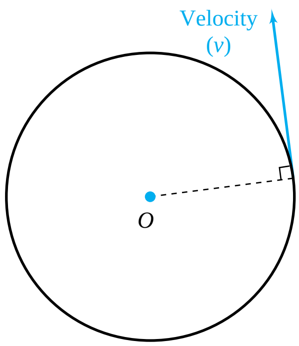


FIGURE 18

Linear and angular speed relationship

For a particle moving in a circular path of radius r at a uniform rate, the linear speed v and angular speed ω are related by the equation

$$v = r\omega. \qquad \textbf{(4)}$$

Linear speed and angular speed are sometimes called linear velocity and angular velocity, but we reserve the term *velocity* for directed speeds. Velocity is a vector quantity, meaning that it has both a magnitude and a direction (see Section 7.5). In uniform circular motion, the magnitude of the linear velocity vector is the linear speed defined above, in a direction tangent to the circular path of motion, as indicated by the arrow in Figure 18.

Strategy: First convert r and ω to units that are consistent with feet per second.

►EXAMPLE 10 Linear and angular speed The wheel of a grindstone with a radius of 8 inches is rotating at 300 rpm.

(a) Find the angular speed in radians per second.
(b) Find the linear speed of a point on the circumference of the wheel in feet per second.
(c) For a certain job, it is desirable to have the linear speed of the grinding edge of the wheel at 30 ft/sec. What change in angular speed (in revolutions per minute) is required?

Solution

Given that r is 8 inches and ω is 300 rpm, follow the strategy and express r and ω in units of feet and seconds.

$$r = 8 \text{ in} \times \left(\frac{1 \text{ ft}}{12 \text{ in}}\right) = \frac{2}{3}\text{ft}$$

and

$$\omega = \frac{300 \text{ rev}}{\text{min}} \times \left(\frac{2\pi \text{ rad}}{\text{rev}}\right) \times \left(\frac{1 \text{ min}}{60 \text{ sec}}\right) = 10\pi \text{ rad/sec}$$

(a) The angular speed is $10\pi \frac{\text{rad}}{\text{sec}}$.
(b) Using Equation (4),

$$v = r\omega = \left(\frac{2}{3}\right)(10\pi) = \frac{20\pi}{3} \approx 20.9.$$

Hence, the linear speed of a point on the grinding edge of the wheel is 20.9 ft/sec.

(c) For v to be $30 \frac{\text{ft}}{\text{sec}}$, specify $r\omega = 30$, or

$$\omega = \frac{30}{r} = \frac{30}{\frac{2}{3}} = 45 \text{ rad/sec}.$$

To express ω in revolutions per minute, convert radians per second.

$$\omega = 45\frac{\text{rad}}{\text{sec}} \times \left(\frac{1\text{ rev}}{2\pi\,\text{rad}}\right) \times \left(\frac{60\text{ sec}}{\text{min}}\right) = \frac{1350\text{ rev}}{\pi\,\text{min}} \approx 430\text{ rpm}$$

For a linear speed of 30 ft/sec, the wheel speed must increase to about 430 rpm, almost half again as fast as the present angular speed. ◀

Does this answer make sense? Always ask yourself if a solution is reasonable and make an independent check or a simple estimate, if possible. In Example 10, for instance, we found in part **(b)** that when ω is 300 rpm, the speed is near 20 ft/sec. Therefore, in part **(c)** an angular speed of 30 ft/sec should correspond to about $\frac{3}{2}$ of 300 rpm, giving an estimate of about 450 rpm. The result of 430 rpm in part **(c)** is entirely reasonable.

EXERCISES 5.1

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. An angle of $\frac{22}{7}$ radians is equal to an angle of 180°.
2. An angle of 180° is greater than an angle of 3.16 radians.
3. The sector of a circle with a central angle of 1 radian and radius r cm has an area of $\frac{1}{2}r^2$ cm².
4. If the central angle θ of a circle measures 1 radian, then the length of the arc subtended by θ is equal to the length of the radius.
5. Assume that the planets travel in circular orbits about the sun. If the planet Venus takes 225 days for one revolution about the sun and the Earth takes 365 days, then the angular speed of Venus is less than the angular speed of the Earth.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. An angle of $\dfrac{5\pi}{4}$ is equal to an angle of ______ degrees.
7. An angle of 210° is equal to an angle of ______ radians.
8. In a circular sector, if $s = 48$ and $r = 24$, then $\theta =$ ______ radians.
9. In a circular sector, if $s = 12$ and $\theta = 60°$, then $r =$ ______ .
10. An angular speed of 5 rpm is equal to ______ $\frac{\text{rad}}{\text{min}}$.

Develop Mastery

Unless otherwise specified, results given as decimal approximations should be rounded off to the number of significant digits consistent with the given data.

Exercises 1–4 Draw a diagram to show the angle. Include a curved arrow to indicate the amount and direction of rotation from the initial side to the terminal side.

1. **(a)** $A = 240°$ **(b)** $B = 720°$ **(c)** $C = -210°$
2. **(a)** $A = 540°$ **(b)** $B = -135°$ **(c)** $C = 67°30'$
3. **(a)** $A = \dfrac{2\pi}{3}$ **(b)** $B = -\dfrac{7\pi}{4}$ **(c)** $C = 1.8$
4. **(a)** $A = \dfrac{5\pi}{3}$ **(b)** $B = -3\pi$ **(c)** $C = -2.36$

Exercises 5–6 Sketch an angle θ that satisfies the inequality. Include a curved arrow and also illustrate the range of position for the terminal side with dashed rays.

5. **(a)** $\dfrac{\pi}{2} < \theta < \pi$ **(b)** $-\pi < \theta < -\dfrac{\pi}{2}$
   (c) $1.7 < \theta < 2.5$
6. **(a)** $\dfrac{3\pi}{4} < \theta < \pi$ **(b)** $-\dfrac{5\pi}{5} < \theta < -\pi$
   (c) $0.79 < \theta < 1.05$

Exercises 7–8 ***DMS to Decimal*** *Express the angle as a decimal number of degrees rounded off to three decimal places.*

7. **(a)** 23°38′ **(b)** 143°16′23″ **(c)** −95°31′
8. **(a)** 57°34′ **(b)** 241°15′51″ **(c)** −73°43′

Exercises 9–10 Find the radian measure of the angle and give the result in both exact form (involving the number π) and decimal form rounded off to two decimal places.

9. **(a)** 60° **(b)** 330° **(c)** 22°30′ **(d)** 105°
10. **(a)** 90° **(b)** 450° **(c)** 67°30′ **(d)** −165°

Exercises 11–12 ***Radians to Degrees*** *Express the angle in decimal degree form, rounded off, if necessary, to one decimal place.*

11. **(a)** $\frac{2\pi}{3}$ **(b)** $\frac{5\pi}{12}$ **(c)** 4π **(d)** 3.6

12. **(a)** $\frac{7\pi}{4}$ **(b)** $\frac{11\pi}{12}$ **(c)** -5π **(d)** 5.4

Exercises 13–16 ***Ordering Angles*** *Order angles α, β, and γ from smallest to largest (as, for example, $\alpha < \gamma < \beta$).*

13. $\alpha = 47°24'$, $\beta = 47.48°$, $\gamma = 0.824$

14. $\alpha = 154°35'$, $\beta = 154.32°$, $\gamma = 2.705$

15. $\alpha = \frac{22}{7}$, $\beta = \frac{355}{113}$, $\gamma = \pi$

16. $\alpha = 120°36'$, $\beta = 120.53°$, $\gamma = \frac{21}{10}$

Exercises 17–20 ***Triangle Angles*** *Two of the three angles A, B, and C of a triangle are given. Find the third angle. Remember that the sum of the three angles of any triangle is equal to* 180° *(or π).*

17. $A = 58°$, $B = 73°$

18. $B = 37°41'$, $C = 84°37'$

19. $A = \frac{\pi}{4}$, $C = \frac{5\pi}{12}$

20. $A = \frac{2\pi}{3}$, $B = \frac{\pi}{15}$

Exercises 21–26 ***Arc Length, Area*** *The radius r and the central angle θ of a circular sector are given. Draw a diagram that shows the sector and determine* ***(a)*** *the arc length s and* ***(b)*** *the area A for the sector.*

21. $r = 24$, $\theta = 30°$

22. $r = 32.1$, $\theta = 96.3°$

23. $r = 164$, $\theta = 256°$

24. $r = 47$, $\theta = \frac{3\pi}{5}$

25. $r = 36$, $\theta = 4.3$

26. $r = 16.2$, $\theta = \frac{7\pi}{8}$

27. The radius of a circular sector is 12.5 centimeters and its area is 182 square centimeters. Find the central angle **(a)** in radians and **(b)** in degrees.

28. What is the radius of a circular sector with central angle 37.5° and area 6.80 square feet?

29. Assume that the Earth travels a circular orbit of radius 93 million miles about the sun and that it takes 365 days to complete an orbit.
(a) Through what angle (in radians) will the radial line from the sun to the Earth sweep in 73 days?
(b) How far does the Earth travel in its orbit about the sun in 73 days?

30. The diameter of a bicycle wheel is 26 inches. Through what angle does a spoke of the wheel rotate when the bicycle moves forward 24 feet? Give your result in radians to two significant digits.

31. What is the measure in degrees of the smaller angle between the hour and minute hands of a clock **(a)** At 2:30? **(b)** At 2:45?

32. At what times to the nearest tenth of a minute between 1:00 and 2:00 is the smaller angle between the hour and minute hands 15°?

33. The minute hand of a clock is 6 inches long.
(a) How far does the tip of the hand travel in 15 minutes?
(b) How far does the tip of the hand travel between 8:00 A.M. and 4:15 P.M. of the same day?

34. **(a)** What is the linear speed (in inches per hour) of the tip of the minute hand in Exercise 33?
(b) What is the linear speed of a point 1 inch from the tip of the minute hand?

35. What is the angular speed in radians per minute of
(a) the hour hand of a clock?
(b) the minute hand?

36. ***Nautical Mile*** A nautical mile is the length of an arc of a great circle subtended on the surface of the Earth by an angle of one minute $(1')$ at the center of the Earth. Assuming that the Earth is a sphere of radius 3960 miles, a nautical mile is equal to how many ordinary miles (5280 ft)?

37. ***Speed in Knots*** A ship is traveling along a great circle route at a speed of 20 knots.
(a) How fast is it moving in miles per hour?
(b) How far does it travel in 4 hours in nautical miles? In ordinary miles?
(c) Through what angle does a line from the center of the Earth to the ship revolve in 4 hours? (*Hint:* If you are not familiar with the word "knots," look it up in the dictionary.)

38. A circular sector with central angle 90° is cut out of a circular piece of tin of radius 15 inches. The remaining piece is formed into a cone (see Example 9). Find the volume of the cone.

39. Repeat Exercise 38 reducing the central angle of the sector cut out of the piece of tin to 60°.

40. A circular piece of tin of radius 12 inches is cut into three equal sectors, each of which is then formed into a cone.
(a) What is the height of each cone?
(b) What is the volume of each cone?

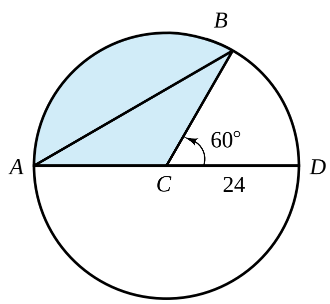
41. In Example 9 find the value of x (2 decimal places) for which $V = 20.7$ cubic inches.

42. ***Looking Ahead to Calculus*** Using calculus, we can show that the x-value in Example 9 giving a maximum volume is a root of the equation

$$3x^2 - 12\pi x + 4\pi^2 = 0.$$

Solve for x. Does this agree with the value of x found in Example 9?

43. In the diagram C is the center and AD is a diameter of the circle with radius 24 cm. $\angle BCD$ measures 60°. Find in exact form the area of **(a)** $\triangle ABC$, **(b)** circular sector BCD, **(c)** the shaded region.



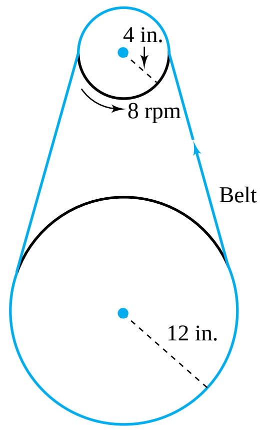
44. A satellite travels in a circular orbit 140 miles above the surface of the Earth. It makes one complete revolution every 150 minutes.
(a) What is its angular speed in revolutions per hour and in radians per hour?
(b) What is its linear speed? Assume that the radius of the Earth is 3960 miles.

45. The face of a windmill is 4.0 meters in diameter and a wind is causing it to rotate at 30 rev/min. What is the linear speed of the tip of one of the blades (in meters per minute)?

46. Assume that the moon follows a circular orbit about the Earth with a radius of 239,000 miles and that one revolution takes 27.3 days. Find the linear speed (in miles per hour) of the moon in its orbit about the Earth.

47. Assume that the Earth travels about the sun in a circular orbit with a radius of 93 million miles and that one revolution takes 365 days. Find the linear speed (in miles per hour) of the Earth in its orbit about the sun.

48. Two pulleys, one with a radius of 4 inches and the other with a radius of 12 inches, are connected by a belt (see the diagram). If the smaller pulley is being driven by a motor at 8 rev/min,
(a) determine the angular speed of the larger pulley (in revolutions per minute).
(b) What is the linear speed of a point on the belt?



49. The diameter of a bicycle wheel is 26 inches. When the bicycle moves at a speed of 30 mph, determine the angular speed of the wheel in revolutions per minute.

50. To measure the approximate speed of the current of a river, a circular paddle wheel with a radius of 3 feet is lowered into the water just far enough to cause it to rotate. If the wheel rotates at a speed of 12 rev/min, what is the speed of the current in miles per hour?

51. The blade of a rotary lawnmower is 34 cm long and rotates at 31 rad/sec.
(a) What is the blade's angular speed in revolutions per minute?
(b) What is the linear speed (in kilometers per hour) of the tip of the blade?

52. A record was set in rope turning with 49,299 turns in 5 hours and 33 minutes.
(a) What is the average angular speed of the rope in revolutions per minute?
(b) Assuming that the rope forms an arc so that its midpoint travels in a circular path of radius 3.5 feet, what is the average linear speed (in feet per minute) of the midpoint of the rope?
(c) How far (in miles) did the midpoint travel during the record-setting turning session?

5.2 TRIGONOMETRIC FUNCTIONS AND THE UNIT CIRCLE

> Since Plato, it has not been uncommon to regard mathematics as composed of divine, eternal, perfect, absolute, certain, infallible, immutable, necessary, a priori, exact, and self-evident truths or ideal forms existing in their own world.
>
> *W. G. Holladay*

> *Somehow I obtained some popular books on mathematics, and about a year later I was helping my brother and sisters with their math homework. When Sylvia started college and was taking trigonometry, she would ask me for help. I would read the section and then figure out how to do the problems.*
>
> Paul Cohen

Chapter 3 was devoted to polynomial functions and Chapter 4 to exponential and logarithmic functions. Now we introduce a third major class of functions called the **trigonometric functions.** Evidence of the importance of trigonometric functions appears on scientific calculators, which have keys that allow evaluation of trigonometric functions and their inverses. Similarly, computer programming languages have built-in capabilities to handle the same functions.

Historically, trigonometry was developed to solve problems in navigation, agriculture, and surveying using triangles. The very word *trigonometry* refers to triangle measurement. The use of triangles is still vitally important throughout physics, engineering, and other disciplines, but what one author called the "ingenious and enduring usefulness" of trigonometric functions depends on much broader applications, many of which have no relation at all to triangles. Analysis of wave motion in electronics, engineering, and quantum mechanics requires trigonometric functions as does the study of economic cycles and other cyclical phenomena.

Leonhard Euler is responsible for the modern concept of a function, which he introduced in his book *Introductio in Analysin Infinitorum* (1748). His ideas have continued to gain importance through the centuries, and we define the trigonometric functions in much the same way as Euler did, by the use of the unit circle.

The Unit Circle

The circle with its center at the origin and a radius of 1 is called the **unit circle.** Its equation is

$$x^2 + y^2 = 1.$$

We define trigonometric functions in terms of coordinates of points on the unit circle. The point $A(1, 0)$ is called the **initial point.** Starting at point A, think of a point moving around the unit circle along an arc of length θ. $P(\theta)$ is called the **terminal position point.** Each real number θ determines $P(\theta)$ uniquely according to the following. See Figure 19.

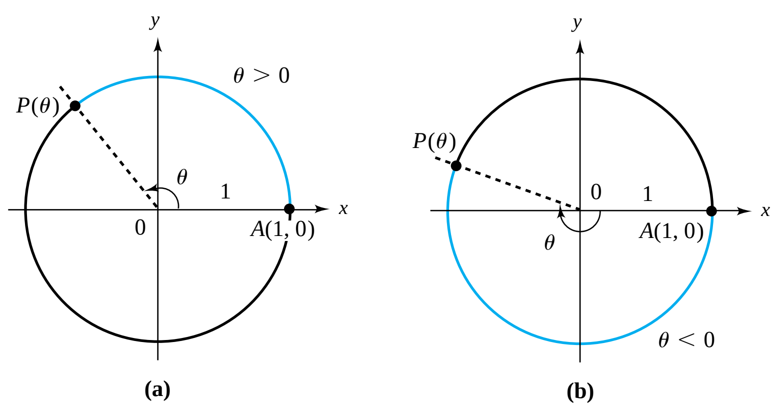


FIGURE 19

Let θ be any given real number.

If $\theta > 0$, then $P(\theta)$ is the point reached by moving counterclockwise on the unit circle a distance of θ units from $A(1, 0)$.

If $\theta < 0$, then $P(\theta)$ is the point reached by moving clockwise along the unit circle a distance of $-\theta$ units from $A(1, 0)$.

If $\theta = 0$, then $P(\theta)$ is point $A(1, 0)$.

As the position point P moves around the unit circle, the ray OP rotates through a central angle, $\angle AOP$. The radian measure of $\angle AOP$ is the ratio of arc length to radius, and in the unit circle, $r = 1$. Thus, when the arc length is some number θ, the radian measure of $\angle AOP$ is given by

$$\angle AOP = \frac{\theta}{r} = \frac{\theta}{1} = \theta.$$

That is, *on the unit circle, arc length* and the *central angle* are measured by the *same number*. Therefore we can think about the location of $P(\theta)$ in either of two ways:

(i) $P(\theta)$ is the point obtained by moving around the unit circle the **directed distance $\boldsymbol{\theta}$** from $A(1, 0)$.

(ii) $P(\theta)$ is the point where the terminal side of $\angle AOP$ of **radian measure $\boldsymbol{\theta}$** meets the unit circle.

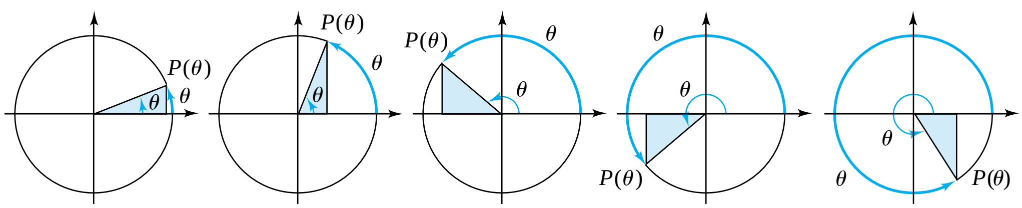


FIGURE 20
Position points and associated reference triangles.

As a position point moves around the unit circle, we want to visualize a triangle moving along with it. If $P(\theta)$ is not on one of the coordinate axes, a perpendicular dropped from $P(\theta)$ to the x-axis forms a right triangle with a hypotenuse of the unit radius OP. The triangle thus formed is called a **reference triangle** for θ. Figure 20 shows several position points with shaded reference triangles.

To make things more concrete, we suggest that you visualize in your mind how the reference triangle changes shape as $P(\theta)$ moves around the circle. Beginning at $A(1, 0)$, the reference triangle starts very flat, becomes isosceles at $\theta = \pi/4$, gets taller and skinnier as $P(\theta)$ nears the y-axis (where the reference triangle disappears), and then reappears on the other side. After $P(\theta)$ crosses the negative x-axis, the reference triangle lies below the x-axis until $P(\theta)$ has made a complete circuit around to $A(1, 0)$ again, as suggested in Figure 20.

Since the distance around the unit circle is the circumference of the circle, which is 2π, we can easily identify the coordinates of $P(\theta)$ for a number of values of θ. The distance around a quarter-circle is $\frac{2\pi}{4}$, or $\frac{\pi}{2}$. If we label the points $B(0, 1)$, $C(-1, 0)$, and $D(0, -1)$ where the unit circle meets the coordinate axes, then we know that $P(\frac{\pi}{2})$ is point B, and $P(\frac{-\pi}{2})$ is point D (see Figure 21a). Similarly, the arc length along the circle from A to C, in either direction, is π and so both $P(\pi)$ and $P(-\pi)$ are point C as in Figure 21b.

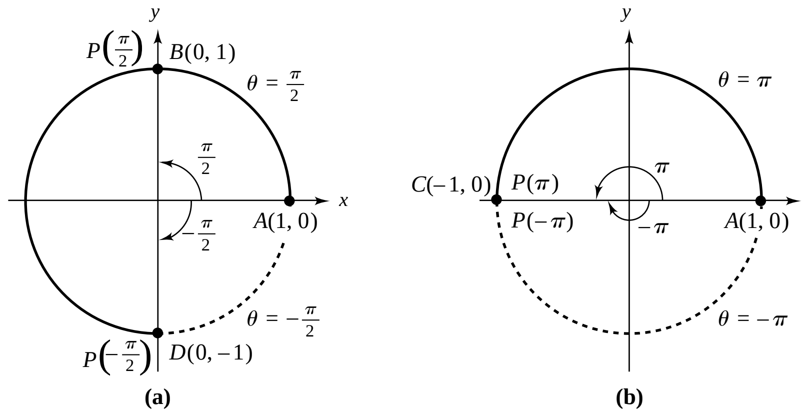


FIGURE 21

When we know that the coordinates of a particular $P(\theta)$ are, say (x_1, y_1), then $P(\theta) = (x_1, y_1)$. Thus, for the points shown in Figure 21,

$$P\left(\frac{\pi}{2}\right) = (0, 1) \qquad P\left(-\frac{\pi}{2}\right) = (0, -1) \qquad P(\pi) = P(-\pi) = (-1, 0).$$

It is clear that point $A(1, 0)$ is the terminal position for $\theta = 0$, that is, $P(0) = (1, 0)$, but notice also $(1, 0) = P(2\pi) = P(-4\pi)$ going around counterclockwise once or clockwise twice. In fact, $P(\theta) = (1, 0)$ whenever θ is an even multiple of π, that is, for every value of θ in the set

$$\{0, \pm 2\pi, \pm 4\pi, \ldots\}.$$

In addition to the points on the coordinate axes, there are other special points on the unit circle for which we can identify the coordinates of $P(\theta)$ in exact form. To do this it is helpful first to recall some information concerning certain right triangles.

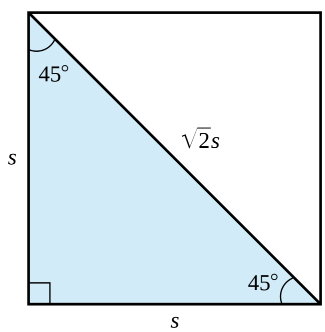


FIGURE 22

Special Right Triangles

When one of the angles of a triangle is 90°, the triangle is a right triangle. We will often indicate that the length of a side of a triangle is k by the phrase "the side is (or equals) k." Two special right triangles occur frequently in trigonometry.

The **45°–45° right triangle** is related to a square. If the square has side s, then its diagonal (the hypotenuse of two triangles) is $\sqrt{s^2 + s^2}$ or $\sqrt{2}\,s$ (see Figure 22).

The **30°–60° right triangle** is related to an equilateral triangle. The key relationship is that the shorter leg is half the length of the hypotenuse (which is the side of the equilateral triangle). If the hypotenuse is s, the short leg is $\frac{1}{2}s$, and by the Pythagorean theorem, the other leg is $\frac{\sqrt{3}}{2}s$ (see Figure 23).

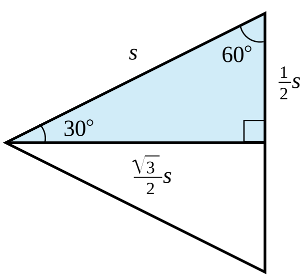


FIGURE 23

Special right triangles

In a 45°–45° right triangle, if each leg has length a, then the hypotenuse has length $\sqrt{2}a$.

In a 30°–60° right triangle, if the hypotenuse has length c, then the shorter leg is $\frac{c}{2}$ and the longer leg is $\frac{\sqrt{3}}{2}c$. The hypotenuse is always *twice as long as the shorter leg.*

▶EXAMPLE 1 ***Using 45°–45° triangles*** Find the coordinates of $P(\frac{\pi}{4})$ and of $P(\frac{-5\pi}{4})$.

Solution

The first step should always be to draw a diagram to locate the given points on the unit circle. Recall that $\frac{\pi}{2}$ is a right angle, so $\frac{\pi}{4}$ is 45°. Thus, $P(\frac{\pi}{4})$ is on the line $y = x$, and $P(\frac{-5\pi}{4})$ is five-eighths of the way around the circle in the clockwise direction, that is, $P(\frac{-5\pi}{4})$ is in the second quadrant on the line $y = -x$ (see Figure 24). If we draw perpendiculars from the two position points to the x-axis, we have 45°–45° reference triangles, as shown in color in the diagrams. The hypotenuse of each triangle is 1, so the legs have length $\frac{1}{\sqrt{2}}$. Therefore,

$$P\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

In the second quadrant, the x-coordinate is negative, so

$$P\left(\frac{-5\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right). \quad ◀$$

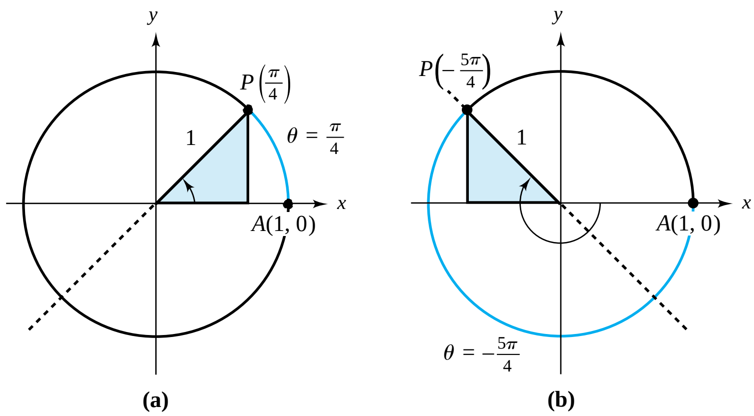


FIGURE 24

Strategy: Consider $\frac{7\pi}{6}$ as $\pi + \frac{\pi}{6}$. To locate $P(\frac{7\pi}{6})$ on the unit circle, move counterclockwise from $A(1, 0)$ an arc of π (one-half rotation) plus $\frac{\pi}{6}$, giving $P(\frac{7\pi}{6})$ in the third quadrant and a 30°–60° reference triangle.

▶EXAMPLE 2 ***A 30°–60° reference triangle*** Draw a diagram and find the coordinates of $P(\frac{7\pi}{6})$.

Solution

Follow the strategy. Point $P(\frac{7\pi}{6})$ is shown in Figure 25. As in Example 1, draw a perpendicular to the x-axis and get a right triangle, in this case a 30°–60° triangle. The short leg is $\frac{1}{2}$ and the other leg is $\frac{\sqrt{3}}{2}$. In the third quadrant both coordinates are negative, so

$$P\left(\frac{7\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right). \quad ◀$$

Quadrantal and Coterminal Numbers

Since we can identify distances around a unit circle with the measure of central angles, we can also use the language of angles to describe certain numbers and distances. If, for a given value of θ, point $P(\theta)$ is in the first quadrant, then we say

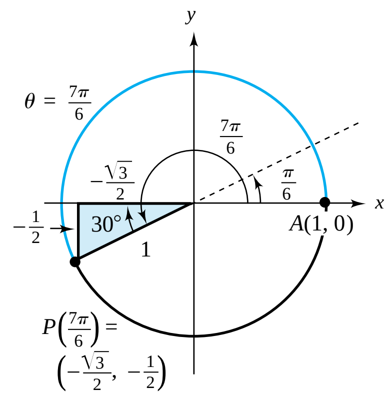


FIGURE 25

that θ is in the first quadrant, and similarly for the other quadrants. For instance, if θ is 2.34, since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$, we have $\frac{\pi}{2} < 2.34 < \pi$, so 2.34 is in the second quadrant. When $P(\theta)$ is located on one of the coordinate axes, we say that θ is a **quadrantal number,** meaning that θ is not in any quadrant. For example, 0, $\frac{\pi}{2}$, and $\frac{-3\pi}{2}$ are quadrantal numbers.

Because we think of $P(\theta)$ as the terminal position of a moving point, two numbers are said to be **coterminal** if they have the same terminal position point. That is, numbers θ_1 and θ_2 are coterminal if $P(\theta_1) = P(\theta_2)$. In general, the set of numbers coterminal with any given θ_1 is $\{\theta_1 + k \cdot 2\pi \mid k \text{ is any integer}\}$.

Trigonometric Functions

The first two trigonometric functions we encounter, the sine and cosine, are defined directly in terms of the unit circle and a position point. The remaining trigonometric functions are built from the sine and cosine.

Sine and cosine functions. Every real number θ determines a unique point $P(\theta)$ on the unit circle, and so the coordinates of $P(\theta)$ are also uniquely determined. That means that both the x- and y-coordinates of $P(\theta)$ are functions of θ. These functions, the **cosine** and **sine,** are often called **circular,** or **trigonometric functions.**

Definition: cosine and sine functions

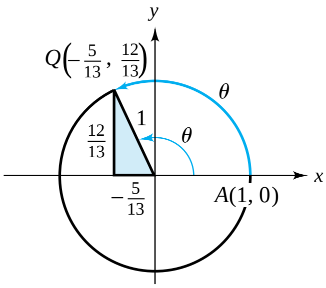
Suppose θ is any real number and $P(\theta)$ is the corresponding terminal position point on the unit circle. Then the functions cosine and sine are defined by

cosine ($\boldsymbol{\theta}$) is the ***x*-coordinate of *P*($\boldsymbol{\theta}$),**
sine ($\boldsymbol{\theta}$) is the ***y*-coordinate of *P*($\boldsymbol{\theta}$).**

We abbreviate cosine (θ) by cos θ, and sine (θ) by sin θ. If follows that every point on the unit circle has coordinates of the form

$$\boldsymbol{P(\theta) = (\cos\theta, \sin\theta).}$$

Reference triangles and circular functions. Traditional definitions of trigonometric functions use ratios of sides of right triangles. There are all sorts of mnemonic devices (such as SOHCAHTOA, whose first three letters are a reminder, "Sine: Opposite over Hypotenuse") for remembering such definitions. Reference triangles relate our definition of circular functions to the triangle definitions, using



$\cos\theta = -\frac{5}{13}$, $\sin\theta = \frac{12}{13}$

FIGURE 26

a very handy convention: we label the legs of a reference triangle by their *signed lengths.* That is, the labels on the legs of the reference triangle are the *same as the coordinates* of $P(\theta)$. See Figure 26. Since the hypotenuse of a reference triangle in the unit circle is always 1, $\sin\theta$, the y-coordinate of $P(\theta)$, is also equal to ratio of the vertical leg to the hypotenuse, the same relation as suggested by "Opposite over Hypotenuse."

Exact form evaluation and irrational numbers. Because sine and cosine are defined as coordinates of a terminal position point on the unit circle, we can evaluate these functions exactly for any point whose coordinates we know exactly. There is a profound irrationality in trigonometric functions, however, that we cannot avoid. Except for positions on the coordinate axes and those values of θ for which the reference triangle is one of the special triangles above (45°–45° or 30°–60°), no trigonometric functions of rational multiples of π are rational numbers. We know, for example, that the point $Q(\frac{-5}{13}, \frac{12}{13})$ is a point on the unit circle because its coordinates satisfy the equation $x^2 + y^2 = 1$. It follows that for the arc length θ shown in Figure 26, we have $\cos\theta = \frac{-5}{13}$ and $\sin\theta = \frac{12}{13}$. Since θ is a number whose trigonometric functions are rational, θ is not a rational multiple of π. We have no way of finding a value for θ without a calculator; we will see how to use a calculator for that purpose later in this chapter.

Strategy: **(a)** With no units indicated, 3.6 must be radians. Since $3.6 > \pi$ (≈ 3.1), 3.6 radians is about one-half radian ($\approx 30°$) more than π, so $P(3.6)$ is in QIII.
(b) $\frac{-\pi}{5}$ is less than a rotation of $\frac{\pi}{2}$ in the negative direction, so $P(\frac{-\pi}{5})$ is in QIV.

►EXAMPLE 3 ***The function sign*** Determine the sign (positive or negative) of **(a)** $\sin 3.6$ **(b)** $\cos\left(-\frac{\pi}{5}\right)$.

Solution

Draw central angles of 3.6 and $-\frac{\pi}{5}$, as shown in Figure 27. From the figure, **(a)** $\sin 3.6$ (the y-coordinate) is negative and **(b)** $\cos(\frac{-\pi}{5})$ (the x-coordinate) is positive. ◄

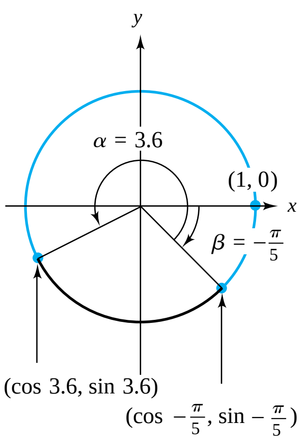


FIGURE 27

Other trigonometric functions. In addition to the cosine and sine, there are four other trigonometric functions, each of which we define in terms of cosine or sine: **tangent, cotangent, secant,** and **cosecant,** abbreviated, respectively, by tan, cot, sec, and csc.

Definition: trigonometric functions

Suppose θ is any real number, and the corresponding position point $P(\theta)$ has coordinates (x, y) on the unit circle. The six trigonometric functions of θ are

$$\cos\theta = x \qquad \sin\theta = y \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{1}{y} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}$$

Domain and range. Because $P(\theta)$ is defined for every real number θ, the cosine and sine are also defined for all real numbers. Hence, the domain for both functions is the set of all real numbers. Furthermore, since $\cos\theta$ and $\sin\theta$ are coordinates of points on the unit circle, the range of both functions is the interval $[-1, 1]$. That is, for every real number θ, $-1 \leq \cos\theta \leq 1$ and $-1 \leq \sin\theta \leq 1$.

Cosine: Domain is R (all real numbers); range is $[-1, 1]$.
Sine: Domain is R (all real numbers); range is $[-1, 1]$.

Each of the other four trigonometric functions involves the reciprocal of one coordinate of $P(\theta)$. Thus, the tangent, cotangent, secant, and cosecant all have restricted domains. Each is defined for all real numbers except those for which the denominator is 0. For instance, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, so $\tan\theta$ is not defined when $\cos\theta$ is zero, such as when θ is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ (or any odd multiple of $\frac{\pi}{2}$). For each quadrantal number, exactly two trigonometric functions are undefined.

▶EXAMPLE 4 ***Exact form evaluation*** Evaluate all trigonometric functions of θ where θ is **(a)** $\frac{5\pi}{6}$ **(b)** $-\frac{2\pi}{3}$.

Solution

Always begin with a diagram.

(a) Since $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$, the reference triangle is the 30°–60° triangle shown in Figure 28a. Therefore, the coordinates of the point $P(\frac{5\pi}{6})$ are $(\frac{-\sqrt{3}}{2}, \frac{1}{2})$. From the definition of trigonometric functions,

$$\cos\frac{5\pi}{6} = x, \text{ so } \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \qquad \sin\frac{5\pi}{6} = y, \text{ so } \sin\frac{5\pi}{6} = \frac{1}{2}$$

$$\tan\frac{5\pi}{6} = \frac{y}{x}, \text{ so } \tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}} \qquad \cot\frac{5\pi}{6} = \frac{x}{y}, \text{ so } \cot\frac{5\pi}{6} = -\sqrt{3}$$

$$\sec\frac{5\pi}{6} = \frac{1}{x}, \text{ so } \sec\frac{5\pi}{6} = -\frac{2}{\sqrt{3}} \qquad \csc\frac{5\pi}{6} = \frac{1}{y}, \text{ so } \csc\frac{5\pi}{6} = 2$$

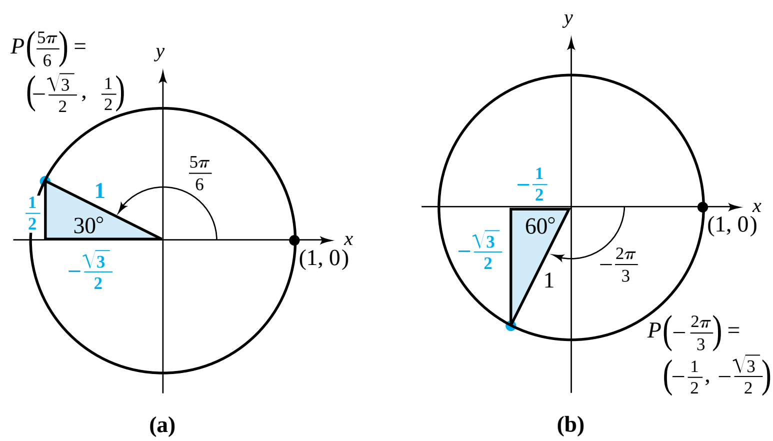


FIGURE 28

(b) The number $\frac{-2\pi}{3}$ corresponds to a clockwise rotation of $\frac{2\pi}{3}$, or 120°. When we draw a perpendicular to the x-axis, the reference triangle is a 30°–60° triangle as shown in Figure 28b. Both x and y values for $P(\frac{-2\pi}{3})$ are negative, so $P(-\frac{2\pi}{3})$ is $(\frac{-1}{2}, \frac{-\sqrt{3}}{2})$. From the definitions of the trigonometric functions,

$$\cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} \qquad \sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2} \qquad \tan\left(-\frac{2\pi}{3}\right) = \sqrt{3}$$

$$\sec\left(-\frac{2\pi}{3}\right) = -2 \qquad \csc\left(-\frac{2\pi}{3}\right) = -\frac{2}{\sqrt{3}} \qquad \cot\left(-\frac{2\pi}{3}\right) = \frac{1}{\sqrt{3}}$$ ◀

By using reference triangles and symmetry, we can get the coordinates of all the points shown in Figure 29. In the preceding section we stressed the importance of learning to think in radians. You should be able to see in your mind's eye the reference triangle for each nonquadrantal point, but the figure is an excellent reference and we recommend its use.

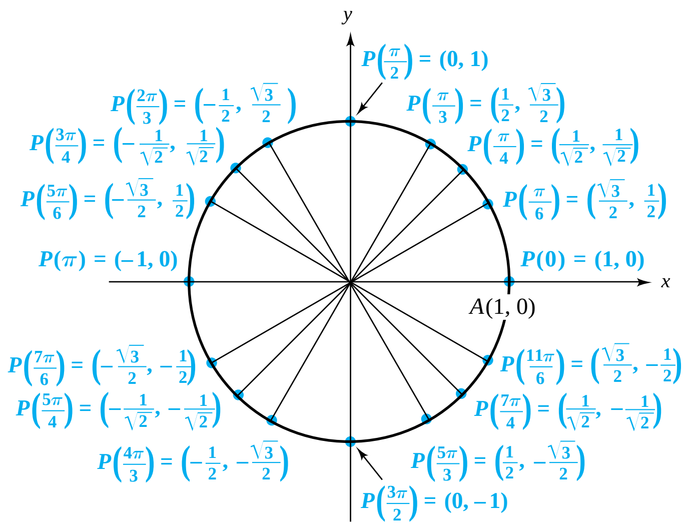


FIGURE 29

▶EXAMPLE 5 ***Using Figure 29*** Evaluate all trigonometric functions of $\frac{11\pi}{4}$.

Solution

Since $\frac{11\pi}{4}$ is greater than 2π, find a coterminal number. Note that $\frac{11\pi}{4} - 2\pi = \frac{3\pi}{4}$, so $\frac{11\pi}{4}$ is coterminal with $\frac{3\pi}{4}$. From Figure 29, read the coordinates $P(\frac{3\pi}{4}) = (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$; $P(\frac{11\pi}{4})$ has the same coordinates. Using the coordinates of $P(\frac{11\pi}{4})$ in the definitions of the trigonometric functions,

$$\cos\frac{11\pi}{4} = -\frac{1}{\sqrt{2}} \qquad \sin\frac{11\pi}{4} = \frac{1}{\sqrt{2}} \qquad \tan\frac{11\pi}{4} = -1$$

$$\sec\frac{11\pi}{4} = -\sqrt{2}, \qquad \csc\frac{11\pi}{4} = \sqrt{2} \qquad \cot\frac{11\pi}{4} = -1. \quad ◀$$

EXERCISES 5.2

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. If both $\sin\theta$ and $\cos\theta$ are negative, then $\tan\theta$ is also negative.
2. There is no number θ for which $\cos\theta$ is negative and $\sec\theta$ is positive.
3. There is no number θ for which $\sin\theta > 1$.
4. The point $(\frac{-5}{13}, \frac{12}{13})$ is not on the unit circle.
5. If θ is any number in the interval $(\frac{-\pi}{2}, 0)$, then $\cos\theta < 0$.

Note on Notation Although we used the symbol θ in defining the trigonometric functions (suggesting an angle), it is important to understand that θ-values are real numbers and that we can use any other symbol as well; $y = \sin\theta$, $y = \sin t$, and $y = \sin x$ all represent the same function.

Exercises 6–7 Fill in the blank so that the resulting statement is true.

6. If $\sin\theta < 0$ and $\cos\theta > 0$, then $P(\theta)$ must be in Quadrant ______.

7. If $\cos\theta < 0$ and $\sec\theta < 0$, then $P(\theta)$ could be in Quadrants ______.

Exercises 8–10 Enter <, >, or = in the blank.

8. $\cos\frac{3\pi}{4}$ ______ $\cos\frac{5\pi}{4}$.

9. $\tan\frac{3\pi}{4}$ ______ $\sin\frac{\pi}{6}$.

10. $\sin 3$ ______ $\cos 3$.

Develop Mastery

Exercises 1–10 ***Point on Unit Circle*** ***(a)*** *Draw a diagram to show the approximate location of $P(\theta)$ on the unit circle and also show the reference triangle.* ***(b)*** *Determine the sign (positive or negative) of $\sin\theta$, $\cos\theta$, and $\tan\theta$.*

1. $\theta = \frac{\pi}{3}$ **2.** $\theta = \frac{3\pi}{4}$ **3.** $\theta = -\frac{\pi}{6}$

4. $\theta = \frac{9\pi}{4}$ **5.** $\theta = -\frac{7\pi}{6}$ **6.** $\theta = -\frac{2\pi}{3}$

7. $\theta = -2$ **8.** $\theta = 2.6$ **9.** $\theta = \frac{9}{4}$

10. $\theta = -\frac{5}{4}$

Exercises 11–16 ***Quadrantal Numbers*** *Locate $P(\theta)$ on the unit circle for the quadrantal number θ. Evaluate all of the trigonometric functions that are defined for θ.*

11. $\theta = \frac{5\pi}{2}$ **12.** $\theta = -4\pi$ **13.** $\theta = -3\pi$

14. $\theta = 7\pi$ **15.** $\theta = -\frac{15\pi}{2}$ **16.** $\theta = \frac{3\pi}{2}$

Exercises 17–24 ***Reference Triangle*** *The reference triangle is one of the special right triangles described in this section. Sketch the reference triangle and evaluate all six trigonometric functions for θ.*

17. $\theta = \frac{5\pi}{6}$ **18.** $\theta = \frac{5\pi}{3}$ **19.** $\theta = \frac{7\pi}{4}$

20. $\theta = -\frac{3\pi}{4}$ **21.** $\theta = -\frac{11\pi}{6}$ **22.** $\theta = \frac{11\pi}{4}$

23. $\theta = \frac{13\pi}{3}$ **24.** $\theta = -\frac{13\pi}{6}$

Exercises 25–36 Describe the set of all real numbers t satisfying the given condition. Figure 29 may be helpful.

25. $\cos t = 1$ **26.** $\tan t = 1$ **27.** $\sec t = 2$

28. $\sec t = -2$ **29.** $\cot t = \sqrt{3}$ **30.** $\sin t = 0$

31. $\cot t = -1$ **32.** $\csc t = 1$

33. $\sin t = \frac{1}{2}$, $\cos t < 0$

34. $\cos t = \frac{-\sqrt{3}}{2}$, $\sin t > 0$

35. $\tan t = -1$, $\cos t < 0$

36. $\sec t = -2$, $\sin t < 0$

37. Evaluate in exact form $\sin t$, $\cos t$, and $\tan t$ for each value of t between $\frac{\pi}{2}$ and π shown in Figure 29. Use complete sentences to write your answers; for instance $\cos\frac{2\pi}{3} = \frac{-1}{2}$, not just $-\frac{1}{2}$.

38. Follow the instructions in Exercise 37 for $\pi < t < \frac{3\pi}{2}$.

Exercises 39–42 ***Quadrant*** *Determine the quadrant or quadrants in which $P(t)$ lies, where both inequalities are satisfied.*

39. $\cos t > 0$, $\sin t < 0$ **40.** $\tan t > 0$, $\cos t < 0$

41. $\sec t < 0$, $\cos t < 0$ **42.** $\csc t < 0$, $\sin t < 0$

Exercises 43–46 ***Sign*** *Determine the sign (positive or negative). First draw a diagram to show the quadrant in which the terminal position point is located.*

43. **(a)** $\cos 3$ **(b)** $\cot 3$

44. **(a)** $\tan\left(\frac{8\pi}{5}\right)$ **(b)** $\sec\left(\frac{8\pi}{5}\right)$

45. **(a)** $\sec(-2.3)$ **(b)** $\tan(-2.3)$

46. **(a)** $\csc(-0.01)$ **(b)** $\cos(-0.01)$

Exercises 47–50 ***Function Values*** *For the terminal position point $P(t)$, find all trigonometric functions of t. First show that $P(t)$ is a point on the unit circle.*

47. $P(t) = \left(-\frac{3}{5}, \frac{4}{5}\right)$

48. $P(t) = \left(-\frac{8}{17}, \frac{15}{17}\right)$

49. $P(t) = \left(\frac{7}{25}, -\frac{24}{25}\right)$

50. $P(t) = \left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right)$

Exercises 51–56 ***Point on Unit Circle*** *If $P(t)$ is a terminal position point on the unit circle, find* ***(a)*** *all possible values of x or y and* ***(b)*** *$\cos t$ and $\sin t$.*

51. $P(t) = \left(\frac{1}{2}, y\right)$ **52.** $P(t) = \left(x, \frac{1}{3}\right)$

53. $P(t) = (x, -x)$ **54.** $P(t) = (x, x + 1)$

55. $P(t) = (2y, y)$ **56.** $P(t) = \left(\frac{y}{2}, y\right)$

Exercises 57–61 Find all numbers t that satisfy the conditions.

57. $\sin t = \sin \frac{7\pi}{6}$ and $-\frac{\pi}{2} < t < 0$

58. $\sin t = \cos \frac{3\pi}{4}$ and $\frac{3\pi}{2} < t < 2\pi$

59. $\cos t = \cos \frac{5\pi}{6}$ and $\pi < t < 2\pi$

60. $\cos t = \sin \frac{3\pi}{4}$ and $\pi < t < 2\pi$

61. $\sin t = \sin \frac{\pi}{2}$ and $-2\pi < t < 0$

Exercises 62–65 Find the smallest positive number t satisfying the conditions.

62. $\cos t = -\frac{1}{2}$ and $\tan t < 0$

63. $\cot t = 1$ and $\sin t < 0$

64. $\sec t = 2$ and $\sin t < 0$

65. $\tan t = 1$ and $\cos t > 0$

Exercises 66–69 Find cos *t and* sin *t if the terminal position point P(t) on the unit circle satisfies the conditions.*

66. The x-coordinate of $P(t)$ is $\frac{3}{5}$ and $P(t)$ is in the fourth quadrant.

67. The y-coordinate of $P(t)$ is $-\frac{3}{4}$ and $P(t)$ is in the third quadrant.

68. The y-coordinate of $P(t)$ is $-\frac{\sqrt{2}}{2}$ and the x-coordinate of $P(t)$ is positive.

69. The x-coordinate of $P(t)$ is $-\frac{3}{5}$ and t is between 0 and π.

Exercises 70–73 For the given values of t, evaluate $(\sin t)^2 + (\cos t)^2$. *Based on your answers, make a guess about the value of the expression* $(\sin t)^2 + (\cos t)^2$ *for any number t.*

70. $t = \frac{\pi}{4}$; $t = \frac{5\pi}{6}$

71. $t = \frac{\pi}{3}$; $t = -\frac{5\pi}{4}$

72. $t = \frac{\pi}{2}$; $t = \frac{-7\pi}{6}$

73. $t = \frac{3\pi}{2}$; $t = \frac{-5\pi}{6}$

*Exercises 74–77 For the given value of θ, evaluate **(a)*** $\sin 2\theta$ ***(b)*** $2 \sin \theta$ ***(c)*** $2 (\sin \theta)(\cos \theta)$. *Based on your answers, make a guess about* $\sin (2\theta)$ *for any number θ.*

74. $\theta = \frac{\pi}{4}$

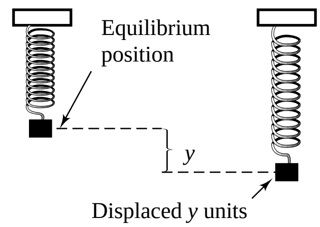
75. $\theta = \frac{-5\pi}{4}$

76. $\theta = \frac{5\pi}{6}$

77. $\theta = \frac{\pi}{2}$

78. A weight is suspended on a spring and rests in equilibrium position. It is then pulled downward and allowed to oscillate. The formula that gives the displacement y from the equilibrium position t seconds after release is

$$y = 3 \cos(4\pi t), \text{ where } y \text{ is in inches.}$$



Find the displacement for each of the following times.

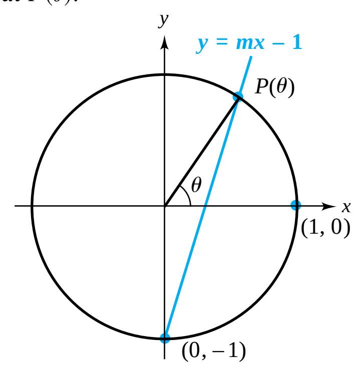
(a) $t = 0$ **(b)** $t = \frac{1}{8}$ **(c)** $t = \frac{1}{4}$

(d) $t = \frac{3}{8}$ **(e)** $t = \frac{1}{2}$

79. Give a verbal description of the oscillation motion in Exercise 78.

80. Repeat Exercise 78 with a formula that includes a damping effect due to friction: $y = 3e^{-t} \cos (4\pi t)$.

81. ***The unit circle and Pythagorean triples*** In the diagram, the line $y = mx - 1$, where $m > 1$, intersects the unit circle $x^2 + y^2 = 1$ at $(0, -1)$ and in the first quadrant at $P(\theta)$.



(a) To find $P(\theta)$, replace y by $mx - 1$ in the equation $x^2 + y^2 = 1$ and solve for x. Then use $y = mx - 1$ to find y. Show that $P(\theta)$ is $\left(\frac{2m}{1 + m^2}, \frac{m^2 - 1}{m^2 + 1}\right)$.

(b) If m is a rational number, $m = \frac{a}{b}$, where a and b are positive integers and $a > b$, then show that $P(\theta)$ is $\left(\frac{2ab}{a^2 + b^2}, \frac{a^2 - b^2}{a^2 + b^2}\right)$.

(c) Since $P(\theta)$ satisfies $x^2 + y^2 = 1$, show that $(2ab)^2 + (a^2 - b^2) = (a^2 + b^2)^2$ giving $2ab$, $a^2 - b^2$, and $a^2 + b^2$ as Pythagorean triples where a and b are any positive integers, $a > b$.

(d) Find Pythagorean triples for $a = 7$, $b = 3$ and for $a = 12$, $b = 5$. Find others of your own choice.

5.3 EVALUATION OF TRIGONOMETRIC FUNCTIONS

> The importance of the limit concept in mathematics lies in the fact that many numbers are defined only as limits. This is why the field of rational numbers, in which such limits may not exist, is too narrow for the needs of mathematics.
> *Courant and Robbins*

> *Initially I thought I was going to become a chemist because in the little high school that I was going to, it was not clear that any other scientific careers were open. I had read a book . . . which said that chemistry was a great field. It was only after I went to college that I shifted to mathematics.*
> Saunders MacLane

The values of the six trigonometric functions at any real number θ are determined by the coordinates of the terminal position $P(\theta)$ on the unit circle. Except for comparatively few values of θ, however, we have no direct way to find a simple form for the coordinates of $P(\theta)$. In Section 5.2 we learned how to evaluate the trigonometric functions for all angles that are coterminal with integer multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$. Representatives of all such angles appear in Figure 29.

In this section we consider the problem of evaluating the trigonometric functions for arbitrary angles. We begin by considering angles in standard position. If we know the coordinates in exact form for a point on the terminal side of such an angle, then we show how to find exact values for all trigonometric functions of the angle. For most angles, however, the calculator is the most convenient way to evaluate the trigonometric functions. This section concludes with a discussion of calculator evaluations.

Angles in Standard Position

An angle in **standard position** has its vertex at the origin and the positive x-axis as its initial side (see Figure 30). The advantage of having an angle in standard position is that all trigonometric functions of the angle are determined by the coordinates of any point, other than the origin, on the terminal side.

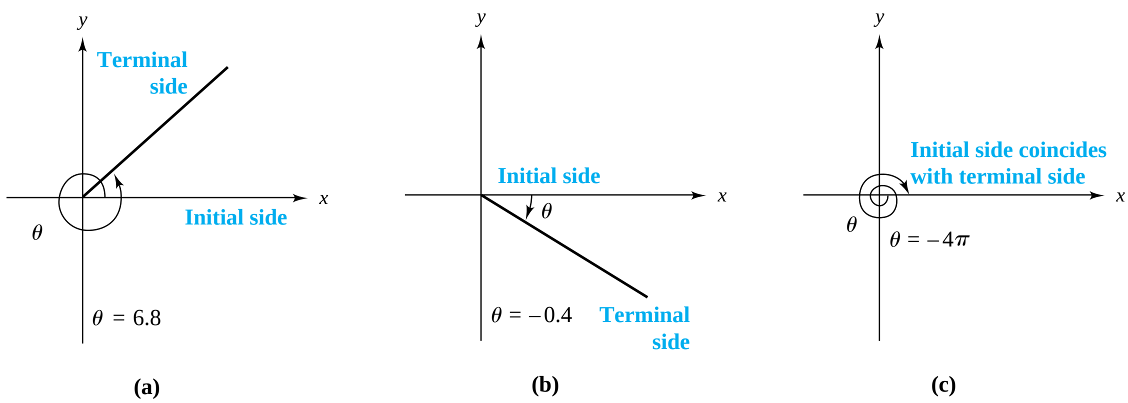


FIGURE 30

Suppose $Q(a, b)$ is an arbitrary point on the terminal side of angle θ in standard position. The point where the terminal side of θ intersects the unit circle is $P(\theta)$. To see how to get the coordinates of $P(\theta)$ from the coordinates of $Q(a, b)$, draw a perpendicular from Q to the x-axis. We get a triangle OQR that is similar to the reference triangle for $P(\theta)$, triangle OPS (see Figure 31). We label the legs of triangle OQR with the coordinates of $Q(a, b)$ to remind us of appropriate signs in the various quadrants. We call $\triangle OQR$ a reference triangle for the standard position angle θ.

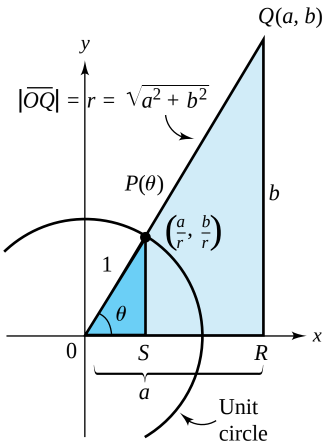


FIGURE 31

Definition: reference triangle for standard position angle

Given a nonquadrantal angle θ in standard position and any point Q on the terminal side of θ, draw a perpendicular from Q to the x-axis. If R is the foot of the perpendicular and O is the origin, then the right triangle OQR is a **reference triangle** for θ.

The legs of the reference triangle are labeled with the **signed coordinates of Q.**

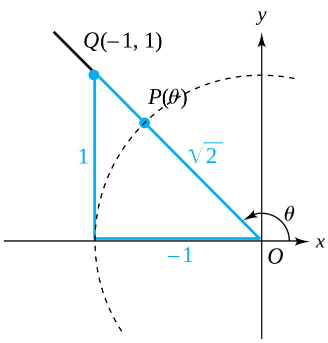
The length of the hypotenuse of triangle OQR is the distance from the origin to Q, the positive number given by $r = \sqrt{a^2 + b^2}$. Since triangles OQR and OPS in Figure 31 are similar, corresponding sides are in proportion. If $P(\theta) = (x, y)$, then we have

$$\frac{b}{r} = \frac{y}{1}, \qquad y = \frac{b}{r}$$

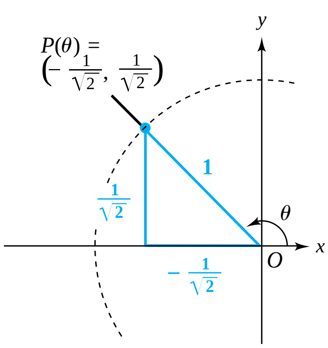
$$\frac{a}{r} = \frac{x}{1}, \qquad x = \frac{a}{r}$$

Thus, the coordinates of $P(\theta)$ are given by

$$P(\theta) = (x, y) = \left(\frac{a}{r}, \frac{b}{r}\right).$$



(a)



(b)

FIGURE 32

Figure 31 shows a first-quadrant angle, but the same relations hold for any angle in any quadrant. For instance, see Figure 32.

Once we have the coordinates of the terminal position $P(\theta)$ on the unit circle, we immediately have the values for the trigonometric functions at θ. The cosine and sine of θ are the coordinates of $P(\theta)$, the remaining four functions are defined as in Section 5.2.

Trigonometric functions of an angle in standard position

Suppose θ is an angle in standard position, and $Q(a, b)$ is any point on the terminal side of θ, other than the origin. The distance r from the origin to Q is $\sqrt{a^2 + b^2}$, and the trigonometric functions of θ are given by

$$\cos\theta = \frac{a}{r} \qquad \sin\theta = \frac{b}{r} \qquad \tan\theta = \frac{b}{a}$$
$$\sec\theta = \frac{r}{a} \qquad \csc\theta = \frac{r}{b} \qquad \cot\theta = \frac{a}{b}. \qquad \textbf{(1)}$$

It may help you remember these definitions to think in terms of the x-coordinate and y-coordinate rather than a and b. The cosine is expressed as the x-coordinate divided by r, etc. It is understood, also, that the tangent and secant are undefined when $a = 0$ (when the terminal side is on the y-axis), and that the cosecant and cotangent are undefined when $b = 0$ (when the terminal side is on the x-axis).

►EXAMPLE 1 ***Point on terminal side*** Let θ be an angle in standard position with $Q(-3, 4)$ on the terminal side. Evaluate all trigonometric functions at θ.

Strategy: Draw a diagram with a reference triangle by dropping a perpendicular from Q to the x-axis. Find the distance $r = |\overline{OQ}|$ and use the formulas in Equation (1).

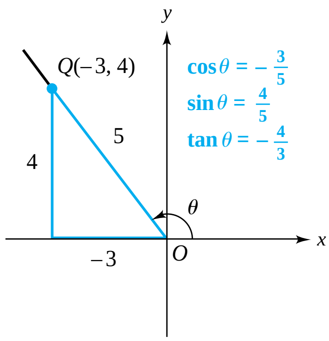


FIGURE 33

Solution

Follow the strategy. Figure 33 shows a reference triangle for θ. First, we find r.

$$r = \sqrt{(-3)^2 + (4)^2} = \sqrt{25} = 5.$$

With -3 for a, 4 for b, and 5 for r, use Equation (1) to get

$$\cos\theta = \frac{a}{r} = -\frac{3}{5} \qquad \sin\theta = \frac{b}{r} = \frac{4}{5} \qquad \tan\theta = \frac{b}{a} = -\frac{4}{3}$$

$$\sec\theta = \frac{r}{a} = -\frac{5}{3} \qquad \csc\theta = \frac{r}{b} = \frac{5}{4} \qquad \cot\theta = \frac{a}{b} = -\frac{3}{4} \quad ◀$$

▶EXAMPLE 2 ***Point on terminal side*** Let ϕ be the angle shown in standard position in Figure 34 whose terminal side contains the point $Q(\frac{7}{\sqrt{53}}, \frac{-2}{\sqrt{53}})$. Find $\cos\phi$, $\sin\phi$, and $\tan\phi$.

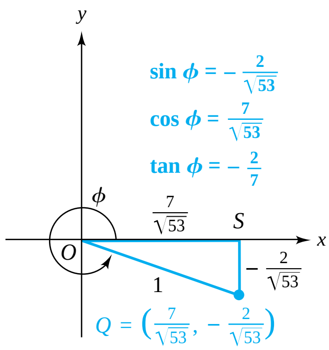


FIGURE 34

Solution

First, draw a perpendicular from Q to the x-axis to obtain a reference triangle, OQS (see Figure 34). Next, determine r.

$$r = \sqrt{\left(\frac{7}{\sqrt{53}}\right)^2 + \left(-\frac{2}{\sqrt{53}}\right)^2} = \sqrt{\left(\frac{49}{53}\right) + \left(\frac{4}{53}\right)} = 1.$$

Since r is 1, point Q is on the unit circle, so

$$\cos\phi = \frac{7}{\sqrt{53}}$$

$$\sin\phi = -\frac{2}{\sqrt{53}}$$

$$\tan\phi = -\frac{2}{7} \quad ◀$$

▶EXAMPLE 3 ***Exact form from one value*** If $\tan\theta = -\frac{7}{24}$ and θ is in the second quadrant, find the exact values of the other trigonometric functions of θ.

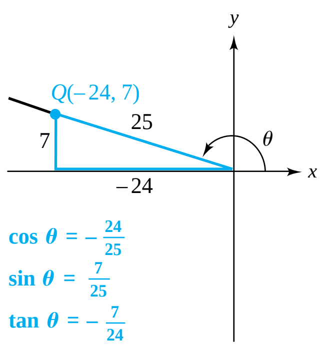


FIGURE 35

Strategy: Begin with a diagram that shows θ and a reference triangle. For a point $Q(a, b)$ on the terminal side of θ, $\tan\theta = \frac{b}{a}$. The fraction $\frac{-7}{24}$ can be written as $\frac{7}{-24}$. Choose Q as $(-24, 7)$.

Solution

Follow the strategy. Draw a diagram (Figure 35), and then find r.

$$r = \sqrt{(-24)^2 + (7)^2} = \sqrt{625} = 25$$

Therefore r is 25, and Equation (1) gives

$$\cos\theta = \frac{a}{r} = -\frac{24}{25} \qquad \sin\theta = \frac{b}{r} = \frac{7}{25} \qquad \tan\theta = \frac{b}{a} = -\frac{7}{24}$$

$$\sec\theta = \frac{r}{a} = -\frac{25}{24} \qquad \csc\theta = \frac{r}{b} = \frac{25}{7} \qquad \cot\theta = \frac{a}{b} = -\frac{24}{7} \quad ◀$$

Calculator Evaluation of Trigonometric Functions—Modes

Before calculators, evaluation of trigonometric functions required tables of some sort. Literally lifetimes of computation were invested in the production of tables of sufficient accuracy for scientific calculations. Now, the touch of a calculator key gives instant access to more accurate information than was ever available previously. Develop Mastery Exercises 70 and 71 illustrate how calculators can be programmed to evaluate the sine and cosine functions.

Graphing calculators have keys labeled SIN, COS, TAN, and they operate in distinct *modes* as well, to accomodate angles measured in either degrees or radians. Technically, we could think of two different sets of trigonometric functions, one for angles measured in degrees and the other for angles measured in radians. We avoid the need for such a distinction by adopting the following convention.

Mode convention

All evaluation is done in **radian mode** unless a degree symbol is specified.

sin 30 means the **sine of 30 radians** (almost five revolutions)

sin 30° means the **sine of 30 degrees.**

Modes. There is nothing more frustrating than going through a long series of calculations and then discovering that all your numbers are wrong (or meaningless) because your calculator is in the wrong mode. Most calculators have a MODE key from which you can select the correct mode for a particular computation. The following Technology Tip gives suggestions on setting modes for different calculators.

TECHNOLOGY TIP ◆

Mode setting

Texas Instruments Press MODE. On the Mode screen, the third line is Radian Degree. Use arrow keys, highlight your choice, and ENTER.
Hewlett-Packard The screen always shows when you are in radian mode by RAD in the upper left corner. Set radian mode on the Modes screen. Highlight the ANGLE MEASURE line, CHOOS, make your selection, OK, and return to the Home screen.
Casio The initial screen shows the mode in use. To change, from the home screen DRG (above 1) displays the menu. The first two choices set Deg, Rad mode. Press F1 or F2 and EXE.

Evaluating trigonometric functions. Since calculators have keys only for the sine, cosine, and tangent functions, we evaluate the remaining trigonometric functions as reciprocals as for instance: $\cot x = \frac{1}{\tan x}$.

WARNING: the functions above the trigonometric function keys (SIN^{-1}, and so on) are inverse function keys, *not reciprocals*. $\tan^{-1} x \neq \frac{1}{\tan x}$.

Before going further, make sure you know how to evaluate trigonometric functions in both modes. Check all of the calculator evaluations in the following table on your calculator.

Evaluate	*Mode*	*Enter*	*Display*	*Comments*
sin 30	Rad	sin 30	−0.9880	
sin 30°	Deg	sin 30	0.5	
tan(−2)	Rad	tan(−2)	2.1850	Change sign, not subtract
$\cos\frac{2\pi}{3}$	Rad	cos(2π/3)	−0.5	See Technology Tip
$\sec\frac{2\pi}{3}$	Rad	1/cos(2π/3)	−2	
$\cot\left(\frac{\pi}{3}\right)^{\circ}$	Deg	1/tan(π/3)	54.7073	$\left(\frac{\pi}{3}\right)^{\circ}$ is just larger than 1°

TECHNOLOGY TIP ◆ ***Always use enough parentheses to be sure***

Calculators differ, but most will evaluate cos 2π/3 as cos 2π divided by 3, giving $\frac{1}{3}$. If you want $\cos\frac{2\pi}{3}$, you must use parentheses, as in the table above. The HP–48 is an exception, since you first evaluate 2π/3 and then apply the cosine function to the result.

Strategy: For a rough sketch of $P(1.85)$ and $P(-9)$, remember that half a revolution is measured by π (just over 3), so that 1.85 is a little more than $\frac{\pi}{2}$. Thus $P(1.85)$ is in the second quadrant. Since 9 is slightly less than $3\pi(\approx 9.42)$ and 3π is $2\pi + \pi$, move clockwise from (1, 0) one complete revolution and then slightly less than half a revolution. Hence, $P(-9)$ is in the third quadrant.

►EXAMPLE 4 ***Calculator approximations*** Draw a rough sketch that shows $P(\theta)$ on the unit circle, and give a five-decimal-place approximation for all six trigonometric functions of **(a)** $t = 1.85$ and **(b)** $t = -9$.

Solution

Follow the strategy and draw the diagrams shown in Figure 36. Using a calculator in radian mode, evaluate the six trigonometric functions when θ is 1.85 and then when θ is −9.

$$\cos 1.85 \approx -0.27559 \qquad \sin 1.85 \approx 0.96128 \qquad \tan 1.85 \approx -3.48806$$
$$\sec 1.85 \approx -3.62858 \qquad \csc 1.85 \approx 1.04028 \qquad \cot 1.85 \approx -0.28669$$
$$\cos(-9) \approx -0.91113 \qquad \sin(-9) \approx -0.41212 \qquad \tan(-9) \approx 0.45232$$
$$\sec(-9) \approx -1.09754 \qquad \csc(-9) \approx -2.42649 \qquad \cot(-9) \approx 2.21085$$

◄

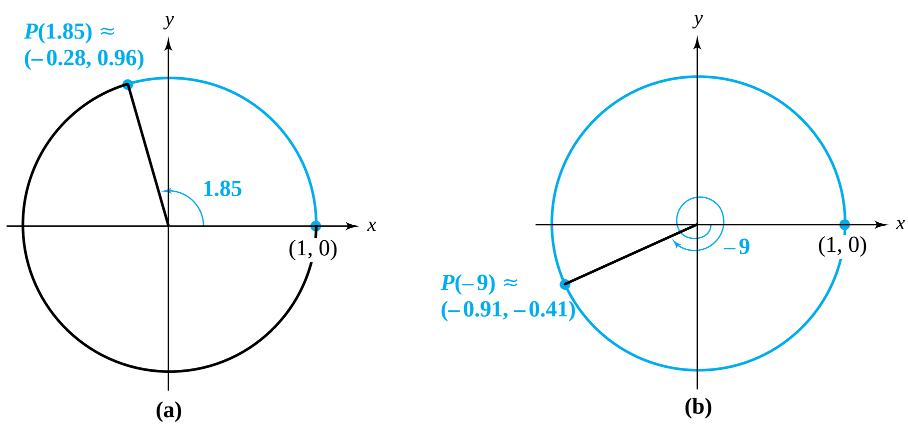


FIGURE 36

Right Triangle Trigonometry

Our original definition of trigonometric functions made use of coordinates of points on the unit circle. We also saw that the trigonometric functions can be defined using coordinates of *any* point on the terminal side of an angle in standard position. In many situations we want trigonometric functions of acute angles in right triangles.

In working with right triangles we do not want to be dependent on any particular orientation of the triangle. Figure 37a shows a right triangle with acute angles α and β, and legs labeled a and b. Since side a is opposite angle α, we denote it by *opp* α, and similarly *opp* β indicates side b, the side opposite angle β. The hypotenuse c is labeled *hyp*. By placing the triangle in a coordinate system with leg b along the positive x-axis and angle α in standard position (see Figure 37b), we see that the point with coordinates (b, a) is on the terminal side of angle α at a distance c from the origin. Hence the definitions on page 278 apply and we can express the trigonometric functions of α in terms of opp α, adj α, and hyp.

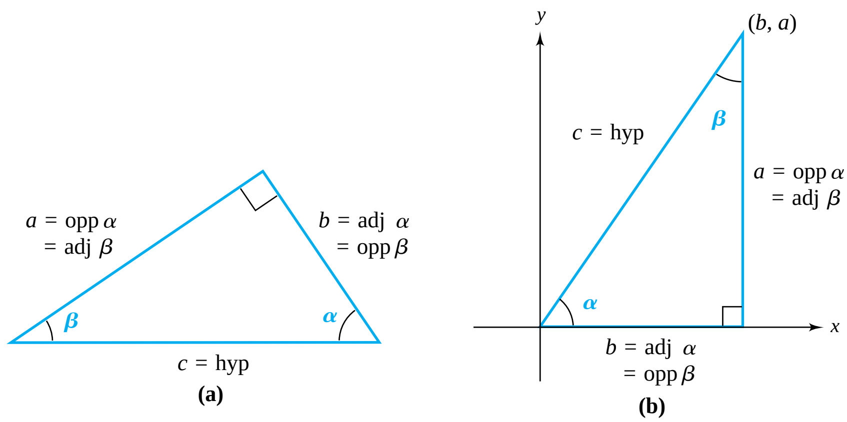


FIGURE 37

Definition: trigonometric functions of an acute angle

Suppose α is an acute angle of a right triangle. The trigonometric functions of α are

$$\sin \alpha = \frac{\text{opp } \alpha}{\text{hyp}} \qquad \cos \alpha = \frac{\text{adj } \alpha}{\text{hyp}} \qquad \tan \alpha = \frac{\text{opp } \alpha}{\text{adj } \alpha}$$

$$\csc \alpha = \frac{\text{hyp}}{\text{opp } \alpha} \qquad \sec \alpha = \frac{\text{hyp}}{\text{adj } \alpha} \qquad \cot \alpha = \frac{\text{adj } \alpha}{\text{opp } \alpha}$$

In a similar manner, for angle β we have

$$\sin \beta = \frac{\text{opp } \beta}{\text{hyp}} \qquad \cos \beta = \frac{\text{adj } \beta}{\text{hyp}} \qquad \tan \beta = \frac{\text{opp } \beta}{\text{adj } \beta}.$$

In Figure 37a, in addition to the right angle, we refer to α, β, a, b, and c, as **parts of the triangle.**

Given information about some parts of a right triangle, we can use trigonometric functions to determine other parts. The process of using given data to solve for remaining parts is called **solving the triangle.** In virtually all instances, to solve a triangle we look for trigonometric functions that relate known information to a

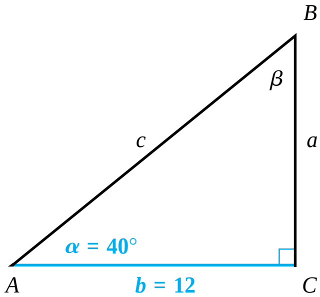


FIGURE 38

single unknown, giving us equations that we can solve for the desired quantities. We illustrate with an example that is typical of the applications of right triangles we explore in more depth in Section 7.1.

▶EXAMPLE 5 ***Solving a right triangle*** In the right triangle in Figure 38, b is 12, and α is 40°. Find β, a, and c.

Solution

Since the sum of the acute angles of a right triangle is 90°, $\beta = 90 - \alpha = 90° - 40° = 50°$. To find the lengths of the unknown sides, look for trigonometric functions that involve just one of the unknowns. In this case,

$$\cos 40° = \frac{\text{adj}}{\text{hyp}} = \frac{12}{c} \quad \text{and} \quad \tan 40° = \frac{\text{opp}}{\text{adj}} = \frac{a}{12}.$$

Solving for c and a, respectively,

$$c = \frac{12}{\cos 40°} \approx 15.66 \quad \text{and} \quad a = 12 \tan 40° \approx 10.07.$$

Rounding off to two significant digits, c is 16 and a is 10. ◀

Strategy: Use an altitude of $\triangle ABC$ to find its area K_1, and subtract K_1 from the area K_2 of the sector of the circle.

▶EXAMPLE 6 ***Area of a circular segment*** The central angle of a circular sector is 50°, and the radius is 12.4 inches as shown in Figure 39. Find the area K_1 of $\triangle ABC$ and the area K of the shaded region.

Solution

Follow the strategy. Let CD be the altitude from C, of length h. In right triangle BCD, $\sin 50° = h/12.4$, or $h = 12.4 \sin 50°$.

$$K_1 = \frac{1}{2}h(12.4) = \frac{1}{2}(12.4)^2 \sin 50° \approx 58.89 \approx 59 \text{ in}^2.$$

For the circular sector, we need the radian measure of the central angle: $50° = 50\left(\frac{\pi}{180}\right) = 5\frac{\pi}{18}$ radians. Using the formula for the area of a circular sector from Section 5.1,

$$K_2 = \frac{1}{2}r^2\theta = \frac{1}{2}(12.4)^2\frac{5\pi}{18} \approx 67.09 \approx 67 \text{ in}^2.$$

Finally, the area of the shaded circular segment is given by

$$K = K_2 - K_1 \approx 8.2 \text{ in}^2. \quad ◀$$

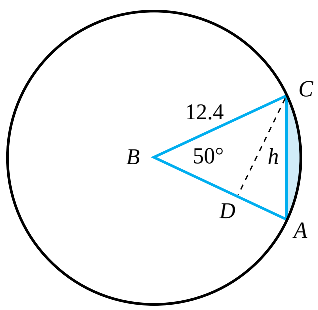


FIGURE 39

▶EXAMPLE 7 ***Rotating wheel*** A wheel of radius 2 is rotating in a counterclockwise direction at a uniform angular speed ω of 12 rev/min. Take a coordinate system with the origin at the center of rotation and designate a point P on the circumference of the wheel.

(a) If P is located at (2, 0) at time $t = 0$, find formulas to give the coordinates of $P(x, y)$ at any time t in seconds.
(b) Give the coordinates of point P to two decimal places at times $t = 1, 2, 4$, and 5 seconds.

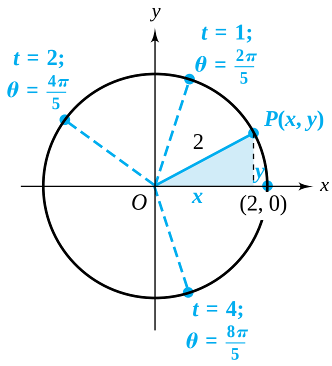


FIGURE 40

Solution

First, draw a diagram. Since t is in seconds, express the angular speed in units of radians per second.

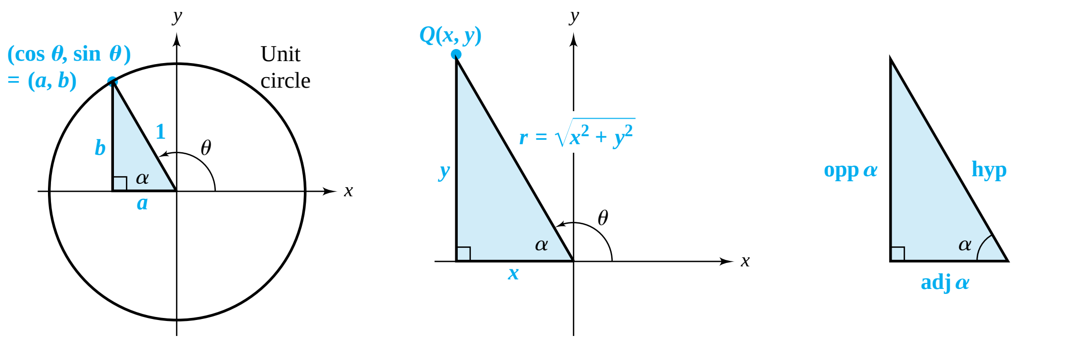
$$\omega = 12\,\frac{\text{rev}}{\text{min}} = \frac{12\text{ rev}}{\text{min}} \cdot \frac{2\pi\text{ rad}}{1\text{ rev}} \cdot \frac{1\text{ min}}{60\text{ sec}} = \frac{2\pi}{5}\frac{\text{rad}}{\text{sec}}$$

Therefore, in t seconds the radial line OP will rotate through an angle θ where θ is $\frac{2\pi}{5}t$.

(a) From the reference triangle OPS in Figure 40, $\cos\theta = \frac{x}{2}$ so $x = 2\cos\theta = 2\cos(\frac{2\pi}{5}t)$. Similarly, $y = 2\sin(\frac{2\pi}{5}t)$.

(b) If $t = 1$, then $x = 2\cos(\frac{2\pi}{5}) \approx 0.618$, $y = 2\sin(\frac{2\pi}{5}) \approx 1.902$. Hence at 1 second, point P is at $(0.62, 1.90)$. When $t = 2$, $x = 2\cos(\frac{4\pi}{5}) \approx -1.62$, $y = 2\sin(\frac{4\pi}{5}) \approx 1.18$, so P is at $(-1.62, 1.18)$. Similarly, when $t = 4$, P is at $(0.62, -1.90)$, and when t is 5, P is at $(2, 0)$, back to the starting point. ◀

Relating trigonometric functions of any angle and right triangle trigonometry. On page 272 we defined trigonometric functions of any angle θ by using coordinates of a point on the unit circle (see Figure 41a). On page 278 we defined trigonometric functions of θ in terms of coordinates of an arbitrary point Q on the terminal side of θ (Figure 41b). In both cases, for nonquadrantal angles we use the reference triangle, the right triangle formed by dropping a perpendicular from a point Q on the terminal side to the x-axis. If we label the sides of the reference triangle with the signed-number coordinates of Q, then we can read all trigonometric functions of θ (including signs) from the right triangle definitions for the reference triangle (Figure 41c).



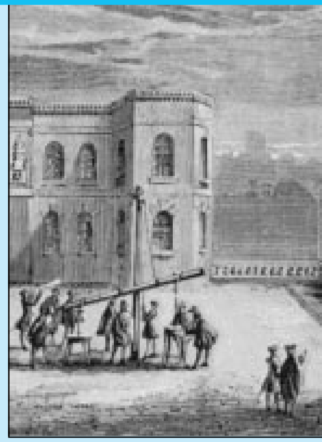
(a) Unit circle definitions **(b)** Reference triangle definitions **(c)** Right triangle definitions

FIGURE 41

Relating reference and right triangle definitions

For any nonquadrantal angle θ with a point Q on the terminal side, if α is the acute angle at the origin in the reference triangle and the legs of the reference triangle are labeled with the signed-number coordinates of Q, then the trigonometric functions of θ are the same as the corresponding right triangle functions for α in the reference triangle.

HISTORICAL NOTE TRIGONOMETRIC TABLES



Astronomers at the Paris Observatory in the 17th century

The urgent need for accurate trigonometric calculations arose from astronomy and navigation. After all, 1 degree of longitude is $\frac{1}{360}$ of the circumference of the earth. To a navigator out in the middle of the unknown, even a minute ($\frac{1}{60}$ of a degree) covers a big chunk of ocean.

Claudius Ptolemy of Alexandria laid out principles of astronomy and geography in the second century A.D. that remained the supreme authority for well over a thousand years. Some of his views of the world were surprisingly modern; it was his idea to divide the equator into 360 equal parts or degrees. So great was the authority of men like Ptolemy that people were unwilling to challenge what was written even when it was contradicted by direct experience.

By the end of the sixteenth century, explorers were pushing ever farther into the unknown and had to rely increasingly on celestial navigation. Astronomers also needed more accurate trigonometric calculations. At that time, sines and cosines were not functions, they were lengths of chords in a circle. Larger circles had larger sines, so to increase accuracy, users increased the size of the radius.

Napier criticized some of his contemporaries for using a radius of only 1 million, writing "the more learned put 10,000,000, whereby the difference of all sines is better expressed." Prodigious (and tedious) efforts went into calculating trigonometric tables. Rheticus (1514–1576) began to compile 15-place tables, an effort that wasn't completed until 20 years after his death.

Valuable as the tables may have been, their use still involved horrendous problems. Just think of multiplying and dividing three or four 15-digit numbers! There was really no alternative until Napier's invention of logarithms in 1614 (see the Historical Note, "Invention of Logarithms" in Section 4.5).

EXERCISES 5.3

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. If point $(3, 4)$ is on the terminal side of θ, then $(-3, -4)$ is on the terminal side of $-\theta$.
2. If point $(-2, 4)$ is on the terminal side of θ, then so is $(-1, 2)$.
3. The smallest integer that is greater than $\tan 5$ is -3.
4. If point $(1, 1)$ is on the terminal side of θ, then θ must be equal to $\frac{\pi}{4}$.
5. The number $\tan(1 + 9\pi)$ is negative.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. If $\theta = 3$, then the terminal side of θ is in Quadrant ______.
7. If $\theta = -4$, then the terminal side of θ is in Quadrant ______.
8. If the terminal side of θ is in Quadrant III, then the terminal side of $\theta - \pi$ is in Quadrant ______.
9. In a right triangle labeled as in Figure 37, if $a = 3$ and $b = 7$, then $\sin \alpha =$ ______.
10. In Exercise 9, $\cos \beta =$ ______.

Develop Mastery

Exercises 1–16 ***Reference Triangle, Function Values*** *Point Q is on the terminal side of angle θ. From a diagram that shows Q and a reference triangle for θ, evaluate the six trigonometric functions of θ in exact form.*

1. $Q(-3, 4)$ **2.** $Q(-6, 8)$ **3.** $Q(5, 12)$
4. $Q(-5, -12)$ **5.** $Q(-7, -24)$ **6.** $Q(3.5, -12)$
7. $Q(3, -3)$ **8.** $Q(-4, 2)$ **9.** $Q(-2, -4)$
10. $Q(-1, 2)$ **11.** $Q(2, 3)$ **12.** $Q(4, -1)$
13. $Q(\sqrt{5}, -2)$ **14.** $Q(\sqrt{3}, \sqrt{6})$ **15.** $Q(-1.5, 2)$
16. $Q(2.5, -6)$

Exercises 17–25 ***Exact and Decimal Values*** *An angle ϕ is specified. From a diagram that shows ϕ and a reference triangle, evaluate* $\cos\phi$, $\sin\phi$, *and* $\tan\phi$ *in exact form, and also in decimal form rounded off to two places.*

17. $\tan\phi = \frac{1}{2}$ and ϕ is in Quadrant III.
18. $\sin\phi = \frac{2}{3}$ and ϕ is in Quadrant II.
19. $\sin\phi = \frac{2}{5}$ and $\cos\phi$ is negative.
20. $\sin\phi = -\frac{2}{5}$ and $\cos\phi$ is negative.
21. $\tan\phi = -\frac{3}{4}$ and $\sin\phi$ is negative.
22. $\cos\phi = \frac{1}{10}$ and $\tan\phi$ is positive.
23. $\sec\phi = 2$ and $\cot\phi$ is negative.
24. $\cot\phi = \frac{2}{3}$ and $\csc\phi = \dfrac{-\sqrt{13}}{3}$.
25. $\tan\phi = -5$ and $\sec\phi$ is positive.

Exercises 26–37 ***Decimal Approximations*** *Give a decimal approximation rounded off to three places.*

26. $\sin 2.41$ **27.** $\cos 13.5$ **28.** $\tan(-1.29)$
29. $\cos 13.5°$ **30.** $\csc 37.2°$ **31.** $\cot 97°23'$
32. $\sin 21°37'$ **33.** $\tan 5$ **34.** $\cot\left(\dfrac{2\pi}{5}\right)$
35. $\sec\left(\dfrac{-\pi}{7}\right)$ **36.** $\cos\left(\dfrac{5\pi}{8}\right)$ **37.** $\csc\left(\dfrac{-8\pi}{11}\right)$

Exercises 38–45 ***Points on Unit Circle*** ***(a)*** *Give the coordinates of point $P(t)$ on the unit circle. Round off to two decimal places. Show $P(t)$ in a diagram.* ***(b)*** *Evaluate the six trigonometric functions at t (rounded off to two decimal places).*

38. $t = -1$ **39.** $t = 8$ **40.** $t = -1.32$
41. $t = \dfrac{-\pi}{5}$ **42.** $t = \sqrt{\pi}$ **43.** $t = \pi + 1$
44. $t = \sqrt{6}$ **45.** $t = e$

Exercises 46–49 ***Decimal Approximations*** *Evaluate and round off to three decimal places. Be certain your calculator is in radian mode.*

46. **(a)** $\sin(3 + 16\pi)$ **(b)** $\cos 31$
47. **(a)** $\cos(2 + 15\pi)$ **(b)** $\tan 36$
48. **(a)** $\tan(2 - 9\pi)$ **(b)** $\sec 30$
49. **(a)** $\sin(2 - 35\pi)$ **(b)** $\csc 40$

Exercises 50–53 ***Right Triangles*** *The angles (α and β) and sides (a, b, and c) of a right triangle are labeled as in Figure 37.*

50. If $a = 24$ and $\alpha = 48°$, find b and c.
51. If $c = 35$ and $\beta = 27°$, find a and b.
52. If $c = 12$ and $a = 4$, find b.
53. If $a = 16$ and $\beta = 65°$, find b and c.

Exercises 54–55 *For each value of θ, evaluate* $\cos\theta$ *and* $\sin(\theta + \frac{\pi}{2})$. *Based on your results, make a guess about a relationship between the values of* $\cos\theta$ *and* $\sin(\theta + \frac{\pi}{2})$ *for any angle θ.*

54. $\theta = \dfrac{\pi}{3}$; $\theta = 4.5$; $\theta = -2.6$

55. $\theta = \dfrac{5\pi}{6}$; $\theta = 4.8$; $\theta = -2.9$

Exercises 56–57 *Evaluate expressions* $1 + (\tan\theta)^2$ *and* $(\sec\theta)^2$. *Based on your results, make a guess about a relationship between the values of the two given expressions for any angle θ.*

56. **(a)** $\theta = 36°$; $\theta = 158°$; $\theta = -215°$
(b) $\theta = \dfrac{3\pi}{5}$; $\theta = 3.8$; $\theta = -6$

57. **(a)** $\theta = 65°$; $\theta = 210°$; $\theta = -115°$
(b) $\theta = \dfrac{5\pi}{8}$; $\theta = 4.8$; $\theta = -7.2$

Exercises 58–59 *For each value of θ, evaluate the expressions* $\cos(2\theta)$, $2\cos\theta$, $(\cos\theta)^2 - (\sin\theta)^2$ *and* $2(\cos\theta)^2 - 1$. *Based on your answers, which expressions appear to be equal for any angle θ?*

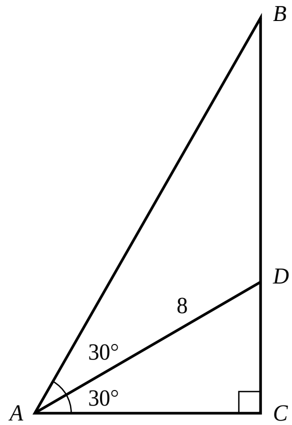
58. **(a)** $\theta = 63°$; $\theta = 258°$; $\theta = -135°$
(b) $\theta = \dfrac{2\pi}{7}$; $\theta = 4.3$; $\theta = -1.5$

59. **(a)** $\theta = 73°$; $\theta = 510°$; $\theta = -135°$
(b) $\theta = \dfrac{5\pi}{8}$; $\theta = 5.3$; $\theta = -1.2$

60. **(a)** Compare $\sqrt{(\tan t)^2 - (\sin t)^2}$ and $(\sin t)(\tan t)$ for four different numbers t from the first and fourth quadrants. What do your results suggest?
(b) Compare values of these expressions for four different second- and third-quadrant numbers t. Modify

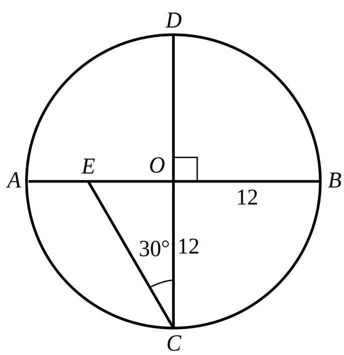
the guess you made in part **(a)**. How should the expression $\sin t \tan t$ be changed so that the expressions are equal for every real number t where $\tan t$ is defined?

61. In the diagram $\triangle ABC$ is a right triangle with $\angle CAD = 30°$, $\angle DAB = 30°$ and $|\overline{AD}| = 8$. Find $|\overline{BD}|$. *Hint:* Use relationships of 30°–60° triangles.



62. Solve the problem in Exercise 61 when each of the two 30° angles is replaced by 20°.

63. In the diagram, $\overline{AB}$ and $\overline{CD}$ are perpendicular diameters of the circle with center at O and radius 12. Find **(a)** $|\overline{CE}|$ and **(b)** the ratio of $|\overline{AE}|$ to $|\overline{EO}|$.



64. Solve the problem in Exercise 63 if $\angle ECO$ is 20°.

65. Solve the problem in Example 6 for a central angle of 40° and a radius of 16.5 inches.

66. A wheel of radius 3 is rotating counterclockwise at a uniform angular speed of 2 rev/min. Take a coordinate system with the origin at the center of rotation and designate a point Q on the circumference of the wheel.

(a) If Q is located at (3, 0) when t is 0, find equations that give the coordinates of $Q(x, y)$ at any time t in seconds.

(b) Give the coordinates (to two decimal places) of point Q when t is 10, 20, 25, and 40 seconds.

67. Repeat Exercise 66 with a wheel of radius 4 whose uniform angular speed is 4 rev/min, with point Q located at (4, 0) when t is 0.

68. One end of a spring is anchored to the ceiling and a weight is attached to the other end. When the weight is at rest, it is in equilibrium position, however, if the weight is pulled downward and released, it oscillates. Its displacement d (in millimeters) at any time t seconds after release is given by the equation $d = 40\cos(1.5\,t)$. What is the displacement (to three significant digits) of the weight when t is **(a)** 1 second, **(b)** 2 seconds, **(c)** 4 seconds?

69. In Exercise 68, if friction is taken into account, we get a damping effect and the formula for the displacement becomes $d = 40\,e^{-t}\cos(1.5\,t)$. Find the displacement when t is **(a)** 1 second, **(b)** 2 seconds, **(c)** 4 seconds.

Exercises 70–71 ***Looking Ahead to Calculus*** *In calculus the sine and cosine functions can be expressed as infinite series as follows, where (x is in radians):*

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

and

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots.$$

70. For small values of x, the functions obtained by taking the first few terms of each series can be used to get good approximations of the sine and cosine. Thus let $S(x) = x - \frac{x^3}{6}$ and $C(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$.

Complete the table (rounding off to four decimal places).

x	0.1	0.2	0.3	0.4	0.5	0.6
$\sin x$						
$S(x)$						

71. Repeat Exercise 70 using $\cos x$ and $C(x)$ in place of $\sin x$ and $S(x)$, respectively.

5.4 PROPERTIES AND GRAPHS

Why does nature require a nontrivial and yet entirely manageable amount of mathematics for the successful description of such a large part of it?

P. W. C. Davies

The trigonometric functions are defined in terms of the coordinates of the point $P(t)$ as it moves around the unit circle. All the properties of the trigonometric functions ultimately derive from this fact. Since we need to understand trigonometric functions thoroughly, we devote this section to an examination of their properties and graphs, with particular emphasis on what the graphs can tell us about functional behavior.

We make extensive use of calculator graphs, but because it is so important to understand the basic properties of trigonometric functions, we keep reminding ourselves of their unit circle definitions.

I was, and have always remained, a problem solver rather than a creator of ideas. I cannot, as Bohr and Feynman did, sit for years with my whole mind concentrating upon one deep question. I am interested in too many things.

Freeman Dyson

Graphs

We begin with the graph of the sine function, $y = \sin x$. From the fact that the coordinates of $P(t)$ on the unit circle are $(\cos t, \sin t)$, we need only consider how the coordinates change as $P(t)$ moves around the unit circle. The point $P(t)$ makes a complete trip around the circle as t increases from 0 to 2π. If we know what happens to both coordinates on the interval $[0, 2\pi]$, essentially we know the behavior of both sine and cosine functions.

The symbol t in $(\cos t, \sin t)$ suggests the possibility of parametric equations. Graphing two curves parametrically allows us to see the unit circle being traced out at the same time we see the graph of $y = \sin x$ being drawn. Many books show a graph such as Figure 42. We want you to see more than the finished graph; we want you to *see* the graphs in action. Graph these curves on your calculator and watch them being drawn (simultaneously if possible). Then trace, watching what happens as you jump from one curve to the other.

TECHNOLOGY TIP ◆ *Unit circle and sine graph*

In parametric mode, enter X1 = COS T, Y1 = SIN T for the unit circle $x^2 + y^2 = 1$, and X2 = T, Y2 = SIN T for the sine function $y = \sin x$. Set the t-range as $[0, 2\pi]$ or $[0, 6.28]$ with t-step of .1. We want an equal scale window, and you may want to experiment, but we like the following. For the TI-82 (or 81), or Casio 7700, try $[-1.4, 6.1]$ (or $[-1.5, 6.1]$) $\times [-2.5, 2.5]$. For the TI-85, HP–38, HP–48, or Casio 9700, try $[-2, 6.2] \times [-2, 2]$.

When you GRAPH, you should see the curves in Figure 42, with the two sets of axes superimposed. As you trace, you should be able to see the portion of the sine curve corresponding to each quadrant as labeled in Figure 42, as $P(t)$ goes around the unit circle, stopping to jump from one curve to the other.

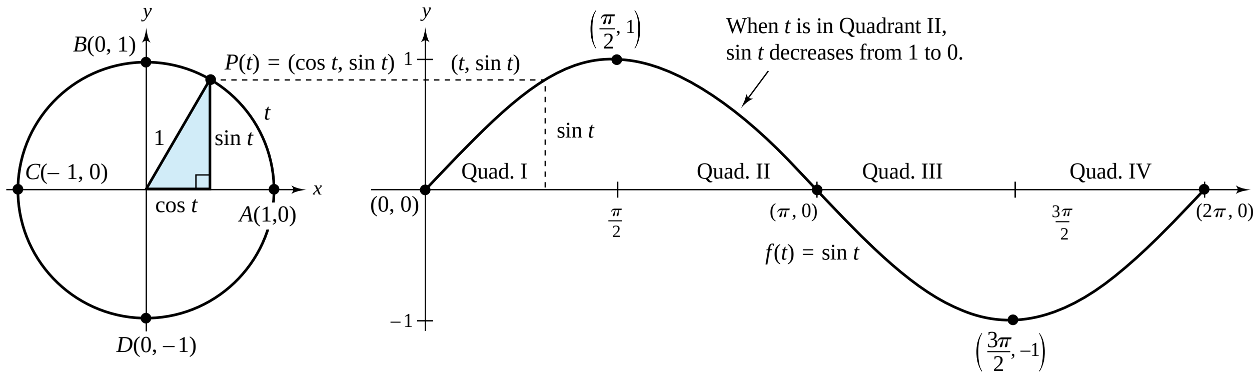


FIGURE 42
Fundamental cycle for $y = \sin t$

Sine Wave, Fundamental Cycle, and Periodic Functions

As t continues on from 2π, or back through negative values from 0, $P(t)$ just repeats its circuit around the unit circle, so the sine curve repeats its values precisely, extending in both directions from what we see in Figure 42. A larger portion of the graph of the sine function appears in Figure 43. The full graph of the sine function is called a **sine wave.** The portion shown in Figure 42, that corresponds to one complete circuit of $P(t)$, is called a **fundamental cycle** of the sine curve.

A function f is called **periodic** if there is some number p such that $f(x + p) = f(x)$ for every x in the domain of f; the *smallest* such positive number p is called the **period** of f. For the trigonometric functions, after the point $P(t)$ makes a complete trip around the unit circle, it retraces its path exactly, returning to the same point after every revolution. The coordinates of $P(t)$ are the same as the coordinates of $P(t + 2\pi)$, which implies that $\sin(t + 2\pi) = \sin t$ for every real number t, that $\sin(t + 4\pi) = \sin(t - 2\pi) = \sin t$, and so on. The same relations hold for the

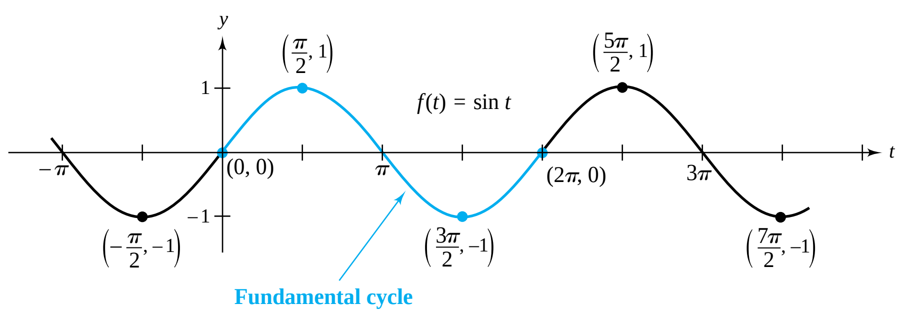
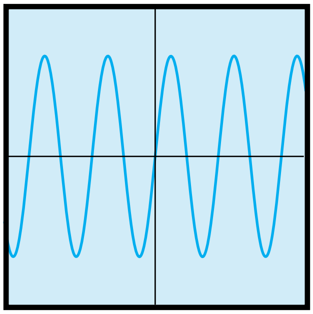


FIGURE 43



[−15, 15] by [−1.5 x 1.5]

$y = \sin x$

FIGURE 44

cosine function as well. Thus we have the following.

Period of sine and cosine

The sine and cosine functions are both periodic, with period 2π.

$$\left.\begin{aligned}\sin(t + 2\pi) &= \sin t\\ \cos(t + 2\pi) &= \cos t\end{aligned}\right\} \text{ for every real number } t.$$

From a fundamental cycle of the sine curve, we get the rest of the graph of $y = \sin x$ by repeating the fundamental cycle in both directions. To see more of the sine wave, return to function mode and graph $y = \sin x$ in a window such as $[-7, 7] \times [-1.5, 1.5]$. This allows us to see just a little more than two fundamental cycles. Now, if we increase the horizontal range to $[-15, 15]$, the calculator squeezes more than four fundamental cycles onto the screen, as in Figure 44.

TECHNOLOGY TIP ◆ ***Trigonometric window***

Most graphing calculators have an automatic setting for a trigonometric window. You should become familiar with yours, whether or not it is a window you like for viewing trigonometric functions. On TI calculators the window is set from the ZOOM menu by Z TRIG. On the HP-48G, the ZOOM menu is only accessible from the screen after drawing a graph. Then ZOOM NXT NXT shows the menu with Z TRIG. On Casio calculators, simply enter Graph y = sin (no x) and EXE, or from the Range screen, press F2.

Drawing calculator graphs of sine curves also reminds us of some limitations of calculators. When graphing $y = \sin x$ with larger and larger x-ranges, at first we see very smooth fundamental cycles repeated, but then the cycles become more jagged. Remember that the graphing calculator draws a graph by dividing up the x-range, computing a function value for each pixel-column, and then connecting pixels in adjacent columns. The calculator is *sampling* the graph and showing us the sampled points. When the x-range becomes larger, the sample points are more widely spaced. To see this effect in action, follow the instructions in the next Technology Tip.

TECHNOLOGY TIP ◆ ***Sampling graphs***

Begin by graphing the function $y = \sin \pi x$ in $[-5, 5] \times [-1.5, 1.5]$. You should see exactly five fundamental cycles of a sine wave, a little jagged, but certainly recognizable as a sine curve. Now increase the x-range to $[-10, 10]$, and then to $[-20, 20]$. On this screen, it wouldn't be obvious to an uninformed observer that we are really seeing a sine curve.

To see the sampling most dramatically, we want to squeeze in just the right number of fundamental cycles, and it differs from calculator to calculator. We want to refer to a special x-range $[A, B]$ for your calculator:

TI-81: $[-47, 48]$; **TI-82:** $[-47, 47]$;
TI-85 and **Casio:** $[-63, 63]$; **HP:** $[-65, 65]$.

Increase your x-range to $[A + 2, B - 2]$ and graph, then to $[A + 1, B - 1]$ and graph. If these graphs had not been seen in order, it is unlikely that a person would recognize the graphs as sine curves. What we are seeing is sometimes called a "resonance" phenomenon, where only a few points are taken from each fundamental cycle. Finally, draw a graph in the interval $[A, B]$. Explain what you see. *Hint:* try tracing, and remember what we know about $P(t)$. How much of the graph is missing between successive pixels?

Graph of the Cosine Function

Since the cosine function is also defined in terms of the coordinates of the point $P(t)$, we expect its graph to be closely related to the graph of the sine function. To see how closely related these two curves are, graph $y = \sin x$ and $y = \cos x$ on the same screen. See Figure 45.

From the graphs in Figure 45, we see that the cosine curve is also a sine wave, meaning that it has exactly the same shape and period. A **fundamental cycle** of the cosine curve is, as for the sine curve, a portion of the graph corresponding to one revolution of $P(t)$. The cosine curve "lags behind" the sine curve by a distance $\frac{\pi}{2}$ in the sense of Figure 46. In equation form, the relation is expressed as $\cos(t - \frac{\pi}{2}) = \sin t$, for every t.

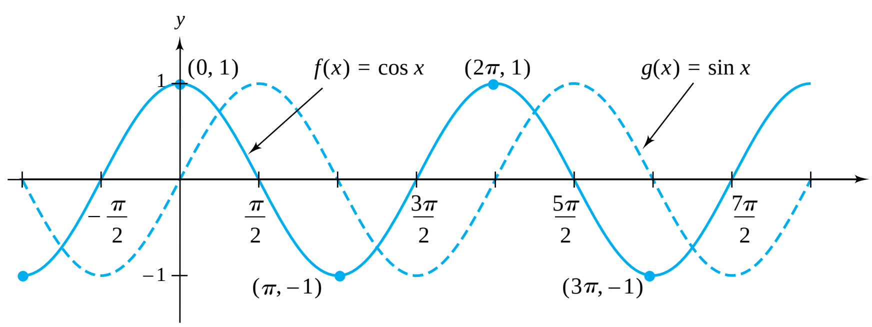


FIGURE 45
Graphs of $y = \sin x$ and $y = \cos x$.

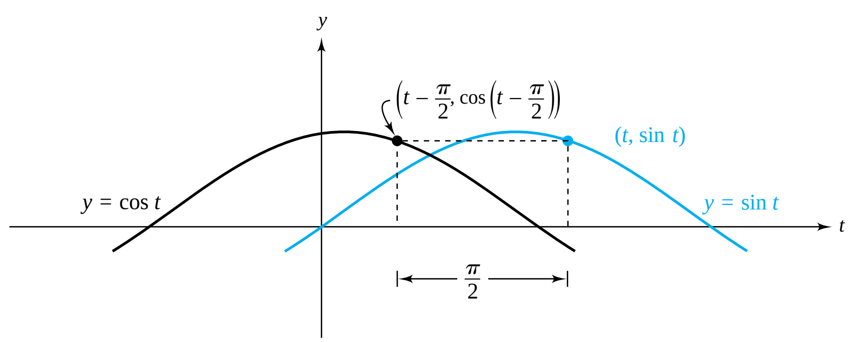


FIGURE 46
$\cos\left(t - \frac{\pi}{2}\right) = \sin t$ for every t.

Reduction Formulas

The equation $\cos(t - \frac{\pi}{2}) = \sin t$ is called a **reduction formula.** Reduction formulas play a significant role throughout trigonometry. Other important relations could be read from the graphs in Figure 45, but most can be seen more easily from the unit circle.

The key to using a unit circle for reduction formulas is finding congruent reference triangles. As an example, the graphs in Figure 45 suggest that the sine curve is symmetric about the origin, while the cosine is symmetric about the y-axis. Using terminology from Chapter 2, it appears that the sine is an odd function and the cosine is an even function, or in equation form, for every real number t,

$$\sin(-t) = -\sin t \quad \text{and} \quad \cos(-t) = \cos t.$$

Strategy: Draw a unit circle diagram that shows angles t and $-t$, with reference triangles for each, then compare coordinates of $P(t)$ and $P(-t)$.

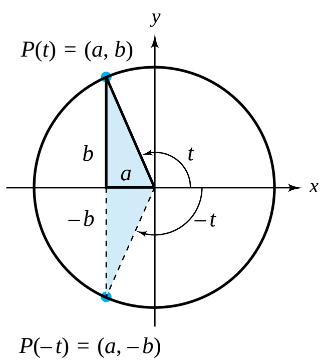


FIGURE 47

In the following example we justify these claims.

▶EXAMPLE 1 *Reduction formulas*

Show that for any real number t,

$$\sin(-t) = -\sin t \quad \text{and} \quad \cos(-t) = \cos t.$$

Solution

Because on the unit circle $P(t) = (\cos t, \sin t)$ and $P(-t) = (\cos(-t), \sin(-t))$, we want to relate the coordinates of $P(t)$ and $P(-t)$. Begin with an arbitrary $P(t)$ in Figure 47. $P(-t)$ is located the same distance around the unit circle in the opposite direction. The reference triangles are clearly congruent, so $P(t)$ and $P(-t)$ have the same x-coordinates and their y-coordinates are equal but have opposite signs. You may find it helpful to draw diagrams that show different $P(t)$, $P(-t)$ pairs. If $P(t)$ has coordinates (a, b), then $P(-t) = (a, -b)$. Expressing the coordinates in terms of cosine and sine,

$$\begin{cases} \cos t = a \\ \sin t = b \end{cases} \quad \text{and} \quad \begin{cases} \cos(-t) = a \\ \sin(-t) = -b \end{cases}$$

Therefore $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$. ◀

Since the sine is an odd function and the cosine is even, we can classify the other trigonometric functions similarly.

▶EXAMPLE 2 *Even-odd functions*

Find formulas that relate $f(-t)$ and $f(t)$, for the tan, cot, sec, and csc functions.

Solution

From Example 1 and the equations defining the other trigonometric functions in terms of sine and cosine,

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin t}{\cos t} = -\tan t$$

$$\cot(-t) = \frac{\cos(-t)}{\sin(-t)} = \frac{\cos t}{-\sin t} = -\cot t$$

$$\sec(-t) = \frac{1}{\cos(-t)} = \frac{1}{\cos t} = \sec t$$

$$\csc(-t) = \frac{1}{\sin(-t)} = \frac{1}{-\sin t} = -\csc t \quad ◀$$

Example 2 demonstrates that the cosine and secant functions are even and the other four trigonometric functions are odd.

▶EXAMPLE 3 ***More reduction formulas*** From a unit circle diagram that shows $P(t)$ and $P(t+\pi)$, find reduction formulas for $\cos(t+\pi)$, $\sin(t+\pi)$, and $\tan(t+\pi)$.

Strategy: For any position point $P(t)$, point $(t+\pi)$ is diametrically opposite, so if $P(t)$ has coordinates (a, b) then $P(t+\pi)$ has coordinates $(-a, -b)$.

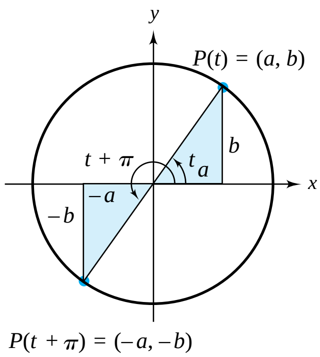


FIGURE 48

Solution

Look at Figure 48. $P(t)$ and $P(t+\pi)$ are end points of a diameter, so their coordinates have opposite signs. Thus, if $P(t) = (a, b)$, then $P(t+\pi) = (-a, -b)$. Therefore

$$\begin{cases} \cos t = a \\ \sin t = b \end{cases} \quad \text{and} \quad \begin{cases} \cos(t+\pi) = -a \\ \sin(t+\pi) = -b \end{cases}$$

This gives the reduction formulas

$$\cos(t+\pi) = -\cos t \quad \text{and} \quad \sin(t+\pi) = -\sin t$$

For $\tan(t+\pi)$,

$$\tan(t+\pi) = \frac{\sin(t+\pi)}{\cos(t+\pi)} = \frac{-\sin t}{-\cos t} = \frac{\sin t}{\cos t} = \tan t. \quad ◀$$

Periods of the Trigonometric Functions

The cosine and sine functions have period 2π. The reciprocals of these two functions, the secant and cosecant, must have the same period. While it is also true that, for every real number t in the domain of the tangent, $\tan(t+2\pi) = \tan t$, 2π is not the *smallest* positive number for which the tangent repeats. It turns out that the period of the tangent function is π. From Example 3, $\tan(t+\pi) = \tan t$ for every t in the domain of the tangent function. From the graph of the tangent function (see Figure 50), we can see that the period of the tangent function is π.

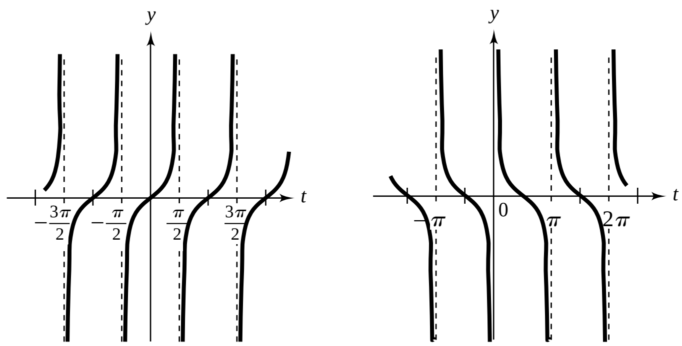
Since the cotangent function is the reciprocal of the tangent function, it must have the same period as the tangent, so the period of the cotangent is also π. Table 1 summarizes these results.

TABLE 1 *Periods of the trigonometric functions*

Function	*Period*	*Function*	*Period*
Sine	2π	Cosecant	2π
Cosine	2π	Secant	2π
Tangent	π	Cotangent	π

Graphs of the Other Trigonometric Functions

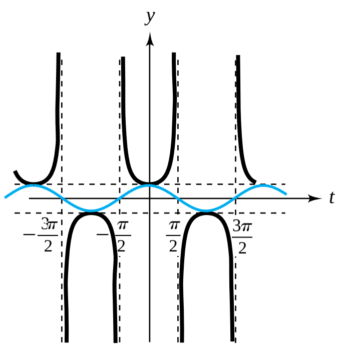
For the graphs of the sine and cosine functions, we paid close attention to the coordinates of $P(t)$ as it moves around the unit circle. The remaining four trigonometric functions are all defined as quotients involving the sine and cosine. From our work with quotients of polynomials, we realize that whenever the denominator of a quotient is zero, the graph of the quotient will have a *vertical asymptote*. The graphs of the other trigonometric functions all have periodically occurring asymptotes.



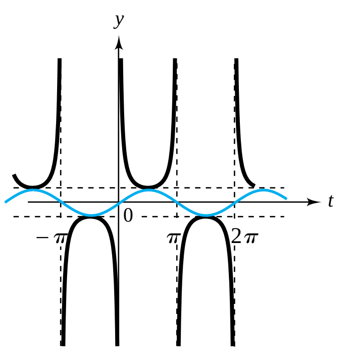
(a) $y = \tan t$
odd; period π

(b) $y = \cot t$
odd; period π

FIGURE 49



(a) $y = \sec t$
even; period 2π



(b) $y = \csc t$
odd; period 2π

FIGURE 50

The tangent function is of sufficient importance that it merits a key of its own on calculators; the cotangent is treated as the reciprocal of the tangent. You should draw the graph of $y = \tan x$ in your trigonometric window, and then add the graph of $y = \cot x (= 1/\tan x)$. The graph of the cotangent may be thought of as the tangent graph reflected in the x-axis and shifted horizontally $\frac{\pi}{2}$ units right or left. That is, $\cot x = -\tan(x \pm \pi/2)$; see Figure 49.

The cosecant and secant are reciprocals of the sine and cosine, respectively, so we can obtain their graphs as we did for reciprocals in Section 3.4. In particular, since the graphs of $y = \sin x$ and $y = \cos x$ lie entirely between the horizontal lines $y = 1$ and $y = -1$, the graphs of their reciprocals are always on or outside the horizontal strip bounded by those lines. The graphs of $y = \csc x$ and $y = \sec x$, with their reciprocal functions, are shown in Figure 50.

►EXAMPLE 4 *Another reduction formula*

(a) Verify that calculator graphs of $y = \sin(x + \frac{\pi}{2})$ and $y = \cos x$ are identical.
(b) Use a unit circle diagram to establish the appropriate reduction formula.

Solution

(a) The graphs of both functions look like the one in Figure 51. To verify that the graphs are identical, recall the suggestions in the Technology Tip in Section 4.4.
(b) Drawing a diagram in a unit circle, reference triangles for the angles x and $x + \frac{\pi}{2}$ are congruent. Thus in Figure 52, if we label the coordinates of $P(x)$ as (a, b), then the coordinates of $P(x + \frac{\pi}{2})$ are $(-b, a)$.

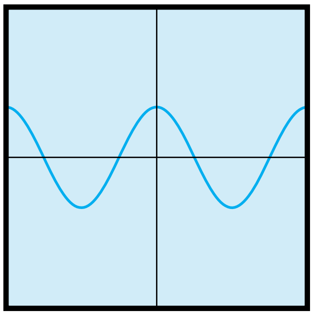
$$\sin\left(x + \frac{\pi}{2}\right) = a \quad \text{and} \quad \cos x = a, \quad \text{so } \sin\left(x + \frac{\pi}{2}\right) = \cos x. \quad ◄$$

►EXAMPLE 5 *Yet another reduction formula*

Repeat Example 4 with the functions $y = \csc(x + \frac{\pi}{2})$ and $y = \sec x$.

Solution

(a) For the graphs we enter Y1 = 1/SIN(X + π/2), Y2 = 1/COS X. Again, the graphs appear identical.



$[-2\pi, 2\pi]$ by $[-3, 3]$

FIGURE 51

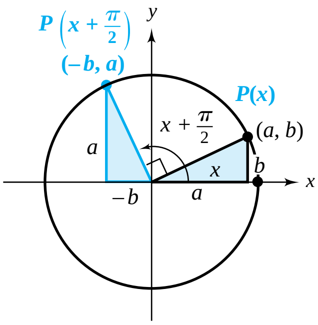


FIGURE 52

(b) Referring again to Figure 52, or using the reduction formula from Example 4, we have

$$\csc\left(x+\frac{\pi}{2}\right) = 1/\sin\left(x+\frac{\pi}{2}\right) = 1/\cos x = \sec x. \blacktriangleleft$$

More Reduction Formulas

So far we have derived several reduction formulas, but there are many more. Table 2 gives a number of the most commonly used reduction formulas, from which many others can be derived, as Example 6 shows.

TABLE 2 *Reduction formulas*

	$-t$	$\frac{\pi}{2}-t$	$\frac{\pi}{2}+t$	$\pi-t$	$\pi+t$	$\frac{3\pi}{2}-t$	$\frac{3\pi}{2}+t$
sin	$-\sin t$	$\cos t$	$\cos t$	$\sin t$	$-\sin t$	$-\cos t$	$-\cos t$
cos	$\cos t$	$\sin t$	$-\sin t$	$-\cos t$	$-\cos t$	$-\sin t$	$\sin t$
tan	$-\tan t$	$\cot t$	$-\cot t$	$-\tan t$	$\tan t$	$\cot t$	$-\cot t$

►EXAMPLE 6 ***Using Table 2*** Use Table 2 to simplify $\csc(t-\frac{\pi}{2})$.

Solution

Remember that $\csc(t-\frac{\pi}{2}) = \dfrac{1}{\sin(t-\frac{\pi}{2})}$. Table 2 has $\sin(\frac{\pi}{2}-t)$, but not $\sin(t-\frac{\pi}{2})$, so first use the fact that sine is an odd function to get,

$$\sin\left(t-\frac{\pi}{2}\right) = \sin\left(-\left(\frac{\pi}{2}-t\right)\right) = -\sin\left(\frac{\pi}{2}-t\right)$$

From Table 2, $\sin(\frac{\pi}{2}-t) = \cos t$. Putting all this together,

$$\csc\left(t-\frac{\pi}{2}\right) = \frac{1}{\sin\left(t-\frac{\pi}{2}\right)} = \frac{1}{-\sin\left(\frac{\pi}{2}-t\right)} = \frac{1}{-\cos t} = -\sec t.$$

Thus, $\csc(t-\frac{\pi}{2}) = -\sec t$ is a reduction formula (an identity). ◄

Using Graphs in Problem Solving

The next two examples illustrate how helpful a graph can be in solving a problem.

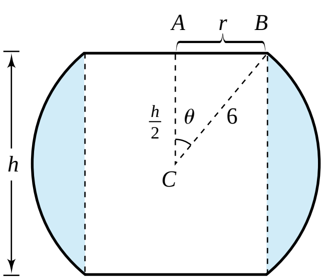
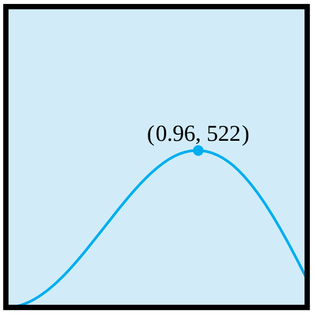


FIGURE 53

►EXAMPLE 7 ***A hole in a sphere*** We want to drill a hole through a wooden sphere of radius 6 inches, leaving a bead-shaped shell surrounding a hollow cylinder in which flowers can be displayed. We want the largest possible cylinder. Figure 53 shows a cross-section with C at the center and angle θ between the axis of the hole and the cylinder rim. The radius of the cylinder is r and the height is h.

(a) Express r and h, and then the volume V of the cylinder, as functions of θ.
(b) Draw a graph of V and find the maximum volume and the corresponding value of θ and the radius of the hole to drill.



[0, 1.5] by [0, 1000]

FIGURE 54

Solution

(a) $\triangle ABC$ is a right triangle with hypotenuse 6 and legs of length r and $h/2$, so $\sin\theta = \dfrac{r}{6}$ and $\cos\theta = \dfrac{h/2}{6}$, from which $r(\theta) = 6\sin\theta$, $h(\theta) = 12\cos\theta$. The volume of a cylinder is $\pi r^2 h$, so the volume we want is given by

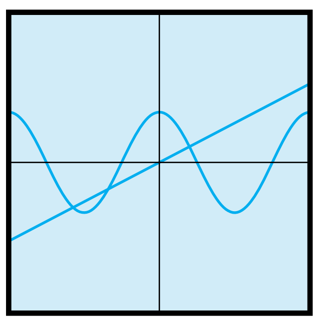
$$V = \pi[r(\theta)]^2\, h(\theta) = 432\pi\,(\sin\theta)^2\cos\theta.$$

(b) We graph the volume as Y = 432π(SIN X)² COS X. For a suitable window, we see that θ is certainly between 0 and $\pi/2$, so an x-range of $[0, 1.5]$ should work. The maximum y-value must be less than 432π, so we will try $[0, 1000]$. The calculator graph is shown in Figure 54. The maximum volume is just over 522 cubic inches, when $\theta \approx 0.96$. The hole giving a maximum volume has radius of about $r(0.96) = 6\sin(0.96) \approx 4.9$. ◀

►EXAMPLE 8 ***Counting intersections*** In how many points do the graphs of f and g intersect if $f(x) = \cos x$ and **(a)** $g(x) = x/4$, **(b)** $g(x) = x/20$?

Solution

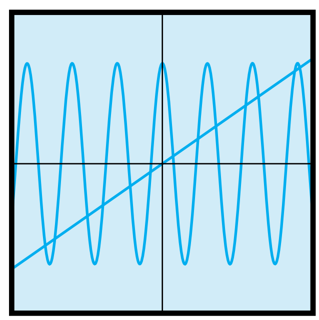
(a) A calculator graph of f and g in a trigonometric window is shown in Figure 55. The question, of course, is whether this window shows all the intersections. We know that the cosine function always lies between the horizontal lines $y = 1$ and $y = -1$. Since g is an increasing function (a line with positive slope), after $g(x) > 1$ there can be no more intersections, and similarly for $g(x) < -1$. Tracing along g, we verify that Figure 55a shows all intersections; there are three.



$[-2\pi, 2\pi]$ by $[-3, 3]$

$f(x) = \cos x$, $g(x) = x/4$

(a)



$[-21, 21]$ by $[-1.5, 1.5]$

$f(x) = \cos x$, $g(x) = x/20$

(b)

FIGURE 55

(b) When we change the denominator of g to 20 and graph f and g in a trigonometric window, it is obvious that we do not have all intersections. Following the same reasoning as in **(a)**, $g(x) = 1$ when $x = 20$, so we must have a window going at least to 20. We graph f and g in $[-21, 21] \times [-1.5, 1.5]$ and get the picture in Figure 55b. There are seven intersections in Quadrant I and six in Quadrant III, for a total of thirteen. ◀

EXERCISES 5.4

Check Your Understanding

Use graphs whenever helpful.

Exercises 1–5 True or False. Give reasons.

1. For every real number x, $\cos|x| - \cos x = 0$.
2. For every real number x, $\sin|x| - \sin x \geq 0$.
3. The graph of $y = \cos|x|$ is the same as the graph of $y = \cos x$.
4. The function $f(x) = |\sin x|$ is an even function.
5. The graph of $y = \cos(\frac{\pi}{2} - x)$ is the same as the graph of $y = \sin x$.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. For $0 \leq x \leq \pi$, the highest point on the graph of $y = 2\sin(\pi - x)$ is ______.
7. The period of $f(x) = 2\sin(\frac{\pi x}{2})$ is ______.
8. The period of $f(x) = |\sin x|$ is ______.
9. The graph of $f(x) = -\cos x$ and $y = x$ intersect in Quadrants(s) ______.
10. The number of points common to the graphs of $y = \cos 2x$ and $y = \frac{x}{2}$ is ______.

Develop Mastery

Exercises 1–3 Use Table 2 to simplify the expression.

1. **(a)** $\sin\left(t - \dfrac{3\pi}{2}\right)$ **(b)** $\csc\left(t - \dfrac{\pi}{2}\right)$
2. **(a)** $\cos\left(t - \dfrac{\pi}{2}\right)$ **(b)** $\cot\left(t - \dfrac{\pi}{2}\right)$
3. **(a)** $\tan(t - \pi)$ **(b)** $\sec\left(\dfrac{\pi}{2} + t\right)$

Exercises 4–6 Given the coordinates of a point $Q(a, b)$ on the terminal side of an angle θ in standard position, draw a diagram to evaluate in exact form the six trigonometric functions of the indicated angle.

4. $Q(-3, 4)$; $\theta + \pi$
5. $Q(-3, 4)$; $\theta + \dfrac{\pi}{2}$
6. $Q(12, -5)$; $\theta - \dfrac{\pi}{2}$

Exercises 7–10 ***Reduction Formula*** *Determine whether the equation is a valid reduction formula. First draw graphs of the left and right sides. If the graphs appear to be identical then use algebra.*

7. $\cos\left(\dfrac{5\pi}{2} - t\right) = \cos t$
8. $\sin(3\pi + t) = -\sin t$
9. $\cos\left(t - \dfrac{3\pi}{2}\right) = \sin t$
10. $\sec(t - 3\pi) = \sec t$

11. Draw a calculator graph of $y = \sin \pi x$ for each of the windows suggested in the Technology Tip ("Sampling Graphs," page 290). Explain the appearance of the graph of $y = \sin \pi x$ for the specified window. (*Hint:* In each case TRACE and look at the (x, y) values to see what points the calculator is trying to connect.)
12. Repeat Exercise 11 for $y = \cos \pi x$.

Exercises 13–16 ***Graph*** *Without using a calculator sketch a graph of f for x in $[-2\pi, 2\pi]$.*

13. $f(x) = \cos\left(\dfrac{\pi}{2} - x\right)$
14. $f(x) = \sin\left(\dfrac{3\pi}{2} + x\right)$
15. $f(x) = \tan(\pi + x)$
16. $f(x) = \sec(\pi - x)$
17. Draw graphs of the unit circle and $y = \sin x$ using parametric equations as suggested on page 288. Watch as the graphs are being drawn. What point on the unit circle is used as a starting point for the graph, and in what direction is it traced? (*Hint:* Trace and use the up–down arrow keys to see the correspondence between points on the unit circle and points on the sine graph.)
18. Repeat Exercise 17, except draw a graph of the unit circle and $y = \cos x$.

Exercises 19–20 ***Graph*** *Sketch a graph of the equation for x in $[-\pi, \pi]$. Describe how to obtain the graph by basic transformations of a graph in Figures 43, 45, 49, or 50.*

19. **(a)** $y = 2\sin x$ **(b)** $y = -2\cos x$
20. **(a)** $y = -\tan x$ **(b)** $y = -\csc x$

Exercises 21–24 ***Simplify Formula*** *Use appropriate reduction formulas (identities) from Table 2 to find a simpler equation to describe the function. In each case, give the domain of the function.*

21. $f(x) = \dfrac{\sin\left(x - \dfrac{\pi}{2}\right)}{\cos x}$
22. $g(x) = \dfrac{\cos\left(x + \dfrac{3\pi}{2}\right)}{\sin x}$
23. $f(x) = \dfrac{\tan(x + \pi)}{\tan x}$
24. $g(x) = \dfrac{1}{2}\left[\sin x - \cos\left(x + \dfrac{\pi}{2}\right)\right]$

Exercises 25–26 ***Graphs with Absolute Value*** *Draw graphs of f and g for x in $[-2\pi, 2\pi]$.*

(a) For what values of x do the graphs coincide?
(b) Is f even or odd? Is g even or odd?
(c) Is f periodic? Is g periodic?

25. $f(x) = |\sin x|, \quad g(x) = \sin|x|$

26. $f(x) = |\cos x|, \quad g(x) = \cos|x|$

Exercises 27–30 *Sketch a graph of f for x in $[-\pi, \pi]$. (Hint: First use an appropriate reduction formula to get a simpler equation for $f(x)$. Show points that are not included with open circles.)*

27. $f(x) = \dfrac{\cos(-x)}{\cos x}$ **28.** $f(x) = \dfrac{\cos\left(x + \frac{\pi}{2}\right)}{\sin x}$

29. $f(x) = \dfrac{\sin(x + \pi)}{\cos x}$

30. $f(x) = \dfrac{1}{2}\left[\sin x + \cos\left(\dfrac{\pi}{2} - x\right)\right]$

Exercises 31–34 ***Related Graphs*** *Draw graphs of the two functions on the same screen. Explain how the two graphs are related.*

31. $f(x) = \sin x$, $g(x) = 2 \sin x$

32. $f(x) = \cos x$, $g(x) = -2 \cos x$

33. $f(x) = \sin x$, $g(x) = \sin\left(x - \dfrac{\pi}{2}\right)$

34. $f(x) = \cos\left(x - \dfrac{\pi}{2}\right)$, $g(x) = \sin x$

Exercises 35–36 ***Intercept Points from Graph*** *Use a graph to approximate the x-intercept points in the interval $[-2\pi, 2\pi]$ (1 decimal place).*

35. $f(x) = 0.5 + \sin x$ **36.** $g(x) = \cos x - \sin x$

Exercises 37–38 ***Points of Intersection*** *Draw graphs of the two functions on the same screen. Use the graphs to approximate the solutions to the equation $f(x) = g(x)$ in the interval $[0, \pi]$ (1 decimal place).*

37. $f(x) = \sin x$, $g(x) = \cos x$

38. $f(x) = \cos x$, $g(x) = \tan x$

Exercises 39–40 ***Number of Intersections*** *On the same screen draw graphs of f and g. At how many points do the graphs intersect? Draw as many periods of the sine or cosine function as needed to be certain that you have all points of intersection.*

(a) $g(x) = \frac{x}{2}$ ***(b)*** $g(x) = \frac{x}{4}$ ***(c)*** $g(x) = \frac{x}{10}$

39. $f(x) = \sin x$ **40.** $f(x) = \cos x$

Exercises 41–43 ***Maximum Value*** *Find the maximum value of f in the window $[0, \pi] \times [-1, 3]$, (1 decimal place).*

41. **(a)** $f(x) = 2 \sin x + \cos \dfrac{x}{2}$

(b) $f(x) = 2^{\sin x}$

42. **(a)** $f(x) = \sin x + \left(\cos \dfrac{x}{2}\right)(\sin x)^2$

(b) $f(x) = e^{\sin x}$

43. **(a)** $f(x) = 3\left(\cos \dfrac{x}{2}\right)(\sin x)^2$ **(b)** $f(x) = x^{\sin x}$

Exercises 44–46 ***Using Graphs*** *Draw a graph of f for x in $[-2\pi, 2\pi]$. **(i)** If f is periodic give its period. **(ii)** State the domain and range of f.*

44. **(a)** $f(x) = 2 + \cos\left(x - \dfrac{\pi}{4}\right)$

(b) $f(x) = 2^{\cos x}$

45. **(a)** $f(x) = 2 + 3 \sin \dfrac{x}{2}$ **(b)** $f(x) = e^{\sin x}$

46. **(a)** $f(x) = 0.5 \tan \dfrac{\pi x}{2}$ **(b)** $f(x) = \sin(\cos x)$

47. **(a)** ***Parentheses*** Do you get the same graph if you enter $Y_1 = (\sin x)^2$ as for $Y_1 = \sin x^2$?
(b) Draw a graph of $f(x) = (\sin x)^2 + (\cos x)^2$. What do you observe?
(c) Do the same for $f(x) = (\sec x)^2 - (\tan x)^2$.

Exercises 48–49 *Draw graphs of f and g. Based on the graphs, for what values of x does it appear that $f(x) = g(x)$?*

48. $f(x) = \sin 2x, \quad g(x) = 2 \sin x \cos x$

49. $f(x) = \sin x \tan x, \quad g(x) = \sec x - \cos x$

Exercises 50–52 *Find the smallest positive root of the equation (2 decimal places). (Hint: Graphs will help.)*

50. **(a)** $\cos x = 0.658$ **(b)** $\tan x = \sqrt{x}$

51. **(a)** $1 + \cos 2x = \cos(x - 1)$
(b) $\sin(x - 0.5) = \cos x$

52. **(a)** $\sin(\cos x) = \sin x$ **(b)** $1 - (\tan x)^2 = 2 \sin x$

53. Find the largest positive root of $\sin x = \frac{x}{10}$. (2 decimal places).

Exercises 54–55 ***Period and Graph*** *Determine the period of f and then draw a graph that shows exactly two periods. Give the window you are using for your graph.*

54. $f(x) = \cos \dfrac{\pi x}{2}$ **55.** $f(x) = 2 - \cos 2x$

56. Draw the graph of $y = \sec x$ with y-range $[0, 3]$ and each of the given x-ranges.
$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right], \left[\dfrac{3\pi}{2}, \dfrac{5\pi}{2}\right], \left[\dfrac{7\pi}{2}, \dfrac{9\pi}{2}\right]$. Give two additional windows that will show a similar graph.

57. **(a)** Without using a calculator, for x in $[-2\pi, 2\pi]$, find the points that are on the graphs of both $f(x) = x \sin x$ and $g(x) = x$. (*Hint:* Solve $x \sin x = x$ for x.)
(b) Use graphs of f and g as a check.

Exercises 58–60 ***Solution Set*** *Find the solution set for $0 < x < \frac{\pi}{2}$ (2 decimal places).*

58. **(a)** $\cos x = 0.25$ **(b)** $\cos x < 0.25$.

59. **(a)** $\sin x = 0.64$ **(b)** $\sin x < 0.64$.

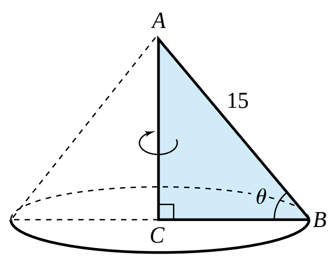
60. **(a)** $\tan x = 1.5$ **(b)** $\tan x < 1.5$.

Exercises 61–62 ***(a)*** *Are the graphs of f and g identical?* ***(b)*** *If not, adjust the formula for g so that they are.*

61. $f(x) = \sin x$, $g(x) = \cos(x + \frac{\pi}{2})$

62. $f(x) = \cos x$, $g(x) = \sin(x - \frac{\pi}{2})$

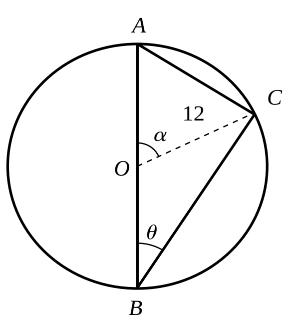
63. Right triangle ABC shown in the diagram is revolved about side $\overline{AC}$, giving a cone. If θ is small, the cone is shallow and has a small volume, if θ is almost $\frac{\pi}{2}$, the cone is slender and has a small volume. Between these extremes there must be a value of θ that gives a cone of maximum volume.



(a) Express the volume V as a function of θ.
(b) Use a graph of the function to find the value of θ for which V is a maximum (1 decimal place).
(c) What is the maximum value of V?

64. ***Alternate Solution for Exercise 63:*** Solve the problem by finding a formula for V as a function of x, where $|\overline{BC}| = x$. Use a graph of the function to find the value of x that gives the maximum value of V.

65. ***Maximum Area*** Points A, B, and C are on a circle of radius 12 as shown in the diagram, where point O is the center of the circle.
(a) Use geometry to show that $\alpha = 2\theta$ and $\angle ACB$ is a right angle. As point C moves along the circle we get different right triangles ABC.



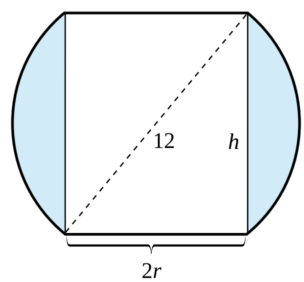
(b) Of all such possible triangles find θ so that the area is a maximum.
(c) What is the maximum area?

66. ***Maximum Area*** The length of the equal sides $\overline{AB}$ and $\overline{AC}$ of an isosceles triangle ABC is 24. Let θ denote each of the two equal angles.
(a) Draw several such triangles from small values of θ to nearly $\frac{\pi}{2}$, and describe what happens to the area of $\triangle ABC$.
(b) For what θ is the area a maximum? (2 decimal places).
(c) What is the maximum area?

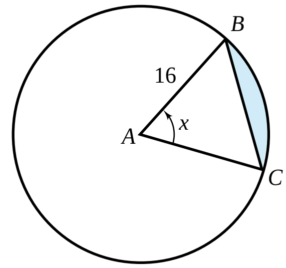
67. In Example 7, what value of θ will give a cylinder whose volume is one-half the volume of the sphere.

68. Repeat Example 7 for a sphere of radius 4 inches.

69. ***Alternate Solution:*** Solve the problem in Example 7 by getting a formula for V as a function of r. See diagram showing a cross-section of the cylinder.



70. In the diagram A is the center of a circle of radius 16 inches and $\angle BAC = x$ radians, $0 < x < \pi$.
(a) Find a formula that gives the area K of the shaded region as a function of x. Use a graph to find
(b) K when $x = 2.4$, and
(c) x when $K = 164$ square inches (2 decimal places).



71. ***Polynomial Approximations*** In Exercises 70 and 71 of Section 5.3 we indicated that the sine and cosine functions can be approximated very closely by polynomials

$$S(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$

$$C(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}.$$

In radian mode draw graphs of $y = \sin x$ and $y = S(x)$ on the same screen $[-5, 5] \times [-2, 2]$. Trace and use the arrow keys to move from one graph to the other to find the values of x for which the two values of y agree to at least four decimal digits.

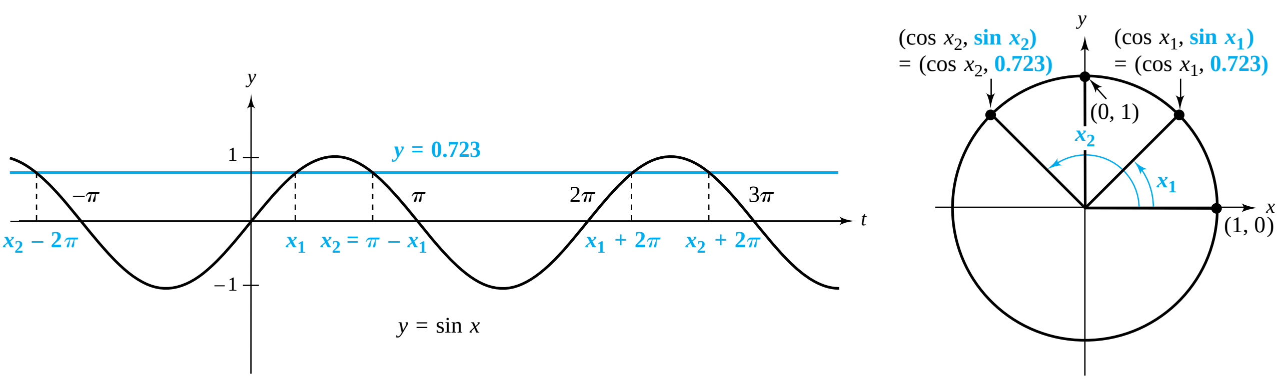
72. Follow instructions similar to those in Exercise 71 for $y = \cos x$, $y = C(x)$.

5.5 INVERSE TRIGONOMETRIC FUNCTIONS

> Quite often [mathematicians] do not deliver a frontal attack against a given problem, but rather they shape it, transform it, until it is eventually changed into a problem that they have solved before.
>
> *Rózsa Péter*

I didn't like the symbols for sine, cosine, tangent, and so on. So I invented other symbols. Now the inverse sine was the same [symbol] but left-to-right reflected . . . NOT $\sin^{-1}$—that was crazy! To me $\sin^{-1}$ meant $\frac{1}{\text{sine}}$, the reciprocal. So my symbols were better.
Richard Feynman

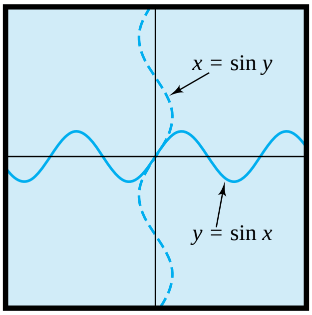
Given the equation $\sin t = 0.723$, how can we find the solution set, that is, the number or numbers whose sine is 0.723? If we look at the graph of the sine function in Figure 56, it is apparent that there are infinitely many numbers whose sine is 0.723. All such numbers are coterminal with one of the numbers x_1 or x_2 in the figure. In most cases, there is little hope of finding an exact solution although we could trace and zoom to approximate any of the intersection points shown in Figure 56a. Calculators are programmed to give an excellent approximation for the number x_1 in Figure 56a. The calculator function is SIN^{-1}, located above the SIN key. When we evaluate SIN^{-1}.723 the calculator returns 0.808134999. From the diagram in Figure 56b, we see that $x_2 = \pi - x_1$. If we zoom in several times on the intersection at x_2, we can read $x_2 \approx 2.333$, in excellent agreement with $\pi - x_1 \approx 2.333458$.



(a) (b)

FIGURE 56

The calculator function SIN^{-1} is an **inverse trigonometric function.** As a function, SIN^{-1} must pick out a single value from the set shown in Figure 56a. In this section we want to understand inverse trigonometric functions, to learn what kinds of values the calculator returns and to learn to find solutions the calculator does not give.



[−9, 9] by [−6, 6]

FIGURE 57

Restricted Domains for Inverse Trigonometric Functions

In Section 2.7 we learned that only inverses of one–one functions are themselves functions. To reemphasize that point, we want to look at pairs, functions and their inverses, beginning with the sine function. The natural approach uses parametric equations. With your calculator in parametric mode, enter the functions as follows.

$$\begin{cases} \text{X1} = \text{T} \\ \text{Y1} = \text{SIN T} \end{cases} \text{for } y = \sin x, \quad \begin{cases} \text{X2} = \text{SIN T} \\ \text{Y2} = \text{T} \end{cases} \text{for the inverse.}$$

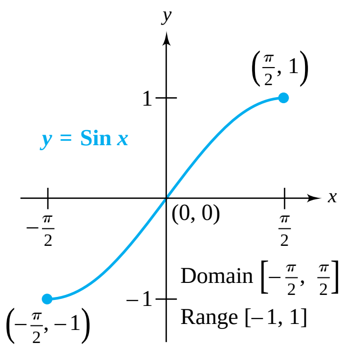
In $[-9, 9] \times [-6, 6]$, with a t-range of $[-9, 9]$, the calculator graph looks like Figure 57, where the graph of the inverse is dotted to make it easier to see.

In order for the inverse to be a function, we must restrict the domain of the sine function to an interval where the graph is either increasing or decreasing. The most natural choice is the interval $(\frac{-\pi}{2}, \frac{\pi}{2})$, as shown in Figure 58. We use the name **Sin *x*** (with a capital letter) for the restricted function in Figure 58a.

You can graph $y = \text{Sin } x$ easily by staying in parametric mode and plotting X = T, Y = SIN T, changing your t-range to $(\frac{-\pi}{2}, \frac{\pi}{2})$, (about -1.57 to 1.57). If, however, you don't turn off the inverse function used for the graph in Figure 57, your screen will show both the function and its inverse, and it is difficult to tell which is which. To make it easier to see, we usually show the graphs of functions and their inverses separately, as in the two panels of Figure 58. The inverse of the restricted sine is the function $y = \text{Sin}^{-1} x$. Since calculators are programmed to graph $y = \text{Sin}^{-1} x$, we do not need parametric equations; the SIN^{-1} key does it nicely. To see this on your calculator, graph $y = \sin^{-1} x$.

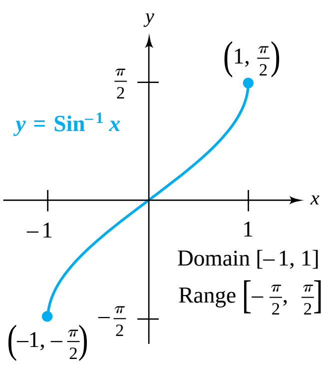
In this text, we denote the inverse of the restricted sine function either by $\text{Sin}^{-1} x$, read "inverse sine," or by **Arcsin** x; they are names for the same function. Casio and Texas Instrument calculators use the label SIN^{-1} but HP uses ASIN for Arcsin. Both names are common, so you should become familiar with both.

The tangent function is increasing on the interval $(\frac{-\pi}{2}, \frac{\pi}{2})$, and so we use a restricted tangent function, Tan x, to define the inverse tangent, $\text{Tan}^{-1} x$ or **Arctan *x***. The graphs, with domains and ranges, are shown in Figure 59.



(a)

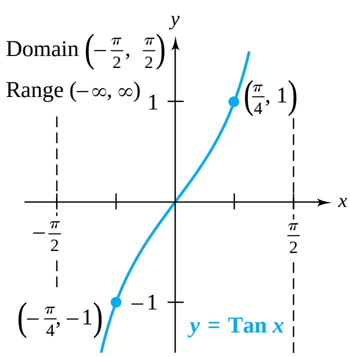
Restricted sine function



(b)

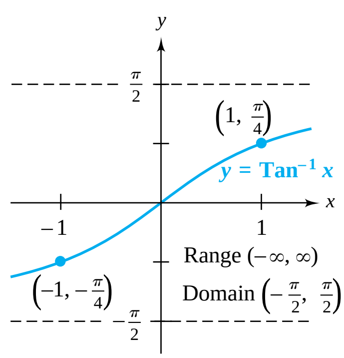
Inverse sine function

FIGURE 58



(a)

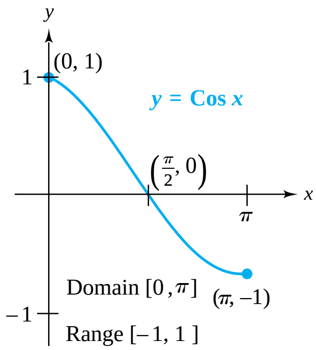
Restricted tangent function,
$y = \text{Tan } x$



(b)

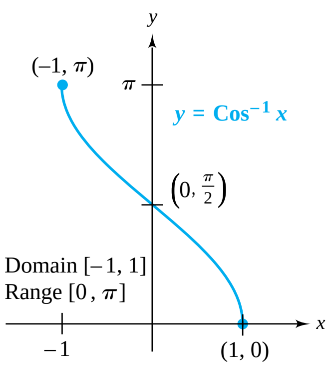
Inverse tangent function,
$y = \text{Tan}^{-1} x$

FIGURE 59



(a)

Restricted cosine function, $y = \text{Cos}\, x$



(b)

Inverse cosine function, $y = \text{Cos}^{-1} x$

FIGURE 60

The cosine function presents a small problem. We can no longer use the interval $(\frac{-\pi}{2}, \frac{\pi}{2})$, because $y = \cos x$ is neither increasing nor decreasing on the interval. To get a workable interval that includes first quadrant values, we use 0 to π for the restricted cosine function. The restricted cosine, Cos x, and its inverse, $\text{Cos}^{-1} x$ or **Arccos** $\boldsymbol{x}$**,** are graphed in Figure 60.

Inverse Function Identities

All of the relationships we observed in previous chapters between a function and its inverse apply to Sin x and $\text{Sin}^{-1} x$, and to the other pairs of inverse trigonometric functions:

The graphs are reflections of each other in the line $y = x$.
Every point (a, b) on one graph corresponds to the point (b, a) on the graph of the inverse.
The domain and range are interchanged.

One of the most important characteristics of inverse function pairs is that they "undo each other" in the sense described in Section 2.7.

$f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}

$f^{-1}(f(x)) = x$ for all x in the domain of f.

The interval where each inverse trigonometric function identity holds is given in the box on page 306.

The identity $\sin(\text{Sin}^{-1} x) = x$ suggests a way to think about the inverse trigonometric functions that many people find helpful:

$\text{Sin}^{-1} x$ is the number (or angle) whose sine is x.

As in the boxed statement below, the inverse trigonometric functions may be thought of as picking out the single number (or angle) whose sine (or cosine, etc.) has a given value. For example, from a 30°–60° triangle $\sin(\frac{\pi}{6}) = \frac{1}{2}$, so we know that $\frac{\pi}{6}$ is the angle whose sine is $\frac{1}{2}$; that is, $\text{Sin}^{-1}(\frac{1}{2}) = \frac{\pi}{6}$.

We began this section with the problem of solving $\sin t = 0.723$. One solution is given by $t = \text{Sin}^{-1}\, 0.723 \approx 0.808135$. Because trigonometric functions are periodic, there are always infinitely many solutions (if any) to an equation such as $\sin t = 0.723$. To use inverse trigonometric functions intelligently, we must know what kinds of numbers they pick out and then how to go from the single value they give to get a solution satisfying whatever conditions may apply in a given problem. You should make the information in the following box *very* familiar.

Ranges of inverse trigonometric functions

$\text{Sin}^{-1} x$ and $\text{Cos}^{-1} x$ are only defined when $-1 \le x \le 1$;
Tan^{-1} is defined for all real numbers.

$x \ge 0$ — $\text{Sin}^{-1} x$, $\text{Tan}^{-1} x$, $\text{Cos}^{-1} x$ are *positive* numbers, **all in Quadrant I, between 0 and $\pi/2$.**

$x < 0$ — $\text{Sin}^{-1} x$ and $\text{Tan}^{-1} x$ are *negative* numbers, between $\frac{-\pi}{2}$ **and 0, coterminal with Quadrant IV numbers.**

$\text{Cos}^{-1} x$ is a number **in Quadrant II, between $\frac{\pi}{2}$ and π.**

Calculator Evaluation and Graphs

In general, we need calculators to find solutions to equations involving trigonometric functions, but for many familiar values, we can evaluate inverse functions without a calculator. In working with such equations, here are some useful guidelines.

Keep in mind the angles displayed in Figure 29 from Section 5.2.
Sketch a quick graph of the sine or cosine curve for reference.
Draw unit circle diagrams or reference triangles.

And always check your results by calculator.

▶EXAMPLE 1 ***Evaluating inverse functions*** Find all calculator solutions for the equation and show the result on a graph of the appropriate inverse trigonometric function. **(a)** $\sin x = \frac{1}{3}$ **(b)** $(2\cos x - 1)(\cos x + 1) = 0$

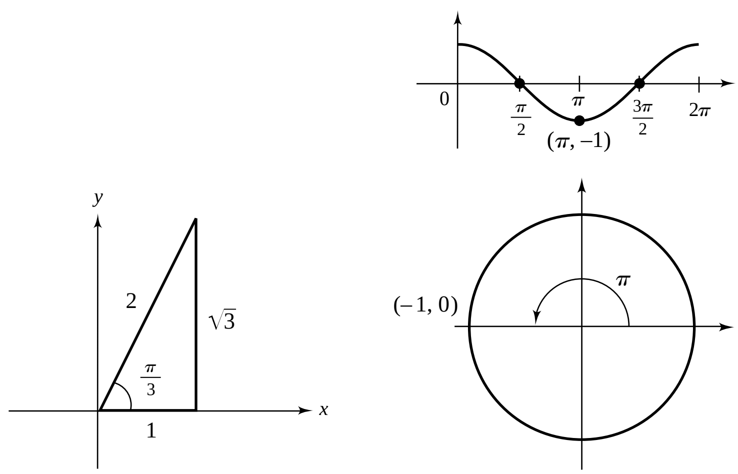
Solution

(a) Evaluating $\text{Sin}^{-1}(\frac{1}{3})$, the calculator displays .33983690945.
(b) By the zero-product principle, the given equation is equivalent to two:

$$2\cos x - 1 = 0, \qquad \cos x + 1 = 0, \text{ or}$$
$$\cos x = \frac{1}{2}, \qquad \cos x = -1.$$

Here is a place where we may recognize exact form solutions. For the first equation, the first quadrant angle whose cosine is $\frac{1}{2}$ is $\frac{\pi}{3}$ (see Figure 61a). By calculator, $x = \text{Cos}^{-1}(\frac{1}{2}) = 1.0471975512$, the calculator approximation for $\frac{\pi}{3}$. For the second equation, we can sketch a cosine curve to see where the cosine is -1, or draw a unit circle diagram (Figure 61b). By calculator, $x = \text{Cos}^{-1}(-1) = 3.14159265359$, which we recognize immediately as π.

The points corresponding to each solution are shown in Figure 62. ◀



(a) (b)

FIGURE 61

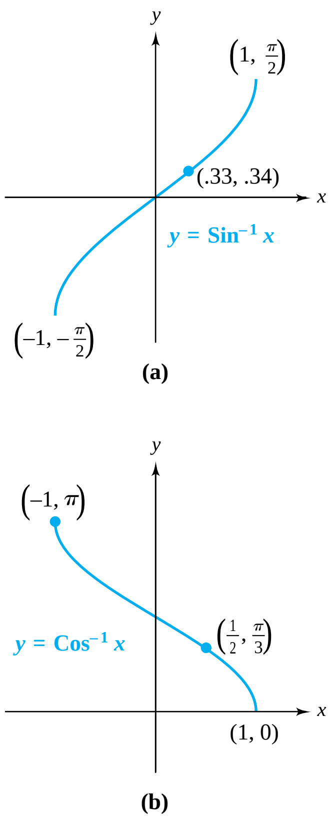


FIGURE 62

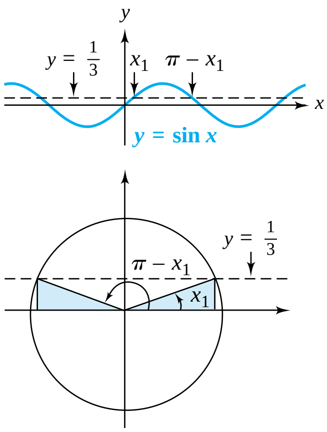


FIGURE 63

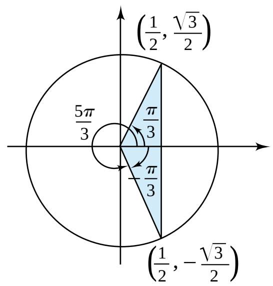
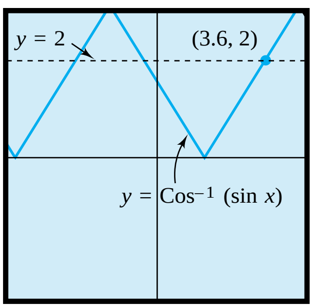
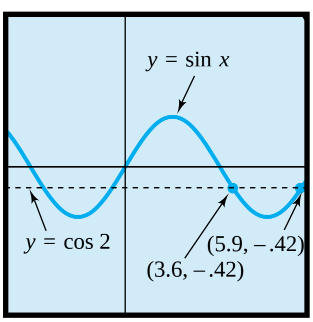


FIGURE 64



[−5, 5] by [−3, 3]

(a)



[−4, 6] by [−3, 3]

(b)

FIGURE 65

►EXAMPLE 2 ***Evaluating inverse functions*** Use graphs of $y = \sin x$ and $y = \cos x$ to find all solutions (four decimal places).

(a) $\sin x = \frac{1}{3}$ **(b)** $(2 \cos x - 1)(\cos x + 1) = 0$

Solution

(a) Either from the graph or from the unit circle diagram in Figure 63, we can see that any solution to the equation must be coterminal with either the number $x_1 \approx 0.3398$ we obtained in Example la for $\text{Sin}^{-1}(\frac{1}{3})$ or with $\pi - x_1 \approx 2.8018$. All numbers coterminal with x_1 are obtained by adding some integer multiple of 2π. The solutions are given by

$$x \approx 0.3398 + 2k\pi \quad \text{or} \quad x \approx 2.8018 + 2k\pi,$$

where k is any integer

(b) In Example 1b we found that $\frac{\pi}{3}$ and π are exact form solutions. While $\text{Cos}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ (in Quadrant I), there are other angles with cosine $\frac{1}{2}$, such as $\frac{5\pi}{3}$, or $-\frac{\pi}{3}$ (see Figure 64). All solutions are coterminal with one of $\frac{\pi}{3}$, π, or $\frac{5\pi}{3}$. In exact form, the solutions are given by

$$x = \frac{\pi}{3} + 2k\pi, \; x = \pi + 2k\pi, \quad \text{or} \quad x = \frac{5\pi}{3} + 2k\pi. \;◄$$

►EXAMPLE 3 ***Using a graph*** Find two solutions of $\text{Cos}^{-1}(\sin x) = 2$ in the interval $[0, 2\pi]$ (two decimal places).

Solution

Graphing `Y1 = COS⁻¹(SIN X)` and `Y2 = 2` in a decimal window gives the graph shown in Figure 65a. Tracing along the sawtooth curve, we find one solution near $x = 3.6$. Since we are asked for all solutions between 0 and 6.3, we must extend our window. (Try tracing to the right; if your calculator has a power scroll feature, the graph extends automatically as your trace cursor reaches the right edge of your screen.) It appears that the sawtooth pattern continues, so that there are infinitely many solutions to the given equation, but there is only one more in the interval $[0, 2\pi]$. To two decimal-place accuracy, the solutions are given by $x = 3.57$ and $x = 5.85$.

Alternate Solution Taking the cosine of both sides of the equation allows us to use the inverse function identity $\cos(\text{Cos}^{-1} u) = u$ for every u in $[-1, 1]$:

$$\cos(\text{Cos}^{-1}(\sin x)) = \cos 2, \quad \text{or} \quad \sin x = \cos 2.$$

Graphing `Y1 = SIN X` and `Y2 = COS 2` gives the graph in Figure 65b. As above, we must extend the window to the right to see the two intersections between 0 and 6.28. The solutions are, of course, the same. ◄

Triangle Diagrams with Inverse Trigonometric Functions

It often helps to remember that $\text{Sin}^{-1}(0.44)$ is the number (angle) in Quadrant I with sine 0.44 or $\text{Cos}^{-1}(-0.012)$ is the number (angle) in Quadrant II with cosine -0.012. We sometimes use a name we associate with an angle, such as $\theta = \text{Sin}^{-1} 0.15$, to help us remember that inverse trigonometric functions name angles. It then becomes natural to draw diagrams showing angles and reference triangles, as in the next example.

HISTORICAL NOTE π AND e, PART II

Because π and e are transcendental, there is no polynomial equation with integer coefficients—not of degree ten or ten million—whose graph has an x-intercept at either number. How are such numbers approximated to thousands of decimal places? Some limiting process is needed, usually an infinite series. Various series differ dramatically in their rates of convergence (the number of terms needed for a good approximation). We list below some series that have actually been used to calculate digits of π and e.

Mathematicians Peter (left) and Jonathan Borwein

Most recent computer calculations use series for the inverse tangent function (the source of Gregory's approximation). The 1986 program on the CRAY-2 supercomputer that produced 29 million digits of π used a new iteration algorithm due to the two Borwein brothers of Dalhousie University in Nova Scotia.

$$\left.\begin{aligned} e &= 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots \\ \frac{\pi^2}{6} &= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \end{aligned}\right\} \quad \text{Euler (The series for } e \text{ is very fast but for } \pi \text{ is quite slow.)}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \qquad \text{Gregory, 1688 (very slow)}$$

$$\frac{\pi}{2} = \frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot \cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot \cdots} \qquad \text{Wallis, 1650}$$

Strategy: (a) First let $\theta = \text{Sin}^{-1}\frac{-2}{3}$, so $\sin\theta = -\frac{2}{3}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Draw a diagram that shows θ in Quadrant IV and a reference triangle with $y = -2$ and $r = 3$. Use the Pythagorean theorem to find x and then use the triangle to evaluate $\tan\theta$.

▶EXAMPLE 4 ***Drawing triangle diagrams*** Evaluate in exact form

(a) $\tan(\text{Sin}^{-1} - \frac{2}{3})$ and **(b)** $\cos(\pi + \text{Tan}^{-1} 2)$.

Solution

(a) Follow the strategy. The diagram in Figure 66 shows that the x value for the reference triangle is $\sqrt{3^2 - (-2)^2}$ or $\sqrt{5}$. From the reference triangle, $\tan\theta = \frac{-2}{\sqrt{5}}$. Therefore, $\tan(\text{Sin}^{-1} - \frac{2}{3}) = \frac{-2}{\sqrt{5}}$.

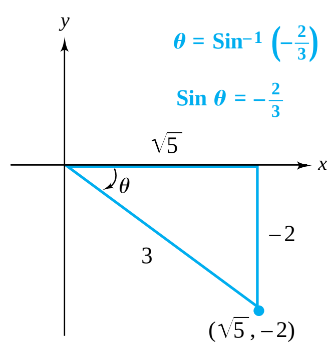


FIGURE 66

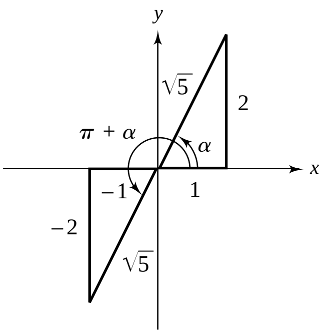


FIGURE 67

(b) If we let $\alpha = \text{Tan}^{-1} 2$, then α is a first quadrant angle whose tangent is 2, as in Figure 67. Adding π gives an angle in Quadrant III with reference triangle as in the diagram. From the diagram we read

$$\cos(\pi + \text{Tan}^{-1} 2) = \cos(\pi + \alpha) = \frac{-1}{\sqrt{5}}.$$

Note that we could have used a reduction formula, $\cos(\pi + \alpha) = -\cos \alpha$, instead of drawing the diagram. However we obtain the results, we should check our answers on the calculator. ◀

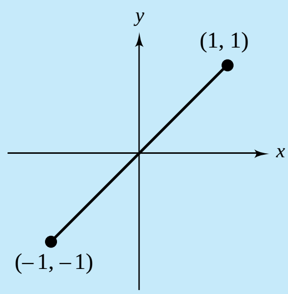
Compositions of Trigonometric Functions and Their Inverses

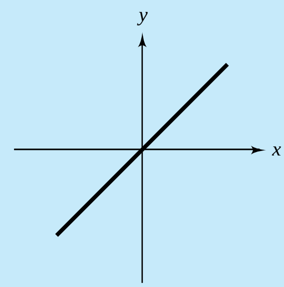
By the inverse function identities, all compositions of the form $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to x for at least some values of x. It follows that the graphs such as $y = \sin(\text{Sin}^{-1} x)$ or $y = \text{Tan}^{-1}(\tan x)$ must look like the graph of $y = x$ on the domain of the *argument* (the inside function). To make it easier to recall where these functions look like $y = x$ and where they differ, we show part of the graphs of each composition. We invite you to duplicate all of these graphs on your calculator and experiment further for yourself.

Graphs of compositions- inverse function identities

$\left.\begin{matrix}\sin(\text{Sin}^{-1}x)\\ \cos(\text{Cos}^{-1}x)\end{matrix}\right\} = x$ on $[-1, 1]$

$\tan(\text{Tan}^{-1}x) = x$ for all real x

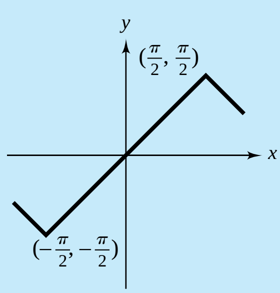


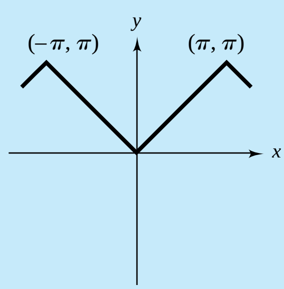


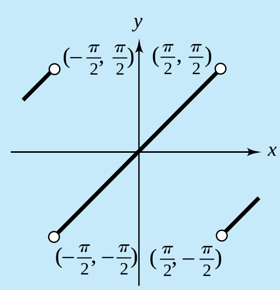
$\text{Sin}^{-1}(\sin x) = x$
on $[-\pi/2, \pi 2]$

$\text{Cos}^{-1}(\cos x) = x$
on $[0, \pi]$

$\text{Tan}^{-1}(\tan x) = x$
on $(-\pi/2, \pi 2)$
(in dot mode)







▶EXAMPLE 5 ***Exact form*** Evaluate in exact form

(a) $\sin(\text{Arcsin } \frac{2}{3})$ and

(b) $\text{Cos}^{-1}(\cos \frac{5\pi}{4})$. Explain why $\text{Cos}^{-1}(\cos \frac{5\pi}{4}) \neq \frac{5\pi}{4}$.

Solution

(a) Since $\frac{2}{3}$ is a number in the interval $[-1, 1]$, we may apply $\sin(\text{Sin}^{-1} x) = x$ directly and obtain $\sin(\text{Arcsin } \frac{2}{3}) = \frac{2}{3}$.

(b) The identity $\text{Cos}^{-1}(\cos x) = x$ applies only when x is in $[0, \pi]$ and $\frac{5\pi}{4}$ is not in that interval. To evaluate **(b)** first evaluate $\cos \frac{5\pi}{4}$.

$$\text{Cos}^{-1}\left(\cos \frac{5\pi}{4}\right) = \text{Cos}^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}.$$

Hence $\text{Cos}^{-1}(\cos \frac{5\pi}{4}) = \frac{3\pi}{4}$.
(See Figure 68.) As a check, evaluate $\text{Cos}^{-1}(\cos \frac{5\pi}{4})$ by calculator. ◀

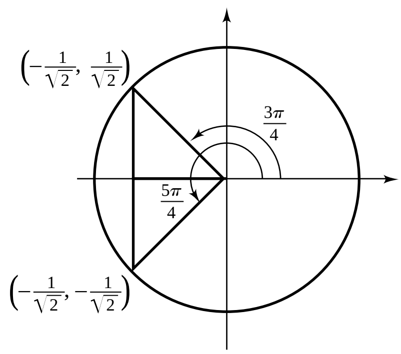


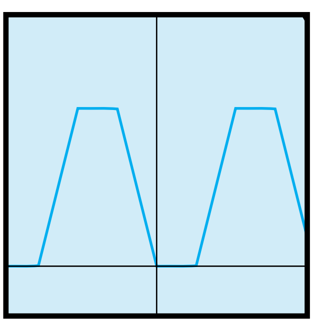
FIGURE 68

▶EXAMPLE 6 ***Difference of compositions***

(a) Draw a calculator graph of $y = \text{Arccos}(\cos x) - \text{Arcsin}(\sin x)$. Explain why there appears to be no graph on certain intervals.

(b) Find the exact coordinates of the "corner" points of the graph on the interval from $x = -\pi$ to $x = \pi$.

Solution



[−6, 6] by [−1, 5]

FIGURE 69

(a) In a shifted decimal window the graph looks like Figure 69. Where the graph seems to disappear, it is really overwriting the x-axis, as we can tell by tracing. When we trace, starting at $x = 0$ and moving to the right, the calculator shows that y is some number less than 10^{-10}; a calculator display of a number that small usually indicates some round-off error from 0.

(b) Referring to the composition graphs in the box above, we see that $\text{Cos}^{-1}(\cos x) = x$ on $[0, \pi]$, and $\text{Sin}^{-1}(\sin x) = x$ on $[\frac{-\pi}{2}, \frac{\pi}{2}]$. Therefore $\text{Cos}^{-1}(\cos x) - \text{Sin}^{-1}(\sin x) = x - x = 0$ on $[0, \frac{\pi}{2}]$. There is a corner at $(0, 0)$ and at $(\frac{\pi}{2}, 0)$. The next corner to the right is at (π, π). Moving to the left, there is a corner at $(\frac{-\pi}{2}, \pi)$ and at $(-\pi, \pi)$. ◀

▶EXAMPLE 7 ***Identical functions?*** Compare the graphs of $f(x) = \sqrt{1 - x^2}$ and $g(x) = \sin(\text{Cos}^{-1} x)$. Determine algebraically if the domains of f and g are the same.

Solution

Graphical If we graph Y1 = $\sqrt{1 - X^2}$, and Y2 = SIN(COS^{-1} X), we see only one curve, a semicircle centered at the origin. By translating the second graph up by one-half unit, we can see both graphs at the same time. Graph

Y1 = $\sqrt{1 - X^2}$, Y2 = SIN(COS^{-1} X) + .5 .

Tracing along the curves, comparing y-values to calculator accuracy, the y-coordinates differ by exactly 0.5. The graphs make it appear that the domain of both functions is the interval $[-1, 1]$.

Algebraic The domain of f is the set of numbers for which $1 - x^2 \geq 0$, $x^2 \leq 1$, or $-1 \leq x \leq 1$. Thus the domain of f is the interval $[-1, 1]$. The sine function is defined for all real numbers, so the domain of $\sin(\text{Cos}^{-1} x)$ is the same as the domain of the argument, Cos^{-1}, namely $[-1, 1]$.

We conclude that f and g have the same domain, and the graph suggests that $\sin(\text{Cos}^{-1} x) = \sqrt{1 - x^2}$ for every x in $[-1, 1]$. ◀

▶EXAMPLE 8 ***Applying the arctan function*** Janet, whose eye-level is 5.2 feet, is walking along Main Street looking up at a movie marquee that is 3.5 feet tall, with its bottom edge 12 feet above the sidewalk. See the diagram in Figure 70. Janet's "view" of the marquee is measured by her viewing angle θ, which in turn depends on the distance x. When she is far away, θ is small; as she approaches, the angle increases, and then gets small again as she nears the marquee.

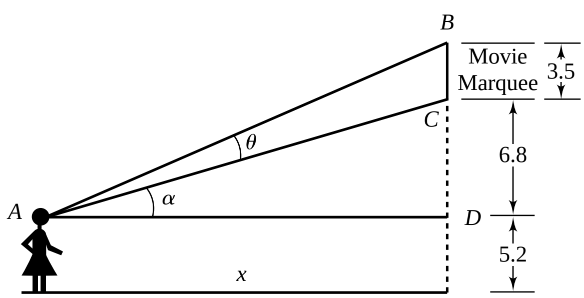
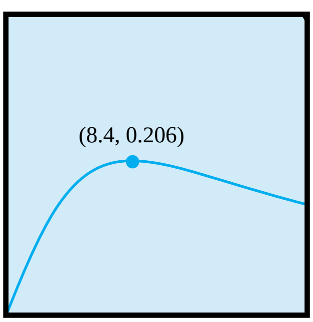


FIGURE 70

(a) Find a formula that gives θ as a function of x.

(b) Use a graph of f to find the distance x giving the best view (the maximum viewing angle).



[0, 20] by [0, 0.4]

FIGURE 71

Solution

(a) From $\triangle ADB$, $\tan(\theta + \alpha) = \frac{10.3}{x}$, so $\theta = \text{Tan}^{-1}(\frac{10.3}{x}) - \alpha$. From $\triangle ADC$, $\tan \alpha = \frac{6.8}{x}$, so $\alpha = \text{Tan}^{-1}(\frac{6.8}{x})$. Putting the two equations together, we have the desired function,

$$\theta = f(x) = \text{Tan}^{-1}\left(\frac{10.3}{x}\right) - \text{Tan}^{-1}\left(\frac{6.8}{x}\right).$$

(b) To select an appropriate window, we try a few values of x and see that θ appears to be less than 0.3. In $[0, 20] \times [0, 0.4]$ we get the graph in Figure 71. Tracing and zooming in as needed, we see that the highest point is near (8.4, 0.206), so Janet's maximum viewing angle is about 0.206, or about 12°, when she is about 8.4 feet from the point directly below the bottom of the marquee. From the graph, however, we see that her view really changes very little while she is anywhere from about 11 feet to about 6.5 feet. ◀

Inverse Functions for Secant, Cosecant, and Cotangent

By suitably restricting the domains of the secant, cosecant, and cotangent, we can define inverse functions. Unfortunately, there is no universal agreement as to which domains are most useful. We are not going to do much with these inverse functions except to recognize that when a need arises, it is almost always possible to translate problems involving Sec^{-1}, Csc^{-1}, or Cot^{-1} into terms of Cos^{-1}, Sin^{-1}, or Tan^{-1}, as in the following example.

▶EXAMPLE 9 ***Inverse secant*** Evaluate $\mathrm{Sec}^{-1} 3$, rounded off to three decimal places.

Solution

Let $\theta = \mathrm{Sec}^{-1} 3$. Then $\sec\theta = 3$, and so $\frac{1}{\cos\theta} = 3$, or $\cos\theta = \frac{1}{3}$. If the range of Sec^{-1} agrees with the range of Cos^{-1}, then use a calculator:

$$\theta = \mathrm{Cos}^{-1}\left(\frac{1}{3}\right) \approx 1.231.$$

Therefore, $\mathrm{Sec}^{-1} 3 \approx 1.231$. ◀

EXERCISES 5.5

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. There is no number (or angle) θ such that $\theta = \mathrm{Sin}^{-1}(\frac{20}{29})$ and $\theta = \mathrm{Cos}^{-1}(\frac{21}{29})$.
2. There is no number (or angle) θ such that $\theta = \mathrm{Sin}^{-1}(\frac{2}{3})$ and $\theta = \mathrm{Cos}^{-1}(\frac{1}{3})$.
3. The function $f(x) = \mathrm{Cos}^{-1} x$ is an increasing function.
4. The point $(\frac{\pi}{2}, 1)$ is on the graph of $y = \mathrm{Sin}^{-1} x$.
5. The range of $f(x) = \mathrm{Cos}^{-1} x$ contains four integers.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. The largest interger in the range of $f(x) = \mathrm{Cos}^{-1} x$ is ______.
7. The largest negative integer that is not in the range of $f(x) = \mathrm{Sin}^{-1} x$ is ______.
8. The smallest positive integer that is not in the range of $f(x) = \mathrm{Tan}^{-1} x$ is ______.
9. The maximum value of $f(x) = \mathrm{Cos}^{-1} x$ is ______.
10. The graph of $y = \mathrm{Tan}^{-1} x$ contains no points in Quadrant(s) ______.

Develop Mastery

Exercises 1–4 ***Exact Form*** *Evaluate in exact form in radians, using π as needed.*

1. **(a)** $\mathrm{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **(b)** $\mathrm{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
2. **(a)** $\mathrm{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **(b)** $\mathrm{Arcsin}\left(\frac{1}{2}\right) + \mathrm{Arccos}\left(-\frac{1}{2}\right)$
3. **(a)** $\mathrm{Arctan}\ 0$ **(b)** $\mathrm{Arcsin}\left(-\frac{1}{\sqrt{2}}\right)$
4. **(a)** $\mathrm{Sin}^{-1}\left(\frac{1}{2}\right) + \mathrm{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **(b)** $\mathrm{Sin}^{-1}\left(-\frac{1}{2}\right)$

Exercises 5–8 ***Decimal Approximations*** *Evaluate by calculator and give results in radians rounded off to three decimal places. If the display indicates an error, explain why.*

5. **(a)** $\mathrm{Cos}^{-1}(0.399)$ **(b)** $\mathrm{Sin}^{-1}\ 0.25$
6. **(a)** $\mathrm{Tan}^{-1}\left(-\frac{\pi}{3}\right)$ **(b)** $\mathrm{Arcsin}(\sin 5.43)$
7. **(a)** $\mathrm{Tan}^{-1}(\sin\sqrt{3})$ **(b)** $\sec\left(\mathrm{Cos}^{-1} - \frac{1}{3}\right)$
8. **(a)** $\sin(\mathrm{Arcsin}\ 1.01)$ **(b)** $\sin\left(\mathrm{Sin}^{-1}\frac{1}{2} - \mathrm{Tan}^{-1}\frac{1}{\sqrt{2}}\right)$

Exercises 9–12 ***Exact Form*** *Evaluate in exact form. In some cases reduction fomulas may be helpful; see Table 2 in section 5.4. In case an expression is undefined, explain why.*

9. **(a)** $\sin(\text{Sin}^{-1} 0.3)$ **(b)** $\text{Sin}^{-1}\left(\sin \dfrac{3\pi}{2}\right)$

10. **(a)** $\sec\left(2\ \text{Sin}^{-1} \dfrac{1}{2}\right)$ **(b)** $\cos\left(\text{Sin}^{-1} \dfrac{1}{2} + \text{Cos}^{-1} \dfrac{1}{2}\right)$

11. **(a)** $\sin\left(\dfrac{\pi}{2} - \text{Cos}^{-1} \dfrac{2}{3}\right)$ **(b)** $\cot\left(\dfrac{\pi}{2} - \text{Arctan}\ \dfrac{1}{3}\right)$

12. **(a)** $\cos\left(\pi - \text{Sin}^{-1} \dfrac{2}{7}\right)$ **(b)** $\sec(\pi + \text{Cos}^{-1} 0.75)$

Exercises 13–14 ***Decimal Approximations*** *Give an approximation rounded off to three decimal places. See Example 9.*

13. **(a)** $\text{Cot}^{-1}\left(\dfrac{1}{10}\right)$ **(b)** $\text{Sec}^{-1} 1.532$

14. **(a)** $\sin(\text{Tan}^{-1} 1 + \text{Sec}^{-1} 1)$ **(b)** $\tan(-2\ \text{Csc}^{-1} 3)$

Exercises 15–16 ***Graph to Solution*** *Use a graph of the inverse function to help you solve for x (2 decimal places). As a check apply an appropriate function to both sides and use an inverse trigonometric function identity. Explain why this could lead to a false "solution."*

15. **(a)** $\text{Sin}^{-1} x = -0.36$ **(b)** $\text{Cos}^{-1} x = 2.4$

16. **(a)** $\text{Tan}^{-1} x = 1.9$ **(b)** $\text{Tan}^{-1} x = -1.8$

Exercises 17–19 ***Translations, Reflections*** *Without using a calculator sketch the graph of f. (Hint: Use translations or reflections of core inverse function graphs where appropriate.)*

17. **(a)** $f(x) = -\text{Cos}^{-1} x$ **(b)** $f(x) = \text{Cos}^{-1}(x + 1)$

18. **(a)** $f(x) = -\text{Sin}^{-1} x$ **(b)** $f(x) = \text{Sin}^{-1}(-x)$

19. **(a)** $f(x) = \text{Sin}^{-1}(\sin x)$ **(b)** $f(x) = \sin(\text{Sin}^{-1} x)$

Exercises 20–22 ***Using Graphs*** *Use graphs to solve the equation for x where $0 \le x \le 1$ (2 decimal places).*

20. **(a)** $\text{Tan}^{-1} x = \text{Cos}^{-1} x$ **(b)** $\text{Cos}^{-1} x = x$

21. **(a)** $\text{Cos}^{-1} x = x^2$ **(b)** $\sin(2.4x) = \text{Tan}^{-1} x$

22. **(a)** $\text{Sin}^{-1} x = \text{Cos}^{-1} x$ **(b)** $\cos x = \text{Cos}^{-1} x$

23. **(a)** Compare graphs of $f(x) = \sin(\text{Cos}^{-1} x)$ and $g(x) = \sqrt{1 - x^2}$. What do you observe? (*Hint:* Draw graphs of $y = f(x) + 1$ and $y = g(x)$ on the same screen and compare.)
(b) In Example 7 we observed that $\sin(\text{Cos}^{-1} x) = \sqrt{1 - x^2}$ is an identity. Use this to solve $\text{Sin}^{-1} x = \text{Cos}^{-1} x$ by first taking the sine of both sides. Check graphically.

Exercises 24–26 ***Related Graphs*** *Without using a calculator, draw a graph. Use the graph to help sketch a graph of g.*

24. **(a)** $f(x) = \text{Sin}^{-1} x$ **(b)** $g(x) = |\text{Sin}^{-1} x|$

25. **(a)** $f(x) = \text{Cos}^{-1} x$ **(b)** $g(x) = \text{Cos}^{-1}|x|$

26. **(a)** $f(x) = \text{Tan}^{-1} x$ **(b)** $g(x) = \text{Tan}^{-1}|x|$

27. For $f(x) = \text{Sin}^{-1} \frac{x}{4} + \text{Cos}^{-1} \frac{x}{4}$,
(a) What is the domain of f?
(b) What can you conclude from the graph of f?
(c) What is the range of f?

28. **(a)** Draw graphs of $y = \tan(\text{Tan}^{-1} x)$ and $y = x$ separately. What do the graphs suggest?
(b) Do the same for $y = \text{Tan}^{-1}(\tan x)$ and $y = x$.

29. **(a)** Draw a graph of $f(x) = \sin(\text{Cos}^{-1} x)$, $[-2, 2] \times [-2, 2]$. Use the graph to help you determine the domain and range of f.
(b) Draw a graph of $g(x) = \text{Cos}^{-1}(\sin x)$, $[-2\pi, 2\pi] \times [-1, 4]$.
(c) Explain why you should not expect to have any points on either graph with negative values of y.

30. Draw a graph of $f(x) = \text{Cos}^{-1}(\sin x)$.
(a) Use the graph to find $\text{Cos}^{-1}(\sin 3.6)$.
(b) Find solutions to $\text{Cos}^{-1}(\sin x) = 1.24$ where x is in $[0, \pi]$, 2 decimal places.
(c) Show that $(\frac{-\pi}{2}, \pi)$, $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, \pi)$ are points on the graph of f. Give a piecewise formula for f in the interval $[\frac{-\pi}{2}, \frac{3\pi}{2}]$.

31. **(a)** Use a graph of $y = \cos x$ to find two solutions (2 decimal places) for the equation $\cos x = -0.64$ in the interval $[0, 2\pi]$.
(b) Evaluate $\text{Cos}^{-1}(-0.64)$ amd compare with solutions in part **(a)**.

32. **(a)** Use a graph of $y = \sin x$ to find two solutions (2 decimal places) for the equation $\sin x = -0.36$, where x is in the interval $[0, 2\pi]$.
(b) Evaluate $\text{Sin}^{-1}(-0.36)$ and see if it is one of the numbers in part **(a)**. Explain.

33. **(a)** For $f(x) = \text{Cos}^{-1} x$ and $g(x) = \sin x$, evaluate $(f \circ g)(1.25)$.
(b) What is the range of $f \circ g$?

34. ***Maximum Value***
(a) For what x is $f(x) = \cos(\text{Tan}^{-1} x)$ a maximum?
(b) Does $g(x) = \sin(\text{Tan}^{-1} x)$ have a maximum value? (*Hint:* Remember that $-\frac{\pi}{2} < \text{Tan}^{-1} x < \frac{\pi}{2}$.)

35. ***Best View*** Solve the best view problem in Example 8 if Janet is replaced by a basketball player whose eye level height is 6 feet 9 inches. How far from Janet is the basketball player when both get their best view?

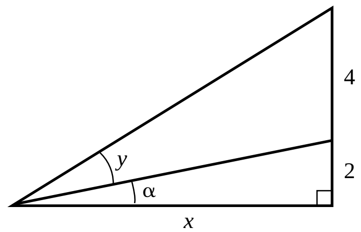
36. Solve the best view problem in Example 8 if the width of the marquee is 4.5 feet rather than 3.5 feet.

37. ***Maximum Value***

(a) In the diagram, find a formula that gives y as a function of x and draw a graph.

(b) For what x is y a maximum?

(c) Using calculus it can be shown that for maximum y the value of x must satisfy the equation $\frac{2}{x^2+4} - \frac{6}{x^2+36} = 0$. Solve for x and compare with part(b).



38. Draw a graph of $f(x) = \frac{6}{\pi}(\mathrm{Sin}^{-1} x + \mathrm{Cos}^{-1} x)$. From the graph what conclusion can be drawn about the function f?

Exercises 39–40 ***Domain, Range*** ***(a)*** *Without using graphs, find the domain and range of f.* ***(b)*** *Use a graph as a check.*

39. $f(x) = \mathrm{Cos}^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$

40. $f(x) = \mathrm{Sin}^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$

Exercises 41–42 ***Solutions*** *Find all solutions (2 decimal places) in the interval* $[0, \pi]$. *Compare the solutions in* ***(a)*** *and* ***(b)***.

41. **(a)** $\sin x = 0.4$ **(b)** $x = \mathrm{Sin}^{-1} 0.4$

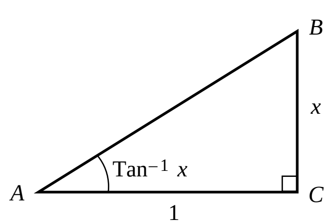
42. **(a)** $\cos x = -0.7$ **(b)** $x = \mathrm{Cos}^{-1}(-0.7)$

Exercises 43–44 Determine the solution set for the equation.

43. **(a)** $\mathrm{Cos}^{-1}(\cos x) = x$ **(b)** $\mathrm{Sin}^{-1}(\sin x) = x$

44. **(a)** $\tan(\mathrm{Tan}^{-1} x) = x$ **(b)** $\mathrm{Tan}^{-1}(\tan x) = x$

45. See that the angle in the diagram is labeled correctly.



(a) Show that $\sin(\mathrm{Tan}^{-1} x) = \dfrac{x}{\sqrt{1+x^2}}$

(b) Use calculator graphs to support a claim that
$$\sin(\mathrm{Tan}^{-1} x) = \frac{x}{\sqrt{1+x^2}} \text{ for every real number } x.$$

Exercises 46–48 Find an equation that does not involve inverse trigonometric functions to describe the function f. In each case, check your work graphically. (Hint: For 46, let $\theta = \mathrm{Sin}^{-1} x$ *and so* $\sin\theta = x$. *Draw a diagram.)*

46. $f(x) = \cos(\mathrm{Sin}^{-1} x)$ **47.** $f(x) = \sin(\mathrm{Tan}^{-1} x)$

48. $f(x) = \cot(\mathrm{Tan}^{-1} x)$

49. Show that Sin^{-1} and Tan^{-1} are odd functions.

50. Show that $\mathrm{Cos}^{-1}(-x) = \pi - \mathrm{Cos}^{-1} x$ for $-1 \le x \le 1$. Is $\mathrm{Cos}^{-1} x$ an even function? An odd function?

51. Show that $\mathrm{Sin}^{-1} x + \mathrm{Cos}^{-1} x = \frac{\pi}{2}$ for $-1 \le x \le 1$. (*Hint:* Show first that $\sin(\frac{\pi}{2} - \mathrm{Cos}^{-1} x) = x$.)

52. Show that $\mathrm{Tan}^{-1} x + \mathrm{Tan}^{-1} \dfrac{1}{x} = \dfrac{\pi}{2}$ for $x > 0$.

53. ***Maximum Value*** For $x > 0$, find the maximum value (2 decimal places) of $f(x) = 6(\mathrm{Tan}^{-1}\frac{8}{x} - \mathrm{Tan}^{-1}\frac{4}{x})$.

54. ***Minimum Value*** For $x > 0$, find the minimum value (2 decimal places) of $f(x) = 8(\mathrm{Tan}^{-1}\frac{2}{x} - \mathrm{Tan}^{-1}\frac{6}{x})$.

55. **(a)** Is $f(x) = \mathrm{Cos}^{-1}(\sin x)$ a periodic function? Explain.

(b) Is $g(x) = \mathrm{Tan}^{-1}(\cos x)$ a periodic function? Explain.

56. **(a)** Draw a graph of $g(x) = \mathrm{Tan}^{-1} x$ to support the claim that g is an odd function.

(b) Is $f(x) = \sin(\mathrm{Tan}^{-1} x)$ an odd funcion? Explain.

57. Determine whether f is odd, even, or neither.

(a) $f(x) = \mathrm{Cos}^{-1}(\cos x)$

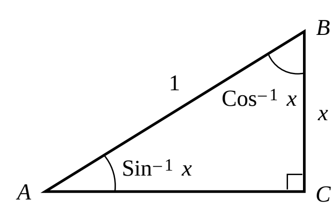
(b) $f(x) = \mathrm{Cos}^{-1}(\sin x)$

(c) $f(x) = \mathrm{Sin}^{-1}(\cos x)$

(d) $f(x) = \mathrm{Sin}^{-1}(\sin x)$

(*Hint:* First draw a graph.)

58. In the diagram explain why the angles are labeled correctly. Show that
$$\cos(\mathrm{Sin}^{-1} x) = \sin(\mathrm{Cos}^{-1} x) = \sqrt{1-x^2}.$$



59. Solve the problem in Example 6 for $f(x) = \mathrm{Sin}^{-1}(\sin x) - \mathrm{Cos}^{-1}(\cos x)$. Give a piecewise formula for f where $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$.

60. Find the solution set for $\tan(\mathrm{Tan}^{-1} x) = \cos(\mathrm{Cos}^{-1} x)$. Remember domain.

61. Give a piecewise formula for $f(x) = \mathrm{Sin}^{-1}(\sin x)$ for $-\pi \le x \le \pi$.

62. Do Exercise 61 for $f(x) = \mathrm{Cos}^{-1}(\cos x)$.

Exercises 63–64 ***Your Choice*** *Give a formula for a linear function f (with nonzero slope) that satisfies the specified conditions. Check by drawing graphs.*

63. **(a)** The graphs of f and $y = \mathrm{Sin}^{-1} x$ intersect at more than one point.
(b) The graphs of f and $y = -\mathrm{Cos}^{-1} x$ intersect at exactly one point which is in Quadrant II.

64. **(a)** The graphs of f and $y = \mathrm{Tan}^{-1} x$ intersect at three points.
(b) The graphs of f and $y = \mathrm{Cos}^{-1} x$ intersect at exactly one point, which is in Quadrant III.

CHAPTER 5 REVIEW

Test Your Understanding

Determine the truth value (T or F). Give reasons. Drawing graphs can be helpful.

1. There is no number x such that

(a) $\sin x = 2$, **(b)** $\cos x = -\dfrac{3}{4}$,

(c) $\tan x = \dfrac{\pi}{2}$, **(d)** $\sec x = \dfrac{1}{2}$.

2. **(a)** $\sin 1 = \dfrac{\pi}{2}$, **(b)** $\cos(-1) = \pi$,

(c) $\tan\left(\dfrac{\pi}{2}\right) = 0$

3. If θ is an angle in the fourth quadrant, then $\cos\theta$ is negative.

4. The numbers $\frac{3\pi}{2}$ and $-\frac{5\pi}{4}$ are coterminal.

5. There is no number x such that

(a) $\mathrm{Sin}^{-1} x = \dfrac{\pi}{4}$, **(b)** $\mathrm{Sin}^{-1} x = \dfrac{3\pi}{4}$,

(c) $\mathrm{Cos}^{-1} x = \dfrac{5\pi}{4}$, **(d)** $\mathrm{Cos}^{-1} x = -\dfrac{\pi}{3}$.

6. The number $\sec 3$ is negative.

7. The point $(\frac{\pi}{2}, 1)$ is on the unit circle.

8. The point$(-\frac{5}{13}, -\frac{12}{13})$ is on the unit circle.

9. $\tan\dfrac{3\pi}{4} < \tan\dfrac{5\pi}{4}$

10. If $(3, -4)$ is on the terminal side of θ, then $(3, 4)$ is on the terminal side of $-\theta$.

11. If $\sin x > 0$ and $\cos x < 0$, then $\tan x < 0$.

12. If $\cos\theta = \frac{3}{5}$, then $\sin(\theta + \frac{\pi}{2}) = \frac{3}{5}$.

13. The smallest prime number that is greater than $\tan 1.5$ is 13.

14. If $\theta = 450°$, then the radian measure of θ is $\frac{5\pi}{2}$.

15. The graphs of $f(x) = \cos x$ and $g(x) = \sin(\frac{\pi}{2} + x)$ are identical.

16. The graphs of $y = \sqrt{\cos x}$ and $y = x$ intersect in the first quadrant.

17. There is no number x for which $\mathrm{Tan}^{-1} x \ge \frac{\pi}{2}$.

18. If θ is an angle in the second quadrant, then $\tan\theta$ is negative.

19. **(a)** $\mathrm{Sin}^{-1}\left(\dfrac{\pi}{2}\right) = 1$, **(b)** $\mathrm{Cos}^{-1}\pi = -1$

20. The range of $f(x) = \mathrm{Cos}^{-1} x$ contains two prime numbers.

21. The graph of $y = 4(\mathrm{Tan}^{-1}\frac{6}{x} - \mathrm{Tan}^{-1}\frac{1}{x})$ has a local maximum point in the first quadrant.

22. **(a)** $\sec\pi$ is undefined. **(b)** $\mathrm{Sin}^{-1}(\frac{\pi}{2})$ is undefined.

23. The graphs of $f(x) = \sin(\mathrm{Cos}^{-1} x)$ and $g(x) = \sqrt{1 - x^2}$ are identical.

24. $\tan(\mathrm{Tan}^{-1} x) = x$ for every real number x.

25. $\sin x > \cos x$ for every x in the second quadrant.

26. If $f(x) = \mathrm{Sin}^{-1} x$, then f is an increasing function.

27. If $f(x) = \mathrm{Cos}^{-1} x$, then f is a decreasing function.

28. The graph of $y = 2^{\cos x}$ has no points in Quadrant III.

29. The function $f(x) = \cos x$ is an even function.

30. The function $f(x) = \sin x$ is neither even nor odd.

31. The function $f(x) = \mathrm{Sin}^{-1} x$ is one–one.

32. The function $f(x) = \tan x$ is one–one.

33. Point $(1, \frac{\pi}{2})$ is on the graph of $y = \sin x$.

34. Point $(\pi, -1)$ is on the graph of $y = \cos x$.

35. Point $(0, \frac{\pi}{2})$ is on the graph of $y = \mathrm{Cos}^{-1} x$.

36. The function $f(x) = \mathrm{Cos}^{-1} x$ is an even function.

37. The graphs of $y = \mathrm{Cos}^{-1} x$ and $y = \mathrm{Tan}^{-1} x$ intersect in the first quadrant.

38. The range of $f(x) = \mathrm{Sin}^{-1} x$ contains only one negative integer.

39. The graph of $y = \mathrm{Cos}^{-1} x$ contains exactly one point for which both coordinates are integers.

40. If $f(x) = \mathrm{Cos}^{-1} x$, then the maximum value of $f(x)$ is π.

41. In the window $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 3]$ the graph of $y = \sec x$ is similar to the graph of $y = \sec x$ when using $[\frac{5\pi}{2}, \frac{7\pi}{2}] \times [0, 3]$.

42. The graphs of $f(x) = \tan(\mathrm{Cos}^{-1} x)$ and $g(x) = \sqrt{1 - x^2}$ are identical.

43. The graphs of $y = \mathrm{Tan}^{-1} x$ and $y = \sin(2.5x)$ do not intersect in the first quadrant.

44. The function $f(x) = \mathrm{Cos}^{-1} x$ is periodic.

45. Planet Mars travels around the sun once in 687 days. Assuming circular orbits, the angular speed of Mars is greater than that of the Earth.

Review for Mastery

Exercises 1–3 ***Circular Sector*** *Refer to a circular sector with radius r, central angle θ, arc length s, and area A. Give results to two significant digits.*

1. If $r = 24$ cm and $\theta = 30°$, find s and A.

2. If $r = 12$ cm and $s = 20$ cm, find θ and A.

3. If $s = 13$ cm and $A = 64$ cm^2, find r and θ.

4. What is the degree measure of the smaller angle between the hour and minute hands of a clock at time 2:20?

Exercises 5–10 ***Points on Unit Circle*** *Point P(t) on the unit circle corresponds to the number t as described in Section 5.2.* ***(a)*** *From a diagram showing P(t), give the coordinates of P(t).* ***(b)*** *Give the values of the six trigonometric functions at t. In Exercises 5 through 7, give results in exact form, and in Exercises 8 through 10, give results rounded off to two decimal places.*

5. $t = \dfrac{3\pi}{4}$ **6.** $t = \dfrac{-2\pi}{3}$ **7.** $t = \dfrac{17\pi}{4}$

8. $t = 4.21$ **9.** $t = -\dfrac{\pi}{5}$ **10.** $t = 8.3$

11. Determine all real numbers t for which $\cos t = -1$.

12. Determine all real numbers t for which $\sin t = -1$.

13. **(a)** Draw a diagram that shows all points $P(t)$ on the unit circle where $0 \le t \le 2\pi$ and $\cos t = \frac{1}{4}$.
(b) What are the coordinates of $P(t)$?

Exercises 14–17 ***Simplify*** *Simplify by using an appropriate reduction formula.*

14. $\cos\left(\dfrac{\pi}{2} - t\right)$ **15.** $\sin\left(t + \dfrac{3\pi}{2}\right)$

16. $\tan\left(t + \dfrac{5\pi}{2}\right)$ **17.** $\sec(\pi - t)$

18. Evaluate in exact form
(a) $\sin\left(\dfrac{\pi}{2}\right)$, **(b)** $\tan\left(\dfrac{5\pi}{3}\right)$,
(c) $\cos\left(-\dfrac{7\pi}{6}\right)$, **(d)** $\sin\left(\pi - \dfrac{5\pi}{4}\right)$,
(e) $\sec\left(\pi + \dfrac{\pi}{3}\right)$.

19. Determine θ.
(a) $\sin\theta = \dfrac{-\sqrt{2}}{2}$ and $\pi < \theta < \dfrac{3\pi}{2}$
(b) $\tan\theta = -\sqrt{3}$ and $\dfrac{\pi}{2} < \theta < \pi$
(c) $\sec\theta = -1$ and $0 < \theta < 2\pi$.

20. If $\cos t = -0.75$ and $\tan t$ is negative, evaluate
(a) $\sin t$, **(b)** $\tan t$,
(c) $\cos\left(t - \dfrac{\pi}{2}\right)$, **(d)** $\tan(t + \pi)$.

21. Evaluate and give results rounded off to three decimal places
(a) $\sin 43°$, **(b)** $\tan 152°$, **(c)** $\cos 57°16'$.

22. Evaluate and give results rounded off to three decimal places
(a) $\sin 1.43$, **(b)** $\tan\left(\dfrac{5\pi}{8}\right)$,
(c) $\sec 1.46 + \cos 1.46$.

23. If point $(-3, 4)$ is on the terminal side of the angle θ in standard position, evaluate in exact form
(a) $\sin\left(\theta + \dfrac{\pi}{2}\right)$, **(b)** $\cos(\theta + \pi)$.

24. Suppose $P(t)$ is point $(\frac{-3}{5}, \frac{4}{5})$.
(a) Show that $P(t)$ is on the unit circle.
(b) What are the coordinates of $P(t + \frac{\pi}{2})$?

25. Suppose $P(t)$ is the point $(\frac{5}{13}, \frac{-12}{13})$.
(a) Show that $P(t)$ is on the unit circle.
(b) What are the coordinates of $P(t + \pi)$?
(c) Evaluate $\sin(t + \pi)$ and $\tan(t + \pi)$.

26. Which of the following points are on the unit circle?
(a) $(1, -1)$, **(b)** $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$, **(c)** $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$,
(d) $\left(\dfrac{\sqrt{3}}{2}, \dfrac{-1}{2}\right)$

Exercises 27–30 ***Reference Triangles, Function Values*** *Point Q is on the terminal side of angle θ in standard position. Assume that* $0 \le \theta \le 2\pi$. *From a diagram that shows a reference triangle for θ, find* $\sin\theta$, $\cos\theta$, *and* $\tan\theta$ *in exact form. Find angle θ in radians rounded off to two decimal places.*

27. $Q(3, 4)$ **28.** $Q(-3, 5)$

29. $Q(-4, -3)$ **30.** $Q(\sqrt{2}, \sqrt{7})$

Exercises 31–38 ***Exact Form*** *Evaluate in exact form.*

31. **(a)** $\mathrm{Sin}^{-1}\left(\frac{\sqrt{2}}{2}\right)$ **(b)** $\mathrm{Cos}^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

32. **(a)** $\mathrm{Tan}^{-1}(-\sqrt{3})$ **(b)** $\tan\left(\mathrm{Sin}^{-1}\frac{-2}{5}\right)$

33. **(a)** $\cos(\mathrm{Tan}^{-1} -2)$ **(b)** $\sin\left(\mathrm{Sin}^{-1}\frac{\pi}{4}\right)$

34. **(a)** $\mathrm{Cos}^{-1}\left(\cos\frac{-\pi}{6}\right)$ **(b)** $\sin(\pi + \mathrm{Cos}^{-1} 0.5)$

35. **(a)** $\mathrm{Tan}^{-1}\left(\tan\frac{5\pi}{6}\right)$ **(b)** $\sec(\pi - \mathrm{Tan}^{-1} - 2)$

36. **(a)** $\cos\left(\pi - \mathrm{Cos}^{-1}\frac{4}{7}\right)$

(b) $\sin\left(\frac{\pi}{2} + \mathrm{Cos}^{-1}\frac{3}{7}\right)$

37. **(a)** $\tan\left(\pi - \mathrm{Tan}^{-1}\frac{5}{7}\right)$

(b) $\cos\left(\frac{3\pi}{2} - \mathrm{Cos}^{-1}\frac{-5}{13}\right)$

38. **(a)** $\cos\left(\pi + \mathrm{Tan}^{-1}\frac{3}{4}\right)$

(b) $\sin\left(\pi - \mathrm{Cos}^{-1}\frac{2}{7}\right)$

Exercises 39–40 ***Decimal Approximations*** *Evaluate and round off to two decimal places.*

39. **(a)** $\mathrm{Sin}^{-1} 0.47$ **(b)** $\mathrm{Cos}^{-1} -0.25$

40. **(a)** $\sin(\mathrm{Tan}^{-1} -2.5)$ **(b)** $\sec(\mathrm{Cos}^{-1} 0.48)$

41. For what value(s) of x is $\mathrm{Sin}^{-1} x = \frac{\pi}{3}$?

42. For what value(s) of x is $\mathrm{Cos}^{-1} x = \frac{3\pi}{4}$?

43. For what value(s) of x is $\mathrm{Tan}^{-1} x = \frac{2\pi}{3}$?

Exercises 44–49 ***Graphs*** *Without using a calculator draw a graph of the function for $-\pi \le x \le \pi$.*

44. $f(x) = \sin x$ **45.** $f(x) = \cos x$

46. $f(x) = \tan x$ **47.** $f(x) = 1 + \cos x$

48. $f(x) = 1 - \sin x$ **49.** $f(x) = \tan\left(x - \frac{\pi}{2}\right)$

Exercises 50–53 ***Graph, Domain, Range*** *Without using a calculator draw a graph of the function and find the domain and range of f. Use a calculator as a check.*

50. $f(x) = \sin(\mathrm{Sin}^{-1} x)$ **51.** $f(x) = \cos(\mathrm{Cos}^{-1} x)$

52. $f(x) = \mathrm{Sin}^{-1} x + \frac{\pi}{2}$ **53.** $f(x) = \mathrm{Cos}^{-1} x - \frac{\pi}{2}$

54. Draw a graph of $y = \mathrm{Sin}^{-1} x$.
(a) Show all points on the graph where $y \ge \frac{\pi}{6}$.
(b) Find the solution set for the inequality $\mathrm{Sin}^{-1} x \ge \frac{\pi}{6}$.

55. Draw a graph of $y = \mathrm{Cos}^{-1} x$.
(a) Show all points on the graph where $y \ge \frac{2\pi}{3}$.
(b) Find the solution set for the inequality

$$\mathrm{Cos}^{-1} x \ge \frac{2\pi}{3}.$$

56. Draw a graph of $y = \mathrm{Tan}^{-1} x$.
(a) Show all points on the graph where $y \ge \frac{\pi}{4}$.
(b) Find the solution set for the inequality

$$\mathrm{Tan}^{-1} x \ge \frac{\pi}{4}.$$

57. For what values of x in $[-2\pi, 2\pi]$ do the graphs of $f(x) = \sin x$ and $g(x) = |\sin x|$ coincide?

58. ***Maximum Value*** In the window $[0, 3] \times [0, 5]$, find the maximum value of $f(x) = 2\sin x + 2\cos\frac{x}{2}$ (2 decimal places).

59. **(a)** Determine the domain and range of $f(x) = 3^{\cos x}$.
(b) Is f a periodic function? Give reasons.

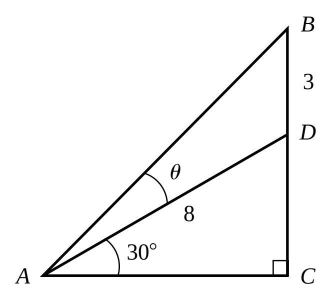
60. Give a verbal description of translations that can be applied to the graph of $f(x) = \cos x$ to get the graph of $g(x) = \cos(x - \frac{\pi}{3}) - 2$. Draw graphs as a check.

61. ***Number of Intersections*** How many points do the graphs of $y = \cos\frac{x}{2}$ and $y = \frac{x}{8}$ have in common? In what quadrant(s) do the graphs intersect?

62. What are the domain and range of $f(x) = \mathrm{Cos}^{-1}\frac{x}{3}$.

63. Find the root of $\sin x = \mathrm{Cos}^{-1} x$ (2 decimal places).

64. In the diagram, $|\overline{AD}| = 8$ and $|\overline{BD}| = 3$. Find θ in degrees.

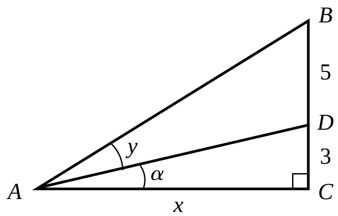


65. ***Volume of a Cone*** A sector with a central angle 45° is cut out of a circular piece of tin of radius 12 inches and the remaining piece is formed into a cone. What is the volume of the cone?

66. ***Maximum Angle***

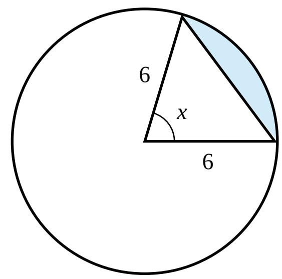
(a) In the diagram, find a formula giving y as a function of x.

(b) For what x is y a maximum (2 decimal places)?



67. (a) If the central angle of a circular sector of radius 6 is denoted by x, find a formula for the area A of the segment (the shaded region in the diagram). We restrict x to $0 < x < \pi$. Why?

(b) Determine the value of x for which $A = 36$ (2 decimal places). (*Hint:* Use a graph.)



6 TRIGONOMETRIC IDENTITIES, EQUATIONS, AND GRAPHS

ONE OF THE IMPORTANT FEATURES of problem solving involves the replacement of a mathematical expression by another expression that is *identically equivalent* to it. This is particularly true in the study of calculus, where solutions to many problems can be made relatively easy by appropriate substitutions.

In this chapter our first task is to establish familiarity with identities that involve trigonometric functions. Several basic identities are introduced in the first three sections, then the remaining sections use these identities whenever appropriate to solve equations and to draw graphs of general trigonometric functions.

6.1 BASIC IDENTITIES

> This is why mathematics is effective: the world exhibits regularities which can be described, independently of the world, by forms which can be studied and then reapplied.
>
> *Saunders MacLane*

Many times in earlier chapters we were able to solve equations that involved polynomial, exponential, or logarithmic functions by using the equivalence operation from Section 1.5 (page 40):

Replace any expression in an equation by an expression *identically* equal to it.

For example, to solve the equation $x^2 + 3x - 4 = 0$, we may replace $x^2 + 3x - 4$ by $(x + 4)(x - 1)$ since $x^2 + 3x - 4 = (x + 4)(x - 1)$ is an **identity.** Thus the

[W]hen I started teaching trigonometric identities, an ingenious student told me of his foolproof way of getting a perfect grade almost every time. If you're told to prove that some expression A is equal to a different-looking B, you put A at the top left corner of the page, B at the bottom right, and using correct but trivial substitutions, keep changing them, working from both ends to the middle. When they meet, stop.

Paul Halmos

original problem is equivalent to solving the equation $(x + 4)(x - 1) = 0$. The zero-product principle yields solutions -4 and 1.

Similarly, to solve $\ln x^2 = 2$, the most common approach is to replace $\ln x^2$ by $2 \ln x$ and solve the equation $2 \ln x = 2$, from which $\ln x = 1$ or $x = e$. It is true that e is a solution, but so is $-e$, so our procedure did not yield the complete solution set. The problem, of course, is that $\ln x^2 = 2 \ln x$ only if $x > 0$, so the two expressions are equal only for $x > 0$. A better procedure is to use equivalence (3) from Section 4.2 to rewrite the equation $\ln x^2 = 2$ in exponential form: $e^2 = x^2$, from which $x = \pm e$.

In this chapter we will see many instances where identities involving trigonometric functions are used to help in problem solving. We already have the notion of the domain of a function, but it is also convenient to talk about the domain of an equation.

Definition: domain of an equation

If f and g are functions, then the **domain D of the equation $f(x) = g(x)$** is the set of all real numbers for which both f and g are defined. That is, D is the **intersection of the domains of f and g.**

►EXAMPLE 1 ***Domains of equations*** Determine the domain of the equation.

(a) $\sin x = 2 \cos x$ **(b)** $\operatorname{Cos} x = \cos x$
(c) $\operatorname{Cos}^{-1} x = \operatorname{Cos} x$ **(d)** $\ln(\cos x \tan x) = \ln \sin x$

Solution

(a) Since $\sin x$ and $2 \cos x$ are both defined for all real numbers, the domain is the set of all real numbers.

(b) The function $\operatorname{Cos} x$ is the restricted cosine function we defined in Section 5.5. Its domain is $[0, \pi]$. Since the domain of $\cos x$ is the set of all real numbers, the domain of the equation is the intersection, $[0, \pi]$.

(c) $\operatorname{Cos}^{-1} x$ is defined for x in $[-1, 1]$ and $\operatorname{Cos} x$ is defined for x in $[0, \pi]$. Therefore $D = [0, 1]$.

(d) $\ln(\cos x \tan x)$ is defined only when the product $\cos x \tan x$ is positive; that is, when $\cos x$ and $\tan x$ have the same sign. Both are positive in Quadrant I and both are negative in Quadrant II. $\ln \sin x$ is defined when $\sin x$ is positive; that is, in Quadrant I and Quadrant II. Therefore D consists of all numbers in the first or second quadrants, not including quadrantal numbers:

$$D = \{x \mid x \text{ is in quadrants QI or QII}\}. \quad ◄$$

An equation that is valid for all numbers in its domain is called an **identity.**

Definition: identity

If the domain of the equation $f(x) = g(x)$ is the nonempty set D and

$$f(x) = g(x) \textbf{ for every } \boldsymbol{x} \textbf{ in } \boldsymbol{D},$$

then the equation $f(x) = g(x)$ is an **identity.**

►EXAMPLE 2 ***Recognizing identities*** Is the equation an identity?

(a) $x^2 - 4 = 3x$ **(b)** $\sqrt{x^2} = x$
(c) $\sqrt{x^2} = |x|$ **(d)** $e^{\ln(x-1)} = x - 1$

Solution

(a) Both sides are defined for every real number x, but equality holds only when x is -1 or 4. Thus $x^2 - 4 = 3x$ is not an identity.

(b) Both sides are defined for every real number x, but equality holds only when x is positive or zero. Hence $\sqrt{x^2} = x$ is not an identity.

(c) Equality holds for every real number, and so $\sqrt{x^2} = |x|$ is an identity.

(d) The set of numbers S for which both sides are defined is $\{x \mid x > 1\}$. Equality holds for all x in S, so the given equation is an identity. When we replace $e^{\ln(x-1)}$ by $x - 1$, we must remember that such a replacement is valid only when x is greater than 1. ◀

The word *identity* is commonly associated with trigonometry even though the concept occurs throughout mathematics, and the proof of a trigonometric identity often involves the use of algebraic identities. We have already seen several trigonometric identities in Chapter 5 (including the reduction formulas). It is not possible to remember, or even list, all the identities that are useful in problem solving. Instead we concentrate on some key identities from which we can easily obtain others. In this section, and in the two that follow, we list basic identities and illustrate how they can be used to derive or prove others. It is important that you learn these not simply by memorization but by working many problems until you become quite comfortable with the key identities and the various forms in which they occur.

Proving Identities: Algebraic Approach

We illustrate two techniques that can be used in showing that an equation is an identity. Example 3 starts with a known identity to which we apply appropriate operations and derive another desired identity. In Example 4 we work only on one side of the equation.

Before illustrating these techniques, we call attention to an important point of logic. Performing the same operation on both sides of a given equation and arriving at an obvious identity does not prove that the original equation is an identity. For instance, if we try to "prove" that $1 + x^2 = 4x - 2$ is an identity we could operate on both sides of the equation, as follows:

$$\begin{aligned} 1 + x^2 &= 4x - 2 \\ x^2 &= 4x - 3 && \text{Subtract 1 from both sides} \\ 0 \cdot x^2 &= 0(4x - 3) && \text{Multiply both sides by 0} \\ 0 \cdot x^2 &= 0 \cdot 4x - 0 \\ 0 \cdot x^2 &= 0 \cdot 4x \end{aligned}$$

Since the last equation is obviously true for all x, can we conclude that the original equation is also an identity? Clearly not, since that equation is satisfied only when $x = 1$ or $x = 3$. All we have accomplished in this "proof" is to show that

"If $1 + x^2 = 4x - 2$ for every x, then $0 = 0$."

which is indeed true, but hardly an enlightening statement.

To begin our work with identities we list several that we encountered in Chapter 5. Some identities listed here are definitions given in Section 5.2 or 5.3. We use labels such as (I-1) and (I-2) to identify identities for easy reference.

Basic identities

$$\cot x = \frac{1}{\tan x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x} \qquad \textbf{(I-1)}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \textbf{(I-2)}$$

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x \qquad \textbf{(I-3)}$$

$$\sin^2 x + \cos^2 x = 1 \quad \text{(Pythagorean identity)} \qquad \textbf{(I-4)}$$

$$1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x \qquad \textbf{(I-5)}$$

Note that we write $\sin^2 x$ and $\cos^2 x$ in place of $(\sin x)^2$ and $(\cos x)^2$, respectively. This common notation is used throughout the rest of the book, although we must still enter (SIN X)^2 on a calculator.

▶EXAMPLE 3 ***Proving an identity*** In Chapter 5 we proved the Pythagorean identity (I-4) using the fact that any point on the unit circle has coordinates of the form $(\cos x, \sin x)$. Use (I-4) to show that the equation in (I-5), $1 + \tan^2 x = \sec^2 x$, is an identity.

Solution

We begin with an established identity (I-4), and divide both sides by $\cos^2 x$.

$$\sin^2 x + \cos^2 x = 1$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\left(\frac{\sin x}{\cos x}\right)^2 + 1 = \left(\frac{1}{\cos x}\right)^2$$

Using (I-1) and (I-2), we have the desired identity

$$\tan^2 x + 1 = \sec^2 x. \quad ◀$$

Strategy: Multiplying the numerator and denominator of the left-hand side by $1 - \cos x$ gives $1 - \cos^2 x$ in the denominator, which can be replaced by $\sin^2 x$.

▶EXAMPLE 4 ***Working with one side*** Prove that $\frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$ is an identity by working with the left-hand side only.

Solution

In general, fractions with a single term in the denominator are simpler. Follow the strategy.

$$\frac{\sin x}{1 + \cos x} \stackrel{?}{=} \frac{1 - \cos x}{\sin x}$$

$$\frac{\sin x(1 - \cos x)}{(1 + \cos x)(1 - \cos x)} \qquad \text{Multiply numerator and denominator by } (1 - \cos x)$$

$$\frac{\sin x(1 - \cos x)}{1 - \cos^2 x} \qquad \text{Algebra}$$

$$\frac{\sin x(1 - \cos x)}{\sin^2 x} \qquad \text{Pythagorean identity (I-4)}$$

$$\frac{1 - \cos x}{\sin x} \qquad \text{Simplify}$$

By the transitivity property of equality, the left-hand side does equal the right; that is, $\frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$ is an identity. ◀

In the previous example, we could just as well have worked with the right side only, by first multiplying the numerator and denominator by $(1 + \cos x)$.

When Is an Equation Not an Identity? Counterexamples

If an equation of the form $f(x) = g(x)$ is not an identity, then there must be at least one number c in the domain of the equation for which $f(c) \neq g(c)$. Any such number is called a **counterexample.** Finding a single value of x for which both sides are defined and for which equality does not hold is all that is necessary to show that an equation is not an identity.

Returning to Example 2 above, to show that $x^2 - 4 = 3x$ is not an identity, we could have simply produced a counterexample, such as 1, which does not satisfy the equation: $1^2 - 4 \neq 3 \cdot 1$. Similarly, -1 is a counterexample showing that the equation $\sqrt{x^2} = x$ in Example 2b is not an identity.

Using Graphs With Identities

Calculator graphs can provide valuable insight about whether or not an equation $f(x) = g(x)$ is an identity. If the graphs of f and g are not identical, then we obviously do not have an identity. Furthermore, nonidentical graphs can help us find a counterexample.

If the graphs do appear identical, what can we conclude? In general, all we can say is that the equation *may* be an identity. The calculator graph only shows function values for about a hundred x-values (at pixel coordinates). Important behavior that distinguishes the functions may not appear in our window, or may be masked by something else on the screen. Nevertheless, calculator graphs are extremely useful in guiding our intuition and may even suggest a way to approach the algebraic problem of proving an identity.

Algorithm for establishing identities

To determine if the equation $f(x) = g(x)$ is an identity:

1. Draw graphs of f and g on the same screen for values of x in D.
2. If the graphs do not appear identical, the equation is not an identity. From the graph you should be able to find a number x for which $f(x) \neq g(x)$, a counterexample.
3. If the graphs appear identical, you may want to try other windows. If the graphs continue to appear the same, then use algebraic techniques and try to show that $f(x) = g(x)$ is an identity.

TECHNOLOGY TIP ◆ **Graphing identical functions**

You may wish to review several methods for graphing identical functions in the Technology Tip in Section 4.4. For trigonometric functions, it is often most useful to graph something like $y = f(x) - g(x) + 1$. Then, if the functions f and g are identically equal, the graph is the horizontal line $y = 1$.

Caution. Since many trigonometric functions are occasionally undefined, it is wise to do more than simply look for the line $y = 1$. As a case in point, in the graph of `Y = Y2 - Y1 + 1` in Example 5b below, the domain of the equation does not include any point where $\tan x$ is undefined, but the calculator graph does not show holes at odd multiples of $\frac{\pi}{2}$.

►EXAMPLE 5 ***Using graphs*** Determine if the equation is an identity. Prove or find a counterexample.

(a) $\sqrt{\tan^2 x - \sin^2 x} = \sin x \tan x$ **(b)** $\sqrt{\tan^2 x - \sin^2 x} = |\sin x \tan x|$

Solution

(a) Follow the algorithm. When we graph Y1 = √((TAN X)² − (SIN X)²), and Y2 = SIN X TAN X in the decimal window, we see two different graphs for much of the domain. Alternatively, graphing Y = Y2 − Y1 + 1, we see the expected horizontal line $y = 1$ only on the interval $[-1.5, 1.5]$ (which should make us suspect that the equation does hold for every x in $(-\frac{\pi}{2}, \frac{\pi}{2})$). For a counterexample, we can try any convenient number outside that interval, say $x = \frac{3\pi}{4}$, where the left side is positive and the right side is negative.

(b) Now when we graph Y1 = √((TAN X)² − (SIN X)²), Y2 = ABS(SIN X TAN X) we see only one graph. Alternatively, graphing Y = Y2 − Y1 + 1 shows the horizontal line $y = 1$ clear across the screen. We suspect that the equation is an identity. To try to prove algebraically that we have an identity, we begin by using (I-2) to express $\tan x$ by $\dfrac{\sin x}{\cos x}$:

$$\begin{aligned}
\sqrt{\tan^2 x - \sin^2 x} &= \sqrt{\frac{\sin^2 x}{\cos^2 x} - \sin^2 x} && \text{By (I-2)}\\
&= \sqrt{\sin^2 x\left(\frac{1}{\cos^2 x} - 1\right)} && \text{Algebra}\\
&= \sqrt{\sin^2 x\,(\sec^2 x - 1)} && \text{By (I-1)}\\
&= \sqrt{\sin^2 x \tan^2 x} && \text{By (I-5)}\\
&= \sqrt{(\sin x \tan x)^2} = |\sin x \tan x|. && \text{Algebra}
\end{aligned}$$

It follows that $\sqrt{\tan^2 x - \sin^2 x} = |\sin x \tan x|$ for every x in the domain, and so the equation is an identity. ◄

►EXAMPLE 6 ***Is it an identity?*** Is $(\sin x + \cos x)^2 = \dfrac{\csc x \sec x + 2}{\csc x \sec x}$ an identity?

Strategy: Follow the algorithm and graph both sides of the equation. If they look identical, expand the left side and express the right side in terms of $\sin x$ and $\cos x$.

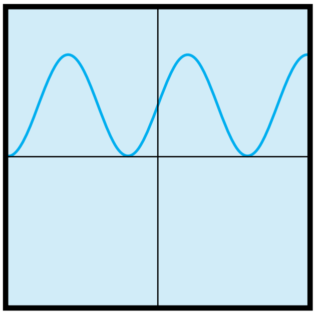
Solution

To graph the right side, we must enter something like

Y = (1/(SIN X COS X) + 2)/(1/(SIN X COS X)),

or we can use identities to simplify the form of the function before graphing. For example,

$$\begin{aligned}
\frac{\csc x \sec x + 2}{\csc x \sec x} &= \frac{1/(\sin x \cos x) + 2}{1/(\sin x \cos x)}\\
&= \left(\frac{1 + 2\sin x \cos x}{\sin x \cos x}\right) \div \frac{1}{\sin x \cos x} && \text{Algebra}\\
&= \left(\frac{1 + 2\sin x \cos x}{\sin x \cos x}\right) \cdot \frac{\sin x \cos x}{1} && \text{Invert and multiply}\\
&= 1 + 2\sin x \cos x.
\end{aligned}$$



$[-4, 4]$ by $[-3, 3]$

FIGURE 1

That is, we can graph the function on the right side as Y2 = 1 + 2 SIN X COS X. Using either form for the right side, it appears that the graphs of the two functions are identical. See Figure 1. For an algebraic proof, we follow the strategy and expand the left side:

$$\begin{aligned}(\sin x + \cos x)^2 &= \sin^2 x + 2 \sin x \cos x + \cos^2 x && \text{Algebra}\\ &= (\sin^2 x + \cos^2 x) + 2 \sin x \cos x && \text{Rearrange}\\ &= 1 + 2 \sin x \cos x. && \text{By (I} - 4)\end{aligned}$$

Each side is identically equal to $1 + 2 \sin x \cos x$ so the two sides are identically equal to each other. The given equation is an identity. ◀

▶EXAMPLE 7 *Inverse trigonometric functions*

(a) Find the domain of the equation $\text{Sin}^{-1} x + \text{Cos}^{-1} x = \frac{\pi}{2}$.
(b) Is the equation an identity?

Solution

(a) Since $\text{Sin}^{-1} x$ and $\text{Cos}^{-1} x$ are defined only for x in the interval $[-1, 1]$ and the right side is defined for all x, then $D = [-1, 1]$.
(b) A calculator graph of the left side shows a horizontal line segment on the interval $[-1, 1]$, whose trace value is clearly an approximation of $\frac{\pi}{2}$, strongly suggesting that the equation is an identity. For an algebraic proof, we consider the equivalent equation, $\text{Sin}^{-1} x = \frac{\pi}{2} - \text{Cos}^{-1} x$, and take the sine of both sides. That is, we want to see if

$$\sin(\text{Sin}^{-1} x) = \sin\left(\frac{\pi}{2} - \text{Cos}^{-1} x\right) \quad \text{for all } x \text{ in } [-1, 1].$$

We know that $\sin(\text{Sin}^{-1} x) = x$ for all x in $[-1, 1]$, and for the right side we have a reduction formula, $\sin(\frac{\pi}{2} - \theta) = \cos \theta$, where $\theta = \text{Cos}^{-1} x$. Thus

$$\sin\left(\frac{\pi}{2} - \text{Cos}^{-1} x\right) = \cos \text{Cos}^{-1} x = x, \text{ again for all } x \text{ in } [-1, 1].$$

Therefore the given equation is an identity. ◀

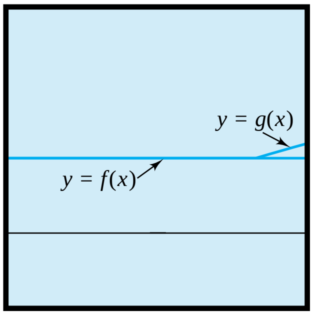
In the next example we illustrate the importance of using care in depending on calculator graphs. Graphs of f and g can coincide over a large interval even though the functions are not identical.

▶EXAMPLE 8 *Functions identical for large intervals*

(a) Find the domain of the equation
$\cos 2x + 2 \sin^2 x = 0.01(\sqrt{x} + |\sqrt{x} - 100|)$.
(b) Is the equation an identity?

Solution

(a) Since $\sqrt{x}$ is defined only for nonnegative real numbers; $D = [0, \infty)$.
(b) In most reasonable windows, if we graph $f(x) = \cos 2x + 2 \sin^2 x$ and $g(x) = 0.01(\sqrt{x} + |\sqrt{x} - 100|)$, it appears that both functions are identically equal to 1. The function f is defined for all real numbers, g only for $x \geq 0$,



[0, 12,000] by [–1, 3]

FIGURE 2

but the calculator graphs make it appear that the functions are identically equal on the domain of the equation. With the absolute value function in g, though, we have reason to be careful. A piecewise formula for g changes where $\sqrt{x} - 100 = 0$; that is, where $x = 100^2 = 10{,}000$. Setting a window such as $[0, 12{,}000] \times [-1, 3]$ shows that the functions clearly differ when $x > 10{,}000$ (see Figure 2). We conclude that the equation is not an identity. Any number greater than 10,000 is a counterexample. ◀

Using Identities

In many situations we are interested in simplifying a mathematical expression. We make no attempt to define a simplest form, but in most cases it will be clear when one form is simpler than another. The next example shows how to use identities to simplify expressions that involve trigonometric functions.

▶EXAMPLE 9 ***Simplifying formulas*** Use identities to simplify the formula for the function f.

(a) $f(x) = \dfrac{\tan(-x)}{\sin(-x)} + \sec x.$ **(b)** $f(x) = (\sin x + \cos x)^2 - 2 \sin x \cos x$

Solution

Strategy: **(a)** Begin by replacing $\tan(-x)$ by $-\tan x$ and $\sin(-x)$ by $-\sin x$, then replace $\tan x$ by $\dfrac{\sin x}{\cos x}$ and simplify.

(a) Follow the strategy.

$$\begin{aligned} f(x) &= \frac{\tan(-x)}{\sin(-x)} + \sec x \\ &= \frac{-\tan x}{-\sin x} + \sec x && \text{By (I-3)} \\ &= \frac{1}{\cos x} + \sec x && \text{Use (I-2) and simplify} \\ &= \sec x + \sec x && \text{By (I-1)} \\ &= 2 \sec x \end{aligned}$$

Therefore, the function f is also given by $f(x) = 2 \sec x$. Keep in mind also that we must exclude values of x for which the original formula for f is not defined, namely $0, \pm\frac{\pi}{2}, \pm\pi, \ldots$ since $\tan(-x)$ is undefined at $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}, \ldots$, and $\sin(-x)$ is zero when x is $0, \pm\pi, \pm 2\pi$.

(b) Expand the right side and use (I-4).

$$\begin{aligned} f(x) &= (\sin x + \cos x)^2 - 2 \sin x \cos x \\ &= \sin^2 x + 2 \sin x \cos x + \cos^2 x - 2 \sin x \cos x \\ &= \sin^2 x + \cos^2 x = 1. \end{aligned}$$

Therefore $f(x) = 1$ for every real number x. ◀

▶EXAMPLE 10 ***Solve an equation*** Find the solution set for the equation

$$\frac{\sin(-x)}{\tan(-x)} + \cos x = 2.$$

Solution

Use (I-2) and (I-3) to simplify the left-hand side.

$$\begin{aligned}
\frac{\sin(-x)}{\tan(-x)} + \cos x &= \frac{-\sin x}{-\tan x} + \cos x && \text{By (I-3)}\\
&= \frac{\sin x}{\dfrac{\sin x}{\cos x}} + \cos x && \text{By (I-2)}\\
&= \sin x \frac{\cos x}{\sin x} + \cos x && \text{Algebra}\\
&= \cos x + \cos x\\
&= 2 \cos x.
\end{aligned}$$

Thus the original equation can be written as $2 \cos x = 2$, or $\cos x = 1$. The set of numbers satisfying the equation $\cos x = 1$ consists of all numbers coterminal with 0; that is, all integer multiples of 2π. Returning to the original equation, however, we discover that numbers coterminal with 0 are not in the domain of the equation. (Why?) Hence, the solution set is the empty set. ◀

Using Parametric Equations for Graphs of Circles

In Section 1.4 we saw how to draw a calculator graph of a circle. To graph the circle $(x - 2)^2 + (y + 3)^2 = 9$, we had to solve for y and use two equations, Y1 = −3 + √(9 − (X − 2)²) and Y2 = −3 − √(9 − (X − 2)²). Even with all of this work, the calculator graph shows a circle with gaps (see Figure 1.20, page 33).

Now, with trigonometric functions at our disposal, we can use parametric equations to get satisfactory graphs more easily, as illustrated in the next example.

▶EXAMPLE 11 *A parametric circle*

Strategy: Solve the parametric equations for $\sin t$, $\cos t$. Then square both sides and add. Use the Pythagorean identity.

(a) Show that the parametric equations $x = 2 + 3 \sin t$, $y = -3 + 3 \cos t$ represent a circle. Identify the center and radius, and use parametric mode to draw a calculator graph.

(b) Use the parametric equations to find the coordinates of the y-intercept points in exact form. Compare the approximate trace coordinates of the y-intercept points on your graph.

Solution

(a) Follow the strategy. From the first equation, $\sin t = \frac{x-2}{3}$, and from the second, $\cos t = \frac{y+3}{3}$. Now we square both sides and add.

$$(\sin t)^2 + (\cos t)^2 = \left(\frac{x-2}{3}\right)^2 + \left(\frac{y+3}{3}\right)^2.$$

By the Pythagorean identity the left side equals 1:

$$1 = \frac{(x-2)^2}{9} + \frac{(y+3)^2}{9}.$$

Multiplying through by 9, we obtain an equation we recognize as a circle with center at $(2, -3)$ and radius 3, shown in Figure 3,

$$(x-2)^2 + (y+3)^2 = 3^2.$$

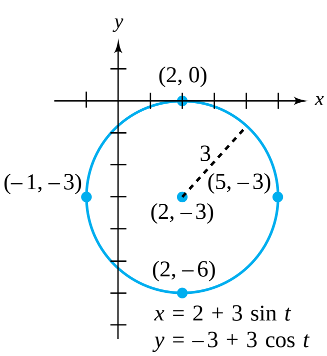


FIGURE 3

With the calculator in parametric mode we enter X = 2 + 3 SIN T, Y = −3 + 3 COS T and graph in a shifted decimal window with a t-range of $[0, 6.3]$ or $[0, 2\pi]$ (see the Technology Tip that follows), we should see a calculator graph that closely approximates the circle shown in Figure 3. Tracing around the graph, there is no t-value that makes $x = 0$, but we come close near $t = 3.9$ and $t = 5.5$ or 5.6. The y-coordinates of the intercept points are roughly -5.2 and -0.8.

(b) To find the y-intercept points in exact form, we can use either the rectangular or parametric equations defining the circle. In either case, we want to find y when $x = 0$. Using the parametric equations and setting $x = 0$, we have

$$\sin t = \frac{-2}{3}.$$

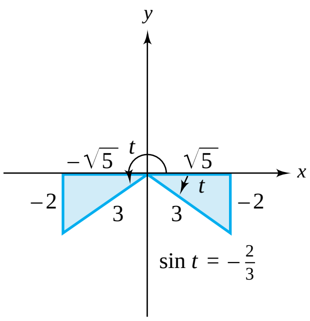


FIGURE 4

Two values of t for which $\sin t = \frac{-2}{3}$ are shown in the diagram in Figure 4. From the reference triangles, we can read the corresponding values of the cosine, $\cos t = \pm\frac{\sqrt{5}}{3}$. Since $y = -3 + 3\cos t$, the y-intercepts are where $y = -3 \pm \sqrt{5} \approx -5.236$ or -0.764. Check by tracing. (You might compare this solution with the steps involved in setting $x = 0$ and solving the standard-form equation of the circle for y.) ◀

TECHNOLOGY TIP ◆ ***Shifting a decimal window***

Decimal windows are nice for many purposes, but the automatic (or default) decimal setting may not show the window you need for a particular graph. The circle in Example 11 doesn't fit in the decimal window. From Figure 3, we can see that we need a window that contains at least $[-1, 5] \times [-6, 0]$, but such a window is not an equal scale window (so the circle appears "squashed") and does not have nice pixel coordinates.

We can shift a window by adding (or subtracting) a constant for either the x- or y-range, as from $[a, b] \times [c, d]$ to $[a + k, b + k] \times [c + l, d + l]$. As a specific example, the decimal window of the TI-82 is $[-4.7, 4.7] \times [-3.1, 3.1]$. By adding 3 to both ends of the x-range and adding -3 to both ends of the y-range, we get a shifted (equal scale) decimal window $[-1.7, 7.7] \times [-6.1, .1]$ in which the whole circle shows up nicely.

EXERCISES 6.1

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. $\sin x = \cos x \tan x$ is an identity.
2. $\sin(2x) = 2\sin x$ is an identity.
3. $\tan(\pi + x) = \tan \pi + \tan x$ is an identity.
4. If $f(x) = 1 + x^2$ and $g(x) = \tan x$, then $f(g(x)) = \sec^2 x$.
5. The graphs of $y = \cos x \tan x$ and $y = \sin x$ are identical.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. The domain of $\text{Sin } x = \sin x$ is ______.
7. The domain of $\dfrac{\sin(-x)}{\tan x} = -\cos x$ is ______.
8. For every real number x, $\ln(\sin^2 x + \cos^2 x) =$ ______.
9. The equation $\dfrac{\sin(-x)}{\cos(-x)} =$ ______ is an identity.
10. The equation $\cos x \sec x =$ ______ is an identity.

Develop Mastery

Exercises 1–2 Determine whether or not the equation is an identity. Give the values of x for which equality holds.

1. **(a)** $x^3 - x^2 = x^2(x-1)$
(b) $|x+2| = |x| + 2$
(c) $e^{2\ln x} = x^2$

2. **(a)** $x^3 - x^2 - 2x = (x^2 - 2x)(x+1)$
(b) $\sqrt{x^2+1} = x + 1$
(c) $\dfrac{x^3-1}{x-1} = x^2 + x + 1$

3. Start with identity (I-4) and show that $1 + \cot^2 x = \csc^2 x$ is an identity.

4. From identity (I-4), show that $\dfrac{\sin^2 x}{1-\cos x} = 1 + \cos x$ is an identity.

5. From identity (I-2), prove that $\cos x \tan x = \sin x$ is an identity.

6. From identity (I-4), show that $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$ is an identity. (*Hint:* Multiply both sides of (I-4) by an appropriate quantity.)

7. From identity (I-4), by adding an appropriate quantity to both sides, show that $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$ is an identity.

8. From the identity $1 + \tan^2 x = \sec^2 x$, show that $\tan^2 x = (\sec x - 1)(\sec x + 1)$ is an identity.

Exercises 9–14 Prove that the equation is an identity. Work only with the left-hand side.

9. $\dfrac{\tan x}{\sin x} = \sec x$

10. $\dfrac{\sin x \csc x}{\cot x} = \tan x$

11. $\dfrac{\cot x}{\sec x} + \sin x = \csc x$

12. $\dfrac{\sin(-x)}{\cos(-x)} = -\tan x$

13. $\tan x \csc x = \sec x$

14. $\sin x \cos x(\tan x + \cot x) = 1$

Exercises 15–20 Show that the equation is an identity. Work only with the right-hand side.

15. $\csc x = \dfrac{\cot x}{\cos x}$

16. $\cot x = \dfrac{\cos x \sec x}{\tan x}$

17. $\tan x + \sec x = \dfrac{1 - \sin(-x)}{\cos(-x)}$

18. $\cos x = \dfrac{\csc(-x)}{\cot(-x) + \tan(-x)}$

19. $\sec x \csc x = \tan x + \cot x$

20. $\csc x = \cos x(\tan x + \cot x)$

Exercises 21–24 ***Domain*** *Determine the domain of the equation.*

21. **(a)** $\cos x = 4\sin x$ **(b)** $\tan x = \dfrac{\sin x}{\cos x}$

22. **(a)** $\mathrm{Sin}^{-1} x = \sin x$ **(b)** $\mathrm{Cos}^{-1} x = \mathrm{Sin}^{-1} x$.

23. **(a)** $\sqrt{1 - \cos x} = x$
(b) $\ln(\tan x) = \ln(\sin x) - \ln(\cos x)$

24. **(a)** $\sqrt{1 - \sin x} = \cos x$
(b) $\ln(\csc x) = -\ln(\sin x)$

Exercises 25–34 ***Prove Identities*** *Show that the equation is an identity.*

25. $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

26. $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$

27. $\dfrac{\sin x}{1 - \cos x} = \dfrac{1 + \cos x}{\sin x}$

28. $\dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\csc x} = 1$

29. $\cos x \tan x = \sin x$

30. $\dfrac{\sin^2 x}{\cos x} = \sec x - \cos x$

31. $\sin x \tan x = \sec x - \cos x$

32. $\sin x + \cos x = (\cos x)(1 + \tan x)$

33. $\sec x + \cot x = \dfrac{\cos x + \tan x}{\sin x}$

34. $\dfrac{\sin x}{1 - \cos^2 x} = \csc x$

Exercises 35–48 ***Is It an Identity?*** *Determine whether or not the equation is an identity. Give either a proof or a counterexample.*

35. $(\sin x - \cos x)^2 = \sin^2 x - \cos^2 x$

36. $\sqrt{1 - \cos^2 x} = \sin x$

37. $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$

38. $\sqrt{1 + \tan^2 x} = \sec x$

39. $\dfrac{\csc x}{\tan x + \csc x} = \cos x$

40. $\dfrac{\csc x}{\sin x} + \dfrac{\sec x}{\cos x} = \sec^2 x \csc^2 x$

41. $(1 - \tan x)^2 = \sec^2 x - 2\tan x$

42. $\sqrt{\sin^2 x + \cos^2 x} = |\sin x| + |\cos x|$

43. $\sqrt{1 + 2\cos x + \cos^2 x} = 1 + \cos x$

44. $\sqrt{2 - 2\cos x - \sin^2 x} = 1 - \cos x$

45. $\sqrt{\sec^2 x - 1} = |\tan x|$

46. $\sqrt{\cot^2 x - \cos^2 x} = |\cot x \cos x|$

47. $\dfrac{1 + \sin(-x)}{\cos(-x)} = \sec x - \tan x$

48. $\dfrac{\sin x}{\cos x - \sin x} = \dfrac{\tan x}{1 - \tan x}$

Exercises 49–58 ***Simplify*** *Find a simpler equation to describe the function.*

49. $f(x) = \dfrac{\cos^2 x - 1}{\sin(-x)}$

50. $f(x) = \sin^4 x + \sin^2 x \cos^2 x$

51. $f(x) = \dfrac{\tan(-x)}{\sin(-x)}$

52. $f(x) = \dfrac{1}{\tan x + \cot x}$

53. $f(x) = \sin^3 x \cos x + \cos^3 x \sin x$

54. $f(x) = \dfrac{\sec x}{\tan x + \cot x}$

55. $f(x) = \dfrac{\sin x + \tan x}{1 + \sec x}$

56. $f(x) = \sqrt{1 - \sin^2 x}$

57. $f(x) = \sin x\left(\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x}\right)$

58. $f(x) = \sin(-x) + \cot(-x)\cos(-x)$

Exercises 59–66 ***Solution Set*** *Find the solution set.*

59. $\text{Cos}\, x = \cos x$

60. $\text{Sin}\, x = \sin x$

61. $\sqrt{1 - \sin^2 x} = \cos x$

62. $\sin^2 x + \cos^2 x = x - 2$

63. $\ln(\tan x) = \ln(\sin x) - \ln(\cos x)$

64. $\ln(1 + \tan^2 x) = -2\ln(\cos x)$

65. **(a)** $\text{Sin}^{-1}\dfrac{x}{4} + \text{Cos}^{-1}\dfrac{x}{4} = \dfrac{\pi}{2}$

(b) $\text{Sin}^{-1}\dfrac{x}{100} + \text{Cos}^{-1}\dfrac{x}{100} = \dfrac{\pi}{2}$

66. $\sqrt{x} + |\sqrt{x} - 7| = \dfrac{14}{\pi}\left(\text{Sin}^{-1}\dfrac{x}{40} + \text{Cos}^{-1}\dfrac{x}{40}\right)$

67. Determine the domain and range of f.
(a) $f(x) = x + \text{Cos}^{-1} x$ **(b)** $f(x) = x + \text{Sin}^{-1} x$

68. Is $0.01(\sqrt{x} + |\sqrt{x} - 100|) = 1$ an identity? See Example 8.

69. Is $\sin^2 x + \cos^2 x = 0.01(\sqrt{x} + |\sqrt{x} - 100|)$ an identity? What do you observe when you draw a graph of each side of the equation? See Exercise 68.

Exercises 70–72 ***Composition*** *For functions f and g, find an equation to describe the composite function $f \circ g$. Simplify when possible.*

70. $f(x) = \sqrt{1 - x^2}$, $g(x) = \cos x$

71. $f(x) = \dfrac{|x|}{(x^2 - 1)^{1/2}}$, $g(x) = \sec x$

72. $f(x) = \dfrac{\sqrt{1 - x^2}}{|x|}$, $g(x) = \sin x$

Exercises 73–74 *Determine whether the statement is true or false. Give reasons.*

73. If the graphs of functions f and g are identical, then $f(x) = g(x)$ is an identity.

74. If $f(x) = g(x)$ is an identity, then the graphs of functions f and g are identical.

Exercises 75–76 ***Parametric Equations*** ***(a)*** *Show that the curve given in parametric equations is a circle by writing an equation in standard form.* ***(b)*** *Draw a calculator graph. See Example 11.*

75. $x = 1 + \sin t$, $y = -2 + \cos t$

76. $x = -3 + \cos t$, $y = 4 + \sin t$

Exercises 77–78 *Draw a graph of the curve given in parametric equations and explain how the graph is related to the graph of the given nonparametric equation.*

77. $x = 3\sin^2 t$, $y = 3\cos^2 t$;
$x + y = 3$

78. $x = \cos t$, $y = \sin^2 t$;
$y = 1 - x^2$

79. Evaluate the sum $S = \ln(\tan \frac{\pi}{180}) + \ln(\tan \frac{2\pi}{180}) + \ln(\tan \frac{3\pi}{180}) + \cdots + \ln(\tan \frac{89\pi}{180})$. (*Hint:* Use the reduction formula $\tan(\frac{\pi}{2} - \theta) = \cot\theta = \frac{1}{\tan\theta}$. Thus $\tan\frac{89\pi}{180} = \frac{1}{\tan(\pi/180)}$.)

6.2 SUM, DIFFERENCE, AND DOUBLE-ANGLE IDENTITIES

> [Mathematics] is as incapable of being restricted within assigned boundaries . . . as the consciousness of life, which . . . is forever ready to burst forth into new forms of . . . existence.
>
> *J. J. Sylvester*

One important property of numbers is that multiplication is distributive over addition, which means that $a(b + c) = ab + ac$ for all numbers a, b, and c. Our first concern in this section is whether or not functions distribute over addition, that is, is $f(x + y) = f(x) + f(y)$ an identity? Such functions are given a name.

I've worked in so many areas . . . . Basically, I'm not interested in doing research and I never have been. I'm interested in understanding, which is quite a different thing.
David Blackwell

Definition: additive functions

A function f is said to be an **additive function** if, and only if,

$$f(x + y) = f(x) + f(y)$$

is an identity; that is, equality holds for all numbers x and y for which both sides are defined.

Most functions we have already considered in this text are not additive. For instance, the natural logarithm function, $f(x) = \ln x$, is not additive because $\ln(x + y)$ is not equal to $\ln x + \ln y$ for all positive numbers x and y. The function $g(x) = x^2$ is not additive since $(x + y)^2$ is not identically equal to $x^2 + y^2$. The function $h(x) = \sqrt{x}$ is not additive either, because $\sqrt{x + y}$ is not equal to $\sqrt{x} + \sqrt{y}$ for all nonnegative x and y.

It should come as no surprise that trigonometric functions are not additive. For instance, to see that the sine function is not additive consider $\sin(\frac{\pi}{2} + \frac{\pi}{2})$ and $\sin \frac{\pi}{2} + \sin \frac{\pi}{2}$.

$$\sin\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \sin \pi = 0 \quad \text{but} \quad \sin \frac{\pi}{2} + \sin \frac{\pi}{2} = 1 + 1 = 2.$$

Thus, $\sin(\frac{\pi}{2} + \frac{\pi}{2}) \neq \sin \frac{\pi}{2} + \sin \frac{\pi}{2}$, so the sine function is not additive. We may wonder if, indeed, there are any additive functions. For some possibilities see Develop Mastery Exercises 2 and 3.

Although trigonometric functions are not additive, some important identities allow us to express functions of sums and differences in relatively simple terms. Collectively, these identities are called the **sum and difference formulas.**

Sum and difference identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \qquad \textbf{(I-6)}$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \qquad \textbf{(I-7)}$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \qquad \textbf{(I-8)}$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \qquad \textbf{(I-9)}$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \qquad \textbf{(I-10)}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \qquad \textbf{(I-11)}$$

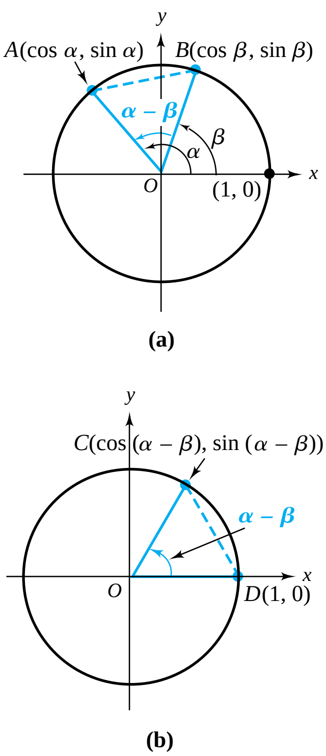


FIGURE 5

Proofs of the Sum and Difference Identities

It is convenient first to give a proof of (I-9) and then proceed to use (I-9) to prove (I-6) and (I-10). Proofs of the other identities are left for the reader. See Develop Mastery Exercise 1.

Proof of (I-9). To prove (I-9), we wish to relate $\cos(\alpha - \beta)$ to functions of α and β. It will be helpful to have a diagram that shows each of the angles α, β, and $\alpha - \beta$. We lose no generality in supposing that α, β, and $\alpha - \beta$ are positive. Placing α and β in standard position on a unit circle, $P(\alpha) = (\cos \alpha, \sin \alpha)$ and $P(\beta) = (\cos \beta, \sin \beta)$, as shown in Figure 5a. Let us denote $P(\alpha)$ by A and $P(\beta)$ by B. In triangle AOB, angle AOB is $\alpha - \beta$. Angle $\alpha - \beta$ appears in standard position in Figure 5b, where C is the point $(\cos(\alpha - \beta), \sin(\alpha - \beta))$.

In Figure 5, triangles AOB and COD are congruent. In fact, we may visualize simply rotating triangle AOB clockwise until B coincides with D. In particular, the segments $\overline{AB}$ and $\overline{CD}$ have the same length. We may use the distance formula to calculate the length of each in terms of the coordinates of their endpoints. We leave details as a worthwhile exercise for the reader (using identity (I-4) as needed) to get the following:

$$\begin{aligned}|\overline{AB}|^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2\\ &= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta).\\ |\overline{CD}|^2 &= [\cos(\alpha-\beta) - 1]^2 + [\sin(\alpha-\beta)]^2\\ &= 2 - 2\cos(\alpha-\beta)\end{aligned}$$

Since $|\overline{AB}| = |\overline{CD}|$, we get $2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 - 2\cos(\alpha-\beta)$. After simplifying, we have identity (I-9).

Proof of (I-6). To prove identity (I-6), use (I-9) and these reduction formulas:

$$\cos\left(\frac{\pi}{2} - t\right) = \sin t$$

$$\sin\left(\frac{\pi}{2} - t\right) = \cos t$$

In the first reduction formula, replace t with $(\alpha + \beta)$. Reading from right to left,

$$\begin{aligned}\sin(\alpha+\beta) &= \cos\left[\frac{\pi}{2} - (\alpha+\beta)\right] = \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right]\\ &= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta && \text{By (I-9)}\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta && \text{By Reduction formulas}\end{aligned}$$

Therefore $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ is an identity. In a similar manner, (I-8) follows from (I-9); see Develop Mastery Exercise 1.

Proof of (I-10). Now we proceed to (I-10). As a first step, use (I-2), $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, where θ is replaced by $(\alpha+\beta)$.

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \qquad \text{By (I-6) and (I-9)}$$

Divide each term in the numerator and denominator by $\cos\alpha\cos\beta$ and simplify:

$$\tan(\alpha+\beta) = \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

▶EXAMPLE 1 ***Using (I-10)*** Prove that $\tan\left(x + \dfrac{\pi}{4}\right) = \dfrac{\cos x + \sin x}{\cos x - \sin x}$ is an identity.

Strategy: Begin by using (I-10) and then replace $\tan x$ by $\dfrac{\sin x}{\cos x}$ and simplify.

Solution

Starting with the left-hand side,

$$\tan\left(x + \frac{\pi}{4}\right) = \frac{\tan x + \tan\left(\frac{\pi}{4}\right)}{1 - \tan x \tan\left(\frac{\pi}{4}\right)} = \frac{\tan x + 1}{1 - \tan x} \qquad \text{By (I-10) and } \tan\frac{\pi}{4} = 1$$

$$= \frac{\frac{\sin x}{\cos x} + 1}{1 - \frac{\sin x}{\cos x}} = \frac{\sin x + \cos x}{\cos x - \sin x} \qquad \text{By (I-2) and algebra}$$

Therefore, the given equation is an identity. ◀

▶EXAMPLE 2 ***Using (I-7)*** Evaluate $\sin \frac{\pi}{12}$ in exact form. Use a calculator to check your result.

Strategy: First express $\frac{\pi}{12}$ as a sum or difference of angles for which we know exact values. $\frac{\pi}{12} = 15° = 45° - 30° = \frac{\pi}{4} - \frac{\pi}{6}$.

Solution

Follow the strategy and use (I-7).

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

Therefore, in exact form, $\sin \dfrac{\pi}{12} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$. With the calculator in radian mode

$$\sin\frac{\pi}{12} \approx 0.2588190451 \quad \text{and} \quad \frac{\sqrt{6} - \sqrt{2}}{4} \approx 0.2588190451. \quad ◀$$

▶EXAMPLE 3 ***More sums and differences*** Suppose α and β satisfy

$$\sin\alpha = \frac{4}{5} \text{ and } \frac{\pi}{2} < \alpha < \frac{3\pi}{2}, \qquad \cos\beta = \frac{5}{13} \text{ and } -\pi < \beta < 0.$$

Evaluate in exact form.

(a) $\tan(\alpha - \beta)$ **(b)** $\sec(\alpha + \beta)$ **(c)** $\sin 2\alpha$.

Strategy: Using the given information, first draw diagrams showing reference triangles for α and β, from which we can get the trigonometric function values of α and β, then use **(a)** (I-11), **(b)** (I-8) **(c)** (I-6).

Solution

Follow the strategy and draw diagrams shown in Figure 6.

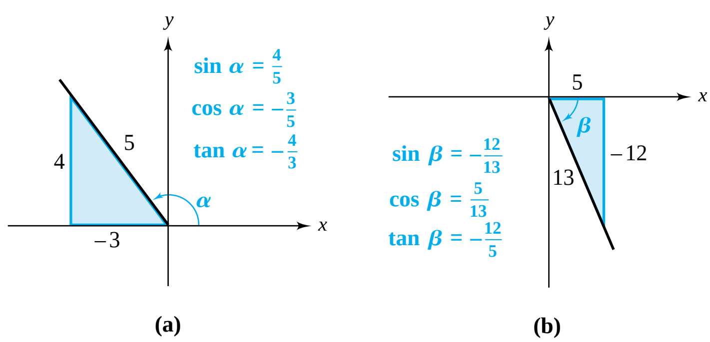


FIGURE 6

(a)
$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \qquad \text{By (I-11)}$$
$$= \frac{\left(\frac{-4}{3}\right) - \left(\frac{-12}{5}\right)}{1 + \left(\frac{-4}{3}\right)\left(\frac{-12}{5}\right)} = \frac{16}{63} \qquad \text{From Figure 6 and arithmetic}$$

(b) By (I-1), $\sec(\alpha + \beta)$ is the reciprocal of $\cos(\alpha + \beta)$, so evaluate $\cos(\alpha + \beta)$ and then take the reciprocal

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \text{By (I-8)}$$
$$= \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) = \frac{33}{65} \qquad \text{From Figure 6}$$

Therefore $\sec(\alpha + \beta) = \frac{65}{33}$.

(c) First write 2α as $(\alpha + \alpha)$ and then use (I-6).

$$\sin 2\alpha = \sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha \qquad \text{By (I-6)}$$
$$= 2\sin\alpha\cos\alpha = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}.$$

Hence, $\sin 2\alpha = -\frac{24}{25}$. ◀

The solution to Example 3c suggests an identity for the sine of twice an angle. We can get $\sin 2\theta$ simply by replacing both α and β by θ in identity (I-6). Similar replacements in (I-8) and (I-10) give **double-angle identities.**

Double-angle identities

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \textbf{(I-12)}$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \textbf{(I-13)}$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} \qquad \textbf{(I-14)}$$

The double-angle identities are three more key identities with which you should become very familiar. In addition, two other forms of (I-13) are worth remembering. We may replace $\cos^2\theta$ by $1 - \sin^2\theta$, or replace $\sin^2\theta$ by $1 - \cos^2\theta$ to get the alternate forms.

Alternate forms of (I-13)

$$\cos 2\theta = 1 - 2\sin^2\theta \quad \text{or} \quad \cos 2\theta = 2\cos^2\theta - 1$$

▶*EXAMPLE 4 Using double-angle identities* Suppose θ is a number (or angle) between 0 and π where $\tan\theta = -\frac{5}{12}$. Evaluate in exact form **(a)** $\cos 2\theta$, **(b)** $\tan 2\theta$, **(c)** $\sec 2\theta$.

HISTORICAL NOTE IDENTITIES IN APPLICATION

Trigonometric identities are important because they can sometimes make feasible tasks that otherwise might be difficult or impossible. Consider the strange case of Zacharias Dase, born in Germany in 1824. Dase apparently had very limited abilities in many areas, but he was one of the most remarkable mental calculators who ever lived. He once calculated the product of a pair of hundred-digit numbers *in his head* in nearly nine hours of intense concentration. He enters our story because he was another calculator of the number π.

James Gregory

Not long after the invention of calculus, James Gregory, a Scottish mathematician, found a way to express the function Arctan x as the sum of an infinite series.

$$\text{Arctan } x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Since $\tan \frac{\pi}{4} = 1$, we may substitute 1 for x in Gregory's series to obtain

$$\frac{\pi}{4} = \text{Arctan } 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

While this series could theoretically be used to approximate π, it has little practical value because it would require 10,000 terms to get four-place accuracy and a million terms for six places. Getting a good approximation with Gregory's series would exceed even Dase's capabilities. Clever use of trigonometric identities, however, put an approximation within reach.

If we let $\alpha = \text{Arctan } \frac{1}{2}$, $\beta = \text{Arctan } \frac{1}{5}$, and $\gamma = \text{Arctan } \frac{1}{8}$, then $\tan \alpha = \frac{1}{2}$, $\tan \beta = \frac{1}{5}$, and $\tan \gamma = \frac{1}{8}$. By using identity (I-10), we can show that $\alpha + \beta + \gamma = \frac{\pi}{4}$. This gives the identity $\pi = 4(\text{Arctan } \frac{1}{2} + \text{Arctan } \frac{1}{5} + \text{Arctan } \frac{1}{8})$. Substituting the values $\frac{1}{2}$, $\frac{1}{5}$, and $\frac{1}{8}$ into Gregory's series yields a manageable sum. In a still prodigious calculating feat, Dase added up hundreds of terms to obtain 200 digits of the expansion of π.

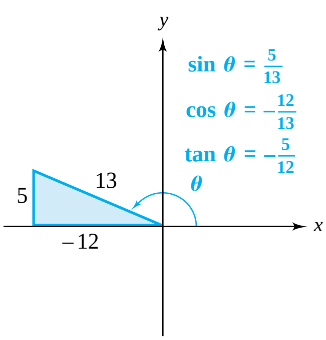


FIGURE 7

Solution

Begin by drawing a diagram to show θ and a reference triangle (see Figure 7).

(a)
$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta \qquad \text{By (I-13)}$$
$$= \left(-\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{119}{169} \qquad \text{From Figure 3}$$

(b) Using (I-14) and Figure 7,

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2\left(-\frac{5}{12}\right)}{1 - \left(-\frac{5}{12}\right)^2} = -\frac{120}{119}$$

(c) Since $\sec 2\theta = \dfrac{1}{\cos 2\theta}$, and $\cos 2\theta = \frac{119}{169}$, so $\sec 2\theta = \frac{169}{119}$. ◄

More Identities

In the preceding section we observed that calculator graphs may be helpful in determining whether or not an equation is an identity. Here we consider some further examples involving identities.

►EXAMPLE 5 ***Is it an identity?*** Determine whether or not the equation is an identity. Refer to the algorithm in Section 6.1.

$$\sin(x - \mathrm{Cos}^{-1} 0.8) = 0.2(4 \sin x - 3 \cos x)$$

Solution

Graphical If $f(x) = \sin(x - \mathrm{Cos}^{-1} 0.8)$, $g(x) = 0.2(4 \sin x - 3 \cos x)$, and we graph $y = f(x) + 1$ and $y = g(x)$ in the same window, the graphs of f and g appear to be identical. On the basis of the graphs, we suspect that the equation is an identity and look for a proof.

Algebraic To simplify the formula for f, let $\theta = \mathrm{Cos}^{-1} 0.8$ and use (I-7).

$$f(x) = \sin(x - \theta) = \sin x \cos \theta - \cos x \sin \theta.$$

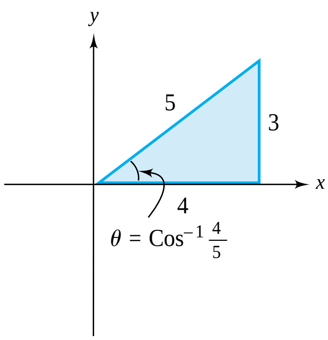


FIGURE 8

Since $\theta = \mathrm{Cos}^{-1} 0.8$, we know that $\cos \theta = 0.8 = \frac{4}{5}$, and so we can draw a reference triangle for θ. See Figure 8. From the diagram in Figure 8 we can read $\sin \theta = \frac{3}{5}$. Returning to the formula for f,

$$\begin{aligned} f(x) &= \sin x \cos \theta - \cos x \sin \theta \\ &= \sin x \cdot \frac{4}{5} - \cos x \cdot \frac{3}{5} \\ &= \frac{1}{5}(4 \sin x - 3 \cos x) \\ &= 0.2(4 \sin x - 3 \cos x). \end{aligned}$$

Therefore $f(x)$ is identically equal to $g(x)$. ◄

►EXAMPLE 6 ***An odd function*** Draw a graph of the function

$$f(x) = \sin(\mathrm{Sin}^{-1} x + 2\,\mathrm{Cos}^{-1} x)$$

and guess a simpler formula for f. Is f an odd or even function (or neither)? What is the domain of f? Prove your simpler formula for f.

Solution

Graphing Y = SIN(SIN^{-1} X + 2 COS^{-1} X) in a decimal window gives a graph like that shown in Figure 9. Tracing along the graph indicates that the x- and y-coordinates are the same, so we have the graph of $y = x$ on the interval $[-1, 1]$. The most reasonable guess is that $f(x) = x$ on that interval, or that

$$\sin(\mathrm{Sin}^{-1} x + 2\,\mathrm{Cos}^{-1} x) = x, \qquad -1 \le x \le 1$$

is an identity.

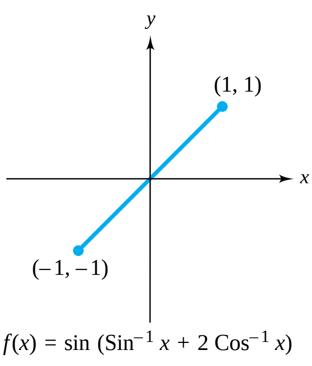
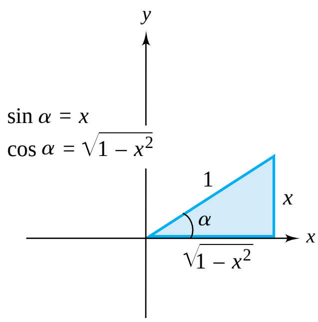
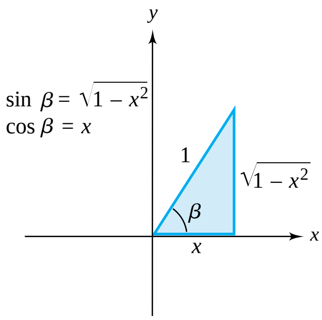


FIGURE 9

To confirm that the domain is actually the interval $[-1, 1]$ we need to take a more careful look at the function. The sine function is defined for all real x, so the domain of f is the same as the domain of the argument. In Section 5.5 we learned that both $\mathrm{Sin}^{-1} x$ and $\mathrm{Cos}^{-1} x$ are defined for $-1 \le x \le 1$, so the domain we read from the graph is correct, $D = [-1, 1]$.



(a) $\alpha = \mathrm{Sin}^{-1} x$



(b) $\beta = \mathrm{Cos}^{-1} x$

FIGURE 10

To verify that f is identically equal to x on the domain, it is probably easiest to draw reference triangles for the inverse function angles. Thus let $\alpha = \mathrm{Sin}^{-1} x$ and $\beta = \mathrm{Cos}^{-1} x$ and draw the reference triangles in Figure 10, from which we observe that $\sin \alpha = x$, $\cos \alpha = \sqrt{1 - x^2}$, $\cos \beta = x$, $\sin \beta = \sqrt{1 - x^2}$. We then use (I-6), and then (I-12) and (I-13).

$$\begin{aligned}
\sin(\mathrm{Sin}^{-1} x + 2\,\mathrm{Cos}^{-1} x) &= \sin(\alpha + 2\beta) \\
&= \sin \alpha \cos 2\beta + \cos \alpha \sin 2\beta \\
&= \sin \alpha(\cos^2 \beta - \sin^2 \beta) + \cos \alpha(2 \sin \beta \cos \beta) \\
&= x(x^2 - (1 - x^2)) + \sqrt{1 - x^2}(2\sqrt{1 - x^2} \cdot x) \\
&= x(2x^2 - 1) + 2x(1 - x^2) \\
&= x(2x^2 - 1 + 2 - 2x^2) = x(1) = x.
\end{aligned}$$

This argument is valid when the reference triangles in Figure 10 apply, that is, for x in $(0, 1)$. To complete the proof, note that the reference triangles in Figure 10 can be reflected through the axes (x-axis for Figure 10a, y-axis for Figure 10b) and everything else remains the same. Checking 0 and ± 1 directly, $\sin(\mathrm{Sin}^{-1} x + 2\,\mathrm{Cos}^{-1} x) = x$ is an identity on $[-1, 1]$, and $f(x) = x$ is an odd function. ◀

In Example 8 of Section 5.5 we learned that the "view" of a vertical marquee is expressible in terms of the inverse tangent function. The same problem arises in many different settings. In the next example we revisit the problem of obtaining the best view, and we give a solution involving identity (I-10).

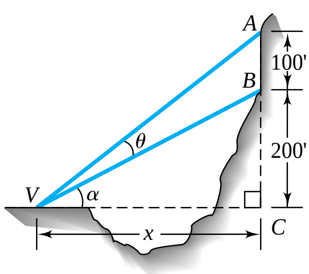


FIGURE 11

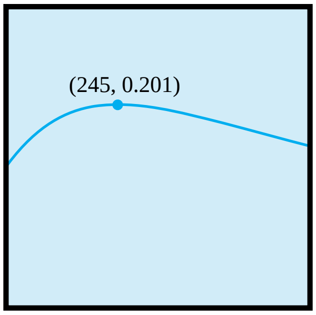
▶*EXAMPLE 7* ***"Best view"*** The Forest Service is going to locate a new visitors' center where the primary attraction is a 100-foot section of a vertical cliff (think of Mt. Rushmore). The only reasonable location is a level area some 200 feet lower than the base of the cliff and several hundred feet away, as in Figure 11.

(a) Show that the viewing angle θ from the point V is given by $\theta = \mathrm{Tan}^{-1} \frac{100x}{x^2 + 60000}$, where x is the distance from V to a spot directly below the cliff. **(b)** If it is decided that the visitor's center V must have a viewing angle of at least 0.2 radians (about 11.5°), what x-interval is allowable? What x gives a maximum θ-value?

Solution

(a) From the diagram, in ΔAVC, $\tan(\theta + \alpha) = \frac{300}{x}$ and in ΔBVC, $\tan \alpha = \frac{200}{x}$. We can use (I-10) to evaluate $\tan \theta = \tan((\theta + \alpha) - \alpha)$:

$$\begin{aligned}
\tan \theta = \tan((\theta + \alpha) - \alpha) &= \frac{\tan(\theta + \alpha) - \tan \alpha}{1 + \tan(\theta + \alpha)\tan \alpha} \\
&= \frac{\dfrac{300}{x} - \dfrac{200}{x}}{1 + \dfrac{300}{x}\dfrac{200}{x}} = \frac{\dfrac{100}{x}}{\dfrac{x^2 + 60{,}000}{x^2}} \\
&= \frac{100x}{x^2 + 60{,}000}.
\end{aligned}$$



[100, 500] by [0, .3]

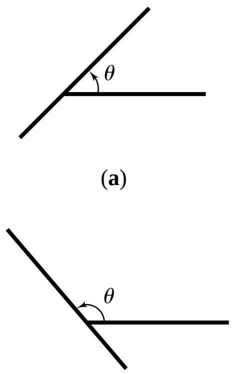
FIGURE 12

Therefore $\theta = \operatorname{Tan}^{-1} \dfrac{100x}{x^2 + 60{,}000}$. We note in passing that this also establishes another identity. Since $\theta + \alpha = \operatorname{Tan}^{-1} \dfrac{300}{x}$, $\theta = \operatorname{Tan}^{-1} \dfrac{100x}{x^2 + 60{,}000}$, and $\alpha = \operatorname{Tan}^{-1} \dfrac{200}{x}$, we have $\theta = (\theta + \alpha) - \alpha$, or

$$\operatorname{Tan}^{-1} \frac{100x}{x^2 + 60{,}000} = \operatorname{Tan}^{-1} \frac{300}{x} - \operatorname{Tan}^{-1} \frac{200}{x}.$$

(b) Graphing Y = TAN^{-1} (100X/(X^2 + 60000)) in $[100, 500] \times [0, .3]$ gives the graph in Figure 12. Tracing along the curve, we find that the maximum viewing angle occurs when $x \approx 245$ feet, but that θ is at least 0.2 radians for any value of x from about 218 to 275 feet. Given the accuracy of the initial data, we aren't justified in assuming more than that the best view occurs when x is approximately 250 feet, and that we have an allowable view anywhere from about 220 to 280 feet away. ◀

Angle Between Lines: Looking Ahead to Calculus



(a)

(b)

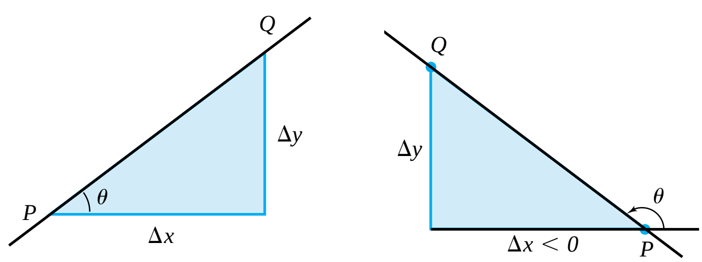
FIGURE 13
Angle of inclination

Identity (I-11) also leads to a formula for the angle between two lines, a topic that finds a number of applications in calculus and differential equations. To get into the formula we first need to relate the slope of a line to its *angle of inclination.* At any point on a line we can measure the angle from the positive horizontal direction to the line, as in Figure 13.

The slope m of a nonvertical line containing points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

We can take P and Q so that the differences Δy and Δx are the signed sides of a reference triangle for the angle of inclination. See Figure 14.



(a) Positive slope. **(b)** Negative slope.

FIGURE 14
$m = \dfrac{\Delta y}{\Delta x} = \tan \theta$

The observation we want is that in both cases shown in Figure 14, the tangent of the angle of inclination is equal to the slope of the line. For a horizontal line, both the slope and the angle of inclination are zero, and $m = 0 = \tan 0$. Thus, in all cases we have the relation given in the box.

Angle of inclination and slope

If a nonvertical line has slope m and an angle of inclination θ, then

$$m = \tan\theta \tag{1}$$

That is, the *tangent of the angle of inclination* is equal to the *slope of the line*.

▶EXAMPLE 8 ***Angle of inclination*** Find the angle of inclination of the line.

(a) L_1: $y = 3x - 2$ **(b)** L_2: $2x + 5y = 10$.

Solution

(a) The equation of the line is already in slope–intercept form, so L_1 has y-intercept point $(0, -2)$ and slope $m = 3$. The angle of inclination is the angle θ in Figure 15, and by Equation (1), $\tan\theta = 3$. Therefore $\theta = \text{Tan}^{-1}\,3 \approx 1.249$ radians, or about 71.6°.

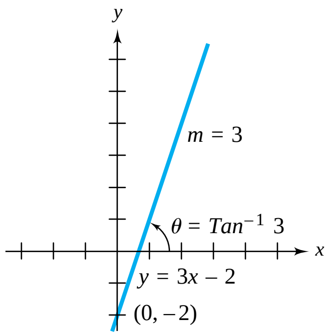


FIGURE 15

(b) To express the equation for L_2 in slope-intercept form, we solve for y:

$$y = -\frac{2}{5}x + 2.$$

Thus $m = -\frac{2}{5}$, and so $\tan\theta = -\frac{2}{5} = -0.4$. In Figure 16a we show L_2 and its angle of inclination θ. Figure 16b shows θ in standard position with a reference triangle. $\text{Tan}^{-1}(-0.4)$ is the negative angle shown in Figure 16b, so $\theta = \pi + \text{Tan}^{-1}(-0.4) \approx 2.761$ radians, or about 158.2°. ◀

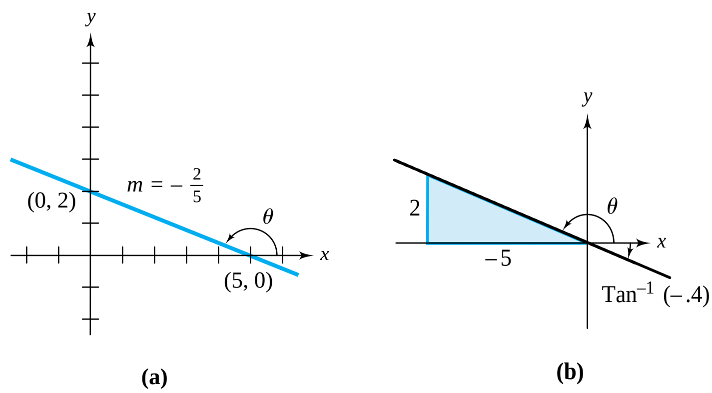


FIGURE 16

Two intersecting lines form four angles, and as in Figure 17, two pairs of vertical angles. The sum of any two adjacent angles of the four is a straight angle. If the lines are perpendicular, then all four angles are right angles; otherwise one of the pairs is acute.

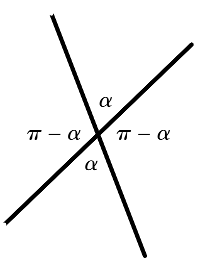


FIGURE 17

Definition: angle between intersecting lines

If two nonperpendicular lines intersect, then the angle between them is the **acute angle** formed by their intersection.

If two lines are perpendicular, the angle between them is $\dfrac{\pi}{2}$.

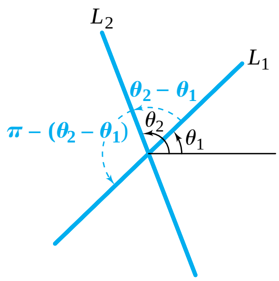


FIGURE 18

Equation (1) and identity (I-11) can be combined to get a convenient way to find the angle between intersecting lines. Suppose nonperpendicular intersecting lines L_1 and L_2 have angles of inclination θ_1 and θ_2 respectively and that $\theta_2 \geq \theta_1$. Then the angle between L_1 and L_2 is either $\theta_2 - \theta_1$ or $\pi - (\theta_2 - \theta_1)$ as in Figure 18. In the first case, where $\alpha = \theta_2 - \theta_1$, identity (I-11) yields

$$\tan\alpha = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2\tan\theta_1} = \frac{m_2 - m_1}{1 + m_2m_1}.$$

If, on the other hand, $\alpha = \pi - (\theta_2 - \theta_1)$, then

$$\tan\alpha = \tan(\pi - (\theta_2 - \theta_1)) = -\tan(\theta_1 - \theta_2) = -\frac{m_2 - m_1}{1 + m_2m_1}.$$

Since the angle between L_1 and L_2 is acute, its tangent is positive, and we have the following.

Formula for the angle between intersecting lines

If two nonperpendicular lines L_1 and L_2 intersect and have slopes m_1 and m_2, respectively, then the angle α between them is the angle whose tangent is the *positive* one of

$$\pm\frac{m_2 - m_1}{1 + m_2m_1}.$$

Equivalently, $\tan\alpha = \left|\dfrac{m_2 - m_1}{1 + m_2m_1}\right|$.

▶EXAMPLE 9 ***Angle between two lines*** Find the angle between the lines of Example 8, L_1: $y = 3x - 2$ and L_2: $2x + 5y = 10$.

Solution

In Example 8 we found that the slopes of the two lines are given by

$$m_1 = 3 \text{ and } m_2 = -\frac{2}{5}.$$

Then

$$\frac{m_2 - m_1}{1 + m_2m_1} = \frac{-\frac{2}{5} - 3}{1 + \left(-\frac{2}{5}\right)3} = \frac{-\frac{17}{5}}{\frac{5}{5} - \frac{6}{5}} = \frac{-\frac{17}{5}}{-\frac{1}{5}} = 17.$$

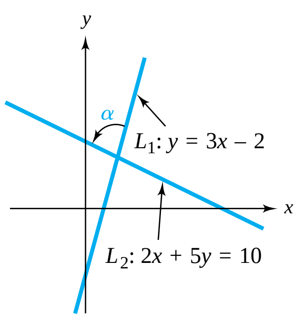


FIGURE 19

Since 17 is positive, the angle α between L_1 and L_2 is given by $\tan\alpha = 17$; from which $\theta = \mathrm{Tan}^{-1}17 \approx 1.512$ radians. As a check, we found that $\theta_2 \approx 2.761$ and $\theta_1 \approx 1.249$, so that $\theta_2 - \theta_1 \approx 1.512$. See Figure 19. ◀

EXERCISES 6.2

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. $\sin(1 + \sqrt{3}) = \sin 1 + \sin \sqrt{3}$
2. There is no number x for which $\sin 2x = 2 \sin x$.
3. If $0 < x < \frac{\pi}{2}$ and $\sin x = \frac{\sqrt{3}}{2}$, then $\sin 2x = \frac{\sqrt{3}}{2}$.
4. If $0 < x < \pi$ and $\cos x = \frac{\sqrt{3}}{2}$, then $\cos 2x = \frac{\sqrt{3}}{2}$.
5. $\tan\left(x + \frac{3\pi}{4}\right) = \frac{\tan x - 1}{\tan x + 1}$ is an identity.

Exercises 6–10 Fill in the blank (use exact form) so that the resulting statement is true.

6. $\sin\left(\pi - \text{Sin}^{-1}\frac{1}{2}\right) = $ ______.
7. $\cos\left(\pi - \text{Cos}^{-1}\frac{1}{2}\right) = $ ______.
8. $\sin\left(\frac{\pi}{2} - \text{Cos}^{-1} 0.4\right) = $ ______.
9. $\tan\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = $ ______.
10. $\tan\left(\frac{3\pi}{2} - \frac{\pi}{4}\right) = $ ______.

Develop Mastery

1. Replace β by $-\beta$ in each of (I-6), (I-9), and (I-10), and then use (I-3) to get identities (I-7), (I-8), and (I-11).
2. Try various simple functions such as $f(x) = x + 1$, $f(x) = 2x - 1$, $f(x) = x^2 + 1$, $f(x) = 2x$, $f(x) = -3x$, and so on, to see if any are additive functions.
3. Based on your findings in Exercise 2, with other examples as needed, make a guess about some types of functions that are additive. Briefly explain why you think that the kinds of functions you have identified are additive and why certain other classes of functions are not additive.

Exercises 4–6 ***Reduction Formulas*** *Use identities given in this section to verify the reduction formulas.*

4. **(a)** $\sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta$
   (b) $\cos(\theta + \pi) = -\cos \theta$
5. **(a)** $\cos\left(\theta - \frac{3\pi}{2}\right) = -\sin \theta$
   (b) $\tan(\pi - \theta) = -\tan \theta$
6. **(a)** $\sin(\theta - \pi) = -\sin \theta$
   (b) $\sec\left(\frac{3\pi}{2} + \theta\right) = \csc \theta$

Exercises 7–10 ***Exact Form*** *Evaluate in exact form and then check by using a calculator to obtain a decimal approximation for your result and for the given expression.*

7. $\cos \frac{5\pi}{12}$, use **(a)** $\frac{5\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$ and **(b)** $\frac{5\pi}{12} = \frac{2\pi}{3} - \frac{\pi}{4}$
8. $\tan \frac{13\pi}{12}$, use **(a)** $\frac{13\pi}{12} = \frac{5\pi}{6} + \frac{\pi}{4}$ and
   (b) $\frac{13\pi}{12} = \frac{5\pi}{4} - \frac{\pi}{6}$
9. $\sin(-15°)$; use **(a)** $-15° = 30° - 45°$ and
   (b) $-15° = 45° - 60°$
10. $\sec 255°$; use **(a)** $255° = 135° + 120°$ and
    (b) $255° = 315° - 60°$

Exercises 11–12 ***Exact Form*** *Evaluate in exact form. First express the given angle as a sum or difference of angles whose trigonometric functions you can evaluate in exact form.*

11. **(a)** $\sin 105°$ **(b)** $\tan 105°$
12. **(a)** $\sin \frac{17\pi}{12}$ **(b)** $\csc \frac{17\pi}{12}$

Exercises 13–18 ***Exact Form*** *For angles α and β, where*

$$\sin \alpha = -\frac{3}{5}, \quad 0 < \alpha < \frac{3\pi}{2}$$

$$\cos \beta = -\frac{5}{13}, \quad \frac{\pi}{2} < \beta < \pi,$$

draw diagrams to show α and β in standard position with reference triangles, then evaluate in exact form. See Example 3.

13. $\sin(\alpha + \beta)$
14. $\cos(\alpha - \beta)$
15. $\tan(\beta - \alpha)$
16. $\sin 2\alpha$
17. $\sin\left(\frac{\pi}{2} - 2\alpha\right)$
18. $\cos\left(\frac{3\pi}{2} + 2\alpha\right)$

Exercises 19–24 ***Exact Form*** *For angle θ that satisfies*

$$\sin \theta = -\frac{3}{4}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

draw a diagram to show θ in standard position with a reference triangle, then evaluate in exact form.

19. $\sin 2\theta$
20. $\sec 2\theta$
21. $\tan\left(\frac{\pi}{3} + \theta\right)$
22. $\tan\left(\frac{2\pi}{3} - \theta\right)$
23. $\sin\left(\frac{5\pi}{4} + \theta\right)$
24. $\cos\left(\frac{5\pi}{6} + \theta\right)$

Exercises 25–27 ***Exact Form*** *Use the identities from this section to simplify the expression, then evaluate in exact form.*

25. **(a)** $\sin 20° \cos 40° + \cos 20° \sin 40°$

(b) $\cos \frac{\pi}{10} \cos \frac{2\pi}{5} - \sin \frac{\pi}{10} \sin \frac{2\pi}{5}$

26. **(a)** $\cos^2 15° - \sin^2 15°$ **(b)** $\sin \frac{5\pi}{12} \cos \frac{5\pi}{12}$

27. **(a)** $\sin 80° \cos 50° - \cos 80° \sin 50°$

(b) $\frac{1}{2} \sec 15° \csc 15°$

Exercises 28–34 ***Prove Identities*** *Prove that the equation is an identity.*

28. $\tan\left(x - \frac{\pi}{4}\right) = \frac{\sin x - \cos x}{\sin x + \cos x}$

29. $\sin\left(\frac{\pi}{6} - x\right) = \frac{1}{2}(\cos x - \sqrt{3} \sin x)$

30. $\frac{\cos 2x}{\cos x + \sin x} = \cos x - \sin x$

31. $(\sin x + \cos x)^2 = 1 + \sin 2x$

32. $(\sin x - \cos x)^2 + \sin 2x = 1$

33. $\tan\left(\frac{5\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$

34. $\sin\left(x - \frac{\pi}{6}\right) = \cos\left(x - \frac{2\pi}{3}\right)$

35. $\frac{\sin 2x}{\cos 2x} = \frac{2 \tan x}{1 - \tan^2 x}$

36. $\frac{\sin 2x}{\tan 2x} = 2 \cos^2 x - 1$

Exercises 37–41 ***Is It an Identity?*** ***(a)*** *Give the domain of the equation.* ***(b)*** *Determine whether or not the equation is an identity.*

37. $\sin(x + \text{Sin}^{-1} 0.6) = 0.2(4 \sin x + 3 \cos x)$

38. $\cos(x - \text{Sin}^{-1} 0.6) = 0.2(3 \sin x + 4 \cos x)$

39. $\cos(x - \text{Cos}^{-1} 0.4) = \cos x - 0.4$

40. $\sin(\text{Sin}^{-1} x + \text{Cos}^{-1} x) = x^2 + |x^2 - 1|$
(*Hint:* Use the identity $\text{Sin}^{-1} x + \text{Cos}^{-1} x = \frac{\pi}{2}$.)

41. $\cos 2x + 2 \sin^2 x = 0.1(\sqrt{x} + |\sqrt{x} - 10|)$

Exercises 42–45 ***Simplify*** *Use a graph of f to help you guess a simpler formula for f. Remember to give domain of f. You might want to use a decimal window.*

42. $f(x) = \sin(\text{Sin}^{-1} x + 2 \text{Cos}^{-1} x)$; line segment

43. $f(x) = \cos(2 \text{Sin}^{-1} x + \text{Cos}^{-1} x)$; line segment

44. $f(x) = \sin(2 \text{Sin}^{-1} x + \text{Cos}^{-1} x)$; half circle.

45. $f(x) = \cos(\text{Sin}^{-1} x + 2 \text{Cos}^{-1} x)$; half circle.

46. ***Explore*** For $f(x) = \sin(\text{Sin}^{-1} x + k \text{Cos}^{-1} x)$, try several integer values of k (positive and negative) and draw graphs.

(a) Describe any interesting observations, such as the number of x-intercept points and local extrema.

(b) How are the graphs for $k = 4$ and $k = -2$ related?

(c) Find other pairs of values of k for which properties are similar to those in part (b).

47. Repeat Exercise 46 for $f(x) = \cos(k \text{ Sin}^{-1} x + \text{Cos}^{-1} x)$.

Exercises 48–50 Use identities (I-10) and (I-11). Give answers in exact form.

48. If $\tan \alpha = 2$ and $\tan(\alpha - \beta) = 3$, find $\tan \beta$.

49. If $\tan x = \frac{3}{4}$ and $x + y = \frac{\pi}{4}$, find $\tan y$.

50. If $\sin x = \frac{3}{5}$, x is in the first quadrant, and $x + y = \frac{3\pi}{4}$, find $\tan y$.

Exercises 51–52 ***Graphs*** *Describe the graph of f as a horizontal translation of the graph of g. Check with graphs.*

51. $g(x) = \sin x, f(x) = \sin(x - \text{Sin}^{-1} 0.5)$.

52. $g(x) = \cos x, f(x) = \cos(x + \text{Cos}^{-1} 0.5)$.

Exercises 53–54 Draw graphs to support the claim that the equation in part ***(a)*** *is an identity. Use part* ***(a)*** *to solve the equation in part* ***(b)****.*

53. **(a)** $\sin(2 \text{Sin}^{-1} x + \text{Cos}^{-1} x) = \sqrt{1 - x^2}$
(b) $\sin(2 \text{Sin}^{-1} x + \text{Cos}^{-1} x) = 0.5$

54. **(a)** $\cos(\text{Sin}^{-1} x + 2 \text{Cos}^{-1} x) = -\sqrt{1 - x^2}$
(b) $\cos(\text{Sin}^{-1} x + 2 \text{Cos}^{-1} x) = -0.4$

Exercises 55–56 ***Angle of Intersection*** ***(a)*** *The graphs of f and g are lines l_1 and l_2. Find the angle θ at the intersection of l_1 and l_2. Give the answer to the nearest tenth of a degree.* ***(b)*** *Find the angle of inclination for each line.* ***(c)*** *Use a decimal window and draw the graphs of f and g on the same screen. Does the angle between the lines and the angles of inclination appear to be consistent with your answer in parts* ***(a)*** *and* ***(b)****? Find the point of intersection of the lines.*

55. $f(x) = 0.5x + 2, \quad g(x) = 1.5x$

56. $f(x) = x + 2, \quad g(x) = -2x - 7$

Exercises 57–58 Solve the equation (Hint: Apply tan *to both sides.)*

57. $\text{Tan}^{-1} x = \text{Tan}^{-1}\left(\frac{1}{4}\right) + \text{Tan}^{-1}\left(\frac{1}{2}\right)$

58. $\text{Tan}^{-1}(2x + 1) = \text{Tan}^{-1} 1 - \text{Tan}^{-1}\left(\frac{1}{2}\right)$

59. Solve $\text{Sin}^{-1} x = \text{Tan}^{-1}\left(\frac{3}{4}\right) + \text{Cos}^{-1}\left(\frac{3}{5}\right)$

60. Solve $\text{Cos}^{-1} x = \text{Tan}^{-1}\left(\frac{4}{3}\right) + \text{Sin}^{-1}\left(\frac{4}{5}\right)$.

61. **(a)** Prove that $\sin 3\theta = 3 \sin \theta \cos^2 \theta - \sin^3 \theta$ is an identity. (*Hint:* $\sin 3\theta = \sin(\theta + 2\theta)$; use (I-6).)
(b) Prove that $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ is an identity.

62. **(a)** Prove that $\cos 3\theta = \cos^3 \theta - 3 \sin^2 \theta \cos \theta$ is an identity. (*Hint:* $\cos 3\theta = \cos(\theta + 2\theta)$; use (I-8).)
(b) Prove that $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$ is an identity.

63. **(a)** Show that $\sin 10°$ is a root of $8x^3 - 6x + 1 = 0$. (*Hint:* Since the equation in Exercise 61(b) is an identity we can replace θ by any value and the result will be a true statement. Show that replacing θ by $10°$ is equivalent to replacing x by $\sin 10°$.)
(b) Show that $\sin 50°$ and $\sin 250°$ are also roots of $8x^3 - 6x + 1 = 0$.
(c) Get 10-decimal place approximations to the three roots of $8x^3 - 6x + 1 = 0$. Check by drawing a graph. (*Hint:* Use your calculator to evaluate $\sin 10°$.)

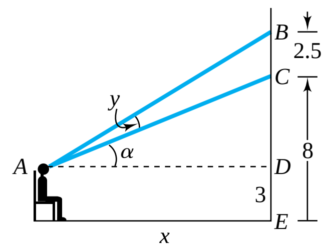
64. Repeat Exercise 63 for the equation $8x^3 - 6x - 1 = 0$ with $\cos 20°$, $\cos 100°$, and $\cos 140°$ as roots.

65. Let $x + \frac{1}{x} = 2 \cos \theta$, where $0 < \theta < \frac{\pi}{2}$.
(a) Show that $x^2 + \frac{1}{x^2} = 2 \cos 2\theta$. (*Hint:* Square both sides of the equation.)
(b) Show that $x^3 + \frac{1}{x^3} = 2 \cos 3\theta$. (*Hint:* Cube both sides of the equation and use the identity in Exercise 62(b).)

66. Use the identity $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1} x$ and the fact that Sin^{-1} is an odd function to prove that f is an even function where $f(x) = \sin(2\,\text{Sin}^{-1} x + \text{Cos}^{-1} x)$. Check by drawing a graph. (*Hint:* You might need to use the reduction formula $\sin(\pi - \theta) = \sin \theta$.)

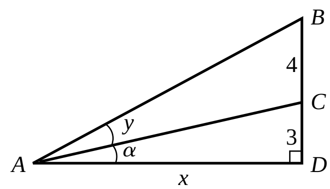
67. Prove that $f(x) = \cos(2\,\text{Sin}^{-1} x + \text{Cos}^{-1} x)$ is an odd function. See Exercise 66.

68. ***Best View*** Bridgect is seated in a chair directly in front of a picture on a wall. Her eye level height is 3 feet from the floor and the picture is 2.5 feet high with the bottom edge 8 feet from the floor (see the diagram). Her "view" of the picture is measured by the size of angle y. She is seated x feet from the wall.
(a) Find a formula for y as a function of x.
(b) How far from the wall should she sit to get the "best view" (largest y)?



69. Solve Exercise 68 for a picture that is 3.5 feet high and 12 feet from the floor.

70. ***Maximum Angle*** In the diagram find the value of x that will make the angle y a maximum (2 decimal places). What is the maximum?



71. Solve Exercise 70 if $|\overline{BC}| = 5$.

72. In Exercise 70 find two different formulas giving y as a function of x. Use these to get an identity. See Example 7. Check Graphically.

73. Give the domain of the equation and show that it is an identity.
(a) $\ln(\sin 2x) = \ln 2 + \ln(\sin x \cos x)$
(b) $\ln(\sin 2x) = \ln 2 + \ln(\sin x) + \ln(\cos x)$

Exercises 74–76 ***Graph, Domain and Range*** *Draw a graph of f; use* $[-20, 20] \times [-2, 2]$. ***(a)*** *From the graph, does it appear that f is odd, even, or neither? Justify your guess.* ***(b)*** *Give the domain and range of f.*

74. $f(x) = 2 \sin\left(\text{Tan}^{-1} \frac{12}{x} - \text{Tan}^{-1} \frac{3}{x}\right)$

75. $f(x) = 3 \sin\left(\text{Tan}^{-1} \frac{8}{x} - \text{Tan}^{-1} \frac{2}{x}\right)$

76. $f(x) = 2 \tan\left(\text{Tan}^{-1} \frac{12}{x} - \text{Tan}^{-1} \frac{2}{x}\right)$

6.3 HALF-ANGLE FORMULAS, PRODUCT–SUM, AND FACTOR IDENTITIES

> The idea that buried among the chaotic data of experience are hidden principles of an exact mathematical nature is far from obvious.
>
> *P. W. C. Davies*

In this section we derive additional key identities.

> *So I learned to do algebra very quickly, and it came in handy in college. When we had a problem in calculus, I was very quick to see where it was going and to do the algebra.*
> Richard Feynman

Half-Angle Formulas

As noted in the preceding section, identity (I-13) can be expressed in alternate forms:

$$\cos 2\alpha = 1 - 2\sin^2\alpha, \qquad \cos 2\alpha = 2\cos^2\alpha - 1.$$

Replacing 2α by θ, and α by $\frac{\theta}{2}$, these equations become

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}, \qquad \cos\theta = 2\cos^2\frac{\theta}{2} - 1.$$

Solving the first of these equations for $\sin\frac{\theta}{2}$ and the second for $\cos\frac{\theta}{2}$ gives the traditional forms of half-angle identities, involving plus-minus signs.

Half-angle identities

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \textbf{(I-15)}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \textbf{(I-16)}$$

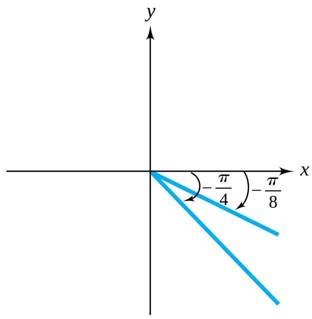
The plus-minus sign can be misleading since it makes it look as if there are two different values for $\sin\frac{\theta}{2}$. We could use graphs to help see that $\sin\frac{\theta}{2}$ coincides with $\sqrt{\frac{1-\cos\theta}{2}}$ when $\sin\frac{\theta}{2}$ is positive, and that $\sin\frac{\theta}{2}$ agrees with $-\sqrt{\frac{1-\cos\theta}{2}}$ when $\sin\frac{\theta}{2}$ is negative. The same relationship holds for $\cos\frac{\theta}{2}$. Although such analysis is possible, it is usually easier to determine the quadrant containing $\frac{\theta}{2}$ and take the proper sign for that quadrant. Thus if θ is an angle between π and $\frac{3\pi}{2}$, then $\frac{\theta}{2}$ is between $\frac{\pi}{2}$ and $\frac{3\pi}{4}$, which tells us that $\sin\frac{\theta}{2}$ is positive and $\cos\frac{\theta}{2}$ is negative.

Half-angle identities sometimes arise in a context where we are explicitly asked for a trigonometric function of $\frac{\theta}{2}$, or we may need an exact form for a function of an angle such as $\frac{3\pi}{8}$, which we can get from our knowledge of functions of $\frac{3\pi}{4}$. If we are only given partial information about θ, we need to exercise care. Keep in mind the following caution, which is illustrated in Example 1.

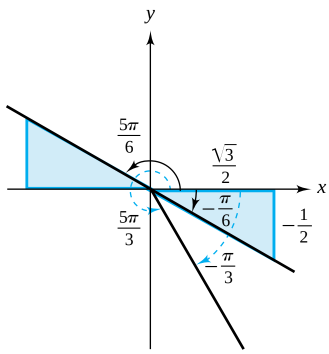
WARNING: The quadrant of θ alone does not determine the quadrant of $\frac{\theta}{2}$; coterminal angles can have different half-angle function values.

►*EXAMPLE 1* ***Half-angle identities*** Evaluate $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ in exact form for the fourth quadrant angle.

(a) $\theta = -\frac{\pi}{4}$ **(b)** $\theta = \frac{5\pi}{3}$ **(c)** $\theta = -\frac{\pi}{3}$



(a)



(b)

FIGURE 20

Solution

(a) Since $\frac{\theta}{2} = -\frac{\pi}{8}$, $\frac{\theta}{2}$ is also in Quadrant IV, where the sine is negative and the cosine is positive. See Figure 20a. Thus

$$\sin\frac{\theta}{2} = -\sqrt{\frac{1-\cos\theta}{2}} = -\sqrt{\frac{1-\cos(-\pi/4)}{2}}$$
$$= -\sqrt{\frac{1-1/\sqrt{2}}{2}} = -\sqrt{\frac{2-\sqrt{2}}{4}}.$$
$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\cos(-\pi/4)}{2}}$$
$$= \sqrt{\frac{1+1/\sqrt{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}}.$$

As always, when evaluating any trigonometric function in exact form, we should get a calculator check.

(b) Half of the fourth quadrant angle $\frac{5\pi}{3}$ is $\frac{5\pi}{6}$, so $\frac{\theta}{2}$ is a second-quadrant angle, where sine is positive and cosine is negative. Either from the reference triangle in Figure 20b or from identities (I-15) and (I-16) we have

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\cos(5\pi/3)}{2}} = \sqrt{\frac{1-1/2}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$
$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\cos(5\pi/3)}{2}} = -\sqrt{\frac{1+1/2}{2}}$$
$$= -\frac{\sqrt{3}}{2}.$$

(c) If $\theta = -\frac{\pi}{3}$, then $\frac{\theta}{2} = -\frac{\pi}{6}$, a fourth-quadrant angle, where sine is negative and cosine is positive. While we can use the half-angle identities, the reference angle for $-\frac{\pi}{6}$ is a special triangle from which we can read the trigonometric functions. See Figure 20b.

$$\sin\frac{\theta}{2} = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2} \qquad \cos\frac{\theta}{2} = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$ ◀

Note that in Example 1, while $\frac{5\pi}{3}$ and $-\frac{\pi}{3}$ are coterminal angles, so that all of their trigonometric functions are the same, dividing by 2 gives different angles with different sine and cosine values.

To get an identity for $\tan\frac{\theta}{2}$ let us first rewrite identities (I-12) and an alternative form of (I-13) by replacing 2θ by θ and θ by $\frac{\theta}{2}$.

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$1+\cos\theta = 2\cos^2\frac{\theta}{2}$$

Now divide the first equation by the second equation and simplify:

$$\frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}.$$

This gives an identity for $\tan \frac{\theta}{2}$:

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}.$$

In Example 4 of Section 6.1 we proved that $\frac{\sin \theta}{1 + \cos \theta}$ is identically equal to $\frac{1 - \cos \theta}{\sin \theta}$, so we have two forms for $\tan \frac{\theta}{2}$.

Another half-angle identity

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} \quad \text{or} \quad \tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} \qquad \textbf{(I-17)}$$

The two forms for (I-17) do not require the plus-minus sign that appeared in identities (I-15) and (I-16) since $1 + \cos \theta \geq 0$ and $1 - \cos \theta \geq 0$, and the signs for $\tan \frac{\theta}{2}$ and $\sin \theta$ agree, as can be seen by graphing the two functions on the same screen.

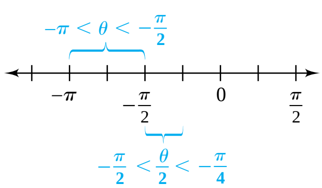
As we noted in Chapter 5, we can get exact-form answers only for certain angles. The important point is that practice is necessary to understand and remember the half-angle formulas.

Strategy: First draw a diagram with a reference triangle for θ from which the functions of θ can be read, then use appropriate identities.

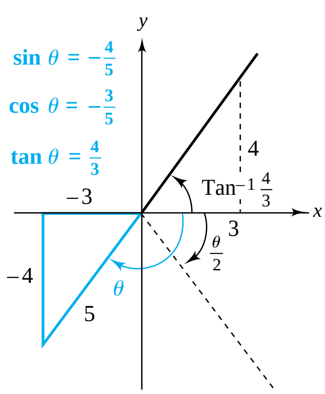
►EXAMPLE 2 ***Half-angle identities*** Angle θ is defined by $\sin \theta = -\frac{4}{5}$ and $-\pi < \theta < -\frac{\pi}{2}$. Evaluate in exact form and use a calculator to get a decimal approximation: **(a)** $\sin \frac{\theta}{2}$ **(b)** $\cot \frac{\theta}{2}$.

Solution

Since $-\pi < \theta < -\frac{\pi}{2}$, dividing by 2 gives $-\frac{\pi}{2} < \frac{\theta}{2} < -\frac{\pi}{4}$. Thus $\frac{\theta}{2}$ is a fourth-quadrant angle as may be seen in Figure 21a. In the fourth quadrant, $\cos \frac{\theta}{2} > 0$ and $\sin \frac{\theta}{2} < 0$.



(a)



(b)

FIGURE 21

(a) Follow the strategy, using (I-15) with a negative sign and $\cos \theta = -\frac{3}{5}$ from Figure 21b,

$$\sin \frac{\theta}{2} = -\sqrt{\frac{1 - \cos \theta}{2}} = -\sqrt{\frac{1 - (-\frac{3}{5})}{2}} = -\sqrt{\frac{8}{10}} = -\frac{2}{\sqrt{5}}.$$

Hence, in exact form, $\sin \frac{\theta}{2} = -\frac{2}{\sqrt{5}}$.

To get a calculator check, first identify θ. From the diagram in Figure 21b, $\theta = -\pi + \text{Tan}^{-1} \frac{4}{3} \approx -2.2142974$. Therefore, $\sin \frac{\theta}{2} \approx \sin(-\frac{2.2142974}{2}) \approx -0.8944272$. Also, $-\frac{2}{\sqrt{5}} \approx -0.8944272$.

(b) Using (I-17) directly with (I-1),

$$\cot \frac{\theta}{2} = \frac{1}{\tan \frac{\theta}{2}} = \frac{1 + \cos \theta}{\sin \theta} = \frac{1 - \frac{3}{5}}{-\frac{4}{5}} = \frac{\frac{2}{5}}{-\frac{4}{5}} = -\frac{1}{2}. \quad ◄$$

In the above examples, we were able to get results in exact form. In many cases this is inconvenient or impossible, as may be seen in the next example.

►EXAMPLE 3 ***Calculator evaluation*** For angle θ where $\sin \frac{\theta}{2} = 0.64$ and $0 < \theta < \pi$, find a decimal approximation for **(a)** θ, **(b)** $\sin 3\theta$, **(c)** $\tan \frac{\theta}{5}$.

Strategy: For decimal approximations, first get a calculator value for θ. Locate the quadrant for $\frac{\theta}{2}$, evaluate $\frac{\theta}{2}$ with Sin^{-1}, and multiply by 2 to get θ. Store θ in memory and recall as needed.

Solution

(a) Follow the strategy. Since $0 < \theta < \pi$, also $0 < \frac{\theta}{2} < \frac{\pi}{2}$, so $\frac{\theta}{2}$ is in the first quadrant. Therefore $\frac{\theta}{2} = \mathrm{Sin}^{-1}(0.64)$ and $\theta = 2\ \mathrm{Sin}^{-1}(0.64) \approx 1.3889965$.

(b) Similarly, $\sin 3\theta = \sin(3 \cdot 2\ \mathrm{Sin}^{-1}\ 0.64)$, so $\sin 3\theta \approx -0.8549202$.

(c) Finally, $\tan \frac{\theta}{5} = \tan(\frac{1}{5} \cdot 2\ \mathrm{Sin}^{-1}\ 0.64)$, so $\tan \frac{\theta}{5} \approx 0.2851732$. ◀

Product-Sum Formulas

We conclude our collection of key identities with two groups of formulas that are useful in a number of special situations. They do not occur as often as (I-1) through (I-17) and they are not easy to remember. However, they are easy to derive from the sum and difference identities (I-6) through (I-9). You should become acquainted with them and their use.

Product-to-sum identities

$$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)] \qquad \textbf{(I-18)}$$

$$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)] \qquad \textbf{(I-19)}$$

$$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \qquad \textbf{(I-20)}$$

To prove (I-18), we add the equations in (I-6) and (I-7):

$$\begin{array}{r}
\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \\
\hline
\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \cdot \sin \alpha \cos \beta \qquad\qquad\quad
\end{array}$$

Identity (I-18) follows if we divide by 2 and read from right to left. Identities (I-19) and (I-20) can be proved similarly. See Develop Mastery Exercise 56.

►*EXAMPLE 4* ***Product to sum*** Use (I-19) to express the product of $\cos \theta$ and $\cos 3\theta$ as a sum. Then draw calculator graphs as a check.

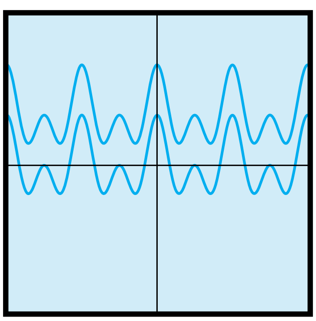
Solution

In (I-19) replace α by θ and β by 3θ.

$$\begin{aligned}
\cos \theta \cdot \cos 3\theta &= \frac{1}{2}[\cos(\theta + 3\theta) + \cos(\theta - 3\theta)] && \text{by (I-19)} \\
&= \frac{1}{2}[\cos(4\theta) + \cos(-2\theta)] \\
&= \frac{1}{2}[\cos 4\theta + \cos 2\theta] && \text{by (I-3)}
\end{aligned}$$

Hence $\cos \theta \cos 3\theta = \frac{1}{2}(\cos 4\theta + \cos 2\theta)$ is an identity.

To see both functions, we translate one up by a unit. In either a decimal window or the trigonometric window, graph `Y1 = COS X COS 3X + 1`, `Y2 = (COS 4X + COS 2X)/2`, See Figure 22. The graphs clearly suggest that the functions are identical. ◀



$[-6.3, 6.3]$ by $[-3, 3]$

FIGURE 22

Identities (I-18) through (I-20) convert products to sums. Another closely related set of identities converts sums to products. These sum-to-product identities are also known as **factoring identities.**

Sum-to-product (factoring) identities

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \qquad \textbf{(I-21)}$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \qquad \textbf{(I-22)}$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \qquad \textbf{(I-23)}$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \qquad \textbf{(I-24)}$$

Identity (I-21) follows from (I-18) by simple substitutions. First write (I-18) and multiply through by 2:

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

Now let $\alpha + \beta = x$ and $\alpha - \beta = y$. Then

$$\alpha = \frac{x+y}{2} \qquad \text{and} \qquad \beta = \frac{x-y}{2}.$$

Substituting these values for α and β into (I-18) and reading from right to left gives identity (I-21). Identities (I-22), (I-23), and (I-24) can be proved similarly (see Exercise 59).

►*EXAMPLE 5* ***Sums to products*** Use sum-to-product identities to simplify the following expression and then draw calculator graphs as a check.

$$\frac{\sin 2x - \sin 4x}{\cos 2x + \cos 4x}$$

Solution

Applying (I-22) to the numerator and (I-23) to the denominator, we have

$$\begin{aligned}\frac{\sin 2x - \sin 4x}{\cos 2x + \cos 4x} &= \frac{2\cos\left(\frac{2x+4x}{2}\right)\cdot\sin\left(\frac{2x-4x}{2}\right)}{2\cos\left(\frac{2x+4x}{2}\right)\cdot\cos\left(\frac{2x-4x}{2}\right)} \\ &= \frac{2\cos 3x\cdot\sin(-x)}{2\cos 3x\cdot\cos(-x)} \\ &= \frac{-2\cos 3x\cdot\sin x}{2\cos 3x\cdot\cos x} = -\tan x.\end{aligned}$$

To see both functions, translate one up by a unit. Graph Y1 = −TAN X + 1, Y2 = (SIN 2X − SIN 4X)/(COS 2X + COS 4X). Because of the steepness with vertical asymptotes, it may not be clear that the functions are identical. You may want to turn off the two functions and graph instead Y3 = Y1 − Y2 and see that the difference appears to be the expected line $y = 1$. ◄

Looking ahead to calculus. Half-angle identities provide the basis for a substitution used in integration in calculus. Quotients that involve sines and cosines can be very difficult to integrate. The substitution illustrated in the next example can change such a quotient into a more manageable rational function in variable u. In Example 6 we show how to deal with $\sin x$. For the substitution for $\cos x$, see Develop Mastery Exercise 65.

▶EXAMPLE 6 A substitution Let $u = \tan\frac{x}{2}$, where $0 < x < \pi$ (so $0 < \frac{x}{2} < \frac{\pi}{2}$.)

Strategy: **(a)** First use the given information to draw a right triangle with angle $\frac{x}{2}$ whose tangent is $\frac{u}{1}$, then find $\sin\frac{x}{2}$ and $\cos\frac{x}{2}$ in terms of u and use identity (I-12).

(a) Express $\sin x$ in terms of u.

(b) Express $\dfrac{\sin x}{1 - \sin x}$ in terms of u.

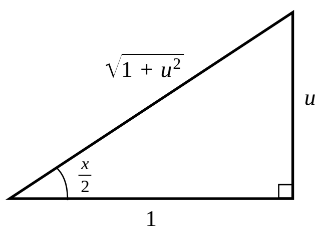


FIGURE 23

Solution

Follow the strategy. See Figure 23.

(a) Express $\sin x$ in terms of $\frac{x}{2}$ by using the double-angle identity (I-12) with θ replaced by $\frac{x}{2}$.

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}.$$

From the right triangle,

$$\sin\frac{x}{2} = \frac{u}{\sqrt{1+u^2}} \quad \text{and} \quad \cos\frac{x}{2} = \frac{1}{\sqrt{1+u^2}},$$

so

$$\sin x = 2\frac{u}{\sqrt{1+u^2}} \cdot \frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}.$$

(b) Replace $\sin x$ by $\dfrac{2u}{1+u^2}$.

$$\frac{\sin x}{1 - \sin x} = \frac{\dfrac{2u}{1+u^2}}{1 - \dfrac{2u}{1+u^2}} = \frac{2u}{1 + u^2 - 2u} = \frac{2u}{(u-1)^2}. \quad ◀$$

EXERCISES 6.3

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. If θ is an angle in the first quadrant, then $\dfrac{\theta}{2}$ must also be in the first quadrant.

2. $2\sin\dfrac{x}{2} = \sin x$ is an identity.

3. $2\cos^2\dfrac{x}{2} = 1 + \cos x$ is an identity.

4. $2\sin^2\dfrac{x}{2} = 1 + \cos x$ is an identity.

5. For every real number x, $\sin^2\dfrac{x}{2} = 1 - \cos^2\dfrac{x}{2}$.

Exercises 6–10 Fill in the blank with $<$, $>$, or $=$ so that the resulting statement is true.

6. For every real number x, $2\cos^2\frac{x}{2} + 1$ ______ $\cos x$.

7. For every real number x, $2\sin\frac{x}{2}\cos\frac{x}{2}$ ______ $1 + \sin x$.

8. For every x in $[3.5, 4.5]$, $\cos\frac{x}{2}$ ______ 0.

9. For every x in $\left[\pi, \frac{3\pi}{2}\right]$, $\sin\frac{x}{2}$ ______ 0.

10. $\cos 1$ ______ $\cos 0.5$.

Develop Mastery

In all exercises, express numerical results in exact form unless otherwise specified.

Exercises 1–4 ***Half-Angle Evaluation*** *Use half-angle formulas to evaluate the expressions, then make a calculator check.*

1. **(a)** $\sin\left(\frac{\pi}{12}\right)$, **(b)** $\cos\left(\frac{11\pi}{8}\right)$

2. **(a)** $\cos\left(\frac{5\pi}{12}\right)$, **(b)** $\tan\left(\frac{7\pi}{12}\right)$

3. **(a)** $\sin\left(-\frac{3\pi}{8}\right)$, **(b)** $\cos\left(-\frac{3\pi}{8}\right)$

4. **(a)** $\csc\left(-\frac{7\pi}{12}\right)$, **(b)** $\cot\left(\frac{17\pi}{8}\right)$

Exercises 5–8 ***Half-Angle Evaluation*** *An angle θ is specified. Draw a diagram that shows the angle θ, determine the quadrant that contains $\frac{\theta}{2}$, and find* ***(a)*** $\sin\frac{\theta}{2}$, ***(b)*** $\cos\frac{\theta}{2}$, ***(c)*** $\tan\frac{\theta}{2}$.

5. $\cos\theta = -\frac{12}{13}$ and $0 < \theta < \pi$.

6. $\sin\theta = -\frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$

7. $\cos\theta = -\frac{3}{5}$ and $2\pi < \theta < 3\pi$

8. $\sec\theta = -2$ and $\pi < \theta < 2\pi$

Exercises 9–12 For each exercise restrictions on x are given and three expressions are to be considered using these restrictions, ***(i)*** $\cos x$ *is* $\boxed{}$ ***(ii)*** $\sin\frac{x}{2}$ *is* $\boxed{}$ ***(iii)*** $\cos 2x$ *is* $\boxed{}$ *. Enter a + sign in the box if the expression is positive, a − sign if it is negative, or a ± if it might be either positive or negative.*

9. **(a)** $\pi < x < \frac{3\pi}{2}$ **(b)** x is in the third quadrant.
(*Hint:* At first glance it might appear that conditions **(a)** and **(b)** are identical; they are not.)

10. **(a)** $\frac{3\pi}{2} < x < 2\pi$. **(b)** x is in the fourth quadrant. (*Hint:* See Exercise 9.)

11. **(a)** $\frac{3\pi}{4} < x < \pi$. **(b)** $\pi < x < \frac{5\pi}{4}$.

12. **(a)** $\pi < x < 2\pi$ and $\tan x > 0$.
(b) Both $\cos x$ and $\tan x$ are negative.

13. Is there an angle θ in the third quadrant such that
(a) $\sin\frac{\theta}{2}$ is positive and $\cos\frac{\theta}{2}$ is negative? Explain.
(b) $\sin\frac{\theta}{2}$ is negative and $\cos\frac{\theta}{2}$ is positive? Explain.

Exercises 14–16 Evaluate in two ways as indicated. Although the two answers may look different, evaluate each by calculator to see if approximations are equal.

14. Evaluate $\sin\frac{\pi}{12}$ by using
(a) (I-15) with $\frac{\pi}{12} = \frac{1}{2}\left(\frac{\pi}{6}\right)$
(b) (I-7) with $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$.

15. Evaluate $\cos\frac{13\pi}{12}$ by using
(a) (I-16) with $\frac{13\pi}{12} = \frac{1}{2}\left(\frac{13\pi}{6}\right)$,
(b) (I-8) with $\frac{13\pi}{12} = \frac{3\pi}{4} + \frac{\pi}{3}$.

16. Evaluate $\tan -\frac{7\pi}{12}$ by using
(a) (I-17) with $-\frac{7\pi}{12} = -\frac{1}{2}\left(\frac{7\pi}{6}\right)$,
(b) (I-11) with $-\frac{7\pi}{12} = \frac{\pi}{6} - \frac{3\pi}{4}$.

Exercises 17–24 ***Prove Identities*** *Prove that the equation is an identity.*

17. $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

18. $\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2 = 1 + \sin x$

19. $\tan\frac{x}{2} = \csc x - \cot x$

20. $2\sin^2\frac{x}{2} = \frac{\sec x - 1}{\sec x}$

21. $\cot\frac{x}{2} = \frac{\sin x \sec x}{\sec x - 1}$

22. $2\sin^2\frac{x}{2} = \sin x \tan\frac{x}{2}$

23. $\sqrt{\dfrac{1+\cos x}{2}} = \left|\cos\dfrac{x}{2}\right|$

24. $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$

Exercises 25–36 ***Is It an Identity?*** *Use the algorithm in Section 6.1 to determine whether or not the equation is an identity. If it is not, then give at least one counterexample, that is, a value of x for which the equality does not hold.*

25. $\sin\dfrac{x}{2} = \dfrac{\sin x}{2}$

26. $2\cos^2\dfrac{x}{2} = 1 + \cos x$

27. $\sqrt{\dfrac{1-\cos x}{2}} = \sin\dfrac{x}{2}$

28. $\sqrt{\dfrac{1+\cos x}{2}} = \cos\dfrac{x}{2}$

29. $\dfrac{\cos 2x - \cos 4x}{2\sin 3x} = \sin x$

30. $\cos\left(\mathrm{Cos}^{-1}\dfrac{x}{2}\right) = \cos\left(\dfrac{\mathrm{Cos}^{-1}x}{2}\right)$

31. $2\cos x\cos 3x = (2\cos 2x - 1)(\cos 2x + 1)$ (*Hint:* See Example 4.)

32. $\cot\dfrac{x}{2} = \dfrac{1+\cos x}{\sin x}$

33. $2\sin 2x\cos x = \sin 3x + \sin x$

34. $2\cos 2x\cos x = \cos 3x + \cos x$

35. $20\left(\sin\dfrac{x}{2}\right)^2 = 0.1(0.01x + |0.01x - 100|) - 10\cos x$

36. $20\left(\cos\dfrac{x}{2}\right)^2 = 0.1(0.01x + |100 - 0.01x|) + 10\cos x$

Exercises 37–38 ***Decimal Approximation*** *Use the given information to find θ (in radians) and to get a three-place decimal approximation.*

37. $\sin\theta = 0.36$ and $0 < \theta < \dfrac{\pi}{2}$. Find

(a) θ, **(b)** $\sin\dfrac{\theta}{2}$, **(c)** $\cos 3\theta$.

38. $\cos\theta = -0.65$ and $0 < \theta < \pi$. Find

(a) θ, **(b)** $\tan\dfrac{\theta}{2}$, **(c)** $\csc\dfrac{\theta}{2}$.

Exercises 39–40 ***Exact Form*** *Use the given information to evaluate in exact form.*

39. 2θ is between $\dfrac{3\pi}{2}$ and 2π, and $\cos 2\theta = \dfrac{4}{5}$. Find

(a) $\tan 2\theta$, **(b)** $\sin\theta$, **(c)** $\cos\theta$.

40. 2θ is between $-\dfrac{\pi}{2}$ and 0, and $\cos 2\theta = \dfrac{5}{13}$. Find

(a) $\sin 2\theta$, **(b)** $\sin\theta$, **(c)** $\cos\theta$.

Exercises 41–42 Evaluate in exact form. The indicated identity may be helpful. Check by calculator evaluation.

41. **(a)** $\sin\dfrac{5\pi}{12} + \sin\dfrac{\pi}{12}$; (I-21)

(b) $\cos\dfrac{5\pi}{12} + \cos\dfrac{\pi}{12}$; (I-23)

42. **(a)** $\sin\dfrac{\pi}{12}\cdot\cos\dfrac{\pi}{12}$; (I-18)

(b) $\sin\dfrac{5\pi}{12}\cdot\cos\dfrac{\pi}{12}$; (I-18)

Exercises 43–46 ***Increasing, Decreasing*** ***(a)*** *Determine whether f is increasing decreasing or neither.* ***(b)*** *Give the domain and range of f in exact form.* ***(c)*** *Determine the y-intercept point (exact form). (Hint: Draw a graph of f.)*

43. $f(x) = \tan\left(\dfrac{\mathrm{Cos}^{-1}x}{2}\right)$

44. $f(x) = 1 - \tan\left(\dfrac{\mathrm{Cos}^{-1}x}{2}\right)$

45. $f(x) = \sin\left(\dfrac{\mathrm{Cos}^{-1}x}{2}\right)$

46. $f(x) = \cos\left(\dfrac{\mathrm{Cos}^{-1}x}{2}\right)$

Exercises 47–48 ***Graph Intersection*** *Find the coordinates (1 decimal place) of the point of intersection of the graphs of f and g.*

47. $f(x) = \cos\left(\dfrac{x}{2}\right)$, $g(x) = \mathrm{Cos}^{-1}\left(\dfrac{x}{2}\right)$

48. $f(x) = \sin\left(\dfrac{x}{2}\right)$, $g(x) = \mathrm{Cos}^{-1}\left(\dfrac{x}{2}\right)$

49. Find a piecewise formula for **(a)** $f(x) = (-1)^{[\sin x]}$, **(b)** $g(x) = (-1)^{[\cos x]}$. Here [ ] is the greatest integer function.

50. Draw graphs of $f(x) = (-1)^{[\sin x]}$ and $g(x) = \sin x$ on the same screen. Use dot mode. What do you observe? See Exercise 49.

51. Draw graphs of $f(x) = (-1)^{[\cos x]}$ and $g(x) = \cos x$ on the same screen. Use dot mode. What do you observe? See Exercise 49.

Exercises 52–53 ***Maximum Value*** *Find the maximum value* (2 *decimal places*) *of* f.

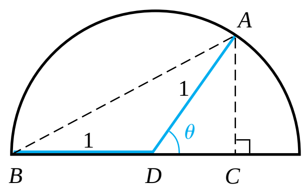
52. $f(x) = 2\cos\left(\dfrac{\mathrm{Cos}^{-1} x}{2}\right) - \sin x.$

53. $f(x) = 2\cos x - \cos\left(\dfrac{\mathrm{Cos}^{-1} x}{2}\right)$

54. We can derive the half-angle tangent formula for $0 < \theta < \frac{\pi}{2}$ from the diagram with a semicircle of radius 1 and center at D and a central angle of θ.

(a) Show that $\angle ABC = \frac{\theta}{2}$, $|\overline{AC}| = \sin\theta$, $|\overline{DC}| = \cos\theta$.

(b) Show that $\tan\dfrac{\theta}{2} = \dfrac{\sin\theta}{1 + \cos\theta}$.



55. If $\sin x + \cos x = u$, then show that

(a) $\sin 2x = u^2 - 1$. (*Hint:* Square both sides.)

(b) $\sin^3 x + \cos^3 x = \frac{1}{2}(3u - u^3)$. (*Hint:* Cube both sides.)

56. A proof of identity (I-18) using (I-6) and (I-7) was given in this section. In a similar manner prove identities (I-19) and (I-20).

Exercises 57–58 ***Product to Sum*** *Use identities (I-18) through (I-20) to express the product as a sum.*

57. **(a)** $\sin 3x \cos 2x$ **(b)** $\cos x \cos 3x$

58. **(a)** $\sin 2x \sin 3x$ **(b)** $\cos 2x \sin 3x$

59. A proof of identity (I-21) was given in this section. In a similar manner prove that (I-23) follows from (I-19).

Exercises 60–61 ***Sum to Product*** *Use identities (I-21) through (I-24) to write the sum or difference as a product.*

60. **(a)** $\sin 2x + \sin 4x$ **(b)** $\cos x + \cos 3x$

61. **(a)** $\sin 3x - \sin x$ **(b)** $\cos 5x - \cos x$

Exercises 62–63 Simplify. (Hint: Use (I-21) through (I-24) to write the numerator and denominator as products.)

62. $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x}$

63. $\dfrac{\sin 4x + \sin x}{\cos 4x + \cos x}$

64. Simplify.

(a) $\cos 20° \cos 40° \cos 80°$. (*Hint:* Use identity (I-19).)

(b) $\sin 10° \sin 50° \sin 70°$. (*Hint:* Use **(a)**.)

65. ***Looking Ahead to Calculus*** Using the substitution $u = \tan\frac{x}{2}$, **(a)** express $\cos x$ and $\tan x$ in terms of u, **(b)** express $\frac{1 - \cos x}{\tan x}$ in terms of u. (*Hint:* See Example 6.)

6.4 SOLVING TRIGONOMETRIC EQUATIONS

> [The] characterization of mathematics as a study of the order and relation of . . . abstract patterns, forms, and structures . . . provides mathematics with root and substance, yet allows for the full range of the intuitive, creative, and aesthetic impulse for which mathematics is so justly renowned.
>
> *W. G. Holladay*

> *I was always interested in everything. Finally, when I entered college, [my father] told me in words of one syllable that I would have to earn my living when I graduated, and I had better make use of my four years to prepare myself for a profession. It was at that point that I decided to become a mathematician.*
>
> Garrett Birkhoff

In Section 1.5 we discussed the notion of a domain D for an equation that involves a variable as the nonempty set of real numbers that can be substituted for the variable to yield a numerical statement that is either true or false. The solution set for the equation consists of all numbers in D that yield true statements.

In the preceding three sections of this chapter we considered equations for which the solution set is the entire domain D. Such equations are called *identities*. Equations for which the solution set is not all of D are called **conditional equations.** In this section we are interested in conditional equations that involve trigonometric functions.

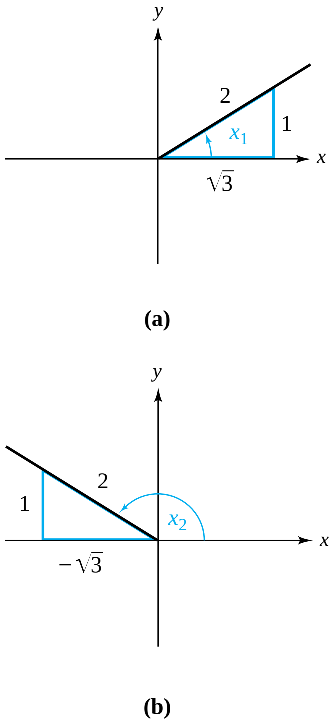


FIGURE 24
$\sin x_1 = \sin x_2 = \frac{1}{2}$

In finding roots of polynomial equations in Chapter 3, we saw that for some equations we are able to find roots in exact form, while for other equations we must settle for approximations. The same observation applies to equations involving logarithmic and exponential functions or trigonometric functions. In the first part of this section we consider equations for which the roots can be expressed in exact form, and then we look at a variety of equations for which we have no tools for exact answers but for which we can nonetheless get good approximations with the aid of technology. Techniques for solving equations involving trigonometric functions can best be explained by considering a number of examples.

Exact Form Solutions

▶***EXAMPLE 1*** ***No domain specified*** Solve the equation $2 \sin x - 1 = 0$.

Solution

With no specified domain, we assume it to be the set of real numbers. The given equation can be written as $\sin x = \frac{1}{2}$. In Figure 24 we show two angles, x_1 and x_2, whose sine equals $\frac{1}{2}$. Each reference triangle is a 30°–60° triangle, so that $x_1 = \frac{\pi}{6}$ and $x_2 = \frac{5\pi}{6}$. The sine function is periodic with period 2π, so we can add any multiple of 2π to either x_1 or x_2. The solution is given by

$$x = \frac{\pi}{6} + 2k\pi, \quad \text{or} \quad x = \frac{5\pi}{6} + 2k\pi,$$

where k is any integer (positive, negative, or zero). ◀

▶***EXAMPLE 2*** ***Limited domain*** Solve the equation $\cos^2 x - 2\cos x - 2 = 0$, where $-\pi \le x \le \pi$. Give the results rounded to two decimal places.

Strategy: The equation is quadratic in $\cos x$. The left side does not factor, so use the quadratic formula.

Solution

Follow the strategy.

$$\cos x = \frac{2 \pm \sqrt{4 + 8}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = \frac{2(1 \pm \sqrt{3})}{2} = 1 \pm \sqrt{3}$$

For $\cos x = 1 + \sqrt{3}$ there is no solution since $1 + \sqrt{3} \approx 2.732$, and $-1 \le \cos x \le 1$ for every x. For $\cos x = 1 - \sqrt{3} \approx -0.732$ we do get solutions as shown in Figure 25. Again we are careful to draw curved arrows, remembering that

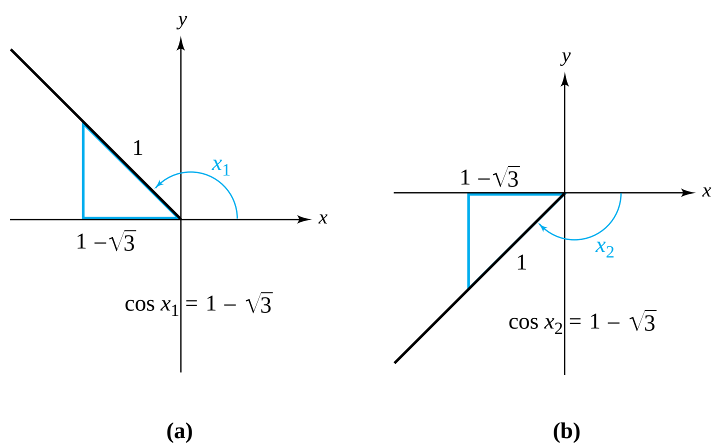


FIGURE 25

$-\pi \le x \le \pi$. From the diagrams it is easy to see that

$$x_1 = \operatorname{Cos}^{-1}(1 - \sqrt{3}) \approx 2.39 \quad \text{and} \quad x_2 = -x_1 = 2.39.$$

Thus, the desired solutions are 2.39 and −2.39. ◀

Strategy: The left side has tan x as a factor. Factor, and then use the zero-product principle.

▶EXAMPLE 3 ***Domain*** $[0, 2\pi]$ Suppose the domain is $[0, 2\pi]$. Find the solution set for the equation $\sin x \cdot \tan x - \tan x = 0$.

Solution

Follow the strategy.

$$(\sin x - 1)\tan x = 0.$$

Therefore, $\tan x = 0$ or $\sin x = 1$. From $\tan x = 0$, $x = 0, \pi, 2\pi$, and from $\sin x = 1$, we get $x = \frac{\pi}{2}$. It would appear that we have four solutions. However, when we check $x = \frac{\pi}{2}$ in the original equation, we recognize that $\tan \frac{\pi}{2}$ is not defined. Hence $x = \frac{\pi}{2}$ is not a solution. Checking the other three numbers we find that they are solutions. The solution set is $\{0, \pi, 2\pi\}$. ◀

Strategy: First simplify the equation. The expression $\sin x \cos x$ brings to mind (I-12): $2 \sin x \cos x = \sin 2x$.

▶EXAMPLE 4 ***Using an identity*** Find the solution set for the equation

$$4 \sin x \cos x - \sqrt{3} = 0 \quad \text{where} \quad -\pi \le x \le \pi.$$

Solution

Following the strategy, the given equation is equivalent to

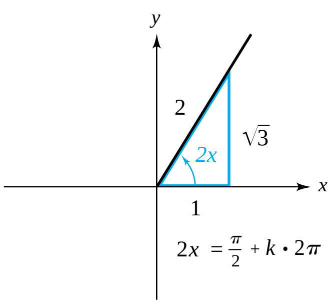
$$2(2 \sin x \cos x) - \sqrt{3} = 0 \quad \text{or} \quad 2 \sin 2x - \sqrt{3} = 0.$$

Thus $\sin 2x = \frac{\sqrt{3}}{2}$. Draw diagrams to show possible values for the angle $2x$ (see Figure 26). From Figure 26(a),

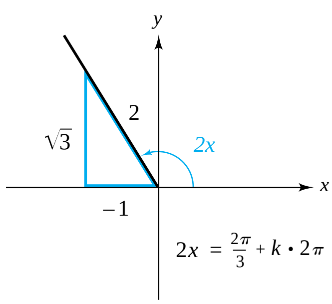
$$2x = \frac{\pi}{3} + k \cdot 2\pi \quad \text{or} \quad x = \frac{\pi}{6} + k \cdot \pi$$

Now pick values for k that give solutions in the interval $[-\pi, \pi]$. When k is -1, x is $-\frac{5\pi}{6}$, and when k is 0, x is $\frac{\pi}{6}$.

In a similar manner, from Figure 26(b), $2x = \frac{2\pi}{3} + k \cdot 2\pi$, or $x = \frac{\pi}{3} + k \cdot \pi$. When k is -1, x is $-\frac{2\pi}{3}$ and when k is 0, x is $\frac{\pi}{3}$. Thus the solution set is $\{-\frac{5\pi}{6}, -\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}\}$. ◀



(a)



(b)

FIGURE 26

$\sin 2x = \frac{\sqrt{3}}{2}$

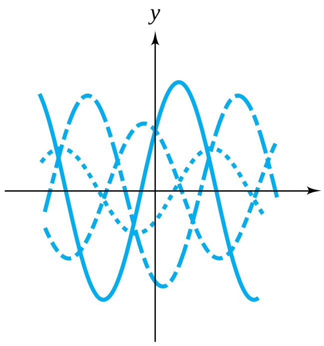
Functions of the Form $f(x) = a \sin x + b \cos x$

Equations of the form $a \sin x + b \cos x = c$ for given number a, b, and c, occur frequently. Before discussing some of the common techniques for solution, we want to get a feeling for the graph of the funcion $f(x) = a \sin x + b \cos x$. We begin by looking at some examples.

▶EXAMPLE 5 $f(x) = a \sin x + b \cos x$ Graph the functions on the same screen, and describe some common features for all four graphs.

(a) $f_1(x) = -\sin x + 2 \cos x$ **(b)** $f_2(x) = \sin x - \cos x$

(c) $f_3(x) = -\sin x - 3 \cos x$ **(d)** $f_4(x) = 3 \sin x + 2 \cos x$



(a) $y = -\sin x + 2\cos x$
(b) $y = \sin x - \cos x$
(c) $y = -\sin x - 3\cos x$
(d) $y = 3\sin x + 2\cos x$

FIGURE 27

Solution

The graphs are shown in Figure 27. All of the graphs look like sine curves (what we called "sinusoidal" in Section 5.4), in that they could belong to the family of sine curves. They all appear as if each could be obtained by shifting and/or dilating the graph of $y = \sin x$. The graphs have different dilation factors and cross the axes at different points, but they all have the same basic shape. ◀

To see that any function of the form

$$f(x) = a \sin x + b \cos x \tag{1}$$

is in fact obtainable as a basic transformation of the sine function, we begin with a *point* and *angle associated* with the function. The point associated with the function in Equation (1) is the point $P(a, b)$, and the associated angle is the angle α (or any of the coterminal angles) containing $P(a, b)$ on its terminal side. See Figure 28. From the diagram in Figure 28 we note that

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \qquad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}. \tag{2}$$

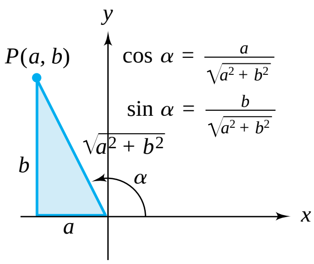


FIGURE 28
Point and angle associated with the function
$f(x) = a\sin x + b\cos x$
$= \sqrt{a^2+b^2}\,\sin(x+\alpha)$.

Having identified the point P and angle α associated with f, we take the equation for f and *multiply and divide* by (or *factor out*) the distance from P to the origin, $\sqrt{a^2+b^2}$. Then rewrite the equation for f as

$$\begin{aligned} f(x) &= \sqrt{a^2+b^2}\left[\sin x \frac{a}{\sqrt{a^2+b^2}} + \cos x \frac{b}{\sqrt{a^2+b^2}}\right] && \text{by (2)} \\ &= \sqrt{a^2+b^2}\,[\sin x\cos\alpha + \cos x \sin\alpha] \\ &= \sqrt{a^2+b^2}\,\sin(x+\alpha). && \text{by (I-6)} \end{aligned}$$

This last equation shows that any function of the form $f(x) = a\sin x + b\cos x$ can also be written in the form $f(x) = \sqrt{a^2+b^2}\sin(x+\alpha)$, whose graph is a transformation of the graph of $y = \sin x$, a horizontal shift (by α) and a dilation (by $\sqrt{a^2+b^2}$). The whole process can be codified as an algorithm.

Algorithm for graphing $f(x) = a\sin x + b\cos x$

Given $f(x) = a\sin x + b\cos x$, where not both of a, b are zero.

1. Draw the associated point $P(a, b)$ on the terminal side of the associated angle α, with $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$.
2. Multiply and divide by $\sqrt{a^2+b^2}$.
3. The equation for f has the form

$$f(x) = \sqrt{a^2+b^2}\,\sin(x+\alpha). \tag{3}$$

▶EXAMPLE 6 $f(x) = a\sin x + b\cos x$ Identify the associated point and angle for the function and write its equation in the form of Equation (3).

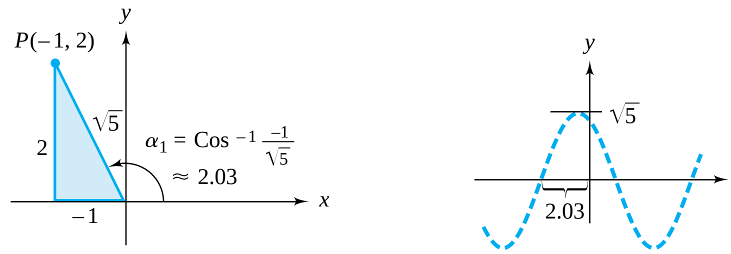
(a) $f_1(x) = -\sin x + 2\cos x$
(b) $f_2(x) = \sin x - \cos x$
(c) $f_3(x) = -\sin x - 3\cos x$

Strategy: Each is a function from Example 5, with graph in Figure 27. For each one follow the steps of the algorithm.

Solution

(a) The point associated with the equation $f_1(x) = -\sin x + 2\cos x$ is $P_1(-1, 2)$ in Figure 29a. From the diagram, the associated angle is $\alpha_1 = \mathrm{Cos}^{-1}\left(-\frac{1}{\sqrt{5}}\right) \approx 2.03$. Multiplying and dividing by $\sqrt{5}$, we have

$$f_1(x) = \sqrt{5}\left(\sin x\left(\frac{-1}{\sqrt{5}}\right) + \cos x\left(\frac{2}{\sqrt{5}}\right)\right) = \sqrt{5}\,(\sin x \cos \alpha_1 + \cos x \sin \alpha_1)$$
$$= \sqrt{5}\sin(x + \alpha_1) \approx \sqrt{5}\sin(x + 2.03).$$



(a) **(b)** $f_1(x) = -\sin x + 2\cos x = \sqrt{5}\sin(x + 2.03)$

FIGURE 29

In Figure 29b we show the graph of f_1, identifying the horizontal shift and dilation of $y = \sin x$.

(b) and **(c)** The functions are as follows.

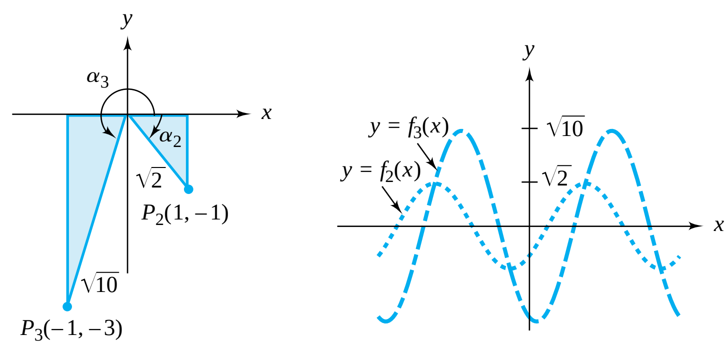
$f_2(x) = \sin x - \cos x$: associated point $P_2(1, -1)$ and angle

$$\alpha_2 = -\frac{\pi}{4} \approx -0.785, \text{ and } \sqrt{a^2 + b^2} = \sqrt{2}.$$

$f_3(x) = -\sin x - 3\cos x$: associated point $P_3(-1, -3)$ and angle

$$\alpha_3 = \pi + \mathrm{Sin}^{-1}\left(\frac{1}{\sqrt{10}}\right) \approx 3.46, \text{ and } \sqrt{a^2 + b^2} = \sqrt{10}.$$

See Figure 30a. For f_2 we multiply and divide by $\sqrt{2}$ and for f_3 by $\sqrt{10}$.



(a) **(b)**

FIGURE 30

$$f_2(x) = \sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$$
$$f_3(x) = -\sin x - 3\cos x = \sqrt{10}\sin(x + 3.46)$$

In the form of Equation (3) we have

$$f_2(x) = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right) \quad \text{and}$$
$$f_3(x) = \sqrt{10}\sin(x + \alpha_3) \approx \sqrt{10}\sin(x + 3.46).$$

The graphs are shown in Figure 30b. ◀

Equations of the Form $a \sin x + b \cos x = c$

With the algorithm above, we have the tools we need for solving equations of the form $a \sin x + b \cos x = c$. Changing the form from $f(x) = a \sin x + b \cos x$ to $f(x) = \sqrt{a^2 + b^2}\sin(x + \alpha)$ gives an equation, one we have already solved earlier in this section, and it also allows us to tell whether or not we can find a convenient exact form solution.

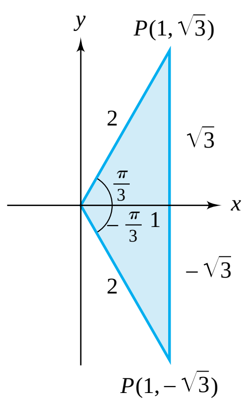


FIGURE 31

▶EXAMPLE 7 *$a \sin x + b \cos x = c$* Solve the equation in exact form.

(a) $\sin x + \sqrt{3}\cos x = 1$ **(b)** $\sin x - \sqrt{3}\cos x = 0$

Solution

(a) With $a = 1$, $b = \sqrt{3}$, the associated point and angle are shown in Figure 31. The algorithm shows how to write $f(x) = \sin x + \sqrt{3}\cos x$ in the form of Equation (3), $f(x) = 2\sin\left(x + \frac{\pi}{3}\right)$. The given equation is equivalent to

$$2\sin\left(x + \frac{\pi}{3}\right) = 1, \quad \text{or} \quad \sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}.$$

The two angles between 0 and 2π whose sine is $\frac{1}{2}$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. Therefore $x + \frac{\pi}{3}$ must be coterminal with one of $\frac{\pi}{6}$ or $\frac{5\pi}{6}$.

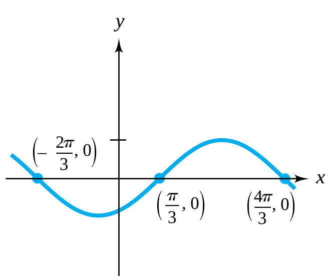
$$x + \frac{\pi}{3} = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad x + \frac{\pi}{3} = \frac{5\pi}{6} + 2k\pi.$$

$$x = -\frac{\pi}{6} + 2k\pi \quad \text{or} \quad x = \frac{\pi}{2} + 2k\pi.$$

(b) The point associated with $\sin x - \sqrt{3}\cos x$ is $P(1, -\sqrt{3})$ with $\alpha = -\frac{\pi}{3}$ as in Figure 31. Following the steps of the algorithm, the equation is equivalent to

$$2\sin\left(x - \frac{\pi}{3}\right) = 0, \quad \text{or} \quad \sin\left(x - \frac{\pi}{3}\right) = 0.$$

The graph of $y = \sin(x - \frac{\pi}{3})$ is the core sine curve shifted $\frac{\pi}{3}$ units to the right, as in Figure 32. The graph crosses the x-axis at $x = \frac{\pi}{3} \pm k\pi$, for every integer k, which thus gives all solutions to the original equation.



$y = \sin\left(x + \frac{\pi}{3}\right)$

FIGURE 32

Alternate Solution Equations of the form $a \sin x + b \cos x = 0$ (where the right side is zero) can be rewritten in a form that does not require the algorithm. In this case, if $\sin x - \sqrt{3}\cos x = 0$, then $\sin x = \sqrt{3}\cos x$. Dividing by $\cos x$, we have the simpler equation $\tan x = \sqrt{3}$, which has the same solution, $x = \frac{\pi}{3} \pm k\pi$, for any integer k. ◀

Using Graphs to Find Roots

In general, it is not possible to find roots of a trigonometric equation in exact form, but graphical techniques allow us to find excellent approximations.

TECHNOLOGY TIP ◆ ***Graphical solutions again***

To solve an equation $f(x) = g(x)$, where f and g are functions, we can use either of two approaches. The second is based on the fact that the equation $f(x) = g(x)$ is equivalent to the equation $f(x) - g(x) = 0$.

1. Draw calculator graphs of Y1 = f(X), Y2 = g(X) on the same screen. The x-coordinates of the points of intersection are the roots of the equation. Trace and zoom as needed to get the desired degree of accuracy. If your calculator has built-in routines to find intersections, you can get more precision. It is usually necessary to move the cursor near to the intersection to give the calculator a good starting point for its approximation routine.

2. Draw a calculator graph of Y = f(X) − g(X). The x-coordinates of the x-intercept points are the roots of the equation. Calculators with routines to find intersections usually also have root-finding routines.

►***EXAMPLE 8*** ***Using graphs*** In Example 8 of Section 5.4 we saw that the equation $\cos x = \frac{x}{4}$ has three roots. Use graphs to find the roots to 2 decimal place accuracy.

Solution

Graphing $y = \cos x$ and $y = \frac{x}{4}$ in the same window (as, say, a decimal window) shows something like the curves in Figure 33. The x-coordinates of the intersections are approximately -3.60, -2.13, and 1.25. ◄

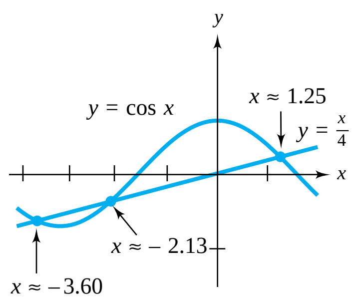


FIGURE 33

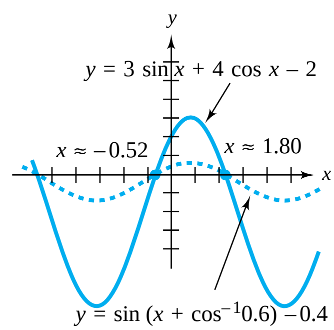


FIGURE 34

►***EXAMPLE 9*** ***Using graphs*** Find the smallest positive root and the largest negative root of the equation (2 decimal place accuracy).

(a) $3 \sin x + 4 \cos x = 2$ **(b)** $\sin(x + \mathrm{Cos}^{-1} 0.6) = 0.4$

Solution

(a) Graphing $y = 3 \sin x + 4 \cos x - 2$ shows something like Figure 34. The first x-intercept point to the right of the origin (the smallest positive root) is located at about $x = 1.80$. The first root to the left is at about $x = -0.52$.

(b) The graph of $y = \sin(x + \mathrm{Cos}^{-1} 0.6) - 0.4$ looks very much like a dilation of the graph of $y = 3 \sin x + 4 \cos x - 2$ in part (a), and the zeros appear to be

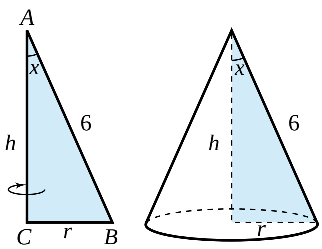
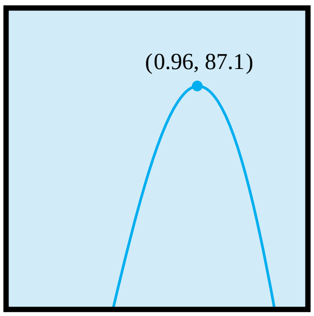
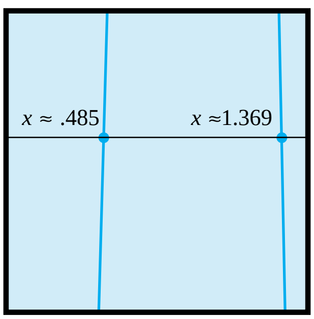


FIGURE 35



[0, 1.5] by [50, 100]
(a)



[0, 1.5] by [40, 46]
(b)

FIGURE 36
$y = 72\pi \sin^2 x \cos x$

the same, $x \approx 1.80, -0.52$. To see if the zeros are identical, we can apply the algorithm above to express $f(x) = 3 \sin x + 4 \cos x - 2$ in the form of Equation (3). The associated point $P(3, 4)$ is 5 units from the origin on the terminal side of the angle $\alpha = \text{Cos}^{-1}\frac{3}{5}$. Multiplying and dividing by 5 gives

$$\begin{aligned} f(x) = 3 \sin x + 4 \cos x - 2 &= 5\left(\frac{3}{5} \sin x + \frac{4}{5} \cos x - \frac{2}{5}\right) \\ &= 5(\sin(x + \alpha) - 0.4), \quad \text{where } \alpha = \text{Cos}^{-1} 0.6. \end{aligned}$$

Thus the graph of $y = 3 \sin x + 4 \cos x - 2$ is a dilation, by a factor of 5, of the graph of $y = \sin(x + \text{Cos}^{-1} 0.6) - 0.4$, and both graphs have exactly the same x-intercept points. ◀

▶EXAMPLE 10 ***Volume of a cone*** Figure 35 shows a right triangle with hypotenuse of 6 inches and $\angle BAC = x$. If the triangle is rotated about the line AC we get a cone of radius r and height h. The volume V of the cone depends on the size of angle x. When x is small, we get a skinny cone of small volume, and when x is near $\frac{\pi}{2}$, the cone is very flat and has little volume.

(a) Express V as a function of x and find the value of x for which V is a maximum. What is the maximum volume?
(b) What values of x give a cone of half the maximum volume?

Solution

(a) The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$, so we need to see how r and h depend on the angle x. In $\triangle ABC$, we have $\sin x = \frac{r}{6}$ and $\cos x = \frac{h}{6}$, so $r = 6 \sin x$, $h = 6 \cos x$. Substituting these values for h and r,

$$V = \frac{\pi}{3}(6 \sin x)^2 (6 \cos x) = 72\pi \sin^2 x \cos x.$$

We want to graph Y = 72π(SIN X)^2 COS X in an appropriate window. From the nature of the problem, x is between 0 and $\frac{\pi}{2}$, so we take an x-range of $[0, 1.5]$. Sampling values of the volume suggests that a y-range of $[50, 100]$ should show us the maximum. We get the graph in Figure 36a. Tracing to find the high point, we find that $V_{max} \approx 87.1$, where $x \approx 0.96$ radians, or just over 54°.

(b) Since half of V_{max} is about 43.5, we add the horizontal line $y = 43.5$ to our graph and find where the curves intersect. In Figure 36b we take a y-range of $[40, 46]$. The two intersections occur where $x \approx 0.485$ or 1.369; that is, just a little less than 28° or just over 78°. ◀

EXERCISES 6.4

Check Your Understanding

Use graphs whenever they might be helpful.

Exercises 1–5 True or False. Give reasons.

1. The equation $\sin x + 1 = 0$ has two solutions in the interval $[0, 2\pi]$.
2. The solution set for the equation $\tan^2 x + 1 = 0$ is the empty set.
3. The solution set for $\sin x \cos x = 1$ is the empty set.
4. The solution set for $\sec^2 x - 2 = 0$ is the same as that for $\tan^2 x - 1 = 0$.
5. The functions $f(x) = \sin x + \cos x$ and $g(x) = 3 \sin\left(x + \frac{\pi}{4}\right)$ have the same zeros.

Exercises 6–10 Fill in the blank so the result will be a true statement.

6. The graphs of $y = 2 \cos x$ and $y = -0.5x$ intersect in Quadrant(s) ______.

7. A number in the interval $[0, \pi]$ that is in the solution set for $\cos x(2 - \sin x) = 0$ is ______.

8. The number of points of intersection of the graphs of $f(x) = 2 \cos x$ and $g(x) = x^2 - 2x - 3$ is ______.

9. The smallest positive number satisfying $\sin x = \cos x$ is ______.

10. The number of zeros for $f(x) = \mathrm{Cos}^{-1} x - \cos x$ is ______.

Develop Mastery

Exercises 1–18 ***Exact Form, Restricted Domain*** *Solve. Give answers in exact form. Use identities as needed. The domain is the interval $[0, 2\pi]$. Use graphs as a check.*

1. $2 \cos x - 1 = 0$

2. $\sec x + 2 = 0$

3. $2 \cos\left(x - \dfrac{\pi}{3}\right) - 1 = 0$

4. $3 \tan x + \sqrt{3} = 0$

5. $\tan\left(x + \dfrac{\pi}{3}\right) - \sqrt{3} = 0$

6. $2 \sin^2 x + \sin x = 0$

7. $2 \sin^2 x - 1 = 0$

8. $4 \cos^2 x + 4 \cos x + 1 = 0$

9. $2 \cos^2 x - 5 \cos x + 2 = 0$

10. $\sin^2 x - 2 \sin x + 1 = 0$

11. $\cos x \cdot \tan x + \sin x = 1$

12. $\cos\left(x + \dfrac{\pi}{2}\right) - \sin x = 1$

13. $\cos^2 x + \cos x = 1 - \sin^2 x$

14. $\sin^2 x + 2 \sin x = 2 - \cos^2 x$

15. $\sin^2 x + \sin x - \cos^2 x = 0$

16. $\sin x - \sqrt{3} \cos x = 0$

17. $\sin 2x - \cos x = 0$

18. $\sin 2x + \sin x = 0$

Exercises 19–24 ***Exact Form, Restricted Domain*** *Solve the equation where the domain is the interval $[-\pi, \pi]$. Give answers in exact form.*

19. $2 \sin x - \sqrt{3} = 0$

20. $\tan\left(x - \dfrac{\pi}{6}\right) - 1 = 0$

21. $2 \cos\left(x + \dfrac{2\pi}{3}\right) - 1 = 0$

22. $\cos^2 x + 2 \cos x + 1 = 0$

23. $\sqrt{3} \tan^2 x + 2 \tan x - \sqrt{3} = 0$

24. $\sin x \tan x + \tan x = 0$

Exercises 25–30 ***Domain Not Restricted*** *No domain is specified. Find the solution set. Give answers in exact form.*

25. $2 \cos x - \sqrt{2} = 0$

26. $\tan^2 x - 3 = 0$

27. $2 \cos^2 x + 5 \cos x + 2 = 0$

28. $2 \cos^2 x + \cos x = 0$

29. $2 \mathrm{Sin}^{-1}(\sin x) - \pi = 0$

30. $\mathrm{Cos}^{-1}(\sin x) - \pi = 0$

Exercises 31–34 ***Use Algorithm*** *The domain is the interval $[0, 2\pi]$. Solve using the algorithm given in this section.*

31. $\sin x - \cos x = 1$

32. $\sqrt{3} \sin x + \cos x = 2$

33. $\sin x + \sqrt{3} \cos x = 2$

34. $-\sqrt{2} \sin x + \sqrt{2} \cos x = 1$

Exercises 35–38 Solve the equation in exact form where the domain is $[0, 2\pi]$. Sum-to-product identities from Section 6.3 may be useful.

35. $\sin 3x + \sin x = 0$

36. $\cos 3x - \cos x = 0$

37. $\sin 3x = \sin x$

38. $\sin 4x - \sin 2x = \sin x$

Exercises 39–48 ***Decimal Approximations*** *Use the Technology Tip in this section to find 1 decimal place approximations for the roots of the equation. The domain is $[-\pi, \pi]$.*

39. $\cos x = x^2$

40. $\sin x = x^2 - 1$

41. $e^{\sin x} = x^2 - x$

42. $4 \cos x + x = 0$

43. $3 \cos x = \sin x$

44. $2 \mathrm{Sin}^{-1}(\sin x) = 2$

45. $2 \cos(\mathrm{Sin}^{-1} x) = 1$

46. $3 \cos(x - \mathrm{Sin}^{-1} 0.4) = 2$

47. $\cos x - 2 \sin x = 0$

48. $\cos 2x + 2 \cos x = 0$

Exercises 49–54 ***Decimal Approximations*** *Find 1 decimal place approximations for the roots of the equation, where the domain is $[0, \pi]$. Use any technique you wish.*

49. $3 \sin x - 2 = 0$

50. $\sin^2 x + 2 \sin x = 3$

51. $\sin^2 x + 2 \sin x = 2$

52. $\tan^2 0.5x - 2 \tan 0.5x = 3$

53. $\cot^2 0.5x - 2 \cot 0.5x = 2$

54. $\sec^2 x + 4 \sec x = -2$

Exercises 55–58 ***Looking Ahead to Calculus*** *In calculus, functions f and g of these exercises are related; for the given function f, function g is called the derivative of f. Assume* $0 < x < \pi$. ***(a)*** *Draw a graph of f; use it to find the coordinates of the local maximum or minimum point P(1 decimal place).* ***(b)*** *On the same screen draw a graph of g and find the zero, c, of g. Evaluate f(c) and let Q be the point (c, f(c)). How are P and Q related?*

55. $f(x) = e^{\sin x}$ $\quad g(x) = (\cos x)e^{\sin x}$

56. $f(x) = 1 - x \sin x$ $\quad g(x) = -\sin x - x \cos x$

57. $f(x) = x \sin x$ $\quad g(x) = \sin x + x \cos x$

58. $f(x) = 2 \sin x - x$ $\quad g(x) = 2 \cos x - 1$

Exercises 59–63 ***Restricted Domain*** *Solve algebraically (2 decimal places) and check graphically. Assume the domain is* $[0, 2]$.

59. $\sin \pi x + \cos \pi x = 0$

60. $\sin \pi x + \cos \pi x = 1$

61. $\sin 2\pi x - \sin \pi x = 0$

62. $\cos 2\pi x + \cos \pi x = 0$

63. $e^{\ln(\cos \pi x)} - \sin \pi x = 0$

Exercises 64–67 ***(a)*** *Express in the form* $f(x) = A \sin (x + \alpha)$. ***(b)*** *Find the largest value that f(x) can assume and* ***(c)*** *find all values of x in* $[0, 2\pi]$ *that yield this maximum value of f(x).*

64. $f(x) = \sin x + \sqrt{3} \cos x$

65. $f(x) = \sin x - \cos x$

66. $f(x) = 2 \sin x + 2 \cos x$

67. $f(x) = \sqrt{3} \sin x - \cos x$

68. Determine the smallest positive root, in exact form, of $8 \sin x \cos x(\cos^4 x - \sin^4 x) = \sqrt{3}$.

69. Find the smallest number in the interval $(50, \infty)$ that satisfies the equation $\sin x - 1 = 0$.

70. Find the smallest number in the interval $[10\pi, 40\pi]$ that satisfies the equation $2 \cos x + 1 = 0$.

71. Find the largest number in the interval $[10\pi, 50\pi]$ that satisfies the equation $\tan x + 1 = 0$.

72. Find the largest number in the interval $(-\infty, -40)$ that satisfies the equation $3 \sin x - 1 = 0$.

Exercises 73–76 *Let S denote the solution set, where D is the domain. Find* ***(a)*** *the smallest integer that is greater than all of the numbers in S, and* ***(b)*** *the largest integer that is less than all of the numbers in S.*

73. $2 \sin x - 1 = 0$; $D = [0, 2\pi]$

74. $\sin 2x - 1 = 0$; $D = [-2\pi, 2\pi]$

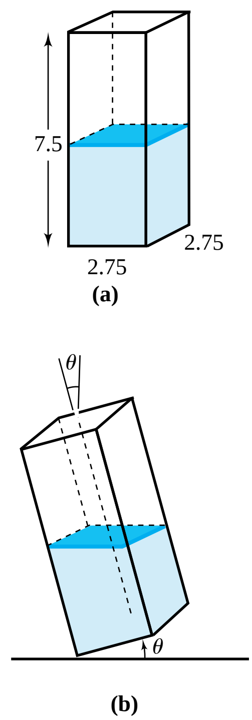
75. $\sin^2 x = 1 - \cos^2 x$; $D = [-\pi, 4\pi]$

76. $\sin x = 2 \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$; $D = [-\pi, 2\pi]$

77. Solve the problem in Example 10 if $|\overline{AB}| = 9$.

78. Solve the problem in Example 10 if triangle ABC is rotated about $\overline{BC}$ to get a cone.

79. A one-quart milk carton has a square bottom and top measuring 2.75 by 2.75 inches. The carton is 7.5 inches tall. Suppose the carton is half full of milk. When it sits upright on a table, the top surface of the milk is a square. However, when the carton is tipped along one of its bottom edges so that the bottom of the carton makes an angle θ with the table, the surface becomes a rectangle (see the diagram). Assume that if the carton is tipped far enough for the milk to reach the top, the milk will spill out.



(a) Find an equation that gives the area A of the rectangle as a function of θ and that is valid up to the time the milk spills.

(b) What is the domain of the function?

6.5 WAVES AND GENERALIZED SINE CURVES

> Mathematical models of biological and social phenomena have traditionally relied on the paradigm of classical physics in the development of their mathematical formalisms. The potency of this paradigm lies in the ability of classical physics to relate cause and effect . . . through a sequence of formal implications and thereby to make predictions.
>
> *B. J. West*

The sine curve is the prototype of the *sinusoidal (wave-like) function.* A sine graph is what we think of as an ideal wave, oscillating and repeating. Wave motion is literally all around us, all the time.

We drop a pebble into a still pond and observe the wave traveling outward in concentric circles; a cross-section would resemble a sine curve. Sound is carried to our ears by pressure waves in the atmosphere; some kind of vibrator sets the air in motion, initiating the sound waves whose variations are received by the elaborate sensing mechanisms of our ears. Electromagnetic radiation is propagated as waves, from alternating current to radio and television signals, through microwaves in our kitchens, through the visible light spectrum on to x-rays and atomic radiation.

> *I think the earliest I remember my father telling me something mathematical was when I was beginning to study Euclidean geometry at school. At that time he also told me about Cartesian geometry. I must have been 12 or 13.*
>
> Cathleen Morawetz

We distinguish varieties of wave motion by two features, their **frequency** and **wavelength,** both of which measure the same kind of behavior (called the **period**) of sinusoidal functions. Count the number of pulses hitting your eardrum or entering your retina in a second and you get the frequency, in units of *cycles per second,* now called *hertz (Hz).* The speed of your computer may be given in megahertz. Measure the distance between successive wave crests and you have the wavelength. For example red light has a different wavelength than blue or green light. Wave motion can also vary in **amplitude,** which is a measure of the intensity that has nothing to do with the frequency or wave-length. A tone of a given frequency can be loud (greater amplitude) or soft (smaller amplitude), or we see a given color as bright or dim.

The generalized sine curve, $y = A\sin(Bx + C)$, gives us tools to model all these phenomena, and many more of the things going on in the physical world. The numbers A, B, C are called **parameters** and are related to the amplitude, frequency, and wavelength of the sine curve. Our goal in this section is to understand the generalized sine curve and the role of each of these parameters in graphs and some applications. Since every cosine curve may be realized as a basic transformation of a sine curve, we usually include both together.

Generalized sine (and cosine) curves

The graph of any function that be written in either of the forms

$$f(x) = A\sin(Bx + C) \quad \text{or} \quad g(x) = A\cos(Bx + C),$$

where A and B are nonzero, is called a ***generalized sine curve.***

The numbers A, B, C affect the amplitude and period and are called *parameters.*

Core Sine and Cosine Curves

Before continuing, we remind you how essential it is to know the graphs of $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$. For convenient reference we reproduce them here, but you want to be able to sketch them both in rough form at any time,

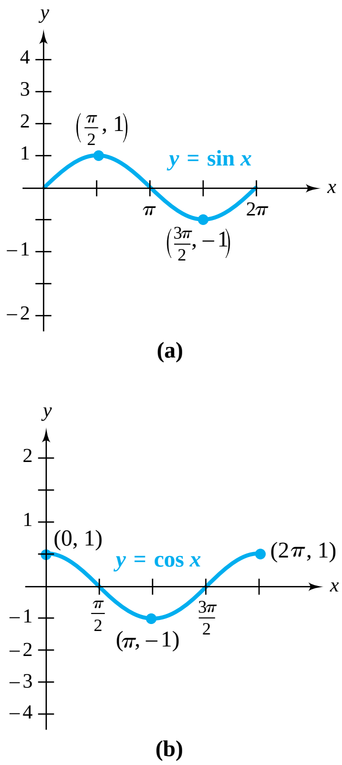


FIGURE 37 Core graphs for $\sin x$ and $\cos x$.

at the top of a page of homework, or whenever you are working with trigonometric functions. We call the graphs of $y = \sin x$ and $y = \cos x$ *core graphs*. They are building blocks from which we can construct any generalized sine curve. The same graphs are also called *fundamental cycles* of the sine and cosine curves. Any portion of a generalized sine curve that corresponds to one of the graphs in Figure 37 is a **fundamental cycle** of the curve. Once we have a fundamental cycle for any generalized sine curve, the rest of the graph simply repeats the fundamental cycle. The parameters A, B, C affect the size, length, and location of a fundamental cycle.

Amplitude (Vertical Dilation, the Parameter *A*)

We are familiar with the fact that multiplication of a function by a nonzero constant dilates the graph vertically. Since core sine and cosine curves oscillate between 1 and -1, multiplying by A affects the magnitude of oscillation; $|A|$ measures the distance the graph of a generalized sine curve reaches from the x-axis about which it oscillates, and hence the maximum value of the function. We call $|A|$ the amplitude of the function. You should see for yourself, with your graphing calculator, the effect of changing A.

▶EXAMPLE 1 ***Amplitude*** Sketch a graph of the function.

(a) $y = 3 \sin x$ **(b)** $y = \frac{1}{2} \sin x$ **(c)** $y = -2 \sin x$

Solution

Graphing the first two functions on the same screen with $y = \sin x$, and $y = -2 \sin x$ with $y = \sin x$ should yield diagrams such as in Figure 38. Observe that the graph of $y = -2 \sin x$ shows a reflection in the x-axis as well as a vertical dilation by a factor of 2. ◀

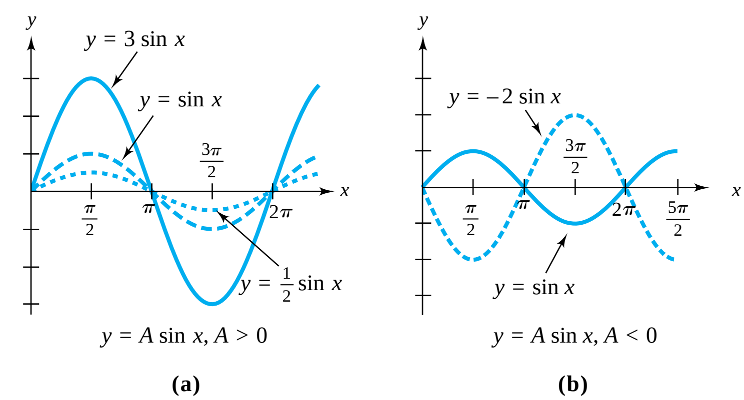


FIGURE 38

Period (Horizontal Dilation, the Parameter *B*)

Multiplying the argument of a function by a nonzero constant dilates a graph horizontally. Thus the graph of $y = \sin Bx$ is a horizontal dilation, stretching or compressing the graph of $y = \sin x$. Again, you should draw a number of graphs yourself until you have a good feeling for the graph of any curve of the form $y = \sin Bx$.

▶EXAMPLE 2 ***Horizontal dilation*** Sketch a graph of the function. Identify a fundamental cycle of the graph (corresponding to the core graph in Figure 38a).

(a) $y = \sin 3x$ **(b)** $y = \sin \frac{1}{2}x$ **(c)** $y = \sin(-2x)$

Solution

Graphing the three functions separately gives the graphs in Figure 39.

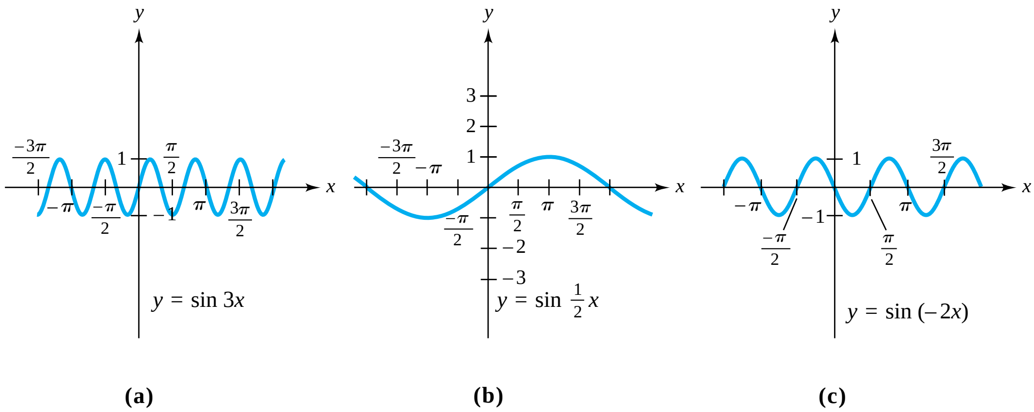


FIGURE 39

(a) The graph oscillates three times as fast as the core sine curve, compressing three cycles into the interval $[0, 2\pi]$. A complete fundamental cycle occurs in the interval $[0, \frac{2\pi}{3}]$ (of length $\frac{2\pi}{3}$).

(b) Since $B(= \frac{1}{2})$ is less than 1, the curve is stretched horizontally, so that it now takes an interval of length $4\pi\left(= \dfrac{2\pi}{\frac{1}{2}}\right)$ for a fundamental cycle. A fundamental interval is $[0, 4\pi]$.

(c) The graph of $y = \sin(-2)x$ shows a reflection as well as a horizontal compression. We can see a fundamental cycle tipped upside down (reflected in the x-axis) on the interval $[0, \pi]$, or we can see a fundamental cycle as shifted to the right, on the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$. Both are correct. Using identity (I-3), we can write $\sin(-2x) = -\sin 2x$, so that the graph of $y = \sin(-2x)$ is a vertical reflection of the graph of $y = \sin 2x$. ◀

In the light of the discussion of Example 2c, it is convenient to *assume that the parameter B is positive* because we can always apply (I-3) before graphing:

$$\sin(-B)x = -\sin Bx, \quad \text{and} \quad \cos(-B)x = \cos Bx.$$

Since the parameter B determines the horizontal dilation, it provides a handy measure of the rapidity or **period,** of oscillation. The length of a fundamental cycle is also the distance between successive wave crests of a sine curve, so the period is the same as the wavelength. The frequency of a sine wave is measured in cycles (or periods) per second and hence is also determined by the parameter B.

The period of the core graph, $y = \sin x$, is the length of the fundamental cycle, 2π. A fundamental cycle of $y = \sin Bx$ is dilated to length of $\frac{2\pi}{B}$, so the period is given by $p = \frac{2\pi}{B}$. If $B > 1$, the fundamental cycle is *compressed;* if $0 < B < 1$, then the fundamental cycle is *stretched.*

Phase Shift (Horizontal Shift, the Parameter *C*)

The graph of $y = f(x \pm c)$ is a horizontal shift of the graph of $y = f(x)$. Thus the graph of $y = \sin(x + 1)$ shifts the sine curve 1 unit left; the graph of $y = \sin(x - \pi)$ is a shift π units to the right, as in Figure 40. Because the sine

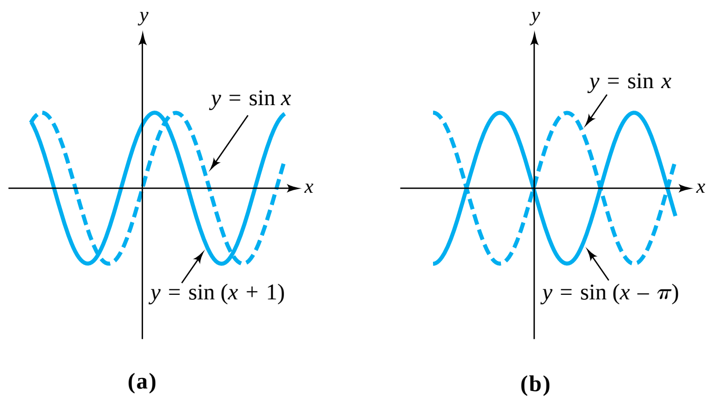


FIGURE 40

curve continues to oscillate in both directions, it helps to look at the shift of a fundamental cycle of the core graph. A fundamental cycle of $\sin(x + 1)$ occurs in the interval $[-1, 2\pi - 1]$, while $\sin(x - \pi)$ has a fundamental cycle in $[-\pi, \pi]$ or in $[\pi, 3\pi]$. The amount of horizontal shift in a generalized sine curve is called the **phase shift.**

The Graph of a Generalized Sine Curve

The parameters B and C together determine the size and location of a fundamental cycle of the curve $y = A\sin(Bx + C)$. As the argument of the sine (or cosine) goes from 0 to 2π, we observe all of the critical behavior of the function. The same is true of any generalized sine function. As the argument $Bx + C$ goes from 0 to 2π, we generate a fundamental cycle giving all the information needed for sketching the graph. The interval containing a fundamental cycle is called a **fundamental interval** and is obtained by solving the inequalities (remember that $B > 0$)

$$0 \le Bx + C \le 2\pi, \quad \text{or} \quad -\frac{C}{B} \le x \le \frac{2\pi - C}{B}.$$

The length of the fundamental interval is the period of the function, $\frac{2\pi}{B}$. Writing the equation for the function in the form

$$y = A\sin B\left(x + \frac{C}{B}\right),$$

identifies the **phase shift,** the horizontal shift to the point where a fundamental cycle begins, as $-\frac{C}{B}$. (Sometimes people refer to the distance from the origin to the beginning of fundamental cycle as the phase shift, in which case they would define the phase shift as $\frac{|C|}{B}$.)

The information discussed so far is put together in the box that follows. The outlined procedure can be used either for sketching a graph by hand or for getting exact information from a calculator graph.

Graph of a generalized sine curve

1. Express equation in the form $y = A\sin(Bx + C)$, where B is positive, using identity (I-3) if needed.
2. Find a fundamental interval (FI) by solving the inequalities
$$0 \le Bx + C \le 2\pi, \text{ so a fundamental interval is } \left[-\frac{C}{B}, \frac{2\pi - C}{B}\right].$$
3. A fundamental cycle for the curve is obtained by drawing a core sine curve on the fundamental interval, dilated vertically by the factor $|A|$ (and reflected in the x-axis if A is negative).
4. The entire graph is a repetition of a fundamental cycle. The function has period, amplitude, and phase shift given by:

Period (or wavelength)	$p = \dfrac{2\pi}{B}$
Amplitude	$\lvert A \rvert$
Phase shift	$\dfrac{C}{B}$.

5. Because of the shape of the core sine curve, the generalized sine curve has a local maximum point one-quarter (and a local minimum three-quarters) of the way across the fundamental interval if A is positive (vice-versa if A is negative). The general form of the graph is shown in Figure 41.

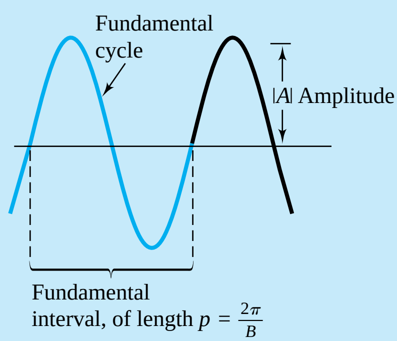
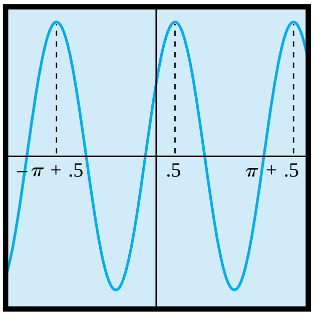


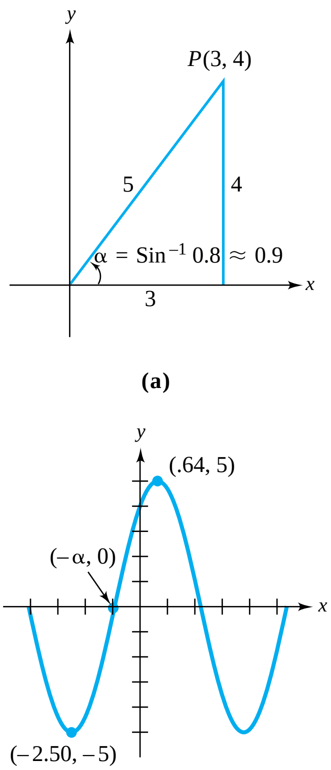
FIGURE 41
Generalized sine curve, $y = A\sin(Bx + C)$.

Graph of $y = A\cos(Bx + C)$. As mentioned above, since the graph of any cosine curve is a horizontal shift of a sine curve, we include functions of the form $y = A\cos(Bx + C)$ in the family of generalized sine curves. We get a fundamental interval in precisely the same way and get the graph by drawing a core cosine curve on the fundamental interval, dilating by $|A|$. The adjustments for locating local extrema should be obvious.



$[-4, 4]$ by $[-4.5, 4.5]$

FIGURE 42
$y = 4\cos(2x - 1)$



$y = 3\sin x + 4\cos x$

(b)

FIGURE 43

▶EXAMPLE 3 ***A generalized sine curve*** Sketch a graph of the function

$$f(x) = 4\cos(2x - 1).$$

Find the smallest positive x-coordinate for which the function has the value 4, and identify a fundamental interval, period, and phase shift. Describe how the graph is obtained from a core graph by basic transformations.

Solution

A calculator graph is shown in Figure 42. The amplitude is 4. Without just the right window, tracing along a calculator graph won't show a y-coordinate of 4, but we know that $\cos u = 1$ when $u = 0$. Thus $f(x) = 4$ when $2x - 1 = 0$, or $x = .5$. Solving $0 \le 2x - 1 \le 2\pi$, we get $.5 \le x \le \pi + .5$, so the fundamental interval is $[.5, \pi + .5]$. The period is $\frac{2\pi}{2}$, or π. Writing $f(x) = 4\cos(2x - 1) = 4\cos 2(x - .5)$, a fundamental cycle begins at $x = 0.5$, for a phase shift of 0.5.

There are several equivalent ways to describe how the graph of f is related to the core cosine curve. For one, the graph of $y = \cos 2x$ is the core graph compressed horizontally so that we have a full cycle in the interval $[0, \pi]$. Multiplying by 4 stretches the graph vertically. Then to get the graph of $y = 4\cos 2(x - 0.5)$, shift the graph of $y = 4\cos 2x$ to the right 0.5 units. ◀

▶EXAMPLE 4 ***Another generalized sine curve*** Let $f(x) = 3\sin x + 4\cos x$.

(a) Use the algorithm from Section 6.4 (page 353) to rewrite the equation in the form of a generalized sine curve.
(b) Find the amplitude, period p, and a fundamental interval (FI).
(c) Find the x-coordinates of the local extrema nearest the origin.

Solution

(a) The point and angle associated with f are $P(3, 4)$ and $\alpha = \text{Sin}^{-1}.8 \approx 0.9273$ (see Figure 43a). Multiplying and dividing by 5 gives

$$f(x) = 5\left(\frac{3}{5}\sin x + \frac{4}{5}\cos x\right) = 5\sin(x + \alpha).$$

A graph is shown in Figure 43b.

(b) The amplitude is 5 and the period is given by $p = \frac{2\pi}{1} = 2\pi$. For the FI, solve $0 \le x + \alpha \le 2\pi$ for x, giving $[-\alpha, 2\pi - \alpha]$ for the number $\alpha = \text{Sin}^{-1}.8$.

(c) The local maximum nearest the origin is located one-fourth of the way across the FI, at $x = -\alpha + \frac{\pi}{2} \approx 0.6435$; the desired local minimum is $\frac{\pi}{2}$ units to the left of the FI, at $x = -\alpha - \frac{\pi}{2} \approx -2.4981$. See graph. ◀

Simple Harmonic Motion

An important class of physical phenomena is characterized by oscillatory behavior. Among such phenomena are all kinds of spring and wave motion—sound waves and electromagnetic waves (from light waves through radio waves, infrared, to X-rays). This oscillatory behavior is called **simple harmonic motion** and is modeled by generalized sine functions.

HISTORICAL NOTE THE ROSETTA STONE AND FOURIER SERIES

When Napoleon led a military expedition to Egypt in 1798, his forces included the mathematican Joseph Fourier. Napoleon's troops discovered a large slab of polished stone covered with three different kinds of writing, including mysterious hieroglyphics which no one in the world knew how to read. When Fourier returned to France, he brought a rubbing of the writing on the Rosetta Stone.

Robert Moog (background) applied Fourier analysis to develop the Moog synthesizer, forerunner of today's synthesizers.

In Fourier's study an 11-year-old boy became intrigued by the strange pictures on the stone. Young Champollion vowed that he would someday read the ancient Egyptian writing. His fascination led him to become an Egyptologist at the University of Grenoble by age 17. He achieved a translation of the whole hieroglyphic panel in 1822. Fourier was thus indirectly responsible for unlocking the mysteries of Egyptian hieroglyphics.

Fourier is much more directly responsible for the use of trigonometric series to understand and unravel virtually all wave forms. In one of the most imaginative and profoundly important applications of trigonometric identities, Fourier showed how to represent a great variety of functions as sums of sines and cosines. Fourier's remarkable theorem says that, assuming an infinite number of terms, any kind of wave function can be written as a sum of sine and cosine waves.

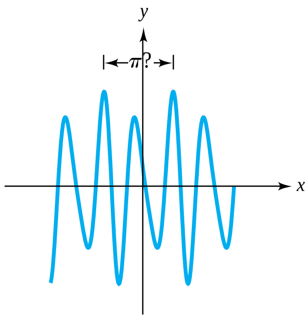
The unique musical signatures of a violin or a tenor saxophon are no more than complex combinations of sound waves. Fourier analysis can break up any such wave form into a sine-cosine combination in essentially one way. Electronic instruments such as the Moog synthesizer depend on Fourier analysis to combine wave forms to duplicate particular musical sounds or create new combinations and sounds never heard before. With increasing sophistication and computer help, synthesizers can combine thousands of tiny pieces of sine waves as brushstrokes to paint almost any sound picture. Similarly, Fourier analysis makes it possible to take discrete optical or radio signals (which are also wave forms) and filter out atmospheric interference or reconstruct coherent images from data of telescopes located thousands of miles apart—all because of trigonometric identities.

For simple harmonic motion defined by a function of the form $f(t) = A\sin(Bt + C)$, we say that the motion has **amplitude** $|A|$ and **frequency** f (in cycles per unit time) given by the reciprocal of the period:

$$f = \frac{1}{p} = \frac{B}{2\pi}.$$

Sinusoidal Functions and Sums

The algorithm from Section 6.4 gives a procedure for rewriting a function $f(x) = a\sin x + b\cos x$ in a form we recognize as a *sinusoidal curve,* a generalized sine



$g(x) = \sin(6x - 1) + 3\cos(4x + 1)$

FIGURE 44

function. What happens if we add two sinusoidal curves? Do we always get a sinusoidal curve? The answer is no, as we show in the next example. Also see Develop Mastery Exercise 53.

►EXAMPLE 5 ***Sum of generalized sine curves*** Is the function g, given by $g(x) = \sin(6x - 1) + 3\cos(4x + 1)$, periodic? sinusoidal?

Solution

Both $y = \sin(6x - 1)$ and $y = 3\cos(4x + 1)$ are generalized sine functions, the first having period $\frac{2\pi}{6}$, and the second with period $\frac{2\pi}{4}$. The graph in Figure 44 appears to be periodic, but it is clearly *not* sinusoidal because the local extrema have different heights. To verify that g is periodic, we need to find a number p for which $g(x + p) = g(x)$ for all real numbers x. It looks as if the graph repeats from one high point to another. Tracing along the curve between the indicated high points, we find that they are about 3.16 units apart, suggesting the possibility that the period is π. From the graph, the period clearly cannot be much less than π. We compute $g(x + \pi)$.

$$\begin{aligned} g(x + \pi) &= \sin(6x + 6\pi - 1) + 3\cos(4x + 4\pi + 1) \\ &= \sin((6x - 1) + 6\pi) + 3\cos((4x + 1) + 4\pi) \\ &= \sin(6x - 1) + 3\cos(4x + 1) = g(x). \end{aligned}$$

Therefore $g(x + \pi) = g(x)$ for all real x, and g is a periodic function with period π. ◄

Envelopes and Damped Oscillations

The generalized sine curve $f(x) = A\sin(Bx + C)$ oscillates repeatedly between the horizontal lines $y = A$ and $y = -A$. The oscillatory behavior continues to be a vital feature of more general functions of the form

$$F(x) = g(x)\sin(Bx + C),$$

where we have replaced the constant A by $g(x)$. $F(x)$ now oscillates between the curves $y = g(x)$ and $y = -g(x)$, just touching $y = g(x)$ when $\sin(Bx + C) = 1$, and touching $y = -g(x)$ when $\sin(Bx + C) = -1$. For convenience, we say that *F oscillates between* $\pm g(x)$, and we call the curves $y = \pm g(x)$ **envelopes** for F. We explore these ideas briefly in the next couple of examples.

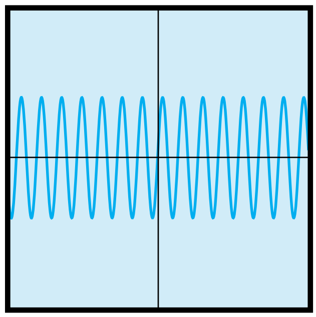
►EXAMPLE 6 ***Envelopes*** Graph each function separately in the window $[-4.7, 4.7] \times [-2.5, 2.5]$.

(a) $y = \sin 10x$ **(b)** $y = 2\cos x \sin 10x$ **(c)** $y = \pm 2\cos x$

Then **(d)** graph $y = 2\cos x \sin 10x$ and $y = \pm 2\cos x$ together. Describe how the graphs in each part are related to each other, and describe what should happen if we replace $10x$ by $15x$. Check by graphing.

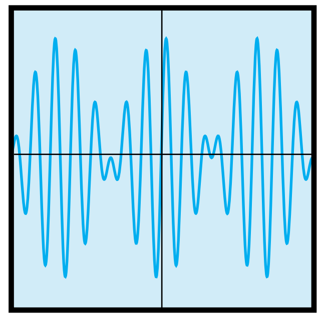
Solution

The graphs are shown in Figure 45. The graph of $y = \sin 10x$ oscillates fairly rapidly between $y = \pm 1$. Multiplying by $2 \cos x$ in part **(b)** gives a graph that oscillates at the same rate, but the size of the oscillations now varies. Comparing the graphs in **(c)** and **(d)**, we can see that the curves $y = \pm 2 \cos x$ bound the oscillation in **(b)**, so that the curves $y = \pm 2 \cos x$ are the *envelope curves* for the graph in **(b)**.



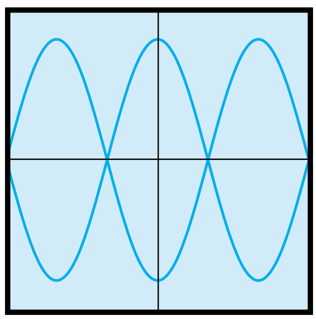
$[-4.7, 4.7]$ by $[-2.5, 2.5]$
$y = \sin 10x$

(a)



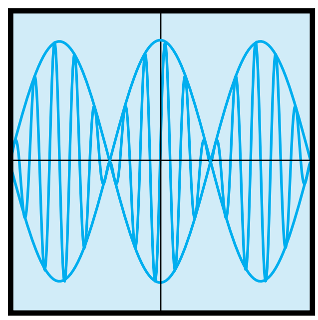
$[-4.7, 4.7]$ by $[-2.5, 2.5]$
$y = 2 \cos x \sin 10x$

(b)



$[-4.7, 4.7]$ by $[-2.5, 2.5]$
$y = 2 \cos x, y = -2 \cos x$

(c)



$[-4.7, 4.7]$ by $[-2.5, 2.5]$
$y = 2 \cos x \sin 10x$
$y = 2 \cos x, y = -2 \cos x$

(d)

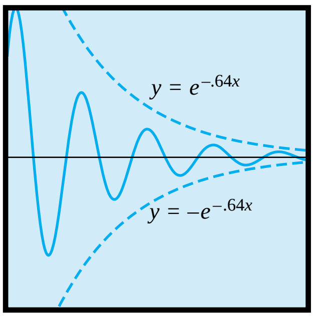
FIGURE 45

Changing $10x$ to $15x$ should simply increase the rate of oscillation, so that the curve $y = 2 \cos x \sin 15x$ is still enveloped by $\pm 2 \cos x$, but the graph oscillates more often within each of the boundary cycles. The graph confirms our prediction (*Check!*), but there are now so many oscillations that the calculator graph of $y = \sin 15x$ begins to look quite ragged instead of showing the smooth sine curve we know is there. ◀

The kind of behavior we observe in the graphs in Figure 45 is typical of many of the sound and electronic wave phenomena that surround us in everyday life. A pulsing sound or the "beats" in closely tuned instruments may indicate a rapidly varying envelope function. Another common experience is damped oscillation, where the envelope functions decrease, as in the following example.

▶EXAMPLE 7 ***Damped oscillation*** A weight hangs from a spring and is pulled downward from the equilibrium point and then released. The weight then goes up and down, but the motion gradually decreases until the weight comes to rest at equilibrium again. The function describing the distance from the rest position for a particular example is given in a book on differential equations as

$$d(t) = e^{-.64t}\left(\frac{4}{9}\sin 4.8t + \frac{1}{3}\cos 4.8t\right).$$



[0, 6] by [–.5, .5]

$y = e^{-.64x}\left(\frac{4}{9}\sin 4.8x + \frac{1}{3}\cos 4.8x\right)$

FIGURE 46

(a) Graph Y = (e^(−.64X))((4/9)SIN 4.8X + (1/3)COS 4.8X) and Y = ±e^(−.64X) in $[0, 6] \times [-0.5, 0.5]$ and explain why $y = \pm e^{-.64t}$ are not the envelope curves for the oscillation.

(b) Use the algorithm from Section 6.4 to express the function $f(t) = \frac{4}{9}\sin 4.8t + \frac{1}{3}\cos 4.8t$ as a generalized sine function and identify the envelope curves for d.

Solution

(a) The graphs are shown in Figure 46, and it is clear that $d(t)$ is not oscillating as far as $y = \pm e^{-.64t}$. If we write $d(t)$ as

$$d(t) = e^{-.64t}f(t), \text{ where } f(t) = \frac{4}{9}\sin 4.8t + \frac{1}{3}\cos 4.8t,$$

then the amplitude of f is not 1. To find the amplitude of f, we need to express f as a generalized sine function.

(b) Using the algorithm of Section 6.4 with $a = \frac{4}{9}$, $b = \frac{1}{3} = \frac{3}{9}$, we find the associated point $P(\frac{4}{9}, \frac{3}{9})$ at a distance $\frac{5}{9}$ from O, and the angle is given by $\alpha = \text{Tan}^{-1}\frac{3}{4} \approx 0.6435$. See Figure 47. Therefore

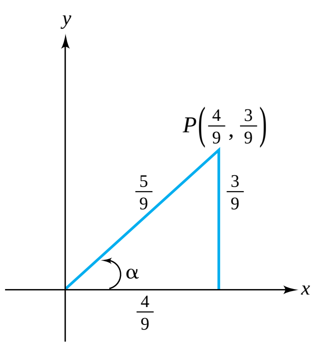


FIGURE 47

$$f(t) = \left(\frac{4}{9}\right)\sin 4.8t + \left(\frac{1}{3}\right)\cos 4.8t = \left(\frac{5}{9}\right)\sin(4.8t + \alpha), \text{ and}$$

$$d(t) = e^{-.64t}f(t) = \frac{5}{9}e^{-.64t}\sin(4.8t + \alpha).$$

The envelope curves for $y = d(t)$ are $y = \pm\frac{5}{9}e^{-.64t}$, as may be confirmed by graphing. (*Check!*) ◀

Calculator Limitations

The ragged-looking calculator graph of $y = \sin 15x$ of Example 6d points up further limitations of technology. Calculator graphs necessarily involve sampling, and sampling a rapidly changing function can produce misleading results. You may have seen ads for compact disc players with a phrase such as "4 Times Oversampling," suggesting that lots of sampling is needed for satisfactory fidelity of sound reproduction, and lots of the waves we live with have a much greater frequency than sound waves. Actually your CD player probably samples the sound many *thousands* of times a second.

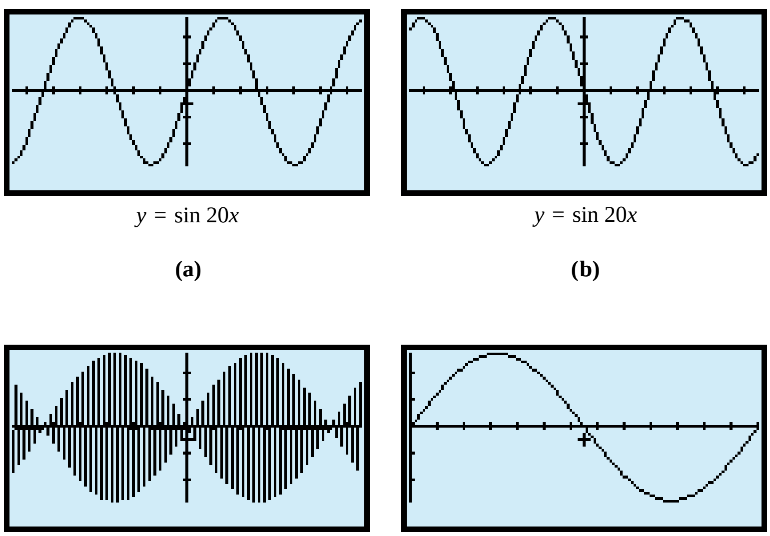
To illustrate how a graph can convey incorrect information, we invite you to experiment with the ideas of the next example. Because the behavior we want to see is so very sensitive to window size, we give instructions in terms of the number of pixel columns. You can tell the number of columns on your screen from your decimal window. For example, pressing ZDECM on the TI-85 gives a window with an x-range of $[-6.3, 6.3]$, in which there are 126 ($= 63 + 63$) tenths, or 126 pixel columns. See the inside front cover.

▶EXAMPLE 8 ***Calculator graphs*** Let $f(x) = \sin 20x$, and set a y-range of $[-1.5, 1.5]$. Graph f on the specified x-ranges in **(a)**, **(b)**, and **(c)**. Keep in mind that each of the graphs in **(a)**, **(b)**, and **(c)** is supposed to be the graph of the same function, $y = \sin 20x$. After comparing your graphs on the three given x-ranges **(a)**, **(b)**, and, **(c)**, **(d)** find and use a fundamental interval for f as the x-range. Explain why the graphs are so very different.

#Cols.	*Calculator*	**(a)**	**(b)**	**(c)**
94	TI-82	± 15.1	± 14.4	± 7.55
	Casio 7700	± 15.1	± 14.4	± 7.55
95	TI-81	± 14.6	± 15.3	$-7.6, 7.8$
126	TI-85	± 20.2	± 19.4	± 10.1
	Casio 9700	± 20.2	± 19.4	± 10.1
130	HP-38, 48	± 20.8	± 20	± 10.4

Solution

A fundamental interval is found by solving $0 \le 20x \le 2\pi$, giving an FI of $[0, \frac{\pi}{10}]$, or about $[0, 0.314]$. The graphs for each part should be similar to those in Figure 48. (On the HP-38, the graph in part **(c)** will not be connected; trace.) You may want to experiment with changing each x-range very slightly, say by one or two tenths, in the first three parts to see the effect. The reason that the graphs are so different is that f goes through a complete cycle in every interval of length $\frac{\pi}{10}$. It follows that in, say, an interval of length 30, there are $\frac{30}{(\pi/10)}$ cycles, *nearly 100*



$y = \sin 20x$

(a)

$y = \sin 20x$

(b)

$y = \sin 20x$

(c)

[0, .314] by [–1.5, 1.5]
$y = \sin 20x$

(d)

FIGURE 48

cycles! Evaluating the function once for each pixel column amounts to picking about one point per cycle and cannot possibly give a very accurate picture of the total behavior of the function. ◀

EXERCISES 6.5

Check Your Understanding

Use graphs whenever they might be helpful.

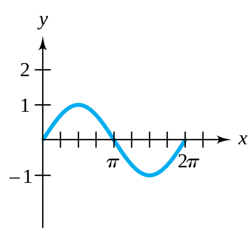
Exercises 1–5 True or False. Give reasons.

1. The graph of $y = 1 + \cos \pi x$ is periodic with a period of 2.
2. Shifting the graph of $y = 4 \sin x$ to the right $\frac{\pi}{3}$ units or to the left $\frac{5\pi}{3}$ units gives the same graph.
3. The graph of $y = \cos x \cdot \tan x$ is the same as the graph of $y = \sin x$.
4. Shifting the graph of $y = 3 \cos(x - \frac{\pi}{3})$ to the left $\frac{\pi}{3}$ units gives the same graph as $y = 3 \cos x$.
5. The graph of $y = 2 \cos x - \sin x$ is sinusoidal.

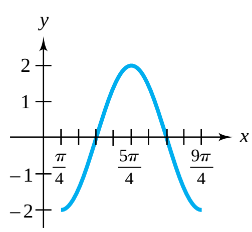
Exercises 6–10 Select from the list below all choices whose graphs contain the cycle shown.

(a) $y = \cos x$ ***(b)*** $y = \sin x$
(c) $y = \sin 2x$ ***(d)*** $y = -\sin \pi x$
(e) $y = 2 \cos 2x$ ***(f)*** $y = -2 \cos\left(x - \dfrac{\pi}{4}\right)$
(g) $y = \cos\left(\dfrac{\pi x}{2}\right)$ ***(h)*** $y = -2 \cos\left(x + \dfrac{\pi}{4}\right)$
(i) $y = 1 + \sin x$

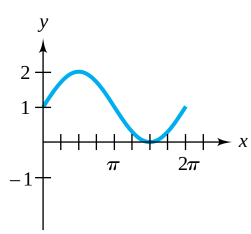
6.



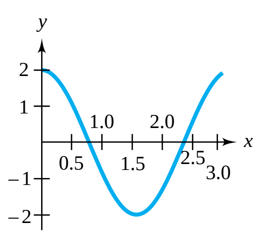
7.



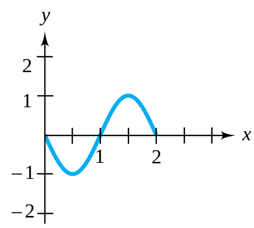
8.



9.



10.



Develop Mastery

Exercises 1–4 Determine a fundamental interval.

1. $y = \sin 3x$
2. $y = \cos 4x$
3. $y = -\sin\left(\pi x + \dfrac{3\pi}{4}\right)$
4. $y = -2 \cos\left(\pi x - \dfrac{\pi}{5}\right)$

Exercises 5–8 ***Simplify*** ***(a)*** *Use an appropriate reduction formula to express the equation in simpler form, then* ***(b)*** *use the simpler form to determine a FI, amplitude, and period.*

5. $y = \sin\left(2x - \dfrac{\pi}{2}\right)$
6. $y = 2 \cos\left(\dfrac{\pi}{2} - x\right)$
7. $y = -2 \cos\left(\dfrac{3\pi}{2} + x\right)$
8. $y = \sin\left(\dfrac{3\pi}{2} - 2x\right)$

Exercises 9–12 Follow the steps outlined in the algorithm of this section to draw a graph of a fundamental cycle. Find the amplitude, period, and phase shift.

9. $y = \sin\left(\dfrac{\pi x}{2}\right)$
10. $y = \cos \pi x$
11. $y = 3 \sin\left(\dfrac{\pi}{4} - 2x\right)$
12. $y = \sqrt{3} \cos\left(\dfrac{3\pi}{4} - x\right)$

Exercises 13–16 ***Amplitude, Period, Phase Shift*** ***(a)*** *Determine the amplitude, period and phase shift for the graph of f. Give a verbal description of phase shift in which you tell what graph is being shifted, by how much, and in what direction, to get the graph of f.* ***(b)*** *Find another formula that will give a phase shift in the opposite direction. Draw a graph.*

13. $f(x) = 2 \sin(\pi x - 3\pi)$
14. $f(x) = -3 \sin(2x - 4)$

15. $f(x) = 3\cos(2x + 3)$

16. $f(x) = -2\cos(\pi x - 4\pi)$

Exercises 17–20 ***Working with Degrees*** *Follow the instructions in Exercises 13–16. Use* $[-180, 180] \times [-5, 5]$, $x_{SCL} = 30$, $y_{SCL} = 1$; *x is in degrees.*

17. $f(x) = 2\sin(3x - 60°)$

18. $f(x) = 3\cos(2x + 30°)$

19. $f(x) = -2\cos(3x - 48°)$

20. $f(x) = -2\sin(2x - 60°)$

Exercises 21–24 ***Using the Algorithm*** ***(a)*** *Apply the algorithm to write an equivalent equation in the form,* $y = A\sin(x + \alpha)$. ***(b)*** *Use this form to sketch a graph. Check by using the original equation to draw a calculator graph.*

21. $y = \sin x + \cos x$

22. $y = \sin x - \cos x$

23. $y = \sin x + \sqrt{3}\cos x$

24. $y = 2(\sqrt{3}\sin x - \cos x)$

Exercises 25–28 ***Zeros*** *Find the smallest positive zero and the largest negative zero of f (2 decimal places).*

25. $f(x) = \sin 2x + 3\cos 2x$

26. $f(x) = \sin(\pi x - 1) + \cos(\pi x + 4)$

27. $f(x) = 2\sin(\pi x) + 2\cos(\pi x)$

28. $f(x) = \cos 2x - 3\sin(2x - 1)$

Exercises 29–32 *Use translations and/or reflections of core graphs to sketch a graph of the equation.*

29. $y = 1 + 2\sin 2x$

30. $y = 2 - \cos\left(\dfrac{\pi x}{2}\right)$

31. $y = \sin \pi x - 2$

32. $y = 3 - \cos 2x$

Exercises 33–42 ***Using Identities*** *Use appropriate identities to get a simpler equivalent equation, then sketch a graph of that equation. Compare with a calculator graph of the original equation.*

33. $y = (\sin x + \cos x)^2 - 1$

34. $y = \left(\sin\dfrac{x}{2} + \cos\dfrac{x}{2}\right)^2$

35. $y = 1 + (\sin x + \cos x)^2$

36. $y = \cos^2 x - \sin^2 x$

37. $y = \cos^2 2x - \sin^2 2x$

38. $y = 4\cos^2\dfrac{x}{2}$

39. $y = \cos x \tan x$ (*Hint:* First check the domain.)

40. $y = e^{\ln \sin x}$ (*Hint:* Check the domain and then use an appropriate identity from Chapter 4.)

41. $y = e^{\ln \cos x}$ (*Hint:* See Exercise 40.)

42. $y = e^{\ln|\sin x|}$ (*Hint:* See Exercise 40.)

Exercises 43–44 ***Damped Oscillations (a)*** *Draw graphs of the three functions on the same screen. What do you observe about the graphs?* ***(b)*** *Determine the coordinates (in exact form) of the local maximum and minimum points for the graph of f, where* $-1 \le x \le 1$. *Use* TRACE *to support your answers.* ***(c)*** *What are the coordinates of the points where the graphs of f and g, f and h meet* $(-1 \le x \le 1)$?

43. $f(x) = 2^{-x}\sin(2\pi x)$, $g(x) = 2^{-x}$, $h(x) = -2^{-x}$

44. $f(x) = 2^{-x}\cos(2\pi x)$, $g(x) = 2^{-x}$, $h(x) = -2^{-x}$

Exercises 45–48 ***Periodic, Sinusoidal*** ***(a)*** *Is f periodic? If it is, give the period in exact form.* ***(b)*** *Draw a graph. Does it appear that f is sinusoidal? See Example 5.*

45. $f(x) = 2\cos(3x + 4) - \sin(3x - 1)$

46. $f(x) = \sin(3\pi x) + \cos(3\pi x + 2)$

47. $f(x) = \sin(3x - 1) - \cos(2x + 3)$

48. $f(x) = \cos(\pi x) - \sin(0.5\pi x)$

Exercises 49–50 ***Periodic and Sinusoidal*** *Function f is periodic and sinusoidal. Draw a graph to support this claim.* ***(a)*** *Find a formula for f of the form* $f(x) = A\sin(Bx + C)$. ***(b)*** *Determine the amplitude and period. Describe the phase shift. See Example 4.*

49. $f(x) = \sqrt{3}\sin x + \cos x$

50. $f(x) = \sqrt{5}\sin 2x - 2\cos 2x$

51. The graph of function f can be obtained by shifting the graph of $y = \sin 2x$ to the left 1 unit. Give two different formulas for f. Check graphically.

52. The graph of function f can be obtained by shifting the graph of $y = -3\cos 2x$ to the right 2 units, and then translating the resulting graph 3 units upward. Give a formula for f. Check graphically.

53. In Exercises 45 and 46, f is periodic and sinusoidal while in 47 and 48, f is periodic but not sinusoidal. Look carefully at the formulas for f and guess how you can predict when the function will be periodic and sinusoidal. Try a few examples of your own choice to support your guess.

54. ***Strange Calculator Graphs*** Example 8 illustrates how the calculator graph of $f(x) = \sin 20x$ varies with the x-range. Use x-ranges listed in **(a)**, **(b)**, and **(c)** of Example 7 for your calculator and in each case see what the display shows as the graph of $y = \sin 20x$. Do any of these actually represent the graph you would expect?

The graph in **(a)** appears to be a sine graph but trace and see if the length of a cycle is really $\frac{\pi}{10}$ (the actual period).

55. ***Exercise 54 Continued***

(a) For the x-range try $[0, \frac{\pi}{10}]$ and trace to see if the cycle has length $\frac{\pi}{10}$ (≈ 0.31416).

(b) Try $[0, \frac{6\pi}{10}]$ for the x-range. Does the display show six cycles, each of length $\frac{\pi}{10}$? Experiment with other windows of your choice.

56. Repeat Exercise 54 for $f(x) = \sin 10x$ where each number in **(a)**, **(b)**, and **(c)** of Example 8 for the x-range is multiplied by 2. Similarly for $f(x) = \sin 40x$, divide each number by 2.

Exercises 57–62 ***Graph to Formula*** *One cycle of the graph of a general sine or cosine function is shown (possibly translated vertically). Determine a formula for f. The answer is not unique. First describe your strategy.*

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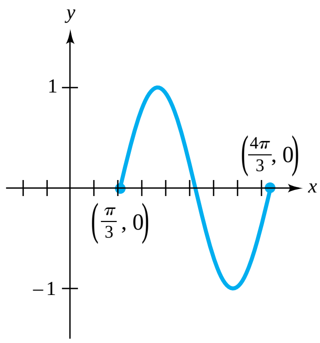

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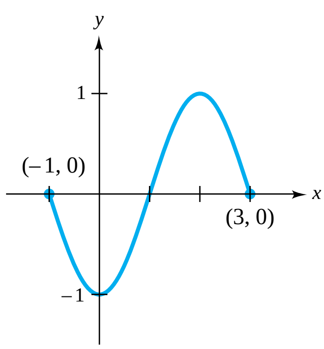

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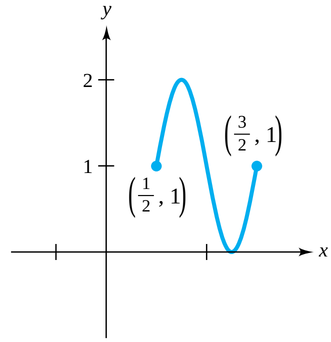

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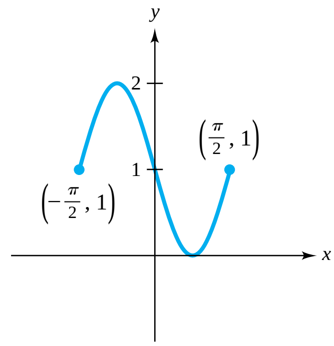

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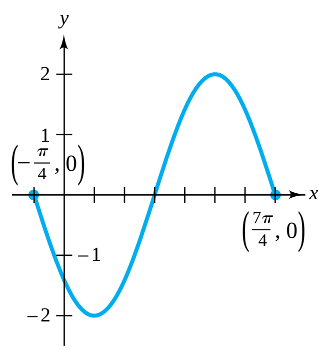

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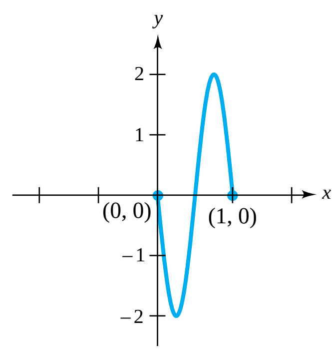


Exercises 63–66 ***Harmonic Motion*** *The function describes simple harmonic motion. Find the amplitude, period, and frequency of the motion, and give the location when t is zero.*

63. $f(t) = 4 \sin 6\pi t$

64. $E(t) = 3 \sin 80\pi t$

65. $E(t) = 2 \cos \dfrac{2\pi t}{3} + 0.4 \sin \dfrac{2\pi t}{3}$

66. $V(t) = 2 \cos 120\pi t - 3 \sin 120\pi t$

Exercises 67–70 ***Envelopes*** *Function f is the product of functions g and h, $f(x) = g(x) \cdot h(x)$. Use $[0, 6.25] \times [-4, 4]$ and on the same screen draw graphs of $y_1 = f(x)$, $y_2 = g(x)$, $y_3 = -g(x)$. Does the display seem to indicate that the graphs of $y = \pm g(x)$ are envelope curves for the graph of f? For x in $[0, \pi]$, at what points does the graph of f touch the envelope graphs? Solve algebraically and graphically.*

67. $g(x) = \sqrt{x}, \quad h(x) = \sin 4x$

68. $g(x) = \sqrt{2x}, \quad h(x) = \sin 4x$

69. $g(x) = \sqrt{2x}, \quad h(x) = \sin(2\pi x)$

70. $g(x) = \sqrt{3x}, \quad h(x) = \sin(1.5\pi x)$

71. ***Envelopes*** For $f(x) = \sin 6x - \sin 4x$, on the same screen draw graphs of $y_1 = f(x)$, $y_2 = 2 \sin x$, $y_3 = -2 \sin x$. Use $[0, 8] \times [-3, 3]$. Do the graphs of y_2 and y_3 appear to be envelope curves for f? Explain by using identity (I-22) in Section 6.3.

72. ***Envelope Curves*** **(a)** For $f(x) = e^{-0.4x}(0.4 \sin 6x + 0.3 \cos 6x)$, on the same screen draw graphs of $y_1 = f(x)$, $y_2 = e^{-0.4x}$, $y_3 = -e^{-0.4x}$. Use $[0, 6] \times [-0.8, 0.8]$. See that the graphs of y_2 and y_3 are not envelope curves for f. Use the algorithm in Section 6.4 to adjust the formula for f so that you do get envelope curves. Give formulas for the envelopes. See Example 7.

73. One of the first astronomical discoveries for measuring distances to other galaxies was a class of stars whose brightness varies regularly. The intensity (brightness) of these Cepheid variables varies according to a function of the form

$$I(t) = A + B \sin \frac{2\pi t}{C},$$

where t is in days, A is the average intensity (magnitude), and the intensity varies by as much as $\pm B$ (magnitudes) every C days.

(a) If the average intensity of a particular Cepheid star is 4.0 and the star becomes as much as 10 percent brighter every 10.8 days, write an equation for its intensity as a function of time.

(b) What is the intensity when $t = 0$? When $t = 4$ days?

Exercises 74–76 ***Your Choice*** *Give a formula for a function f whose graph is sinusoidal and that satisfies the specified conditions. Answer is not unique.*

74. The local extrema points are on the x axis and on the line $y = 2$.

75. The smallest positive zero of f is $\frac{\pi}{4}$ and the largest negative zero is $-\frac{\pi}{4}$.

76. The points $(0, 2)$ and $(\frac{\pi}{2}, 2)$ are local maximum points.

CHAPTER 6 REVIEW

Test Your Understanding

True or False. Give reasons. Use identities or draw graphs when helpful.

1. For every real number x, $-0.5 \le \sin x \cos x \le 0.5$.

2. For every real number x, $-2 \le \sqrt{3} \sin x + \cos x \le 2$.

3. For every real number x, $0 \le \sin^2 x + \cos 2x \le 1$.

4. For every real number x, $-1 \le \cos^2 2x - \sin^2 2x \le 1$.

5. The graph of $y = \sin^2 x + \cos^2 x$ is a horizontal line.

6. The graph of $y = \cos x - 2 \cos^2\left(\dfrac{x}{2}\right)$ is a horizontal line.

7. The graphs of $y = 1 - 2 \sin^2\left(\dfrac{x}{2}\right)$ and $y = \cos x$ are identical.

8. The graphs of $y = \cos 2x + \sin^2 x$ and $y = \cos^2 x$ are identical.

9. The graphs of $y = \sqrt{1 - \sin^2 x}$ and $y = \cos x$ are identical.

10. The graphs of $y = \cos x \tan x$ and $y = \sin x$ are identical.

11. If $\pi \le x \le \dfrac{3\pi}{2}$, then $\cos \dfrac{x}{2} = -\sqrt{\dfrac{1 - \cos x}{2}}$.

12. If $\frac{\pi}{2} \le x \le \pi$, then $\cos \frac{x}{2} = -\sqrt{\frac{1 - \cos x}{2}}$.

13. For every x in the interval $\left[-\frac{\pi}{2}, 0\right]$, $\cos \frac{x}{2} = -\frac{\sqrt{1 + \cos x}}{2}$.

14. The graphs of $y = \sec x \sin 2x$ and $y = 2 \sin x$ are identical.

15. The range of the function $f(x) = \sin x + \cos x$ is the interval $[-1, 1]$.

16. The range of $f(x) = \sqrt{3} \sin x + \cos x$ is the interval $[-2, 2]$.

17. The range of $f(x) = \sqrt{\sin^2 x + \cos^2 x}$ consists of a single number.

18. There is no number x for which $\sin x \cos x$ equals 1.

19. A solution to the equation $\sin 2x - 1 = 0$ is $\frac{\pi}{2}$.

20. A solution to the equation $\cos 2x + 1 = 0$ is $\frac{\pi}{2}$.

21. The equation $1 + \sin^2 x = 0$ has no solution.

22. The equation $\sqrt{1 - \sin^2 x} = \cos x$ is satisfied by every real number x.

23. The function $f(x) = 4 \sin(\pi x + 3)$ is periodic with period 2.

24. The function $f(x) = -2 \cos(4x + \pi)$ is periodic with period π.

25. The graphs of $y = -2 \cos\left(x + \frac{\pi}{2}\right)$ and $y = 2 \sin x$ are identical.

26. The amplitude of the graph of $f(x) = -3 \sin 2x$ is -3.

27. Every point of the form $\left(\frac{(2k + 1)\pi}{2}, 0\right)$, where k is any integer, is an x-intercept point for the graph of $y = \sin 2x$.

28. The graphs of $y = \sin 2x$ and $y = -\sin 2x$ have identical x-intercept points.

29. The graphs of $y = \cos \pi x$ and $y = -3 \cos \pi x$ have identical x-intercept points.

30. Every x-intercept for the graph of $y = \sin x$ is also an x-intercept for the graph of $y = \sin 2x$.

31. Every x-intercept point for the graph of $y = \sin 2x$ is also an x-intercept point for the graph of $y = \sin x$.

32. The graph of $y = 2 + \cos x$ has no x-intercept points.

33. The graphs of $y = e^{\ln \sin x}$ and $y = \sin x$ are identical.

34. The graph of $y = e^{\ln \sin x}$ has no x-intercept points.

35. For every x, $5 \sin(x - \operatorname{Sin}^{-1} 0.8) = 3 \sin x + 4 \cos x$.

36. For every x, $5 \cos(x - \operatorname{Sin}^{-1} 0.6) = 4 \cos x - 3 \sin x$.

Exercises 37–45 Fill in the blank so that the resulting statement is true.

37. For every x, $\log(\sin^2 x + \cos^2 x) =$ ______.

38. $\sin\left(\pi - \operatorname{Sin}^{-1} \frac{3}{5}\right) =$ ______.

39. $\cos\left(\frac{\pi}{2} - \operatorname{Sin}^{-1} \frac{4}{5}\right) =$ ______.

40. $\tan(\pi - \operatorname{Tan}^{-1}(-1)) =$ ______.

41. The domain of $f(x) = \cos(\pi + \operatorname{Sin}^{-1} x)$ is ______.

42. The range of $f(x) = \sin\left(\frac{\pi}{2} + \operatorname{Sin}^{-1} x\right)$ is ______.

43. The graphs of $y = 4 \cos x$ and $y = 0.4x - 1$ intersect at points in Quadrant(s) ______.

44. The number of zeros in the interval $[-\pi, \pi]$ for $f(x) = \sin x + 2 \cos 2x$ is ______.

45. The smallest prime number that is greater than the smallest positive zero of $f(x) = \sin(0.4x) + \cos(0.4x)$ is ______.

Review for Mastery

Exercises 1–12 ***Prove Identity*** *Prove that the equation is an identity.*

1. $\sin x \cot x = \cos x$

2. $\sec\left(\frac{\pi}{2} - x\right) \tan x = \sec x$

3. $\sec x \sin 2x = 2 \sin x$

4. $2 \csc^2 x \cos^2\left(\frac{x}{2}\right) = \frac{1}{1 - \cos x}$

5. $\tan\left(x + \frac{\pi}{4}\right) = \frac{\cos x + \sin x}{\cos x - \sin x}$

6. $(\sin x + \cos x)^2 = 1 + \sin 2x$

7. $\sin x \tan \frac{x}{2} = 1 - \cos x$

8. $4 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2} = \sin^2 x$

9. $2 \sin\left(x - \frac{\pi}{6}\right) = \sqrt{3} \sin x - \cos x$

10. $\sqrt{1 - \sin^2 x} = |\cos x|$

11. $\cos 2x \tan 2x = 2 \sin x \cos x$

12. $\tan \frac{x}{2} + \cot x = \csc x$

Exercises 13–18 ***Is It An identity?*** *Determine whether or not the equation is an identity. Give a counterexample or a proof.*

13. $\sqrt{\sec^2 x - \tan^2 x} = 1$ **14.** $\sqrt{1 + \tan^2 x} = \sec x$

15. $\cos 2x + \cos x = \cos 3x$

16. $\cos^4 x - \sin^4 x = \cos 2x$

17. $13 \sin\left(x + \text{Sin}^{-1}\frac{5}{13}\right) = 12 \sin x + 5 \cos x$

18. $\sin(\text{Sin}^{-1} x + \text{Cos}^{-1} x) = 1$

Exercises 19–22 *Find a simpler formula for function f.*

19. $f(x) = \sin(-x)\tan x + \sec x$

20. $f(x) = \csc x \tan x$

21. $f(x) = (\sin x + \cos x)^2 - \sin 2x$

22. $f(x) = \cot x \sec x \cos\left(x - \frac{\pi}{2}\right)$

Exercises 23–30 ***Exact Form*** *Evaluate in exact form, where angles α, β, and γ satisfy the conditions:*

$\sin\alpha = \frac{4}{5}$ *and* $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$

$\tan\beta = -\frac{5}{12}$ *and* $-\pi < \beta < 0$

$\cos\gamma = \frac{3}{5}$ *and* $0 < \gamma < \frac{\pi}{2}$

23. $\tan\alpha$ **24.** $\sin 2\alpha$

25. $\sin\left(\frac{\beta}{2}\right)$ **26.** $\sin(\alpha - \beta)$

27. $\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)$

28. $\tan\left(\frac{\gamma}{2}\right)$

29. $\tan\left(\gamma + \frac{\pi}{4}\right)$

30. $\sin^2\alpha + \cos^2\beta$

Exercises 31–38 ***Restricted Domain*** *Solve, assuming the domain is the interval $[0, 2\pi]$. Give answers in exact form. Use graphs as a check.*

31. $2 \sin^2 x - 1 = 0$

32. $\sqrt{3} \sin x + \cos x = 0$

33. $2 \sin^2 x - \sin x - 1 = 0$

34. $\sin x \cos x + \cos^2 x = 0$

35. $\sin x - \cos x = \sqrt{2}$

36. $\sqrt{3} \sin x + \cos x = 2$

37. $4 \cos^2 x - 3 = 0$

38. $2 \cos^2 x - 7 \cos x - 4 = 0$

Exercises 39–44 ***Restricted Domain*** *Solve, assuming the domain is the interval $[-\pi, \pi]$. Give answers rounded off to two decimal places.*

39. $2 \cos x + \sin x = 0$ **40.** $4 \sin x \cos x = \cos x$

41. $2 \sin^2 x = 2 \sin x + 3$ **42.** $\sin^2 x = 2 \sin x + 1$

43. $\cos^2 x = 3 \cos x - 2$

44. $\cos^2(\pi x) = 2 \cos(\pi x) - 1$

Exercises 45–50 ***Solution Set*** *Find the solution set (exact form). Draw graphs to support your answer.*

45. $\sqrt{1 - \cos^2 x} = \sin x$

46. $\sin^2 x + \cos^2 x = x^2 - 2x - 3$

47. $\sqrt{1 - \sin^2 x} = \cos x$

48. $\ln(\sec x) + \ln(\cos x) = 0$

49. $\ln(\tan x) + \ln(\cos x) = \ln(\sin x)$

50. $\sin x = \sqrt{\frac{1 - \cos 2x}{2}}$

Exercises 51–56 ***Amplitude, Period, Phase Shift*** *Without using a calculator, draw a graph of f. Give the amplitude and period. Describe the phase shift telling what graph is being shifted, by how much and in what direction.*

51. $f(x) = 2 \sin\left(x - \frac{\pi}{4}\right)$ **52.** $f(x) = -2 \sin \pi x$

53. $f(x) = \sqrt{2}(\sin x - \cos x)$

54. $f(x) = \sqrt{3} \sin x + \cos x$

55. $f(x) = -2 \cos\left(2x + \frac{\pi}{3}\right)$

56. $f(x) = 3 \cos\left(\pi x + \frac{\pi}{2}\right)$

Exercises 57–59 ***Translating Graphs*** *Describe how you would translate the graph of g to get the graph of f.*

57. $g(x) = \cos x$, $f(x) = \cos(x - \text{Sin}^{-1} 0.5)$

58. $g(x) = \sin x$, $f(x) = \sin(x + \text{Cos}^{-1} 0.5)$

59. $g(x) = \tan x$, $f(x) = \tan(x - \text{Tan}^{-1} 1)$

60. Determine the domain and range of $f(x) = x + \text{Sin}^{-1} x$

61. Find the smallest positive zero (2 decimal places) of $f(x) = \sin x + 3 \cos x$.

62. Find the largest negative zero (2 decimal places) of $f(x) = 2 \sin x + \cos x$.

Exercises 63–66 ***Periodic, Sinusoidal*** ***(a)*** *Is the graph of f periodic? If it is, give the period.* ***(b)*** *Is f sinusoidal? Give reasons.*

63. $f(x) = \cos(\pi x - 1) - 2 \sin(\pi x + 4)$

64. $f(x) = \sin \pi x + 2 \cos \pi x$

65. $f(x) = \sin(2x + 3) - \cos 4x$

66. $f(x) = \cos 2x + \sin 3x$

Exercises 67–70 ***Decimal Approximations*** *Find approximations (2 decimal places) for the roots of the equation, where $-\pi \leq x \leq \pi$.*

67. $\cos x = x^2 - 2$

68. $2 \sin x + 3 \cos x = 1$

69. $2 \sin x - 3 \cos 3x = 4$

70. $5(\sin x + \cos x) = 1$

71. Find the maximum value (2 decimal places) of $f(x) = 2 \cos x + 3 \cos(0.5 \operatorname{Cos}^{-1} x)$

72. Find the minimum value (2 decimal places) of $f(x) = 2 \sin x - \cos x$.

Exercises 73–74 ***Angle of Intersection*** ***(a)*** *Find the angle (to the nearest degree) between the lines $y = f(x)$ and $y = g(x)$.* ***(b)*** *Draw the graphs of f and g on the same screen using a decimal window to see if they appear to support the answer in part* ***(a)***. *Give the point of intersection of the two lines.*

73. $f(x) = 0.4x + 1, \quad g(x) = -1.2x - 7$

74. $f(x) = -3x + 1, \quad g(x) = -0.5x - 1.5$

75. Is $f(x) = \pi - \operatorname{Cos}^{-1} x$ an increasing function? If it is, then find a formula for the inverse function f^{-1}, and state the domain and range of f^{-1}.

76. For $f(x) = 2 \sin 2x - 3 \cos 2x$, find A, B, and C such that $f(x) = A \sin(Bx - C)$.

77. Express $f(x) = \sin x + \cos x$ in the form $f(x) = A \sin(Bx + C)$.

78. **(a)** Prove that $f(x) = \sin(2 \operatorname{Sin}^{-1} x + \operatorname{Cos}^{-1} x)$ is an even function. (*Hint:* Use the fact that $y = \operatorname{Sin}^{-1} x$ is an odd function and the identity $\operatorname{Cos}^{-1}(-x) = \pi - \operatorname{Cos}^{-1} x$.)

(b) $\sin(2 \operatorname{Sin}^{-1} x + \operatorname{Cos}^{-1} x) = \sqrt{1 - x^2}$ is an identity. Solve in exact form

$$\sin(2 \operatorname{Sin}^{-1} x + \operatorname{Cos}^{-1} x) = 0.2.$$

Use graphs to support your answer.

7 APPLICATIONS OF TRIGONOMETRIC FUNCTIONS

IN CHAPTER 5 WE INTRODUCED trigonometric functions defined either on the set of real numbers or on measures of angles (in radians or degrees). In this chapter we apply what we have studied to specific types of problems. Many of these applications involve trigonometric functions of angles measured in degrees.

The first three sections of this chapter focus on solving triangles, using trigonometric functions to relate parts of triangles, which allows us to find distances and angles that may not be directly measurable. These techniques illustrate some traditional applications of trigonometry such as surveying and navigation, but similar techniques are needed for all kinds of problem solving in engineering and physics, as well as throughout mathematics.

Section 7.1 focuses on right triangles. In Sections 7.2 and 7.3 we study more general techniques for dealing with more kinds of triangles. In Section 7.4 we use trigonometric functions to represent complex numbers, which supports work in many areas of physics and electrical engineering. Properties of trigonometric functions provide additional insight into complex roots of polynomial equations. Section 7.5 contains a brief introduction to vectors.

7.1 SOLVING RIGHT TRIANGLES

> I invented a set of right triangle problems. But instead of giving the lengths of two of the sides to find the third, I gave the difference of the two sides. A typical example was: There's a flagpole and there's a rope that comes down from the top. When you hold the rope straight down, it's 3 feet longer than the pole, and when you pull the rope out tight, it's 5 feet from the base of the pole. How high is the pole?
>
> *Richard P. Feynman*

In Sections 5.2 and 5.3 we defined trigonometric functions for angles of any size, as long as the angles were in standard position relative to some system of rectangular coordinates. We also defined trigonometric functions for acute angles in a right

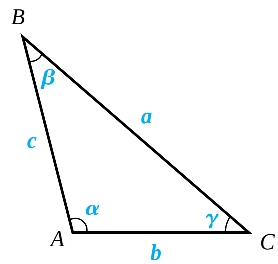


FIGURE 1

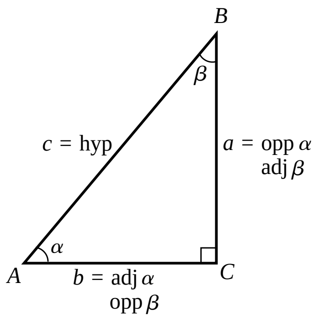


FIGURE 2

> ***My trigonometry teacher . . . taught me about adjacent over hypotenuse (all new) and "solving" triangles by logarithms (a crashing bore). He also taught me about identities (capital fun).***
>
> Paul Halmos

triangle, and at the end of Section 5.3 we related the various definitions to each other. Applications that use right triangle relations to find unknown distances or angles usually are not set in any particular coordinate system, so in this section we concentrate on right triangle definitions.

Throughout this chapter we will follow a consistent method for labeling any triangle. We will frequently use A, B, and C to label vertices, with the opposite sides labeled with the corresponding lower-case letters, a, b, and c. We will occasionally use a vertex label for an angle, but more often we will use Greek letters α, β, and γ, as in Figure 1. When we have a right triangle, we will normally locate the right angle at C, making the hypotenuse c, legs a and b, and the acute angles α and β. We refer to α, β, a, b, and c, as **parts of the triangle** ABC.

For convenient reference, we repeat the right triangle definitions of trigonometric functions from Section 5.2. See Figure 2.

Definition: trigonometric functions of an acute angle

Suppose α is an acute angle of a right triangle. The trigonometric functions of α are defined by

$$\sin\alpha = \frac{\text{opp }\alpha}{\text{hyp}} \qquad \cos\alpha = \frac{\text{adj }\alpha}{\text{hyp}} \qquad \tan\alpha = \frac{\text{opp }\alpha}{\text{adj }\alpha}$$

$$\csc\alpha = \frac{\text{hyp}}{\text{opp }\alpha} \qquad \sec\alpha = \frac{\text{hyp}}{\text{adj }\alpha} \qquad \cot\alpha = \frac{\text{adj }\alpha}{\text{opp }\alpha}$$

In a similar manner, the trigonometric functions for the angle β are:

$$\sin\beta = \frac{\text{opp }\beta}{\text{hyp}} \qquad \cos\beta = \frac{\text{adj }\beta}{\text{hyp}} \qquad \tan\beta = \frac{\text{opp }\beta}{\text{adj }\beta}.$$

Solving Right Triangles

Given information about some of the angles or sides of a right triangle, trigonometric functions can be used to determine the other sides and angles. The process of using given data to solve for unknown parts is called **solving the triangle.** In virtually all instances, we look for trigonometric functions that relate known parts of the triangle to the parts we want to find, giving equations that can be solved for the desired quantities. A number of examples illustrate this point.

▶EXAMPLE 1 ***Given: hypotenuse and an angle*** A right triangle has a hypotenuse of 4.3 meters and an acute angle of 32°. Find the other acute angle and the lengths of the legs.

Solution

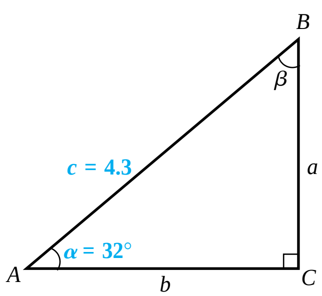


FIGURE 3

Start with a diagram like Figure 3 with c as 4.3 and α as 32°. The sum of the acute angles in a right triangle is 90°, so $\beta = 90° - \alpha = 90° - 32° = 58°$. To find a and b, use trigonometric ratios that relate these sides to known parts:

$$\sin\alpha = \frac{\text{opp }\alpha}{\text{hyp}} = \frac{a}{c} \quad \text{and} \quad \cos\alpha = \frac{\text{adj }\alpha}{\text{hyp}} = \frac{b}{c}.$$

Solve the first equation for a and the second for b:

$$a = c\sin\alpha = 4.3\sin 32° \quad \text{and} \quad b = c\cos\alpha = 4.3\cos 32°.$$

With your calculator in degree mode, obtain decimal approximations for a and b as $a \approx 2.278652836$, $b \approx 3.646606814$. ◀

Measurements and Accuracy. Recording the lengths of the sides to ten significant digits in Exercise 1 requires some explanation. In Example 1, we recorded the full calculator displays as approximations of the lengths of sides a and b. What does this imply? If the given values of $\alpha = 32°$ and $c = 4.3$ come from measurements, as is often the case in applications, then we can only assume that c was measured to the nearest tenth of a meter (to within 10 centimeters). Rounding off the decimal display for a to, say 2.27865, would imply knowing the length of side a to a fraction of a millimeter, an assumption that is surely unjustified.

In most of the work in this chapter we assume that given values represent measurements, (and hence approximations, unless the text specifically indicates otherwise). As a general rule, we cannot justify any more accuracy for calculated values than for the initial data. We use the following rule of thumb as a guideline.

Significant digits guideline

In applied problems that involve measured numbers we are not justified in recording final computed results with any more significant digits than the least precise number given.

Working with triangles often involves both linear and angular measurements. We use the following guidelines for linear–angular measurements:

Length Accuracy of	*Angle Accuracy of*
2 significant digits	nearest 1°
3 significant digits	nearest 10′ or 0.1°
4 significant digits	nearest 1′ or 0.01°
5 significant digits	nearest 10″ or 0.001°

For the problems considered in this chapter, these guidelines should be adequate, and answers should be rounded off to be consistent with the accuracy of the given data. Notation will reflect this convention. When applying these guidelines, we use $=$ instead of $\approx$ and write, for instance, $x = 2.54$ cm rather than the more precise $x \approx 2.54$ cm.

Strategy: Express each of the desired quantities in terms of the given a and c, using the right triangle definitions of trigonometric functions. The area of a right triangle equals $\frac{1}{2}ab$.

►EXAMPLE 2 ***Given: hypotenuse and a leg*** In a right triangle $a = 23.4$ cm and $c = 42.3$ cm. Find b, α, β, and the area of the triangle.

Solution

First draw a diagram to show the given information (see Figure 4). The Pythagorean theorem gives

$$b = \sqrt{c^2 - a^2} = \sqrt{(42.3)^2 - (23.4)^2} = 35.2$$

Thus $b = 35.2$ cm. Trigonometric relations that involve the given parts a and c are $\sin\alpha = \frac{\text{opp } \alpha}{\text{hyp}} = \frac{a}{c}$ and $\cos\beta = \frac{\text{adj } \beta}{\text{hyp}} = \frac{a}{c}$. Therefore,

$$\alpha = \text{Sin}^{-1}\left(\frac{a}{c}\right) = \text{Sin}^{-1}\left(\frac{23.4}{42.3}\right) = 33.6°$$

$$\beta = \text{Cos}^{-1}\left(\frac{a}{c}\right) = \text{Cos}^{-1}\left(\frac{23.4}{42.3}\right) = 56.4°.$$

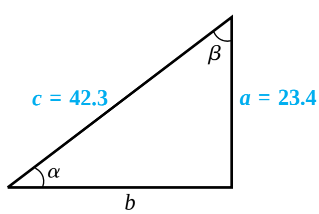


FIGURE 4

Hence α is 33.6° and β is 56.4°. To find the area K of the triangle, use the formula from geometry. The area is half the base times the height. For the example triangle, $K = \frac{ab}{2} = \frac{(23.4)(35.2)}{2} \approx 411.84$. Rounding off to three significant digits, the area is 412 cm^2. ◀

▶EXAMPLE 3 ***Dimensions of a park*** In planning a park, a region of lawn is desired in the shape of a 45°–45° right triangle with a perimeter of 100 yards. How long should the sides of the triangle be?

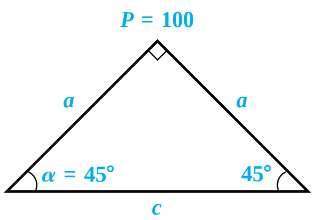


FIGURE 5

Solution

Draw a 45°–45° right triangle and label the sides, as in Figure 5, with the two legs equal. For the perimeter P, $P = a + a + c = 100$. Apply the Pythagorean theorem to get $c = \sqrt{2}\, a$. Therefore, the equation for the perimeter becomes $a + a + \sqrt{2}a = 100$, or $(2 + \sqrt{2})a = 100$. Hence,

$$a = \frac{100}{(2 + \sqrt{2})} \approx 29.3 \quad \text{and} \quad c = \sqrt{2}a \approx 41.4$$

The results are rounded off to three significant digits, assuming that all three digits of the given 100-yard perimeter are significant. Thus, the two equal sides should each be 29.3 yards. ◀

For a more general situation, given the perimeter and angle α, we may use $\sin\alpha = \frac{a}{c}$ and $\tan\alpha = \frac{a}{b}$ to express b and c in terms of a and α:

$$b = \frac{a}{\tan\alpha} \quad \text{and} \quad c = \frac{a}{\sin\alpha}.$$

Substituting into the equation for the perimeter,

$$P = a + \frac{a}{\tan\alpha} + \frac{a}{\sin\alpha} = a\left(1 + \frac{1}{\tan\alpha} + \frac{1}{\sin\alpha}\right).$$

This equation may be readily solved for a in terms of the given quantities P and α.

▶EXAMPLE 4 ***Height of a mountain*** To find the height of a mountaintop, a surveyor locates two accessible points A and B, as shown in Figure 6, and obtains these measurements:

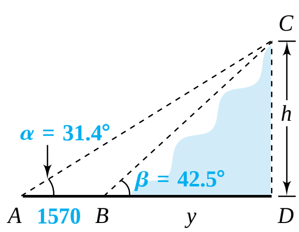


FIGURE 6

Strategy: The diagram contains two right triangles, ACD and BCD. Use both triangles to get equations that involve y and h, then eliminate y and solve the resulting equation for h. Do all of the algebra first and then use your calculator for the final evaluation. Round off to three significant digits.

$$|\overline{AB}| = 1570 \text{ feet}, \ \alpha = 31.4°, \ \beta = 42.5°.$$

Find the height h of the mountain.

Solution

Let $|\overline{CD}| = h$ and $|\overline{BD}| = y$. From right triangle ACD, $\tan\alpha = \frac{h}{1570 + y}$, or

$$h = 1570 \tan\alpha + y \tan\alpha. \tag{1}$$

From the right triangle BCD, $\tan\beta = \frac{h}{y}$, or $y = \frac{h}{\tan\beta}$. Substitute into Equation (1) to get

$$h = 1570 \tan\alpha + h\frac{\tan\alpha}{\tan\beta}.$$

Now solve for h.

$$h - h\frac{\tan\alpha}{\tan\beta} = 1570\tan\alpha$$

$$h\left(\frac{\tan\beta - \tan\alpha}{\tan\beta}\right) = 1570\tan\alpha$$

$$h = \frac{1570\tan\alpha \cdot \tan\beta}{\tan\beta - \tan\alpha} = \frac{1570 \cdot \tan 31.4° \cdot \tan 42.5°}{\tan 42.5° - \tan 31.4°} \approx 2870.4399$$

Rounding off to three significant digits, h is 2870 feet. ◀

It is a good (and efficient) practice to first perform all the necessary algebra, as in Example 4, before doing any calculations. Finding a final formula for h before any evaluation avoids the need to record and use intermediate answers, a practice that often leads to accumulated rounding error.

▶EXAMPLE 5 ***Triangle in a circle*** Triangle ABC is inscribed in a circle of diameter 7.20 cm, as shown in Figure 7, where $\overline{AB}$ is a diameter, O is the center of the circle, and $\alpha = 28.0°$. Find the length of **(a)** chord $\overline{BC}$ and **(b)** circular arc $\overset{\frown}{BC}$.

Strategy: **(a)** Since $\overline{AB}$ is a diameter, $\triangle ACB$ is a right triangle. Solve for $\overline{BC}$. **(b)** For arc length, $s = r\theta$, so we need the central angle θ (in radians). Use isosceles $\triangle AOC$ to relate θ to α. Round off to three significant digits.

Solution

(a) Use the important fact from geometry that any angle inscribed in a half-circle (one that subtends half of the circumference) is a right angle. Since $\overline{AB}$ is a diameter, then $\angle ACB$ is a right angle, so ABC is a right triangle. Then,

$$\sin\alpha = \frac{|\overline{BC}|}{|\overline{AB}|} \quad \text{so} \quad |\overline{BC}| = |\overline{AB}|\sin\alpha$$

Therefore, $|\overline{BC}| = 7.2\sin 28.0° \approx 3.3802 \approx 3.38$.

(b) Let s denote the length of arc $\overset{\frown}{BC}$. From Section 5.1, $s = r\theta$, where θ is measured in radians. First, determine θ using another important fact from geometry, that the measure of any angle inscribed in a circle (such as α) is half the measure of the central angle that subtends the same arc (in this case, θ). This gives $\theta = 2\alpha = 56.0°$. Hence for arc length s,

$$s = r\theta = 3.6\left(56 \cdot \frac{\pi}{180}\right) \approx 3.5186.$$

Rounding off to three significant digits, the length of the chord is 3.38 centimeters and the length of the circular arc is 3.52 centimeters. ◀

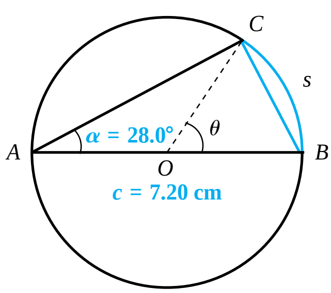


FIGURE 7

▶EXAMPLE 6 ***How much volume is left?*** A conical tank is shaped so that it holds two spheres, one of radius 4 and the other of radius 2, so that they are tangent to the sides of the tank and to each other. A cross-section is shown in the diagram on page 384.

(a) Find the dimensions, r and h, of the tank.
(b) How much will the tank hold when the two spheres are in place?

Strategy: **(a)** Use similar right triangles to solve for x, r, and h. **(b)** The volume remaining equals the volume of the tank less the volume of the two spheres.

Solution

(a) As indicated in the diagram in Figure 8 the height is given by $h = 12 + x$, where x is the distance from the bottom of the smaller sphere to the tip of the

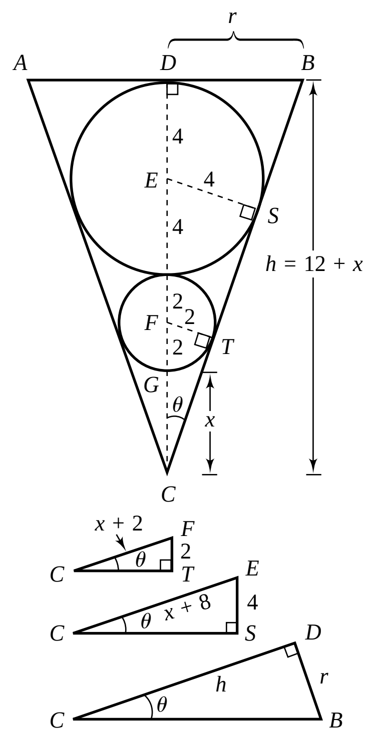


FIGURE 8

cone. In the diagram there are three similar triangles all sharing the acute angle θ, $\triangle CTF$, $\triangle CSE$, and $\triangle CDB$. In the first two, we have

$$\sin\theta = \frac{2}{x+2} = \frac{4}{x+8},$$

from which we can solve for x, $x = 4$. Replacing x by 4, in $\triangle CTF$ the hypotenuse is 6, so the hypotenuse is three times as long as the short leg.

Since all three triangles are similar, the hypotenuse of $\triangle CDB$ must equal $3r$, and $h = 12 + x = 16$. The sides of $\triangle CDB$ thus have lengths 16, r, and $3r$. By the Pythagorean theorem.

$$16^2 + r^2 = (3r)^2,$$
$$8r^2 = 256 \quad \text{or} \quad r = 4\sqrt{2}.$$

The dimensions of the conical tank are given by $r = 4\sqrt{2}$ and $h = 16$.

(b) Follow the strategy. If V_1 is the volume of the tank, then

$$V_1 = \frac{1}{3}\pi r^2 h = \frac{1}{3}512\pi.$$

From the formula for the volume of the sphere, the spheres take up a total volume of

$$\frac{4}{3}\pi(4)^3 + \frac{4}{3}\pi(2)^3.$$

Hence the volume remaining in the tank when the spheres are in place is given by

$$V = \frac{1}{3}512\pi - \frac{1}{3}256\pi - \frac{1}{3}32\pi = \frac{224}{3}\pi. \blacktriangleleft$$

EXERCISES 7.1

Check Your Understanding

In the following exercises, assume the standard notation for the sides and angles of a right triangle, where $\gamma = 90°$.

Exercises 1–5 True or False. Give reasons.

1. There is no right triangle in which one of the angles is 100°.

2. There is a right triangle with sides $a = 2$, $b = 3$, and $c = 4$.

3. There is exactly one right triangle with angles $\alpha = 32°$ and $\beta = 58°$.

4. In a right triangle, if $c = 6$ and $\beta = 32°$, then the area is $9 \sin 64°$.

5. If in a right triangle the length of each leg (a and b) is doubled, then the area of the resulting triangle is also doubled.

Exercises 6–10. Fill in the blank so that the resulting statement is true. For Exercises 6–7, in a right triangle a and α are given.

6. A formula for c is $c =$ ______.

7. A formula for the perimeter P is $P =$ ______.

For Exercises 8–10, in a right triangle, $a = 8$ and $c = 16$.

8. Side $b =$ ______.

9. Angle $\alpha =$ ______.

10. Angle $\beta =$ ______.

Develop Mastery

Use the guidelines stated in this section for rounding off results. In each case before performing computations with your calculator, first write a formula for the desired quantity in terms of the given data. Label angles and sides of a right triangle following the convention used in this section.

Exercises 1–10 ***Solving Right Triangles*** *Two parts of a right triangle are given. Find the remaining angles or sides.*

1. $a = 3.7$, $\alpha = 36°$ **2.** $b = 7.3$, $\beta = 42°$

3. $b = 35$, $\alpha = 27°$ **4.** $a = 56$, $\beta = 48°$

5. $c = 23.7$, $\beta = 65°20'$ **6.** $c = 4.36$, $\alpha = 53°40'$

7. $a = 73$, $b = 56$ **8.** $a = 0.725$, $b = 0.386$

9. $a = 21.4$, $c = 36.8$ **10.** $a = 1648$, $c = 2143$

Exercises 11–16 ***Area*** *Information about a right triangle is given. Find its area.*

11. $\alpha = 34°$, $b = 0.48$ **12.** $\beta = 63°$, $a = 1.4$

13. $c = 1.56$, $\alpha = 52.4°$ **14.** $c = 0.843$, $\beta = 57.3°$

15. $c = 2.53$, $a = 1.36$ **16.** $c = 7.52$, $b = 3.84$

17. A line passes through the two points (2, 6) and (4, 10). Find the acute angle (to the nearest degree) that it makes with the x-axis.

18. A line passes through the points $(-1, 2)$ and (5, 8). Find the acute angle that it makes with the y-axis.

19. Find the perimeter of the right triangle with $a = 1.6$ and $\alpha = 47°$.

20. Find the perimeter of the right triangle with $c = 4.73$ and $\beta = 38.5°$.

21. One angle is 63°15′ in a right triangle with perimeter 43.71 cm. Find the lengths of the two legs.

22. An angle is 26.3° in a right triangle with perimeter 7.45 cm. Find the lengths of the two legs and the hypotenuse.

23. One leg is 3.20 cm in a right triangle with area 5.68 cm^2. Find the length of the other leg and the angle opposite the given leg.

24. If the hypotenuse of a right triangle is c and its area is K,

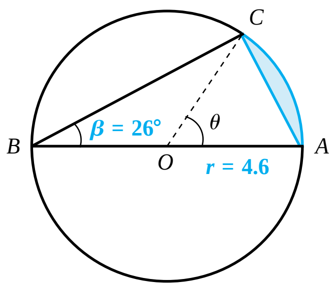
(a) show that $K = \left(\frac{1}{2}\right) c^2 \sin \alpha \cos \alpha$, where one angle is α.

(b) Use identity (I–12) in Section 6.2 to show that $K = \left(\frac{1}{4}\right) c^2 \sin 2\alpha$.

(c) For $K = 25$ cm^2 and $c = 12$ cm, find angle α and the lengths of the two legs.

25. A rope has one end tied to the top of a flagpole. When it hangs straight down it is 2 feet longer than the pole. When the rope is pulled tight with the lower end on the ground, it reaches 8 feet from the base of the pole. How high is the flagpole? (See the epigraph at the beginning of this section.)

Exercises 26–29 Use the diagram where triangle ABC is inscribed in a circle with center O and radius 4.6. $\overline{AB}$ *is a diameter and* $\beta = 26°$.



26. Find angle θ and the lengths of chord $\overline{AC}$ and arc $\overset{\frown}{AC}$.

27. Find the area of triangle ABC.

28. Find the area of **(a)** triangle AOC, **(b)** the circular sector with central angle θ, and **(c)** the shaded segment of the circle.

29. Find the area of **(a)** triangle BOC, and **(b)** the circular sector with arc $\overset{\frown}{BC}$.

30. Find the area of an equilateral triangle with a side of length 16 cm.

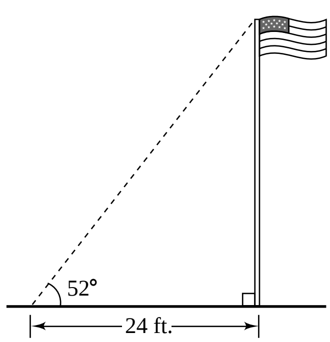
31. The equal sides of an isosceles triangle are 16.0 cm long and the included angle is 58.0°. Find the perimeter and area of the triangle.

32. The equal sides of an isosceles triangle are 3.48 cm and the equal angles are 52.6°. Find the perimeter and the area of the triangle.

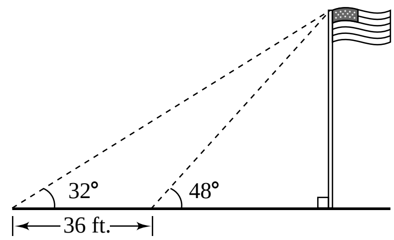
33. A regular polygon (one with n equal sides) is inscribed in a circle of radius 24 cm. Find the area of the region inside the polygon if **(a)** $n = 3$ (an equilateral triangle), **(b)** $n = 4$ (a square), **(c)** $n = 6$ (a hexagon), **(d)** $n = 12$ (a dodecagon).

Exercises 34–35 ***Height of a Flagpole*** *Find the height of the flagpole shown in the diagram.*

34.



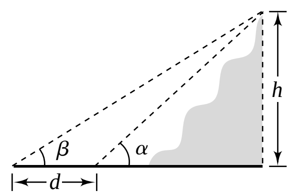
35.



36. ***Height of Mountain***

(a) Find the height h of the mountain peak shown in the diagram in terms of α, β and d.

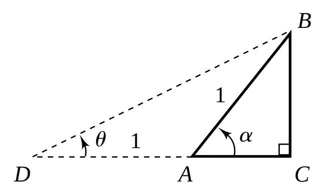
(b) A surveyor finds that $\alpha = 43°$, $\beta = 32°$, and $d = 750$ ft. What is the height of the peak?



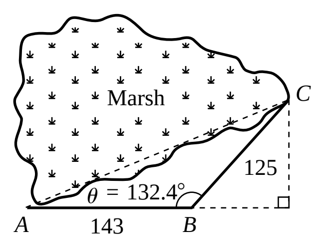
37. ***Half-Angle Formula*** Prove identity (I–17) (Section 6.3) for acute angles using the diagram. The hypotenuse of triangle ABC is 1 and side AC is extended 1 unit to D, so $|\overline{AD}| = 1$.

(a) Show that $\theta = \dfrac{\alpha}{2}$.

(b) Show that $\tan \dfrac{\alpha}{2} = \dfrac{\sin \alpha}{1 + \cos \alpha}$,

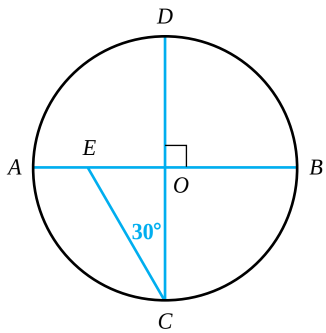


38. ***Inaccessible Distance*** A civil engineer wishes to determine the distance across a marshy area between points A and C. An accessible point B is located and $|\overline{AB}|$, $|\overline{BC}|$, and angle θ are measured. (See the diagram.) $|\overline{AB}| = 143$ feet, $|\overline{BC}| = 125$ feet, $\theta = 132.4°$. Find $|\overline{AC}|$.



39. ***Volume of a Cone*** A sector with a central angle of 60.0° is cut from a circular piece of tin with a radius of 25.0 cm. The edges of the remaining piece are joined together to form a cone. Find the volume of the cone. (See the inside cover for a formula for the volume of a cone.)

40. In the diagram $\overline{AB}$ and $\overline{CD}$ are perpendicular diameters of a circle with center at O, point E is on $\overline{AB}$, and $\angle ECO = 30°$. Find the ratio of $|\overline{EO}|$ to $|\overline{AE}|$.



41. Given the circle with equation $x^2 + y^2 - 4y = 0$ and point $P(5, 2)$, draw a diagram to show the circle and the two lines from P that are tangent to the circle. If the points of tangency are A and B, find the angle between the tangent lines, $\angle APB$.

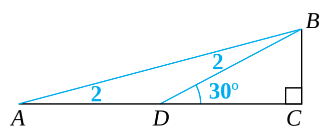
42. Line L is tangent to circle $x^2 + y^2 - 10y = 0$ at point $P(3, 1)$. Find the acute angle that L makes with the x-axis. (*Hint:* Draw a diagram.)

43. In the diagram $|\overline{AD}| = |\overline{DB}| = 2$, and $\angle BDC = 30°$.

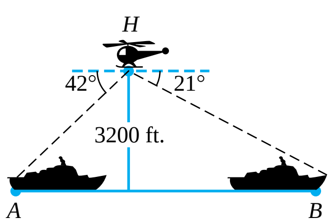
(a) Show that $|\overline{CD}| = \sqrt{3}$ and that $|\overline{AB}| = 2\sqrt{2 + \sqrt{3}}$.

(b) Show that $\angle BAD = 15°$.

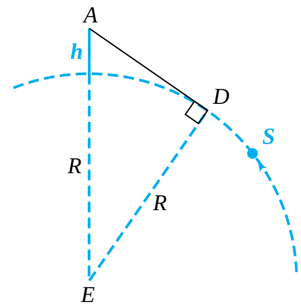
(c) Find sin 15° and cos 15° in exact form. Check your results by calculator.



44. A helicopter H and two ships (A and B) are in the same vertical plane, as shown in the diagram. The pilot finds that the angles of depression of A and B are 42° and 21°, respectively. If the altitude h of the helicopter is 3200 feet, find the distance between the two ships.

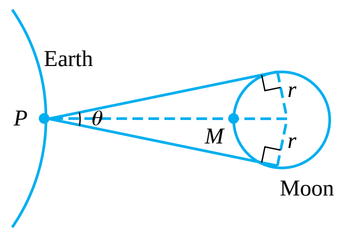


45. ***Distance to Horizon*** A lighthouse is located on the shore of the Atlantic Ocean. The top of the lighthouse (point A in the diagram) is at an elevation of h feet above sea level and a ship S is sailing from Europe toward the lighthouse. Express the distance $d = |\overline{AD}|$ (in miles) at which the ship can first see the light from A (along the tangent line AD) as a function of h. In the diagram E is the center of the earth, and R is the radius of the earth (=3960 miles). Since h is small compared to R, show that a good approximation of d is $1.22\sqrt{h}$ miles. See Develop Mastery Exercise 43, Section 1.1.

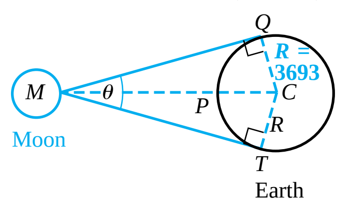


46. The Nauset Light on Cape Cod rises to 114 feet above the shore of the Atlantic Ocean. How far out will a ship be able to see its beacon? Use Exercise 45.

47. ***Size of the Moon*** To determine the radius of the moon, a person on the earth at point P measures the angle θ subtended by the moon to be 0.513°. The distance d from the earth to the moon ($|\overline{PM}|$ in the diagram) is about 239,000 miles. Find the radius r of the moon. $\left(\textit{Hint: } \text{Show that } r = \dfrac{d \sin(\theta/2)}{1 - \sin(\theta/2)}.\right)$



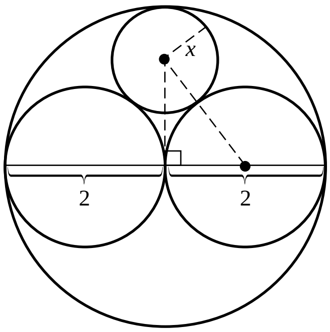
48. ***How Far to the Moon?*** Neil Armstrong and Edwin Aldrin made the first landing on the moon on July 20, 1969. Suppose that these men on the surface of the moon at point M measured the angle θ intercepted by the earth to be $\angle QMT = 1.868°$. (See the diagram, where C is the center of the earth and the radius R of the earth is known to be 3963 miles.) Find the distance d from the moon to earth's surface; that is, find $|\overline{MP}|$.



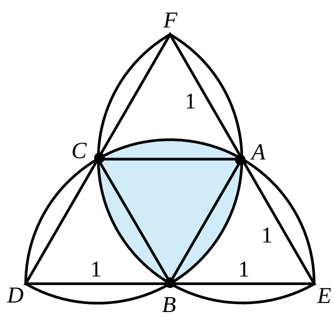
49. In the diagram of the cross-section for Example 6, find the area of the region that is inside $\triangle ABC$ and outside both circles.

50. Solve the problem in Example 6 if the radii of the spheres are 3 and 2.

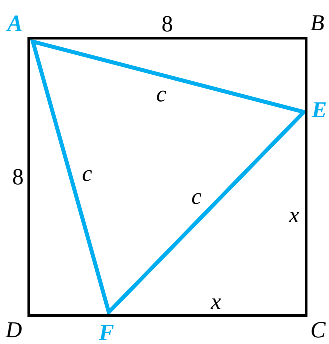
51. Two circles of diameter 2 are tangent to each other and to the inside of a circle of radius 2, and a smaller circle is tangent to all three others as shown. What is the radius of the small circle?



52. The 2″ sides of an equilateral triangle DEF form the diameters of three semicircles. Find the area of the region common to all three semicircles. Give the answer in exact form. (*Hint:* Find the area of $\triangle ABC$ and the area of the segment between the arc $\overset{\frown}{AC}$ and the chord $\overline{AC}$.)

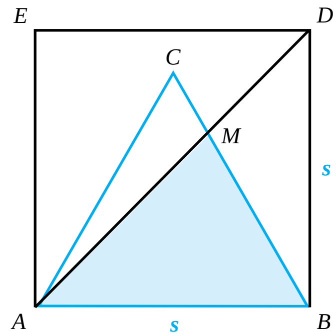


53. An equilateral triangle is inscribed in a square having sides of length 8.0, as in the diagram. Find the length c of the sides of the triangle. Give answer in exact form. (*Hint:* First find x.)



54. In the diagram for Exercise 53, suppose $\triangle AEF$ is isosceles with $|\overline{AE}| = |\overline{AF}|$ and $|\overline{EF}| = 4.0$. Find the length of the equal sides.

55. An equilateral triangle ABC is placed inside a square $ABDE$ with sides of length s. (See the diagram.) The diagonal $\overline{AD}$ of the square intersects $\overline{BC}$ at point M. Find the area of $\triangle ABM$ as a function of s.



Exercises 56–57 In right triangle ABC, suppose the hypotenuse c $(=|\overline{AB}|)$ *and the altitude h, from C to* $\overline{AB}$, *are known. Draw a diagram.*

56. ***Strategy***

(a) Give a verbal description of a strategy you would use to determine the product $a \cdot b$. (*Hint:* Consider area.)

(b) After you know $a \cdot b$, describe a strategy to find the sum $a + b$. (*Hint:* Consider $(a + b)^2$.)

57. ***Explore***

(a) For a given value of c, what values can be assigned to h? For instance if $c = 12$, can h be 4?, 6? 10? . . . .

(b) For given values of c and h, is it possible to determine unique values of a and b?
(*Hint:* Consider $\triangle ABC$ as being inscribed in a circle with $\overline{AB}$ as a diameter. From geometry, what do you know about the angle at C?)

7.2 LAW OF SINES

> Since that summer . . . when I . . . tasted the fruits of discovery, I have not wanted to do anything except mathematics or, more accurately, mathematics and its applications . . . I had been stricken by an acute attack of a disease which at irregular intervals afflicts all mathematicians and, for that matter, all scientists: I became obsessed by a problem.
>
> *Mark Kac*

In Section 7.1 we studied techniques for solving right triangles. However, one frequently encounters triangles in which none of the angles is 90°. Such triangles are usually referred to as **oblique triangles.** Many problems that involve oblique triangles can be solved by using right triangles as the preceding section showed. However, we can derive formulas that provide more efficient methods for solving oblique triangles. For instance, in Example 4 of Section 7.1, we saw how we could determine the height of a mountaintop where we used right triangles but required elaborate algebraic manipulations. After introducing the Law of Sines in this section, we show a simpler solution. See Example 2.

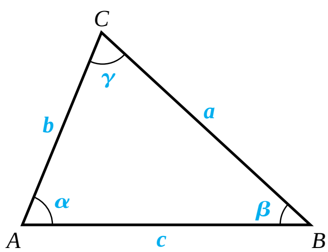


FIGURE 9

We continue to work with the parts of a triangle, the three angles and the three sides. We use the convention for labeling parts of a triangle introduced in Section 7.1, as shown in Figure 9, where a is opposite angle α, b is opposite β, and c is opposite γ.

Solving Triangles

Our primary interest in this section and in the following one, is to develop techniques for solving oblique triangles. As before, to solve a triangle means that we are given sufficient information about its angles and sides to specify a triangle and we determine the remaining angles and sides.

> *Just because my mathematics has its origin in a real problem doesn't make it less interesting to me—just the other way around. I find it makes the puzzle I am working on all the more exciting.*
>
> George Dantzig

One might first ask what information is sufficient to determine a triangle? In general, we need to know three of the six parts, but this does not mean any three parts. For instance, knowing the three angles does not describe a specific triangle, since many triangles have the same three angles. However, three sides uniquely determine a triangle, and we shall see how to proceed to find the three angles. Of course, this assumes that the given numbers a, b, and c are such that the sum of the two smaller sides is greater than the other side. For instance, sides $a = 2$, $b = 3$, and $c = 6$ would not form a triangle.

We can classify problems of solving triangles into the following four cases based on the given parts.

Case 1 One side and two angles (*SAA* or *ASA*)
Case 2 Two sides and the angle opposite one of them (*SSA*)
Case 3 Two sides and the included angle (*SAS*)
Case 4 Three sides (*SSS*)

In this section we develop the Law of Sines and see how it can be used to solve *Case 1* triangles. Then we use right triangle trigonometry to solve *Case 2* triangles. In the next section we will deal with *Cases 3* and *4* when we introduce the Law of Cosines.

Law of Sines and Its Application

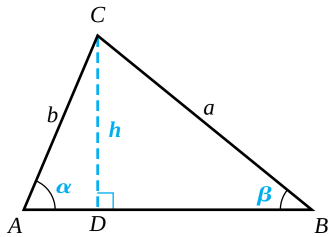


FIGURE 10

Here we derive formulas that relate the angles and sides of a triangle. In triangle ABC we draw a perpendicular (altitude $\overline{CD}$) from vertex C to side $\overline{AB}$ as shown in Figure 10. Let $h = |\overline{CD}|$. From the two right triangles ADC and BDC, we get:

$$\sin\alpha = \frac{h}{b} \quad \text{and} \quad \sin\beta = \frac{h}{a}, \quad \text{so}$$

$$h = b\sin\alpha \quad \text{and} \quad h = a\sin\beta$$

Since $b\sin\alpha$ and $a\sin\beta$ are both equal to h, we get $a\sin\beta = b\sin\alpha$. Dividing both sides by $\sin\alpha\sin\beta$ gives

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}.$$

In a similar manner,

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \quad \text{and} \quad \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

The three equations derived here make up the **Law of Sines,** which can be written in compact form.

Law of sines

Suppose α, β, and γ are the three angles of a triangle, and a, b, and c are the sides opposite those angles, respectively. Then we have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

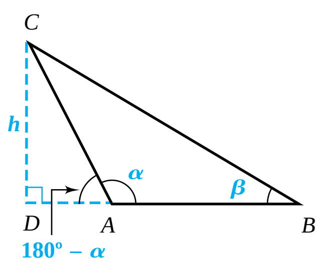


FIGURE 11

The triangle in Figure 11 is such that the altitude from vertex C is inside the triangle. If the altitude $\overline{CD}$ falls outside triangle ABC, as shown in Figure 11, then the derivation of

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

follows from the reduction formula $\sin(180° - \alpha) = \sin\alpha$.

The Law of Sines can be used to solve triangles where the given parts include two angles and a side, or two sides and an angle opposite one of them (Cases 1 and 2). However, the Law of Sines cannot handle problems of the types in Cases 3 and 4. To see this, let us draw circles around the given parts in the Law of Sines equations. For Case 3, suppose, a, b, and γ are given (two sides and the included angle).

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Now we have three equations, but it is clear that each equation involves two unknown parts.

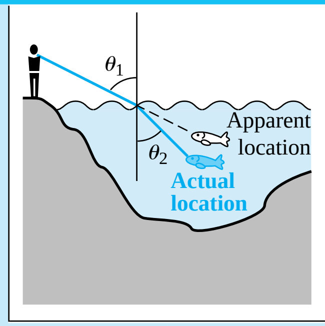
In a similar manner, for Case 4, where we are given the three sides, we have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

HISTORICAL NOTE SNELL'S LAW AND FIBER OPTICS

One exciting area of trigonometric applications links modern technology with experimental observations made more than three hundred years ago.

The speed of light in a vacuum (186,000 miles per second) is one of the important fundamental physical constants, but light slows down when passing through a material such as water or glass. In consequence, a light ray is bent, or refracted, when it passes from one medium to another, say from water to air. A person standing in water appears to have shortened legs, and a fish in water is not located where our eyes see it.



The amount of bending is related to the speed of light in each medium. The relationship discovered by the Dutch mathematician Willebrord Snellius (or Snell) about 1624, may be expressed as

$$\frac{\sin\ \theta_1}{v_1} = \frac{\sin\ \theta_2}{v_2},$$

where the angles are shown in the figure and v_1 and v_2 are the velocities of light in the two materials (air and water in the figure).

When a light ray strikes a surface at a very small angle, the ratio of sines in Snell's law implies that the ray will be completely reflected back into the same medium. This phenomenon, *total internal reflection*, is the basis for the new technology of fiber optics. Light entering one end of a tiny glass fiber is transmitted faithfully to the other end even though the fiber may be bent into curious shapes.

Narrow, flexible "light pipes" allow physicians to examine the interior of a patient's stomach or intestine, or even a beating heart. Knee surgery can now be done with less trauma for the patient by the use of fiber optics in the arthroscope. High fidelity sound can even be transmitted for long distances through optical channels.

Each of the three equations has two unknown parts.

The following examples illustrate techniques for solving triangles where the given parts are described by *SAA* and *SSA* (Cases 1 and 2)

►EXAMPLE 1 ***Two angles and a side*** Suppose $\alpha = 43°$, $\beta = 72°$, and $a = 5.4$. Find γ, b, and c.

Solution

First, it is always helpful to draw a diagram of the triangle, and label the given data as shown in Figure 12. Angle γ can be determined by using $\alpha + \beta + \gamma = 180°$.

$$\gamma = 180° - (\alpha + \beta) = 180° - (43° + 72°) = 65°$$

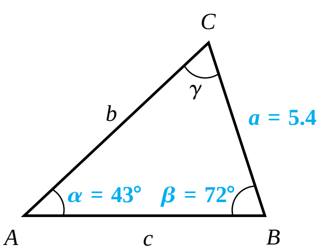


FIGURE 12

To find b, use $\dfrac{b}{\sin\ \beta} = \dfrac{a}{\sin\ \alpha}$, or $b = \dfrac{a\ \sin\ \beta}{\sin\ \alpha}$. Similarly, $c = \dfrac{a\ \sin\ \gamma}{\sin\ \alpha}$.

$$b = \frac{5.4\ \sin\ 72°}{\sin\ 43°} \quad \text{and} \quad c = \frac{5.4\ \sin\ 65°}{\sin\ 43°}$$

Evaluate and round off to two significant digits to get 7.5 for b and 7.2 for c. ◄

► EXAMPLE 2 ***Height of a mountain*** In Figure 13 h represents the height of a mountaintop. A surveyor finds the measurements

$$\theta = 42.5°, \quad \beta = 31.4°, \quad c = |\overline{AB}| = 648 \text{ ft.}$$

Find h. (See Example 4 of Section 7.1 for a different solution to a similar problem.)

Strategy: First find the angles of $\triangle ABC$ and then use the Law of Sines to solve for a, which is the hypotenuse of right $\triangle ADC$. Use right-triangle relations to get h.

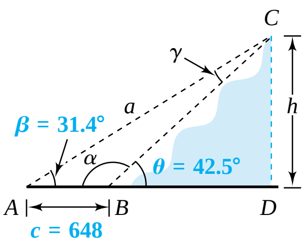


FIGURE 13

Solution

Follow the strategy.

$$\alpha = 180° - \theta = 180° - 42.5° = 137.5°$$
$$\gamma = 180° - (\alpha + \beta) = 180° - 168.9° = 11.1°$$

From the Law of Sines, $a = \dfrac{c \sin \alpha}{\sin \gamma}$. Therefore, from right triangle ADC,

$$h = a \sin \beta = \left(\frac{c \sin \alpha}{\sin \gamma}\right) \sin \beta = \frac{c \sin \alpha \sin \beta}{\sin \gamma}$$

$$h = \frac{648 \sin 137.5° \sin 31.4°}{\sin 11.1°} = 1184.74$$

To three significant digits, h is 1180 feet. ◄

Observe that we did not compute the value of a before calculating h. As noted in Example 4 of Section 7.1, obtaining a complete expression for h before doing any calculations is more efficient and accurate than recording and using any intermediate computations.

► EXAMPLE 3 ***Area of a triangle*** Find the area of the triangle for which $\alpha = 60°$, $\gamma = 48°$, and $c = 25$ cm.

Strategy: The area equals half of the base times the height, where the height is the length of the altitude to any side chosen as base. Draw a diagram with the given parts and altitude h. Express the remaining parts of the triangle and h in terms of the given data using the Law of Sines and a right triangle.

Solution

Follow the strategy. Figure 14 shows an altitude from vertex A to side $\overline{BC}$. The formula for the area of a triangle is

$$\text{Area} = \left(\frac{1}{2}\right) \cdot (\text{base}) \cdot (\text{height}).$$

Therefore, Area $= \frac{1}{2}ah$, where a is the base. First, find a and h.

$$\beta = 180° - (60° + 48°) = 72°.$$

Using the Law of Sines to find a and right triangle ADB to find h,

$$a = \frac{c \sin \alpha}{\sin \gamma} = \frac{25 \sin 60°}{\sin 48°} \quad \text{and} \quad h = c \sin \beta = 25 \sin 72°.$$

Finally,

$$\text{Area} = \frac{1}{2}ah = \frac{1}{2}\left[\frac{25 \sin 60°}{\sin 48°}\right](25 \sin 72°) \approx 346.3.$$

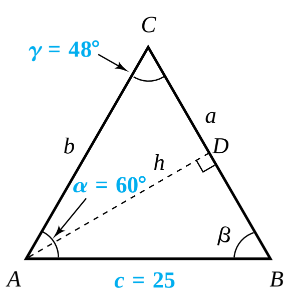


FIGURE 14

To be consistent with the given data, round off to two significant digits, and record the area as 350 cm^2. ◄

Ambiguous Case (*SSA*)

Two sides and a nonincluded angle may determine *one, two,* or *no* triangles. It is the possibility of two different triangles sharing the same sides and angle that has given this case (Case 2, *SSA*) the name **ambiguous case.** Figure 15 illustrates several possibilities for a given set of information, say a, b, and α. It is clear that a given set, *SSA* does not always determine a unique triangle. The figure shows how some people find it helpful to visualize a given side $\overline{AC}$ (length b) making an angle α with the horizontal (opposite C), and then "hinging" the other side, $\overline{CB}$ (length a) at C and letting it "swing," to see if it can reach the horizontal. By comparing $\overline{CB}$ with the altitude $h(=b\sin\alpha)$, we can see the different possibilities.

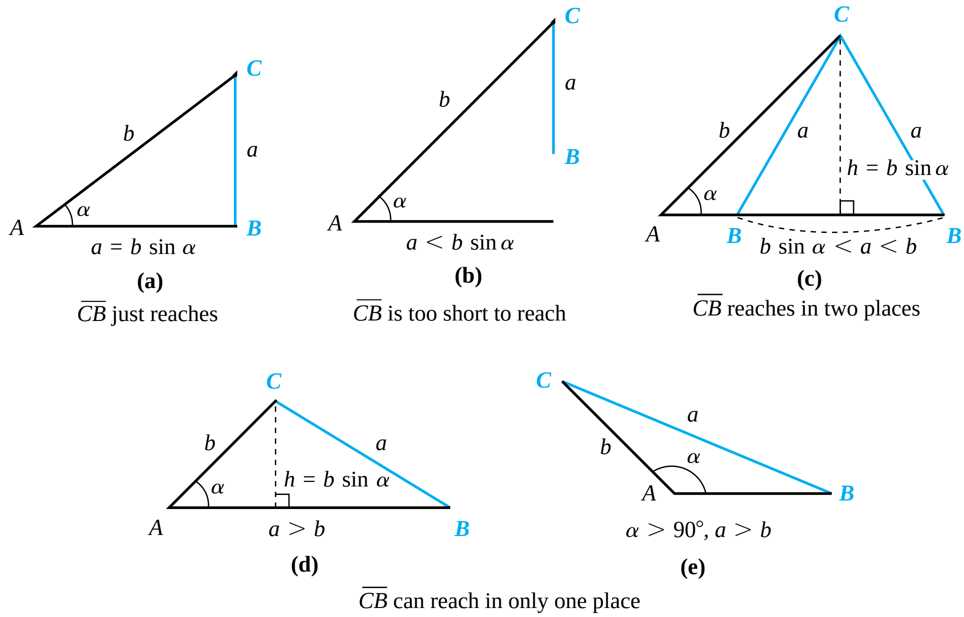


FIGURE 15

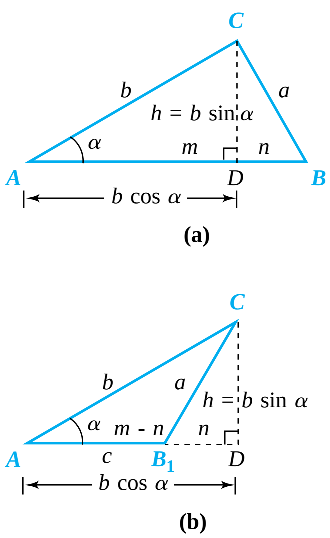


FIGURE 16

To get an algebraic characterization of the possibilities pictured in Figure 15, we look more carefully at the situation in Figure 15c. In Figure 16 we show the two possibilities separately, where $\triangle ABC$ and $\triangle AB_1C$ both have sides of lengths a and b and angle α, and where in both diagrams the right triangle ACD is the same. If $|\overline{CD}| = h$ and $|\overline{AD}| = m$, then $\sin\alpha = h/b$, and $\cos\alpha = m/b$, so that we have

$$h = b\sin\alpha \quad \text{and} \quad m = b\cos\alpha.$$

Similarly $\triangle BCD$ and $\triangle B_1CD$ are congruent, so if we let $n = |\overline{BD}| = |\overline{B_1D}|$, then by the Pythagorean theorem,

$$\begin{aligned} n &= \sqrt{a^2 - h^2} = \sqrt{a^2 - (b\sin\alpha)^2} \\ &= \sqrt{a^2 - b^2\sin^2\alpha}. \end{aligned}$$

$$\begin{aligned} &\text{In } \triangle BCD, \text{ we have } c = m + n = b\cos\alpha + \sqrt{a^2 - b^2\sin^2\alpha}; \\ &\text{in } \triangle B_1CD, \text{ we have } c = m - n = b\cos\alpha - \sqrt{a^2 - b^2\sin^2\alpha}. \end{aligned}$$

These formulas summarize the ambiguous case and give us all the information needed to determine the number of solutions (if any) for any given a, b, α.

Either geometrically or algebraically, it is the comparison of a with $b\sin\alpha$ that determines the nature of the solutions.

Summary of the ambiguous case (SSA)

If a, b, and α are given, then

$$c = b\cos\alpha \pm \sqrt{a^2 - b^2\sin^2\alpha} \tag{1}$$

(i) If $a^2 - b^2\sin^2\alpha < 0$, then there is *no solution* ("too short") (Fig. 15b).
(ii) If $a^2 - b^2\sin^2\alpha = 0$, then there is *one solution* (a right triangle) (Fig. 15a).
(iii) If $a^2 - b^2\sin^2\alpha > 0$, then there are either

two solutions if $b\cos\alpha - \sqrt{a^2 - b^2\sin^2\alpha} > 0$, (Fig. 15c) or

one solution (Fig. 15d, e) if not.

Suggestion: In solving problems for the *SSA* case, rather than memorizing the formula in Equation (1), it is wise to begin with a figure (as we recommend whenever possible anyway) and use right triangle trigonometry. Such an approach is easy to remember and provides greater understanding. In many instances, it is easier to use the Law of Sines for the *SSA* case, as illustrated in the next two examples.

►EXAMPLE 4 **SSA using law of sines** Suppose $a = 75$, $b = 63$, and $\alpha = 54°$. Find β, γ, and c.

Strategy: Draw a diagram. Solve for $\sin\beta$ by the Law of Sines and use Sin^{-1} to find β and then $\gamma = 180° - \alpha - \beta$. Remember that there are two angles between 0° and 180° having the same sine. If only one β fits the diagram, we have a unique solution; if there are two, we have two solutions.

Solution

First draw a diagram that shows the given data; try drawing angle α first (see Figure 17). From the diagram it is clear that we have the situation in Figure 15d, where $a > b$, so there is just one solution. Using the Law of Sines we may find angle β as follows.

$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a}$$

$$\sin\beta = \frac{b\sin\alpha}{a} = \frac{63\sin 54°}{75} \approx 0.67957$$

$$\beta = \mathrm{Sin}^{-1}(0.67957) \approx 42.8°.$$

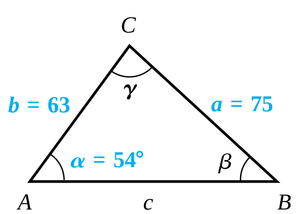


FIGURE 17

Again, we recorded the number 0.67957 for purposes of illustration. When it appears in the calculator display, evaluate Sin^{-1} to get β directly.

Now find angle γ.

$$\gamma = 180° - (54° + 42.8°) = 83.2°$$

Use the Law of Sines again to find c:

$$c = \frac{a\sin\gamma}{\sin\alpha} = \frac{75\sin 83.2°}{\sin 54°} \approx 92.05.$$

As a final step, round off answers to be consistent with the accuracy of the given data: β is 43°, γ is 83°, and c is 92. ◄

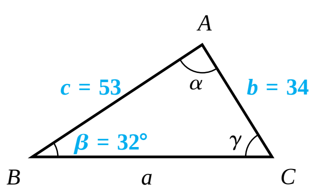


FIGURE 18

►EXAMPLE 5 **SSA: two solutions?** Suppose $b = 34$, $c = 53$, and $\beta = 32°$. Find γ, α, and a.

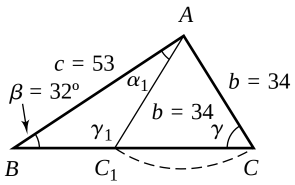


FIGURE 19

Solution

First, draw a diagram to show a triangle with the given parts (see Figure 18). To find γ, apply the Law of Sines:

$$\sin \gamma = \frac{c \sin \beta}{b} = \frac{53 \sin 32^\circ}{34} \approx 0.82605.$$

Hence, $\gamma \approx \text{Sin}^{-1}\, 0.82605 \approx 55.7^\circ$. For $\gamma = 55.7^\circ$ and $\beta = 32^\circ$, $\alpha = 180^\circ - (\beta + \gamma) = 92.3^\circ$.

To find side a, again apply the Law of Sines:

$$a = \frac{c \sin \alpha}{\sin \gamma} = \frac{53 \sin 92.3^\circ}{\sin 55.7^\circ} \approx 64.1.$$

Round off the results to the accuracy of the given data,

$$\gamma = 56^\circ \qquad \alpha = 92^\circ \qquad a = 64.$$

This gives us one solution, but from the diagram in Figure 18 we should be able to see another possible solution. The altitude from A has length $c \sin 32^\circ$, ≈ 28, so we do indeed have two solutions. From Equation (1),

$$a = c \cos \beta \pm \sqrt{b^2 - c^2 \sin^2 \beta} \approx 64.109,\ 25.784.$$

We had already found $a = 64$; for the second solution, $a_1 = 26$. From the diagram in Figure 19, $\triangle AC_1C$ is isosceles, so $\gamma_1 = 180^\circ - \gamma \approx 124.3^\circ$ and $\alpha_1 = 180^\circ - (\beta + \gamma_1) \approx 23.7^\circ$. Rounding off, the second solution is given by

$$\gamma_1 = 124^\circ \qquad \alpha_1 = 24^\circ \qquad a_1 = 26. \blacktriangleleft$$

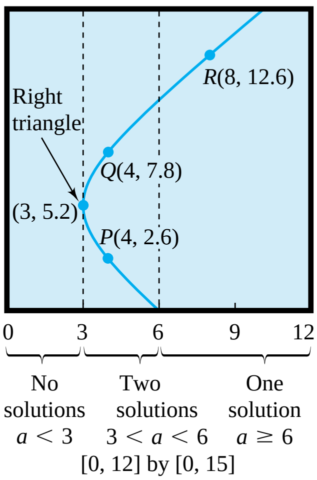


FIGURE 20

▶EXAMPLE 6 ***Ambiguous case in graphical form*** In the *SSA* Case, for fixed b and α, the length c depends on the length a, as given by Equation (1).

$$c = b \cos \alpha \pm \sqrt{a^2 - b^2 \sin^2 \alpha}$$

Suppose $b = 6$ and $\alpha = 30^\circ$.

(a) Express c as a function of a and draw a graph.
(b) For what values of a is there no triangle? one triangle? two triangles?
(c) Draw the triangles corresponding to $a = 4$ and to $a = 8$.

Solution

(a) When $b = 6$ and $\alpha = 30^\circ$,

$$b \cos \alpha = 6(\sqrt{3}/2) = 3\sqrt{3}, \text{ and } b^2 \sin^2 \alpha = 36(1/2)^2 = 9.$$

Thus

$$c = 3\sqrt{3} \pm \sqrt{a^2 - 9}.$$

(b) We want to graph `Y1 = 3√3 +√(X² − 9)` and `Y2 = 3√3 − √(X² − 9)`. In $[0, 12] \times [0, 15]$ we get a graph like Figure 20, where we have indicated the ranges of a for which there is no solution ($a < 3$), one solution ($a = 3$ or $a \geq 6$), or two solutions (when $3 < a < 6$).

(c) When $a = 4$, $c = 3\sqrt{3} \pm \sqrt{7}$, giving the points P and Q in Figure 20, and when $a = 8$, $c = 3\sqrt{3} + \sqrt{55} \approx 12.6$ (point R). The corresponding triangles are shown in Figure 21. ◀

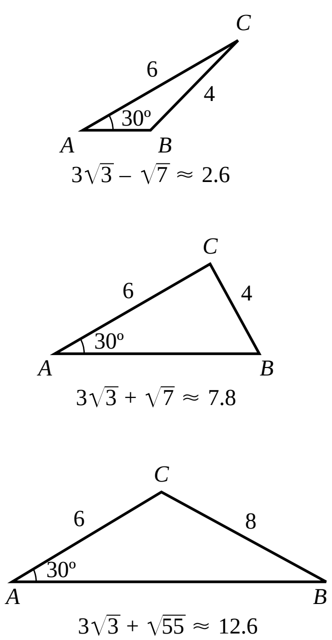


FIGURE 21

Looking Ahead to Calculus

We have mentioned several times throughout this book that exact forms for solutions of some maximum/minimum problems require calculus. Some calculus teachers are convinced that calculus is the only way to approach such problems, even though technology can give excellent approximations. The next example asks for a maximum volume, and we can get a good answer from our hand-held computers (graphing calculator); using calculus to solve the same problem would be at least as difficult, and would still have to involve some of the same kinds of technology.

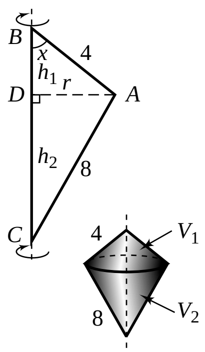


FIGURE 22

►EXAMPLE 7 Maximizing volume Triangle ABC in Figure 22 is revolved about the axis $\overline{BC}$ to give a solid consisting of the union of two cones sharing a common base of radius $|\overline{AD}|$. The volume V of the solid depends on the angle x.

(a) Find a formula for V.
(b) For what value of x is V maximum? What is the maximum volume?

Solution

(a) Thinking of the solid of revolution as the union of two cones, each generated by revolving a triangle, we can find the volume of each and then take their sum for V. The cone obtained by revolving the top triangle, $\triangle ABD$, has radius $r = |\overline{AD}|$ and height $h_1 = |\overline{BD}|$. In terms of the angle x and the hypotenuse 4, $r = 4 \sin x$ and $h_1 = 4 \cos x$, so the volume of the top cone is given by

$$V_1 = \frac{1}{3}\pi r^2 h_1 = \frac{1}{3}\pi(4 \sin x)^2(4 \cos x) = \frac{64\pi}{3}\sin^2 x \cos x.$$

For the lower cone, the Pythagorean theorem for $\triangle ACD$ gives

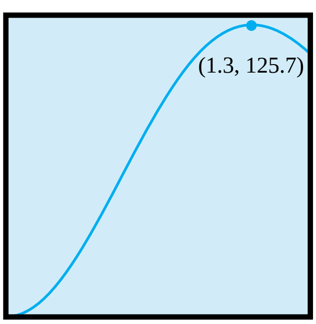
$$h_2 = \sqrt{8^2 - r^2} = \sqrt{8^2 - (4 \sin x)^2} = 4\sqrt{4 - \sin^2 x}.$$

The volume of the lower cone is then given by

$$V_2 = \frac{1}{3}\pi(4 \sin x)^2 \left(4\sqrt{4 - \sin^2 x}\right) = \frac{64\pi}{3}\sin^2 x\sqrt{4 - \sin^2 x}.$$

Putting the pieces together and simplifying, we have a formula for the volume,

$$V = V_1 + V_2 = \frac{64\pi}{3}\sin^2 x\left(\cos x + \sqrt{4 - \sin^2 x}\right).$$



[0, 1.6] by [0, 130]

FIGURE 23

(b) To find the maximum volume, we go to a graph. We are interested only in values of x less than $\pi/2$, and checking a few volume values suggests a window something like $[0, 1.6] \times [1, 130]$. The graph is shown in Figure 23. The maximum volume is about 125.7 and corresponds to an x value of about 1.3 radians, just a little less than 75°. ◄

EXERCISES 7.2

Check Your Understanding

Assume the labeling of the sides and angles of $\triangle ABC$ is as in the text.

Exercises 1–5 *True or False. Give reasons.*

Exercises 1–3 *$\triangle ABC$ has $\alpha = 53°$, $\beta = 31°$, and $b = 12$.*

1. $b < a$ and $a < c$.

2. $a = \dfrac{12 \sin 31°}{\sin 53°}$ **3.** $c = \dfrac{12 \sin 84°}{\sin 31°}$

Exercises 4–5 *A triangle has $a = 4$, $c = 8$, and $\gamma = 64°$.*

4. $\alpha = \text{Sin}^{-1}\left(\dfrac{\sin 64°}{2}\right)$

5. There is exactly one triangle with the given values for a, c, and γ.

Exercises 6–10 *Fill in the blank so that the resulting statement is true.*

6. Given α, β and a, a formula for c is $c =$ ______.

7. Given α, γ and a, a formula for c is $c =$ ______.

8. Given β, γ and a, a formula for c is $c =$ ______.

For Exercises 9–10, find the indicated side in exact form.

9. If $\alpha = 30°$, $\beta = 60°$ and $a = 6$, then $b =$ ______.

10. If $\alpha = 30°$, $\beta = 45°$ and $b = 8$, then $a =$ ______.

Develop Mastery

Round off all calculated results to be consistent with the accuracy of the given data. See guidelines in Section 7.1.

Exercises 1–12 ***Solving a Triangle*** *Three parts of a triangle are given. Find the remaining parts. Begin by drawing a diagram showing a triangle with the given parts labeled.*

1. $\alpha = 24.0°$, $\beta = 75.0°$, $a = 15.0$

2. $\beta = 47.0°$, $\gamma = 36.0°$, $a = 253$

3. $\alpha = 43.0°$, $\beta = 116.0°$, $c = 83.0$

4. $\alpha = 48.7°$, $\beta = 74.2°$, $c = 138$

5. $\beta = 31°30'$, $\gamma = 56°15'$, $b = 7.45$

6. $\alpha = 59°45'$, $\beta = 83°15'$, $a = 65.2$

7. $\alpha = 31.9°$, $\beta = 58.1°$, $b = 45.0$

8. $\alpha = 32.7°$, $\gamma = 81.4°$, $b = 4.57$

9. $a = 57$, $b = 68$, $\alpha = 56°$

10. $a = 46$, $b = 64$, $\beta = 116°$

11. $b = 3.4$, $c = 1.7$, $\beta = 124°$

12. $a = 33.0$, $c = 65.0$, $\alpha = 30.5°$

Exercises 13–16 ***Area of Triangle*** *Find the area of the triangle that has the given measurements.*

13. $\alpha = 43.0°$, $\beta = 72.0°$, $a = 24.0$

14. $\beta = 35.0°$, $\gamma = 68.0°$, $a = 43.0$

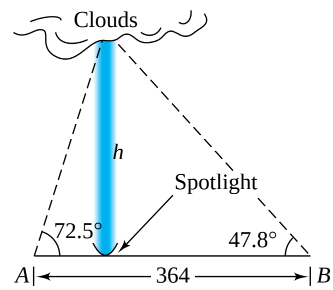
15. $\alpha = 31.9°$, $\beta = 58.1°$, $c = 53.0$

16. $a = 28.0$, $b = 45.0$, $\beta = 58.1°$

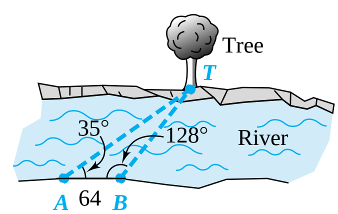
17. ***Height of Clouds*** To measure the height of clouds, a spotlight is aimed vertically. Two observers at points A and B, 364 feet apart and in line with the spotlight, measure angles α and β as shown in the diagram.

$$\alpha = 72.5° \qquad \beta = 47.8°$$

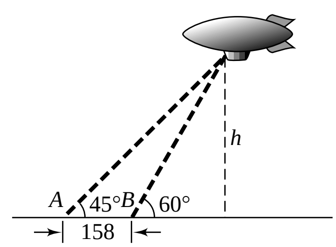
How far from the ground is the bottom of the cloud level?



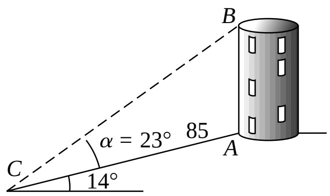
18. ***Width of River*** A surveyor who wishes to determine the width of a river sees a tree on the opposite bank. She selects two accessible points, A and B, as shown in the diagram and takes the following measurements: $|\overline{AB}| = 64$ feet, $\alpha = 35°$, $\beta = 128°$. Find the width of the river.



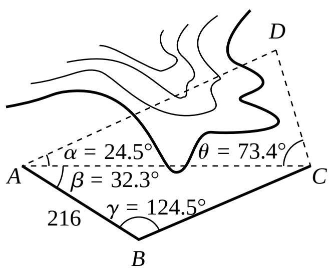
19. ***Height of Blimp*** In order to find the height of the Goodyear blimp, observers at A and B, 158 yards apart, measure the following angles: $\alpha = 45.0°$ and $\beta = 60.0°$. (See the diagram.) How high is the blimp?



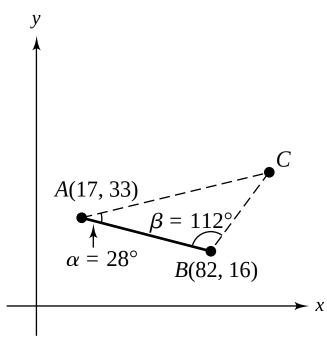
20. ***Height of Tower*** A vertical tower AB is located on a hill that is inclined at 14°. (See the diagram.) From point C, 85 feet downhill from the base A of the tower, angle α is measured and found to be 23°. What is the height of the tower?



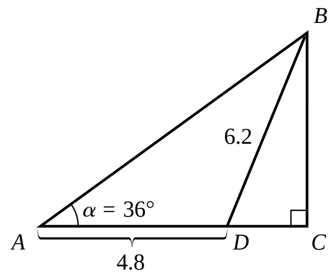
21. ***Inaccessible Distance*** Bridget wishes to find the distance from point A to an inaccessible point D. (See the diagram.) Points B and C are located and the following measurements are found: $|\overline{AB}| = 216$ ft, $\alpha = 24.5°$, $\beta = 32.3°$, $\gamma = 124.5°$, $\theta = 73.4°$. Determine the distance from A to D.



22. ***Remote Locating*** Two forest rangers are stationed at points A and B located on a coordinate system with $A(17, 33)$ and $B(82, 16)$. (See the diagram.) They spot a forest fire at point C and measure angles $\alpha = 28°$, and $\beta = 112°$. Find the coordinates of point C.



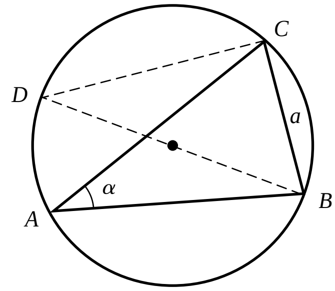
23. Use the following information to find the lengths of $\overline{BC}$ and $\overline{CD}$ in the following diagram: $\alpha = 36°$, $|\overline{BD}| = 6.2$, $|\overline{AD}| = 4.8$.



24. ***Diameter and Law of Sines*** Triangle ABC is inscribed in a circle as shown in the diagram. Show that the diameter d of the circle is given by the ratio in the Law of Sines

$$d = \frac{a}{\sin \alpha}$$

(*Hint:* Take point D on the circle so that $\overline{BD}$ passes through the center; $\overline{BD}$ is a diameter. Recall from geometry that angle BDC is equal to angle BAC, which is angle α. Also triangle BCD is a right triangle.)



25. In a triangle $a = 1.24$, $b = 1.86$, and $\beta = 2\alpha$. Find angle α in degrees rounded off to one decimal place. (*Hint:* Use the Law of Sines and a double-angle identity.)

26. In $\triangle ABC$, $c = 1.64b$ and $\gamma = 2\beta$. Find angle β in degrees rounded off to one decimal place.

Exercises 27–28 Solve the problem in Example 6 for the given value of b and α.

27. $b = 8$, $\alpha = 45°$ **28.** $b = 10$, $\alpha = 60°$

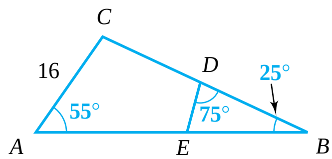
Exercises 29–30 Solve the problem in Example 7 for the given lengths of sides AB and AC.

29. $|\overline{AB}| = 5$, $|\overline{AC}| = 8$ **30.** $|\overline{AB}| = 6$, $|\overline{AC}| = 10$

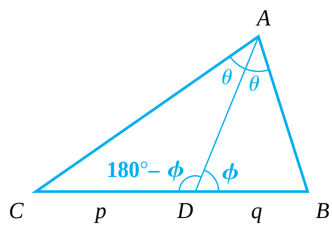
31. ***Maximum Area***
- **(a)** Express the area K of $\triangle ABC$ in Example 7 as a function of x.
- **(b)** For what value of x is K maximum? What is the maximum area?

32. Repeat Exercise 31 if $|\overline{AB}| = 3$, $|\overline{AC}| = 5$.

33. In $\triangle ABC$, D is the midpoint of $\overline{BC}$, $|\overline{AC}| = 16$, and the angles are as shown in the diagram.
(a) Find the area of $\triangle ABC$.
(b) Find the area of $\triangle BDE$.

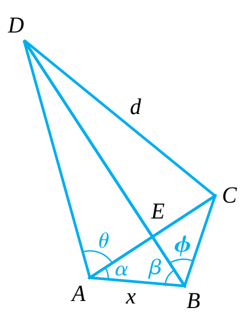


34. In $\triangle ABC$, angle α is bisected by $\overline{AD}$ as shown in the diagram, where $p = |\overline{CD}|$ and $q = |\overline{BD}|$. Show that $\frac{p}{q} = \frac{|\overline{AC}|}{|\overline{AB}|}$. (*Hint:* Use the Law of Sines for $\triangle ACD$ and for $\triangle ABD$. Also recall a reduction formula for $\sin(180° - \phi)$.)
(The result of this exercise is often used in advanced Euclidean geometry, where it is usually stated as a theorem: *Each angle bisector of a triangle divides the opposite side into segments proportional in length to the adjacent sides.*)



35. ***Leaning Tower*** The Leaning Tower of Pisa measures 184.5 feet from its base to its top. When a distance of 137.5 feet is measured along the ground from its base in the direction of its lean, the angle of elevation to the top of the tower is found to be 56.72°. At what angle (measured from the vertical) does the tower lean?

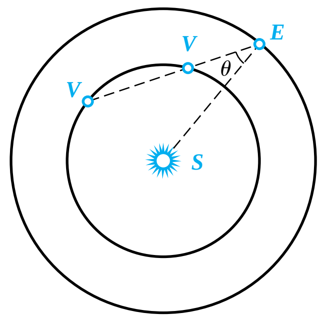
Exercises 36–37 The given information refers to the diagram. Find the length x of the segment $\overline{AB}$.



36. $\alpha = 45°$, $\beta = 45°$, $\theta = 60°$, $\phi = 55°$, and $d = 75.2$.

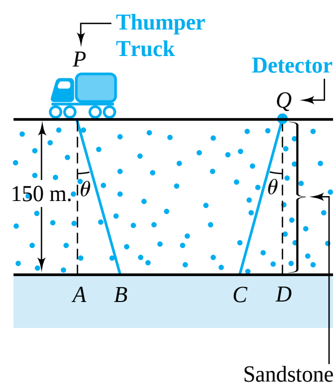
37. $\alpha = 30°$, $\beta = 60°$, $\theta = 70°$, $\phi = 35°$, and $d = 72.6$.

38. ***How Far Is Venus?*** Assume that Earth (E) and Venus (V) rotate about the sun (S) in circular orbits of radii 93 million miles and 67 million miles, respectively. Assume that both orbits lie in the same plane. An astronomer measures angle θ between the lines of sight E to S and E to V. (See the diagram.)
(a) If θ is 15°, how far is Venus from the Earth? There are two possible results.
(b) What is the largest possible value of θ? How many solutions are there for this angle?



39. ***How Far Is Mercury?*** In Exercise 38 replace Venus by the planet Mercury, whose orbital radius is 36 million miles.

Exercises 40–41 ***Geological Exploration*** *Sound and light both travel at different rates in different materials. For example, sound travels* 355 *m/sec in air and about* 1465 *m/sec through water. Thus a sound traveling by paths through two different media will be detected at different times, just as we sometimes hear distinct echoes of a single sound. This fact is used in geological exploration. Suppose a sound is generated by a thumper truck at point P in the diagram, and a detector is located* 1000 *meters away at Q, where there is a (relatively) homogeneous layer of sandstone* 150 *meters deep with another denser layer below.*



40. Express the distances $|\overline{AB}|$ $(= |\overline{CD}|)$, $|\overline{PB}|$, and $|\overline{BC}|$ as functions of angle θ. If the speed of sound through the upper and lower layers is 1500 and 4200 m/sec, respectively, use the relation distance = rate × time to express the time along each of the following paths in terms of the angle θ.

(i) t_1 along $\overline{PB}$, **(ii)** t_2 along $\overline{BC}$,

(iii) total time t_3 along $\overline{PBCQ}$.

Show that the total time can be expressed by $t_3 = \frac{5}{21} + \frac{14 - 5\sin\theta}{70\cos\theta}$, and evaluate the total time for $\theta = 15°, 20°, 25°$, and $30°$. For what value of θ is t_3 a minimum? What is the minimum time?

41. ***Looking Ahead to Calculus*** From your data in Exercise 40, estimate the angle θ for which sound travels fastest from P to Q. What is the approximate time difference between sound traveling the fastest path and sound through the air from P to Q? Through the upper layer of rock from P to Q? The answer obtained in calculus is $\theta = \text{Sin}^{-1}\left(\frac{5}{14}\right)$. How close is your estimate?

42. Use the Law of Sines to establish the identities

$$\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C}$$

$$\frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C}$$

43. Is the following an identity?

$$\frac{a}{b+c} = \frac{\sin A}{\sin B + \sin C}$$

7.3 LAW OF COSINES

> When people ask [what I do and what kind of mathematician I am], I always try to answer them. I say that there are lots of problems in mathematics that are interesting and have not been solved, and every time you solve one you think up a new one. Mathematics . . . expands rather than contracts.
>
> *Mary Ellen Rudin*

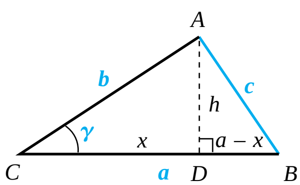


FIGURE 24

In the preceding section we observed that the Law of Sines is not suitable for solving triangles in Cases 3 and 4, where the known information consists of two sides and the included angle or three sides. To handle the *SAS* and *SSS* type problems, we introduce the Law of Cosines.

Consider triangle ABC shown in Figure 24. Suppose angle γ and sides a and b are given. We wish to find side c. Draw the altitude h from vertex A and then use the two right triangles ADC and ADB as follows. From ADC, we have

$$x = b\cos\gamma \quad \text{and} \quad h = b\sin\gamma. \tag{1}$$

Applying the Pythagorean theorem to triangle ADB gives

$$c^2 = h^2 + (a-x)^2 = h^2 + a^2 - 2ax + x^2 \tag{2}$$

Now substitute the expressions for x and h from Equation (1) into Equation (2), and then use identity (I-4):

$$\begin{aligned} c^2 &= (b\sin\gamma)^2 + a^2 - 2a(b\cos\gamma) + (b\cos\gamma)^2 \\ &= a^2 + b^2(\sin^2\gamma + \cos^2\gamma) - 2ab\cos\gamma \\ &= a^2 + b^2 - 2ab\cos\gamma. \end{aligned}$$

> *I got a lot of my geometric approach from [my father], and excitement about mathematics too. Sometimes after dinner we would get off on some topic. [For] example, my brother asked how you would find the area of a triangle in terms of its sides, so we all sat down and spent a lot of time deriving Heron's formula.*
>
> William Thurston

Therefore, c^2 is given by the formula

$$c^2 = a^2 + b^2 - 2ab\cos\gamma. \tag{3}$$

We derived the formula for c^2 in Equation (3) using the triangle shown in Figure 24, where the altitude from vertex A is inside the triangle. If the altitude falls outside the triangle, we still get the same formula; see Exercise 44.

By a process similar to that used to get Equation (3), we can get analogous formulas for a^2 and b^2. The formulas for a^2, b^2, and c^2 are referred to as the **Law of Cosines.**

Law of cosines

Suppose α, β, and γ are the angles of a triangle, and a, b, and c are the sides opposite, respectively:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

The following two examples illustrate two methods for solving the same triangle in which two sides and the included angle are given. Example 1 uses the Law of Cosines exclusively, and Example 2 uses both the Law of Cosines and the Law of Sines.

▶EXAMPLE 1 ***Using the law of cosines*** Suppose $b = 84.0$, $c = 65.0$, and $\alpha = 36.4°$. Find a, β, and γ.

Solution

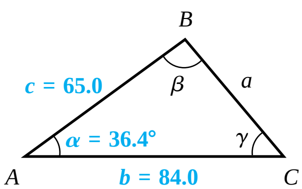


FIGURE 25

First, draw a triangle to show the given data. See Figure 25. To find side a, use the first equation in the Law of Cosines:

$$a = \sqrt{b^2 + c^2 - 2bc\cos\alpha} = \sqrt{84^2 + 65^2 - 2(84)(65)\cos 36.4°}$$
$$a \approx 49.91552597.$$

This is the final result given by a calculator to several decimal places. Since subsequent computations will use a, store the full decimal approximation in the calculator. However, when rounded off to be consistent with the given data, $a = 49.9$.

To find angles β and γ, use the second and third equations of the Law of Cosines:

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac} = \frac{a^2 + 65^2 - 84^2}{2 \cdot a \cdot 65} = -0.052309956.$$
$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} = \frac{a^2 + 84^2 - 65^2}{2 \cdot a \cdot 84} \approx 0.634710394.$$

The computations use the value of a stored in the calculator. Angles β and γ are

$$\beta \approx \mathrm{Cos}^{-1}(-0.052309956) \approx 92.9985°$$
$$\gamma \approx \mathrm{Cos}^{-1}(0.634710394) \approx 50.6015°.$$

Apply the guidelines for linear-angular measurements stated in Section 7.1 and round off to one decimal place to get 93.0° for β and 50.6° for γ.

Here we recorded numbers to several decimal places only for purposes of illustration. In practice we would not even record the values of $\cos\beta$ and $\cos\gamma$, but when their values appear in the calculator display we would evaluate Cos^{-1} and merely record the final rounded-off answers.

Note that after finding β, determining γ would have been easy using $\gamma = 180° - (\alpha + \beta)$. However, it is always wise to have a check on computations. Take the results and see if the sum of the three angles is 180°.

$$\alpha + \beta + \gamma = 36.4° + 93.0° + 50.6° = 180° \quad ◀$$

In the next example we consider the same problem as in Example 1, but we use the Law of Sines to determine the angles.

▶EXAMPLE 2 ***Care must be taken when using the law of sines*** Suppose $b = 84.0$, $c = 65.0$, and $\alpha = 36.4°$. Find a, β, and γ.

Strategy: For *SAS*, use the Law of Cosines to find a, then find the remaining angles using either the Law of Sines or the Law of Cosines. Always check to see that the angle sum is 180°.

Solution

To find a, apply the Law of Cosines as in Example 1 ($a = 49.9$ rounded off, but the full decimal approximation is stored in the calculator).

To find angle β, apply the Law of Sines.

$$\sin \beta = \frac{b \sin \alpha}{a} = \frac{84 \sin 36.4°}{a} \approx 0.998630897.$$

This could lead to the conclusion that

$$\beta \approx \text{Sin}^{-1}(0.998630897) \approx 87.0°.$$

We could then find γ by $\gamma = 180° - (\alpha + \beta) \approx 56.6°$.

Comparing $\beta = 87.0°$, $\gamma = 56.6°$ with the results of Example 1, where $\beta = 93.0°$ and $\gamma = 50.6°$, we see that there is a serious discrepancy. On closer inspection, there are two possible angles β between 0° and 180° for which $\sin \beta = 0.998630897$. Since $\sin(180° - \beta) = \sin \beta$ is an identity, the desired angle is the supplement of that given by the inverse sine. Hence, $\beta = 180° - 87.0° = 93.0°$, which agrees with the value of β determined in Example 1. ◀

The pitfall we encountered in solving Example 2 suggests a word of caution:

WARNING: Exercise care in applying the Law of Sines to determine angles.

Look at the solutions to the same problem in Examples 1 and 2 and notice that the Law of Cosines as in Example 1 gives only one angle β between 0° and 180° that satisfies the equation $\cos \beta = -0.0523$, and that angle is given by the inverse cosine function. Recall from Section 5.5 that the inverse cosine function is always a number in the interval [0, 180°], while the inverse sine gives values in the interval [−90°, 90°].

In conclusion we generally recommend using the Law of Cosines to solve triangles when there is a choice. Before calculators, lengthy computations were performed using tables of logarithms. However, logarithmic computations are not helpful to add or subtract numbers, and for this reason people avoided using the Law of Cosines whenever possible. Calculators eliminate the need to use logarithms for computations; the calculator can handle all needed computations with ease.

In the next example, we illustrate the method for solving triangles in which the three sides are given.

▶EXAMPLE 3 ***Given three sides*** Suppose $a = 53$, $b = 86$, and $c = 62$. Find the three angles.

Strategy: For *SSS*, we must solve an equation in the Law of Cosines for the cosine of an angle. For instance, $\cos \gamma = \dfrac{a^2 + b^2 - c^2}{2ab}$.

Solution

We can solve each of the equations in the Law of Cosines for the cosine of the angle in terms of the three sides. For example,

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc} = \frac{86^2 + 62^2 - 53^2}{2 \cdot 86 \cdot 62}$$

Evaluate by calculator, and with the result in the display, evaluate Cos^{-1} to get $\alpha = 38°$ (rounded off).

Similarly, using the second and third equations from the Law of Cosines, $\beta = 96°$ and $\gamma = 46°$.

As a check, compute the angle sum:

$$\alpha + \beta + \gamma = 38° + 96° + 46° = 180°.$$ ◀

Area of a Triangular Region

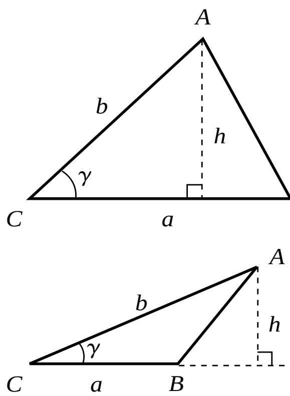


FIGURE 26
Area $= \frac{1}{2}ah = \frac{1}{2}ab \sin \gamma$

In the next example we consider the problem of finding the area of a triangular region. Following common practice, we often refer to the "area of a triangle" rather than the more precise "area of the region enclosed by a triangle," and we will often use K to denote the area. The formula underlying all area questions is

$$\textit{Area} = \frac{1}{2}(\textit{base})(\textit{height}),$$

where any side can be used as the base, and the height is the length of the altitude to the side used as the base.

In the case where we know two sides and the included angle, the area formula has another convenient form. Given sides a and b, if the included angle γ is not $90°$ (in which case the area is $\frac{1}{2}ab$), we have one of the two diagrams in Figure 26. Whether the altitude h from A is inside or outside the triangle, the area equals $\frac{1}{2}ah$, and $\sin \gamma = \frac{h}{b}$, so that $h = b \sin \gamma$. Substituting for h, we have

$$\textit{Area} = \frac{1}{2}ab \sin \gamma. \qquad \textbf{(1)}$$

In words, the area equals half the product of the adjacent sides, times the sine of the included angle.

▶*EXAMPLE 4* ***Area of a triangle*** Suppose $a = 3.4$ cm, $b = 2.7$ cm, and $\gamma = 25°$. Find the area K of the region enclosed by $\triangle ABC$.

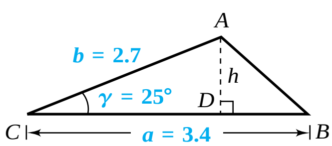


FIGURE 27

Solution

While we can apply Equation (1) without a diagram, it is always wise to draw a picture. See Figure 27. Equation (1) clearly applies, so using the given information, we have

$$K = \frac{1}{2}ab \sin \gamma = \frac{1}{2}(3.4)(2.7)\sin 25° \approx 1.9398.$$

Rounding off to two significant digits, the area is $1.9\ \mathrm{cm}^2$. ◀

Other Kinds of Given Information

In the next example we see how we may use graphs to solve triangles. From the equation relating the area of a triangle to a base and altitude, it might appear that an area, a side, and the altitude to that side should allow us to solve the triangle. There are times, however, when ambiguities arise, as Example 6, in which we illustrate techniques for solving triangles when an altitude is part of the given information. Example 7 illustrates a rather unusual application.

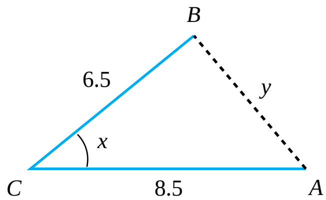


FIGURE 28

►EXAMPLE 5 SAS and graphs In Figure 28 $\triangle ABC$ is shown with $a = 6.5$, $b = 8.5$. The included angle is $x(= \gamma)$, and the opposite side is $y(= c)$.

(a) Express y as a function of x and draw a graph. Use the graph to find
(b) the value of y when $x = 72°$ and
(c) the value of x when $y = 12$.
(d) For what value of x is the area of the triangle a maximum?

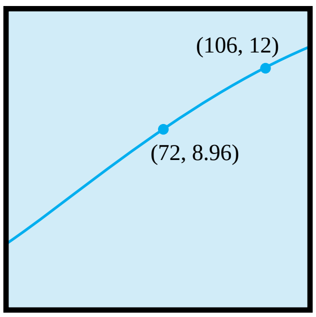
Solution

(a) From the Law of Cosines,

$$y = \sqrt{6.5^2 + 8.5^2 - 2(6.5)(8.5)\cos x} = \sqrt{114.5 - 110.5 \cos x}.$$

A graph, using x in degrees, is shown in Figure 29.

(b) When $x = 72°$, $y \approx 8.96$.
(c) When $y = 12$, $x \approx 106°$.
(d) The area is given by $K = \frac{1}{2}(6.5)(8.5)\sin x = 27.625 \sin x$. The maximum area will occur when $\sin x = 1$, or $x = 90°$. The maximum area is 27.6. ◄



[20, 120] by [0, 15]

FIGURE 29

►EXAMPLE 6 Given angle, side, altitude Given $\beta = 24.6°$, $b = 15.2$, and the altitude from vertex C given by $h = 10.4$. Find a, c, γ.

Solution

Follow the strategy. In both diagrams in Figure 30, we can find a from right triangle BCD:

Strategy: A diagram shows two possibilities. In each, use right triangle BDC and find side a. Then express c in terms of $|\overline{AD}|$ and $|\overline{DB}|$. For γ use Law of Cosines.

$$\sin \beta = \frac{h}{a}, \qquad a = \frac{h}{\sin \beta} = \frac{10.4}{\sin 24.6°} \approx 25.0.$$

For the acute triangle in Figure 30a, using the full decimal value of a in the calculator, $c = |\overline{AD}| + |\overline{DB}| = \sqrt{b^2 - h^2} + \sqrt{a^2 - h^2} \approx 33.8$. Finally, for γ,

$$\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab} \approx -0.37827, \qquad \text{so } \gamma \approx 112.2°.$$

For the obtuse triangle in Figure 30b,

$$c_1 = |\overline{DB}| - |\overline{AD}| = \sqrt{a^2 - h^2} - \sqrt{b^2 - h^2} \approx 11.6,$$

and the Law of Cosines for γ_1 gives $\gamma_1 \approx 18.6°$. ◄

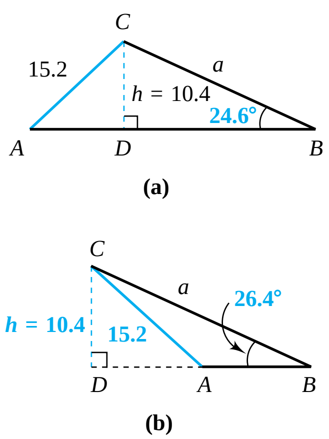
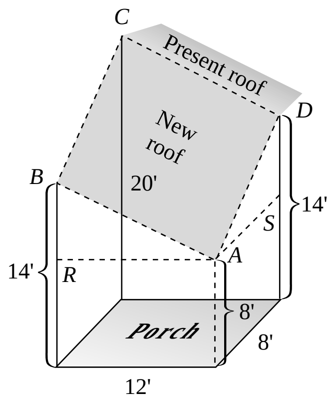
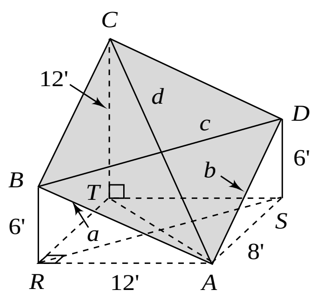


FIGURE 30

►EXAMPLE 7 A roofing application To add a room, we are going to enclose a 12 foot by 8 foot corner porch. The roof is to match the present roof lines as indicated in the diagram, coming to a low point at A, 8 feet above the floor. The new roof will be the parallelogram $ABCD$ shown. It is not a rectangle even though the floor below is. In Figure 31b we cut off everything below A, showing only the parallelogram with diagonals $\overline{BD}$ and $\overline{AC}$. The corners B and D are 6 feet higher than A, and the top corner C is 12 feet higher than A. The diagonal $\overline{AC}$ is the hypotenuse of a vertical right triangle CAT whose bottom leg $\overline{AT}$ is itself the hypotenuse of right triangle ART with legs of lengths 8 feet and 12 feet. Find **(a)** the angles at A and B and **(b)** how much larger the roof area K will be compared to the area of the porch (96 ft^2).



(a)



(b)

FIGURE 31

Solution

(a) The Pythagorean theorem gives us the sides:

$$a = |\overline{AB}| = |\overline{CD}| = \sqrt{6^2 + 12^2} = 6\sqrt{5},$$
$$b = |\overline{AD}| = |\overline{BC}| = \sqrt{6^2 + 8^2} = 10,$$

and the diagonals:

$$c = |\overline{BD}| = |\overline{RS}| = \sqrt{8^2 + 12^2} = 4\sqrt{13},$$
$$d = |\overline{AC}| = \sqrt{12^2 + 8^2 + 12^2} = 4\sqrt{22}.$$

Then from the Law of Cosines,

$$\cos\alpha = \frac{10^2 + (6\sqrt{5})^2 - (4\sqrt{13})^2}{2(10)(6\sqrt{5})}, \quad \text{so } \alpha \approx 74.4°,$$
$$\cos\beta = \frac{10^2 + (6\sqrt{5})^2 - (4\sqrt{22})^2}{2(10)(6\sqrt{5})}, \quad \text{so } \beta \approx 105.6°.$$

(b) The area is twice the area of $\triangle ABC$, or

$$K = 2\left(\frac{1}{2}\right)(6\sqrt{5})(10)\sin\beta \approx 129 \text{ ft}^2.$$

Thus the area of the roof is more than a third larger than the area of the present porch. ◀

EXERCISES 7.3

Check Your Understanding

Assume that in $\triangle ABC$ the sides and angles are labeled as in the text.

Exercises 1–5 True or False. Give reasons.

1. There is no triangle with $b = 12$, $c = 10$, and $\gamma = 100°$. (*Hint:* Draw a diagram.)
2. If $a = 2$, $b = 4$, and $c = 4$, then $\cos\alpha = \frac{7}{8}$.
3. If $a = 12$, $b = 35$, and $c = 37$, then $\gamma = 90°$.
4. There is exactly one triangle ABC for which $b = 10$, $\alpha = 30°$, and $h = 5$, where h is the length of the altitude from C.
5. If $a^2 + b^2 < c^2$, then γ is less than 90°.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. If $a = 2$, $b = 4$, and $\gamma = 30°$, then $c =$ ______.
7. If $b = 5$, $c = 5$, and $\alpha = 120°$, then $a =$ ______.
8. If $a = \sqrt{3}$, $c = \sqrt{5}$, and $b = 2\sqrt{2}$, then $\beta =$ ______.
9. If $b = \sqrt{10}$, $a = 2$ and $c = \sqrt{6}$, then $\beta =$ ______.
10. If $a = 4.35$, $b = \sqrt{18.6}$, $c = 10 \sin 28°$, then the largest angle = ______.

Develop Mastery

Round off calculated results to be consistent with the accuracy of the data. See the guidelines stated in Section 7.1. Begin your solution by drawing a diagram.

Exercises 1–14 ***Solving Triangles*** *Three parts of a triangle are given. Find the remaining parts.*

1. $a = 35$, $b = 68$, $\gamma = 48°$
2. $b = 28$, $c = 54$, $\alpha = 75°$
3. $a = 80.5$, $c = 53.7$, $\beta = 115.4°$
4. $a = 0.43$, $b = 0.55$, $c = 0.68$
5. $a = 53.4$, $b = 42.7$, $c = 68.4$
6. $a = 75.4$, $b = 68.5$, $c = 48.2$
7. $b = 7.45$, $c = 6.31$, $\alpha = 53.7°$

8. $a = 5.73$, $c = 4.58$, $\beta = 23.6°$

9. $a = 53$, $b = 45$, $c = 28$

10. $a = 36$, $b = 81$, $c = 85$

11. $a = 64$, $b = 57$, $c = 88$

12. $a = 49$, $b = 32$, $c = 58$

13. $a = 45$, $b = 28$, $\gamma = 90°$

14. $a = 36$, $b = 81$, $\gamma = 90°$

Exercises 15–18 ***Triangle Area*** ***(a)*** *Find the altitude from vertex A to side* $\overline{BC}$. ***(b)*** *Determine the area of the triangle.*

15. $a = 7.3$, $b = 6.4$, $\gamma = 43°$

16. $a = 3.5$, $c = 5.8$, $\beta = 74°$

17. $a = 5.43$, $c = 7.52$, $\beta = 112.4°$

18. $a = 4.58$, $b = 6.37$, $\gamma = 125.4°$

Exercises 19–22 ***Altitude and Area*** *Find* ***(a)*** angle α, ***(b)*** *the altitude to side b,* ***(c)*** *the area of the triangle.*

19. $a = 37.0$, $b = 62.0$, $c = 45.0$

20. $a = 4.7$, $b = 3.5$, $c = 6.7$

21. $a = 2.45$, $b = 3.41$, $c = 4.36$

22. $a = 3.46$, $b = 5.31$, $c = 4.27$

23. If $a = 43$, $b = 65$, and $c = 52$, find the largest angle.

24. If $a = 7.3$, $b = 6.5$, and $c = 4.2$, find the smallest angle.

25. If $a = 7.2$, $b = 3.8$, and $\gamma = 68°$, find the perimeter of the triangle.

26. If $b = 7.50$, $c = 6.80$, and $\alpha = 53.0°$, find the perimeter of the triangle.

Exercises 27–28 ***Integer Coordinates*** *The coordinates of the vertices of triangle ABC are given. Find* ***(a)*** *the perimeter of the triangle rounded off to the nearest whole number,* ***(b)*** *the largest angle rounded off to the nearest degree.*

27. $A(7, 4)$, $B(-5, 2)$, $C(3, 8)$

28. $A(-5, 3)$, $B(2, -5)$, $C(3, 6)$

Exercises 29–30 ***Coordinates, Vertices, Midpoints*** *The coordinates of the vertices of triangle ABC are given. Find* ***(a)*** *the midpoint M of side* $\overline{BC}$, ***(b)*** *angles BAM and CAM rounded off to the nearest degree.*

29. $A(6, 2)$, $B(-5, 4)$, $C(3, 6)$

30. $A(-3, 4)$, $B(5, -6)$, $C(3, 8)$

Exercises 31–32 Side a, angle γ, *and area K of a triangle are given. Find side b. (Hint: First find a formula for the area in terms of a, b, and* γ.*)*

31. $a = 36$, $\gamma = 45°$, $K = 25$

32. $a = 4.3$, $\gamma = 36°$, $K = 3.8$

Exercises 33–34 ***SAS and Graphs*** *Solve the problem in Example 5 for the given values of a and b.*

33. $a = 12$, $b = 15$

34. $a = 6.5$, $b = 9.3$

Exercises 35–36 ***Side, Angle and Altitude*** *Solve the problem in Example 6 for the given values of* β, *b, and h. Find only the solution for which* $\gamma > 90°$.

35. $\beta = 32.8°$, $b = 12.5$, $h = 8.35$

36. $\beta = 43°$, $b = 8.0$, $h = 5.6$

Exercises 37–38 ***Two Sides and Area*** *From the given information determine the third side. Find only the solution for which* $\gamma > 90°$.

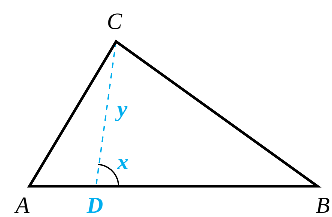
37. $a = 36.0$, $b = 24.0$, $Area = 216$

38. $a = 30.0$, $b = 20.0$, $Area = 150$

39. ***Strategy*** Suppose a, b, and the area K of $\triangle ABC$ are given. Give a verbal description of the strategy you would use to determine c. You should first draw and label a diagram.

40. ***Your Choice*** Using Exercise 39, state a problem of your choice in which you give values a, b, and K. What restrictions are necessary in choosing a value for K? Find c for your problem.

Exercises 41–42 ***Using Graphs*** *Three sides are given, D is any point on side AB,* $x = \angle CDB$, *and* $y = |\overline{CD}|$; *see diagram.* ***(a)*** *Express y as a function of x. What is the domain?* ***(b)*** *From the diagram, estimate the value of x that gives a minimum value of y. What is the minimum value of y? Draw a graph and check your guess. From the graph, find* ***(c)*** *the value of y when* $x = 72°$, ***(d)*** *the values of x for which* $y = 9.5$.

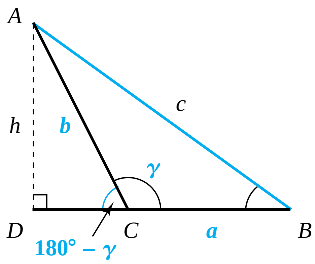


41. $a = 12$, $b = 10$, $c = 16$

42. $a = 14$, $b = 12$, $c = 20$

43. If ABC is a right triangle with $\gamma = 90°$, show that the third equation in the Law of Cosines reduces to $c^2 = a^2 + b^2$, which is consistent with the Pythagorean theorem.

44. In this section the formula for c^2 was derived using the diagram in Figure 24, where the altitude from vertex A was inside the triangle. Derive the formula for c^2 when the altitude is outside the triangle, as shown in the diagram. You should get the same formula.

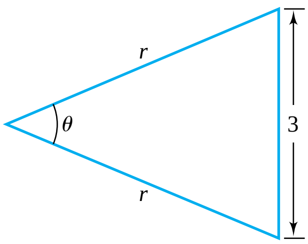


45. Given the isosceles triangle shown in the diagram,

(a) use the Law of Cosines to show that

$$r = \frac{3}{\sqrt{2 - 2\cos\theta}}.$$

(b) Show that r is also given by $1.5 \csc \dfrac{\theta}{2}$.



46. A triangle with sides of lengths 40, 60, and 80 has three altitudes.

(a) What is the length of the shortest altitude?

(b) Find the area of the triangle. Give results to two significant digits.

47. In a triangle with sides of lengths 12, 16, and 23

(a) what is the length of the longest altitude?

(b) What is the area of the triangle?

48. Katharine walks due east for a distance of 3.0 miles, turns 120° to her left and then walks 4.0 miles in the new direction. How far is she from her starting point?

49. Beginning at 8 A.M. Horacio leaves home and walks due east for 1 hour at the rate of 4 mph, turns 120° to his left, and then walks in the new direction for t hours at the rate of 3 mph.

(a) Express his distance d from home as a function of t.

(b) How far is he from home at noon?

(c) At what time will he be 12 miles from home?

Exercises 50–51 The lengths of the three sides of a triangle are related by the equation. Find the angle opposite c.

50. $(a + b + c)(a + b - c) = ab$

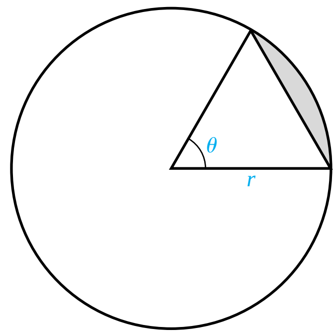
51. $(a + b + c)(a + b - c) = (2 + \sqrt{3})ab$

52. The lengths of two sides of a parallelogram are 45 and 63, and one angle is 68°. Find the lengths of the two diagonals.

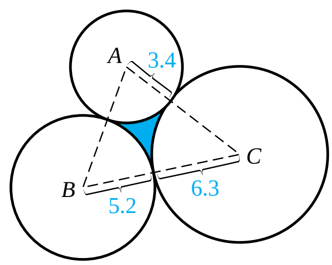
53. **(a)** Find the central angle θ of a sector of a circle where the radius is 24.0 and the length of the chord is 18.0 Give the result in radians.

(b) What is the area of the circular sector?

54. Find an equation for the area of a segment of a circle (the shaded region shown in the diagram) in terms of the radius r and central angle θ, where $0 < \theta < \pi$.

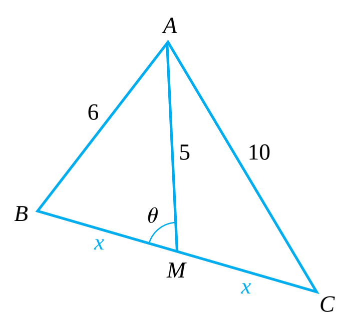


55. ***Tangent Circles*** Three circles are tangent to each other as shown in the diagram where A, B, and C are the centers and the radii are 3.4, 5.2, and 6.3, respectively. Find the largest angle in triangle ABC.

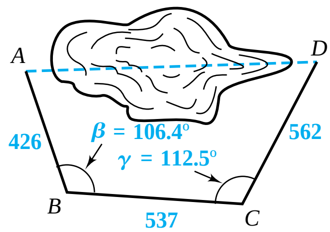


56. In Exercise 55, find the area of the shaded region between the circles in the diagram.

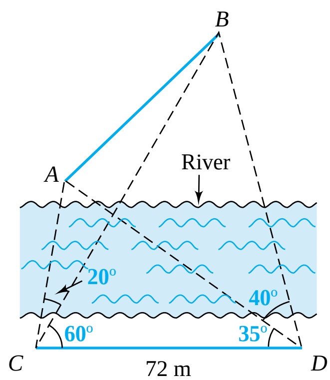
57. For the triangle shown in the diagram, M is the midpoint of side $\overline{BC}$, and $|\overline{AB}| = 6$, $|\overline{AC}| = 10$, $|\overline{AM}| = 5$. Find the length of $\overline{BC}$. (*Hint:* Use the Law of Cosines to get $\cos\theta$ in terms of x from $\triangle ABM$ and get $\cos(180° - \theta)$ from $\triangle ACM$.)



58. ***Inaccessible Distance*** Jennifer wishes to find the distance between two points A and D on opposite sides of a lake. See diagram. She locates two accessible points, B and C, and gets the following measurements: $|\overline{AB}| = 426$ ft, $|\overline{BC}| = 537$ ft, $|\overline{CD}| = 562$ ft, $\beta = 106.4°$, $\gamma = 112.5°$. Find the distance from A to D.

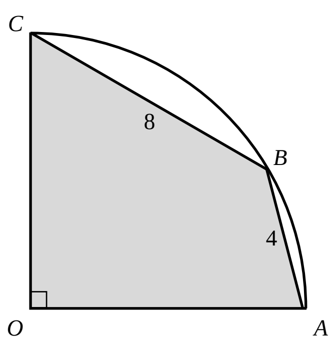


59. ***Inaccessible Distance*** Ashley is on the south bank of a river and wishes to determine the distance between points A and B on the north side. See the diagram. She measures the distance from C to D as 72 meters and the angles as shown in the diagram. Use this information to find the distance from A to B.



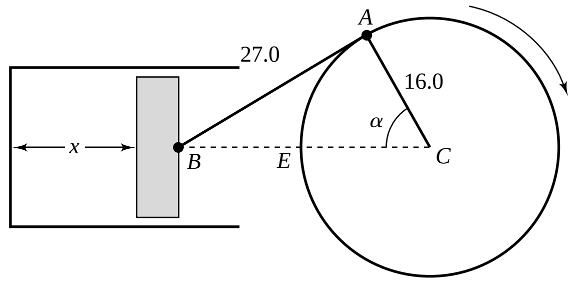
60. In triangle ABC, $a = 4.3$, $b = 5.2$, and $c = 4.1$. If the triangle is inscribed in a circle, find the radius of the circle.

61. In the diagram quadrilateral $OABC$ is inscribed in a quarter circle where $|\overline{AB}| = 4$ and $|\overline{BC}| = 8$. Find the area of the quadrilateral and give your answer in exact form as $a + b\sqrt{2}$ where a and b are whole numbers.



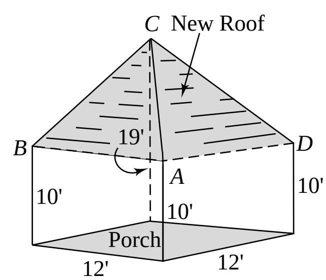
62. ***Piston Displacement*** A piston is driven by a rotating wheel with a radius of 16.0 cm, as shown in the diagram. The driving arm $\overline{AB}$ is 27.0 cm long and is attached to the piston at B, and pivots at A. Suppose the wheel rotates clockwise at the rate of 12° per second. Let t represent the time in seconds and suppose the wheel starts with A located at point E when $t = 0$. If x denotes the displacement of the piston, as shown in the diagram, find x when

(a) $t = 2$ sec, $(\alpha = 24°)$
(b) $t = 10$ sec, $(\alpha = 120°)$
(c) Find an equation that gives x as a function of t.



63. Suppose we want to roof-in a porch as in the diagram, where the new roof consists of two congruent triangles, $\triangle ABC$ and $\triangle ADC$. The porch is 12 feet × 12 feet and the height at the back corner is 19 feet.

(a) Are the angles at B and D right angles?
(b) What is the total area of $\triangle ABC$ and $\triangle ADC$?



7.4 TRIGONOMETRY AND COMPLEX NUMBERS

> [M]athematics is the science of skillful operations with concepts and rules invented just for this purpose. Most more advanced mathematical concepts, such as complex numbers, algebras, linear operators, Borel sets . . . were so devised that they are apt subjects on which the mathematician can demonstrate his ingenuity and sense of formal beauty.
>
> *Eugene P. Wigner*

In Section 1.3 we briefly discussed the system of complex numbers as an extension of the system of real numbers. The only direct application of complex numbers that we have made so far in this book is in connection with the zeros of polynomial functions. Trigonometry allows us another way of viewing complex numbers. In this section we discuss the trigonometric or polar form for representing complex numbers and apply it to the task of finding products and quotients. Then we introduce DeMoivre's theorem to find the roots of complex numbers and to explore some geometric relationships.

Recall from Section 1.3 that we may establish a correspondence between the set of complex numbers and the set of points in the plane by letting the complex number $x + yi$ correspond to the point with coordinates (x, y). This identification deals with the **complex plane,** and a point may be labeled either (x, y) or $x + yi$. Real numbers are associated with points on the x-axis, $x = x + 0i \leftrightarrow (x, 0)$, and pure imaginary numbers of the form yi are associated with points on the y-axis, $yi = 0 + yi \leftrightarrow (0, y)$. In the complex plane the x-axis is called the **real axis** and the y-axis is called the **imaginary axis.**

Each point P in the plane may also be identified by a pair of numbers (r, θ) where r is the distance from P to the origin, $|\overline{OP}|$, and θ is the angle from the positive x-axis to $\overline{OP}$. Since θ is in standard position, the coordinates of any point on the terminal side are expressible as $(r\cos\theta, r\sin\theta)$. Thus if P is identified with the complex number $x + yi$, then P has coordinates (x, y) and

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

See Figure 32. We may also write the complex number $x + yi$ as

$$x + yi = (r\cos\theta) + i(r\sin\theta) = r(\cos\theta + i\sin\theta).$$

> *To my astonishment and dismay high school students do not learn complex numbers nowadays, possibly because high school teachers don't know them. The students I met in a recent graduate course never heard of DeMoivre's theorem; even absolute values and complex conjugates made them feel insecure.*
>
> Paul Halmos

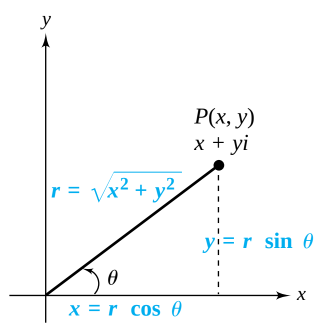


FIGURE 32

The form $r(\cos\theta + i\sin\theta)$ is called the **trigonometric** or **polar form** of $x + yi$. The nonnegative number r is called the **absolute value** or **modulus,** and θ is the **argument** of the complex number. Any angle that is coterminal with θ is also an argument for the same complex number. Because the coordinates (x, y) are rectangular coordinates, the standard form of a complex number, $x + yi$, is its **rectangular form.** We summarize the relations between rectangular and trigonometric forms of a complex number and rectangular and polar coordinates of a point.

Trigonometric form of a complex number

Suppose z is the complex number $x + yi$. The rectangular coordinates of z are (x, y) and the polar coordinates of z are (r, θ), where

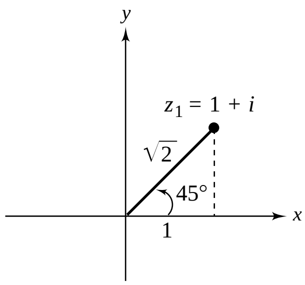
$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

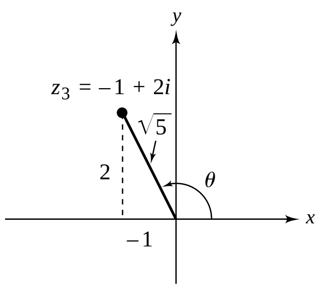
The trigonometric or polar form of z is given by

$$z = r(\cos\theta + i\sin\theta)$$

The nonnegative number r is the modulus of z, and θ is an argument of z. The complex number zero has a modulus of 0, but is not normally assigned any argument.



(a)



(b)

FIGURE 33

►EXAMPLE 1 *Change to trigonometric form*

Sketch in the complex plane and express in trigonometric form using degree measure.

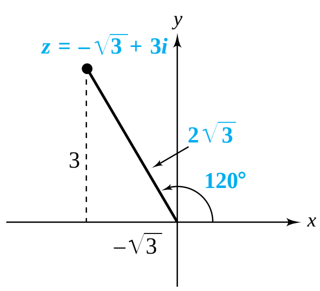
(a) $z = 1 + i$ **(b)** $w = -1 + 2i$

Solution

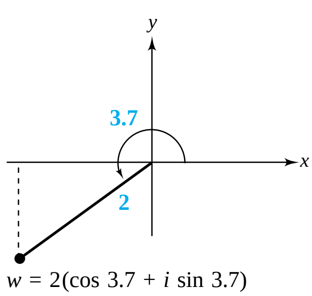
The complex numbers are shown in Figure 33.

(a) For z, the figure shows that $\theta = 45°$ and $r = \sqrt{1^2 + 1^2} = \sqrt{2}$, so a trigonometric form is $z = \sqrt{2}(\cos 45° + i\sin 45°)$.

(b) For the modulus, $r = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$. From the figure, θ is a second-quadrant angle with $\cos\theta = \frac{-1}{\sqrt{5}}$, so $\theta = \text{Cos}^{-1}\left(\frac{-1}{\sqrt{5}}\right) \approx 116.6°$. Therefore, $w \approx \sqrt{5}(\cos 116.6° + i\sin 116.6°)$. ◄



(a)



(b)

FIGURE 34

►EXAMPLE 2 *Change to rectangular form*

Sketch in the complex plane and express in rectangular form.

(a) $z = 2\sqrt{3}(\cos 120° + i\sin 120°)$ **(b)** $w = 2(\cos 3.7 + i\sin 3.7)$

Solution

To sketch a complex number from its polar form, first draw the angle and then locate the point at the distance r along the ray from the origin, as shown in Figure 34.

(a) Since $\cos 120° = -\frac{1}{2}$ and $\sin 120° = \frac{\sqrt{3}}{2}$,

$$z = 2\sqrt{3}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -\sqrt{3} + 3i.$$

(b) In radian mode the calculator returns $\cos 3.7 \approx -0.85$ and $\sin 3.7 \approx -0.53$. Thus an approximate rectangular form for w is $2(-0.85 - 0.53i) \approx -1.7 - 1.1i$. ◄

Trigonometric form gives a very specific representation for a complex number. The modulus r must be nonnegative, and the expression in parentheses must have the precise form $\cos\theta + i\sin\theta$. For example, neither of the complex numbers $-1(\cos 2 - i\sin 2)$ or $1(-\cos 2 + i\sin 2)$ is in trigonometric form. The next example illustrates how to express such complex numbers in trigonometric form.

▶EXAMPLE 3 Trigonometric form Express in trigonometric form.

(a) $z = 3(\cos 60° - i\sin 60°)$ **(b)** $w = -3(\sin 38° + i\cos 38°)$

Solution

Strategy: One possibility is to convert each number to rectangular form and plot, then use the diagram to write each in trigonometric form. Reduction formulas should help identify the quadrants and angles.

(a) First express z in rectangular form.

$$z = 3(\cos 60° - i\sin 60°) = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

Now draw a diagram to show z (see Figure 35), then find r as follows:

$$r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{27}{4}} = \sqrt{9} = 3.$$

The diagram shows a 30°–60° right triangle, so we may take $\theta = -60°$. Therefore, in trigonometric form,

$$z = 3[\cos(-60°) + i\sin(-60°)].$$

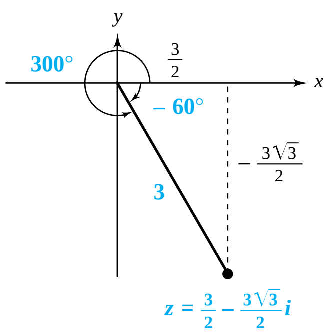


FIGURE 35

(b) $w = -3(\sin 38° + i\cos 38°) = 3(-\sin 38° - i\cos 38°)$. Find an angle θ such that $\cos\theta = -\sin 38°$ and $\sin\theta = -\cos 38°$. Since both $\cos\theta$ and $\sin\theta$ are negative, θ will be a third-quadrant angle (see Figure 36). Use reduction formulas from Section 5.4 to get

$$\cos(270° - t) = -\sin t \quad\text{and}\quad \sin(270° - t) = -\cos t$$

Therefore, take $\theta = 270° - 38° = 232°$. In trigonometric form,

$$w = 3(\cos 232° + i\sin 232°). \quad ◀$$

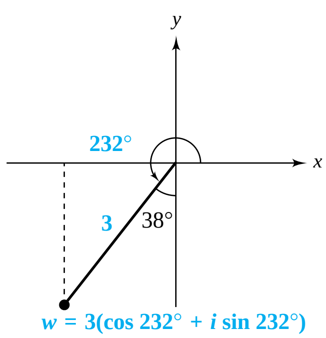


FIGURE 36

Rectangular form is convenient for addition and subtraction of complex numbers, but trigonometric form gives a geometric interpretation of multiplication and division.

Multiplication and Division in Trigonometric Form

Suppose complex numbers z_1 and z_2 are written in trigonometric form:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad\text{and}\quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

To find a trigonometric form for the product $z_1 z_2$, first multiply as complex numbers, and then use the sum identities for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$.

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

The quotient $\frac{z_1}{z_2}$ can be handled in much the same way, and the derivation is left to the exercises (see Exercise 69).

Product and quotient in trigonometric form

Suppose z_1 and z_2 are complex numbers expressed in the form

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2).$$

The product of z_1 and z_2 is

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]. \qquad \textbf{(1)}$$

If $z_2 \neq 0$, then the quotient $\dfrac{z_1}{z_2}$ is

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right)[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]. \qquad \textbf{(2)}$$

When multiplying two complex numbers in trigonometric form, the modulus is the **product of the moduli,** and the argument is the **sum of the arguments.**

When dividing two complex numbers in trigonometric form, the modulus is the **quotient of the moduli,** and the argument is the **difference of the arguments.**

When multiplying, add arguments; when dividing, subtract arguments.

Strategy: Both product and quotients are easier to compute from trigonometric forms, so begin by expressing all three in trigonometric form and then use Equations (1) and (2).

►*EXAMPLE 4 Product and quotient* Given $z_1 = 1 + i$, $z_2 = 2 - 2\sqrt{3}i$, and $z_3 = -\sqrt{3} - i$. Evaluate in both trigonometric form and rectangular form.

(a) $z_1 z_2$ **(b)** $\dfrac{z_2}{z_3}$.

Solution

Follow the strategy. Proceeding in a manner similar to the solution in Example 1,

$$z_1 = \sqrt{2}(\cos 45° + i \sin 45°), \quad z_2 = 4(\cos 300° + i \sin 300°)$$
$$z_3 = 2(\cos 210° + i\, 210°)$$

(a) For $z_1 z_2$, Equation (1) gives

$$\begin{aligned} z_1 z_2 &= \sqrt{2} \cdot 4[\cos(45° + 300°) + i \sin(45° + 300°)] \\ &= 4\sqrt{2}(\cos 345° + i \sin 345°). \end{aligned}$$

To express the product in rectangular form, you can evaluate cos 345° and sin 345°, but to get exact form it is simpler to multiply directly, as in Section 1.3:

$$z_1 z_2 = (1 + i)(2 - 2\sqrt{3}i) = (2 + 2\sqrt{3}) + (2 - 2\sqrt{3})i$$

Therefore, the product $z_1 z_2$ in trigonometric form is $4\sqrt{2}(\cos 345° + i \sin 345°)$. In rectangular form it is $(2 + 2\sqrt{3}) + (2 - 2\sqrt{3})i$.

(b) Using the trigonometric form of z_2 and z_3 and Equation (2),

$$\begin{aligned} \frac{z_2}{z_3} &= \left(\frac{4}{2}\right)[\cos(300° - 210°) + i \sin(300° - 210°)] \\ &= 2(\cos 90° + i \sin 90°). \end{aligned}$$

Since $\cos 90° = 0$ and $\sin 90° = 1$,

$$\frac{z_2}{z_3} = 2(0 + i \cdot 1) = 2i.$$

The two forms for $\frac{z_2}{z_3}$ are $2(\cos 90° + i \sin 90°)$ and $2i$. ◀

DeMoivre's Theorem

By repeated application of Equations (1) and (2), we may derive an important theorem for computing powers and roots of complex numbers. Let $z = r(\cos\theta + i\sin\theta)$. By repeated application of Equation (1), it is easy to see that

$$\begin{aligned} z^2 &= r^2(\cos 2\theta + i \sin 2\theta) \\ z^3 &= r^3(\cos 3\theta + i \sin 3\theta) \\ z^4 &= r^4(\cos 4\theta + i \sin 4\theta). \end{aligned}$$

For negative exponents, Equation (2) gives

$$z^{-1} = \frac{1}{z} = \frac{1(\cos 0° + i \sin 0°)}{r(\cos\theta + i \sin\theta)} = r^{-1}[\cos(-\theta) + i \sin(-\theta)]$$

Similarly,

$$z^{-2} = r^{-2}[\cos(-2\theta) + i \sin(-2\theta)].$$

The pattern exhibited in the above computations holds for every integer n. The result is known as **DeMoivre's theorem.** A formal proof can be made by mathematical induction (see Exercise 34, Section 8.5).

DeMoivre's theorem

Suppose n is any integer and $z = r(\cos\theta + i \sin\theta)$, then

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

▶***EXAMPLE 5*** ***DeMoivre's theorem*** Use DeMoivre's theorem to calculate

(a) $(1 + i)^4$

(b) $\left(\frac{-1 + \sqrt{3}i}{2}\right)^6$.

Solution

(a) If $z = 1 + i$, then, from Example 1, in polar form $z = \sqrt{2}(\cos 45° + i \sin 45°)$. Apply DeMoivre's theorem:

$$\begin{aligned} (1 + i)^4 &= (\sqrt{2})^4[\cos(4 \cdot 45)° + i \sin(4 \cdot 45)°] \\ &= 4(\cos 180° + i \sin 180°) = 4(-1 + i \cdot 0) = -4. \end{aligned}$$

Hence $(1 + i)^4 = -4$.

(b) A trigonometric form for $\frac{-1\sqrt{3}i}{2}$ is 1(cos 120° + i sin 120°). Use DeMoivre's theorem:

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^6 = 1^6[\cos(6 \cdot 120°) + i\sin(6 \cdot 120°)]$$
$$= 1(\cos 720° + i \sin 720°)$$
$$= 1(\cos 0° + i \sin 0°) = 1.$$

Therefore, $\left(\frac{-1+\sqrt{3}i}{2}\right)^6$ is another name for the number 1. ◀

Graphical Display of Powers of a Complex Number

Using parametric equations and an appropriate t-step value, we can get our calculators to display the nth roots of any number, as outlined in the following.

TECHNOLOGY TIP ◆ ***Powers of a complex number***

By DeMoivre's theorem, if $z = r(\cos\theta + i\sin\theta)$, the nth power of z can be expressed in the form $z^n = r^n(\cos n\theta + i \sin n\theta)$, corresponding to the point in the complex plane with coordinates $(r^n \cos n\theta, r^n \sin n\theta)$.

To represent these points with a graphing calculator for given values of r, θ, we can use parametric mode, first storing the values of r and θ, and then setting X = (R^T) COS Tθ, Y = (R^T) SIN Tθ.

DeMoivre's theorem applies only with integer values of n, so how do we get the calculator to show only integer values of t? Probably the easiest way is to use a t-step of 1, so we choose an appropriate t-range and set T step = 1 with x- and y-ranges chosen to allow us to see the points of interest.

▶EXAMPLE 6 ***Graphical representations of powers*** Let $z = 1 + i$. Then display the powers $z^{-1}, z^0, z^1, z^2, z^3, z^4, z^5$ on the complex plane.

Solution

From Example 5, in polar form, $z = \sqrt{2}\,(\cos 45° + i \sin 45°)$. Following the Technology Tip, we put the calculator in parametric (and degree) mode and enter X = (√2^T) COS 45T, Y = (√2^T) SIN 45T.. We want all integer t-values from −1 to 5, so we set $T_{MIN} = -1$, and $T_{MAX} = 5$ and Tstep = 1. The resulting graph is shown in Figure 37. We have shown the graph in connected mode, which connects the points of interest, as labeled. In dot mode, the graph would only show the points corresponding to the powers, but such a graph is difficult to see. It is much easier to see the points connected.

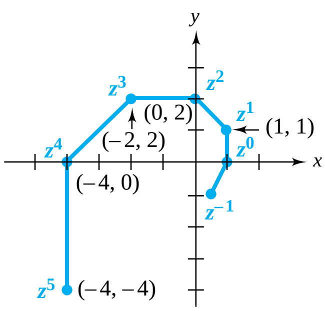


FIGURE 37

Note that the point corresponding to z^4 has coordinates (−4, 0), in agreement with Example 5a, where we found that $z^4 = (1+i)^4 = -4$. ◀

Finding Roots of Complex Numbers

In Example 5, we found that $1 + i$ is a complex number whose fourth power is −4, that is, $1 + i$ is a fourth root of −4. Similarly, $\frac{-1+\sqrt{3}i}{2}$ is a sixth root of 1.

We already know that 1 and −1 are sixth roots of 1, because $1^6 = 1$ and $(-1)^6 = 1$. According to the Fundamental Theorem of Algebra in Chapter 3, the

HISTORICAL NOTE DEMOIVRE'S THEOREM AND EULER'S FUNCTIONS

The theorem credited to Abraham DeMoivre (pictured above) was actually proved by Leonhard Euler.

It soon becomes apparent to anyone who considers the history of mathematics that new discoveries seldom occurred in the order we learn them in school. We observed in the Historical Note in Section 5.3 that sines and cosines did not originate as functions; they were lengths of chords of circles whose value depended on the size of the circles. Likewise, logarithms were not viewed as inverses of exponential functions, but as computational aids for trigonometric calculations.

Credit for the discovery of new theorems or ideas is often given to individuals who may have had little to do with the original discoveries. DeMoivre's theorem was known to, and used by, Abraham DeMoivre (1667–1754), but he never formally proved it. DeMoivre was born in France, but spent most of his life in England. He was closely acquainted with both Newton and Leibnitz. His name as attached to a trigonometric identity that was certainly not as important to him as his work in probability. A probability frequency distribution usually attributed to Laplace or Gauss was first described by DeMoivre and might more properly be called the DeMoivre distribution.

Leonhard Euler, on the other hand, did state and prove DeMoivre's identity in a paper DeMoivre quoted. Euler is the one who first treated sines and cosines as *functions*. He used the unit circle for his definitions, freeing these functions from dependence on the size of the circle. Furthermore, Euler used DeMoivre's identity to derive relationships among trigonometric, exponential, and logarithmic functions. He derived the equation $e^{ix} = \cos x + i \sin x$; thus he saw that exponentials may be viewed as disguised trigonometry, and vice versa. Mathematicians still marvel at Euler's insight and intuition.

polynomial equation $x^6 = 1$ has six roots, so there should be three more in addition to 1, -1, and $\frac{-1+\sqrt{3}i}{2}$. DeMoivre's theorem can be used to find all of the n^{th} roots of any complex number z. For a given complex number z, we can find all roots of the equation $x^n = z$.

Let the given complex number z be written in trigonometric form as $z = r(\cos\theta + i\sin\theta)$ and suppose that w is a solution to the equation $x^n = z$. The number w also has a trigonometric form, say $w = R(\cos\alpha + i\sin\alpha)$. We wish to determine the modulus R and the argument α so that w satisfies the equation $x^n = z$. Using DeMoivre's theorem, $w^n = R^n(\cos n\alpha + i\sin n\alpha)$. If $w^n = z$, then

$$R^n(\cos n\alpha + i\sin n\alpha) = r(\cos\theta + i\sin\theta) \qquad \textbf{(3)}$$

Two complex numbers in trigonometric form are equal if and only if their moduli are equal and their arguments are coterminal. Thus, from Equation (3) we must have $R^n = r$. To find the modulus R, we are interested only in the nonnegative solution to $R^n = r$, so $R = r^{1/n}$. It follows that all roots of the equation $x^n = z$ have the same modulus, namely the positive real number $r^{1/n}$.

To satisfy Equation (3), the angle $n\alpha$ must be equal to one of the angles that are coterminal with θ. This includes any angle of the form $\theta + k \cdot 360°$, where k can be any integer. We must have $n\alpha = \theta + k \cdot 360°$, so, dividing by n,

$$\alpha = \frac{\theta}{n} + k \cdot \frac{360°}{n}.$$

As k ranges through $0, 1, 2, \ldots, n - 1$, we get n distinct arguments. Hence any given nonzero complex number has n distinct complex nth roots.

Roots of a complex number

Suppose n is any positive integer, then the nonzero complex number

$$z = r(\cos\theta + i\sin\theta)$$

has exactly n district n^{th} roots, which are given by

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + k \cdot \frac{360°}{n}\right) + i\sin\left(\frac{\theta}{n} + k \cdot \frac{360°}{n}\right)\right] \qquad \textbf{(4)}$$

where $k = 0, 1, 2, \ldots, n - 1$.

Geometrically, the n roots of a complex number all lie on the circle in the complex plane centered at the origin with radius $\sqrt[n]{r}$. Furthermore, they are equally spaced, like the spokes of a wheel, starting with w_0, at arguments differing by $\dfrac{360°}{n}$. We illustrate this in the next example, in which we find all of the fifth roots of $3 + 4i$.

►EXAMPLE 7 ***Fifth roots*** Find the fifth roots of $3 + 4i$ and locate them in the complex plane.

Solution

First we write $3 + 4i$ in polar form:

$$r = \sqrt{3^2 + 4^2} = 5, \ \theta = \text{Tan}^{-1}(4/3) \approx 53.13°, \text{ so}$$
$$3 + 4i = 5(\cos\theta + i\sin\theta).$$

For the fifth roots ($n = 5$), we use Equation (4) with $\frac{360°}{5} = 72°$:

$$w_k = \sqrt[5]{5}\left(\cos\left(\frac{\theta}{5} + k \cdot 72°\right) + i\sin\left(\frac{\theta}{5} + k \cdot 72°\right)\right),$$

for $k = 0, 1, \ldots, 5$. To get decimal approximations of the roots, we store $\sqrt[5]{5}$ and $\theta/5$ with full calculator accuracy. To three decimal-place accuracy, the roots are

$$w_0 = 1.356 + 0.254i$$
$$w_1 = 0.177 + 1.368i$$
$$w_2 = -1.247 + 0.591i$$
$$w_3 = -0.948 - 1.003i$$
$$w_4 = 0.661 - 1.211i$$

Figure 38 shows these five numbers in the complex plane, located on a circle of radius $\sqrt[5]{5}$. ◄

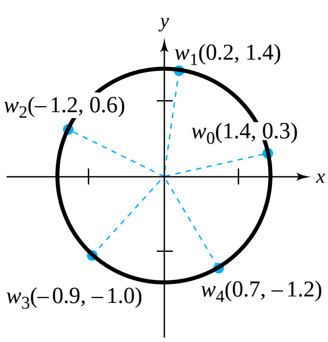


FIGURE 38
The fifth roots of $3 + 4i$

TECHNOLOGY TIP ◆ ***Roots in the complex plane***

Not only can a graphing calculator plot the points corresponding to the roots of a given complex number, but the plot makes the computations *much* simpler. The nth roots of the number $z = r(\cos\theta + i\sin\theta)$ are the numbers

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + 360°\frac{k}{n}\right) + i\sin\left(\frac{\theta}{n} + 360°\frac{k}{n}\right)\right].$$

Using the previous Technology Tip, we can plot the n roots directly. It is also efficient to do some computations on the home screen and store them. We illustrate with the numbers from Example 7. Remember we are working in degree mode.

Given $z = 3 + 4i = 5(\cos\theta + i\sin\theta)$, and $n = 5$.

1. Store the length: 5^(1/5) → R.
   Store the angle $\theta/5$: (TAN^{-1}(4/3))/5 → A.
   Calculate the increment $\frac{360°}{n}$, in this case $\frac{360°}{5} = 72°$.
2. Enter the functions parametrically: X = R COS (A + 72 T), Y = R SIN (A + 72 T).
3. Set T_{min} = 0, T_{max} = 5, and Tstep = 1. (Setting T_{max} = 5 closes the polygon.)
4. The resulting graph connects the n points, representing the n roots $w_0, w_1, w_2, \ldots, w_{n-1}, w_n = w_0$, trace, and read the coordinates.

▶EXAMPLE 8 ***Roots of an equation*** Find graphical approximations (three decimal places) for the roots of $x^4 + x^3 + x^2 + x + 1 = 0$.

Solution

The equation $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$ is an identity. (*Check by multiplying.*) Thus the roots of the given equation are the same as the roots of $x^5 - 1 = 0$ except for 1. To solve the equation $x^5 - 1 = 0$, we rewrite the equation in the form $x^5 = 1 = 1(\cos 0 + i\sin 0)$.

We can used the same pattern as in Example 7 and in the Technology Tip above, with R = 1 and A = 0/5 = 0. Also as above, $\frac{360}{5} = 72$, so we enter the parametric equations X = COS (72 T), Y = SIN (72 T), with t-settings as in the Tip. The graph is shown in Figure 39 and includes the root 1 (= (1, 0)) as well as the other four which are the roots of the given equation. We can read the coordinates as we trace along the graph: $w_1 = 0.309 + 0.951i$, $w_2 = -0.809 + 0.588$, $w_3 = -0.809 - 0.588$, $w_4 = 0.309 - 0.951i$. We observe that the coefficients of the equation are all integers, and there are two pairs of conjugate roots, w_1 and w_4, and w_2 and w_3, as we know there must be by the conjugate zeros theorem from Chapter 3. ◀

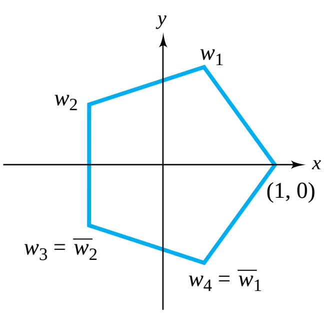


FIGURE 39

EXERCISES 7.4

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. A trigonometric form for $1 - i$ is $\sqrt{2}(\cos 45° - i\sin 45°)$.
2. If z is any point in QI of the complex plane, then $-\overline{z}$ is in QII.
3. A cube root of $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is $\cos 20° + i\sin 20°$.
4. For any angle θ, $(\cos\theta - i\sin\theta)^4 = \cos 4\theta - i\sin 4\theta$.
5. $\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^{16}$ is equal to 1.

Exercises 6–10 Fill in the blank so that the resulting statement is true. Exercises 6–9 refer to the complex plane.

6. The number $2 - 3i$ is in Quadrant ______ .

7. $\cos 150° + i \sin 150°$ is in Quadrant ______ .

8. $\cos 50° + i \sin 50°$ is in Quadrant ______ .

9. The number of 18th roots of 1 in the first quadrant is ______ .

10. The smallest prime number greater than $|3 + 4i|^2$ is ______ .

Develop Mastery

Exercises 1–8 ***Trigonometric Form*** *Sketch in the complex plane and express in trigonometric form using degree measure.*

1. **(a)** -3, **(b)** $-i$

2. **(a)** $1 - i$, **(b)** $1 + \sqrt{3}i$

3. **(a)** $3 + 5i$, **(b)** $2 - 3i$

4. **(a)** $2 + 3i$, **(b)** $4 - i$

5. **(a)** $2i + \sqrt{3}$, **(b)** $i - i^2$

6. **(a)** $\dfrac{i-1}{2}$, **(b)** $i^3 + i^2$

7. **(a)** $\dfrac{1}{i}$, **(b)** $\dfrac{1}{1+i}$

8. **(a)** $\dfrac{1+i}{1-i}$, **(b)** $\dfrac{1-\sqrt{3}i}{1+\sqrt{3}i}$

Exercises 9–14 ***Polar to Rectangular*** *Sketch in the complex plane and express in rectangular form.*

9. $2(\cos 45° + i \sin 45°)$

10. $\sqrt{3}(\cos 150° + i \sin 150°)$

11. $4(\cos 450° + i \sin 450°)$

12. $2[\cos(-270°) + i \sin(-270°)]$

13. $1[\cos(-120°) + i \sin(-120°)]$

14. $\sqrt{2}(\cos 480° + i \sin 480°)$

Exercises 15–20 ***Trigonometric Form*** *Explain why the number is not in trigonometric form, and then express it in trigonometric form.*

15. $2(\cos 45° - i \sin 45°)$

16. $-\sqrt{3}(\cos 150° + i \sin 150°)$

17. $-4(\cos 450° + i \sin 450°)$

18. $2(\cos 36° - i \sin 36°)$

19. $\sin 60° - i \cos 60°$

20. $-(\sin 30° - i \cos 30°)$

Exercises 21–24 ***Complex Number Arithmetic*** *Perform the indicated operation and express the result in both trigonometric and rectangular form. (Hint: To apply the product and quotient formulas, the numbers must first be in trigonometric form.)*

21. $(\cos 17° + i \sin 17°)(\cos 43° + i \sin 43°)$

22. $(\cos 47° - i \sin 47°)(\cos 43° - i \sin 43°)$

23. $\dfrac{\cos 47° + i \sin 47°}{\cos 13° - i \sin 13°}$

24. $\dfrac{8(\cos 150° + i \sin 150°)}{2(\cos 30° + i \sin 30°)}$

Exercises 25–30 ***Complex Number Arithmetic*** *Let $z_1 = \sqrt{3} + i$ and $z_2 = \dfrac{-1+\sqrt{3}i}{2}$. Perform the indicated operation and express the result in both trigonometric and rectangular form. Recall that $\overline{z}$ denotes the complex conjugate of z; that is, if $z = a + bi$, then $\overline{z} = a - bi$.*

25. $z_1 z_2$ **26.** $\overline{z}_1 \overline{z}_2$ **27.** $\dfrac{z_1}{z_2}$

28. $\dfrac{1}{z_2}$ **29.** $(z_1)^3$ **30.** $(z_2)^3$

Exercises 31–44 ***Powers of Complex Numbers*** *Perform the indicated operation and express the result in trigonometric and rectangular form. Give the result in exact form when it is reasonable to do so; otherwise in decimal form with numbers rounded off to two decimal places.*

31. $(\cos 30° + i \sin 30°)^4$

32. $[\cos(-45°) + i \sin(-45°)]^4$

33. $(\cos 40° + i \sin 40°)^{-3}$

34. $(\cos 18° + i \sin 18°)^{-5}$

35. $\dfrac{16}{[2(\cos 45° + i \sin 45°)]^4}$

36. $\dfrac{81}{[3(\cos 15° + i \sin 15°)]^4}$

37. $[2(\cos 15° + i \sin 15°)]^4$

38. $[-2(\cos 45° + i \sin 45°)]^4$

39. $(1 - i)^8$ **40.** $(-\sqrt{2} + \sqrt{2}i)^4$

41. $(-\sqrt{2} + \sqrt{2}i)^{-2}$ **42.** $(2 - i)^4$

43. $\dfrac{1}{(1+\sqrt{3}i)^6}$ **44.** $\dfrac{(1+2i)^4}{(1-2i)^2}$

Exercises 45–50 ***Roots of a Complex Number*** *Without using a calculator graph, find the indicated roots (2 decimal places) of z. Locate the roots in the complex plane. See Example 7.*

45. $z = -i$, cube roots

46. $z = -1 + i$, cube roots

47. $z = 5(\cos 64° + i \sin 64°)$, fourth roots

48. $z = 4(\cos 80° + i \sin 80°)$, fifth roots

49. $z = 4 - 3i$, cube roots

50. $z = 6 + 8i$, fourth roots

Exercises 51–56 ***Roots Using a Graph*** *Use a graph to find the specified roots (2 decimal place accuracy). Show the roots in the complex plane. See Example 8.*

51. Fourth roots of $4 + 3i$

52. Third roots of $4 - 3i$

53. Sixth roots of 1

54. Square roots of $4(\cos 40° - i \sin 40°)$

55. Cube roots of $8(\cos 90° - i \sin 90°)$

56. Fourth roots of $(\cos 160° - i \sin 160°)$

Exercises 57–61 Solve the equation (2 decimal place accuracy).

57. $x^5 - 1 = 0$

58. $x^4 + \sqrt{3} - i = 0$

59. $x^4 + i = 0$

60. $x^2 - 3 + 4i = 0$

61. $x^4 - 2x^2 + 2 = 0$

Exercises 62–65 ***Complex Roots*** *Find the roots of the equation (3 decimal places) using any algebraic or graphical technique. See Example 8.*

62. $x^3 + x^2 + x + 1 = 0$

63. $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$

64. $x^4 - x^3 + x^2 - x + 1 = 0$

65. $x^5 - x^4 + x^3 - x^2 + x - 1 = 0$

Exercises 66–67 ***Patterns*** *For the given z, $E_n = z^n + z^{-n}$ where $n = 1, 2, 3, \ldots$. Evaluate E_1, E_2, E_3, E_4, E_5, and E_6. Find a formula for E_n. What is the value of E_{45}? E_{48}?*

66. $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

67. $z = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$

68. Solve the equation $x^6 - 1 = 0$.

(a) By factoring, $x^6 - 1 = (x^3 + 1)(x^3 - 1)$, and factor further.

(b) Graphically by finding the sixth roots of 1. Compare with answers in part **(a)**.

69. Use identities for $\cos(\theta_1 - \theta_2)$ and $\sin(\theta_1 - \theta_2)$ to establish the quotient form for dividing two complex numbers in trigonometric form (Equation (2)).

7.5 VECTORS

> I was always interested in practical applications . . . . At Westinghouse, Varga and I related vector lattices to nuclear reactors.
>
> *Garrett Birkhoff*

Introduction and Definition

This short introduction foreshadows a very long subject. Vectors are becoming more and more common in applications of mathematics. Entire books are devoted to vector applications; vector analysis forms a significant portion of most calculus books and many introductory physics and engineering texts; courses in linear algebra develop tools to handle vectors and matrices for applications in business and the social and natural sciences as well as the traditional physical sciences. Although our treatment is limited to two-dimensional vectors, the ideas are fundamental to all of vector analysis.

We can describe many of the things we deal with in the real world by a single number; distance, area, mass, volume, time, temperature, etc. These numbers answer questions of magnitude such as how much, how long, how fast, how heavy?

Many questions, however, require more than a single number. Both magnitude and direction may be vital:

> "But Officer, I was only doing 20 miles an hour!" "Yes, but this is a one-way street, and you were going the wrong way."

If Conway's genius is more than one percent inspiration, then it's because he adds up to more than one hundred percent! He does thousands of calculations, looks at thousands of special cases, until he exposes the hidden pattern and divines the underlying structure.

Richard K. Guy, of John Horton Conway

In birling, two competitors get on a floating log and set the log spinning. The object is to use the spin to make your opponent lose balance and end up in the water. The direction of the spin is at least as important as the strength of the thrust applied.

Quantities that have both magnitude and direction are called **vectors.** We restrict ourselves here to vectors that can be specified by ordered pairs of numbers. The vectors considered in this section can all be represented by directed line segments in the plane.

To begin, we give a definition.

Definition: vectors

A **vector v** is an ordered pair of real numbers, denoted $\langle a, b\rangle$. The individual numbers a and b are called the **components** of **v**. The **magnitude** of **v**, denoted by $|\mathbf{v}|$, is the number $\sqrt{a^2 + b^2}$, and the **direction** of **v** is the direction in the plane from the origin to point (a, b). Vector notation looks like

$$\mathbf{v} = \langle a, b\rangle \quad \text{and} \quad |\mathbf{v}| = \sqrt{a^2 + b^2}.$$

Equality of Vectors

For vectors $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, we say that **u** and **v** are **equal** if and only if $a = c$ and $b = d$. Two vectors are equal if and only if their corresponding components are equal.

Geometric Representations of Vectors

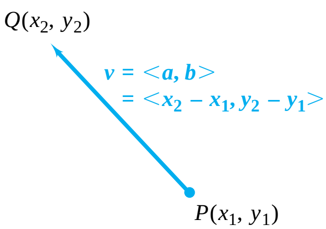


FIGURE 40

It is customary to represent a vector as a directed line segment, as shown in Figure 40, where we may also denote vector **v** by **PQ**. While a vector is technically just a pair of numbers, we will also occasionally use the name vector for a directed line segment that represents the vector, so that we may write $\mathbf{v} = \mathbf{PQ}$. P is the **initial point** (or tail) and Q is the **terminal point** (or tip or head). The length of the directed line segment is the magnitude of the vector, and the arrow indicates the direction. The components of the vector **PQ** are given by

$$\mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle, \text{ for the points } P(x_1, y_1) \text{ and } Q(x_2, y_2). \tag{1}$$

It follows that the vector $\mathbf{v} = \langle a, b\rangle$ can be represented just as well by any other directed line segment **RS** for which the components, the changes in coordinates going from R to S, are a and b, respectively. This means that **v** can be represented by any directed segment that has the same length and direction. If the two directed line segments **RS** and **PQ** represent the same vector, we write $\mathbf{RS} = \mathbf{PQ}$.

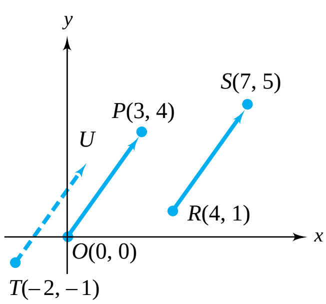


FIGURE 41

In a coordinate system it is often convenient to represent the vector $\mathbf{v} = \langle a, b\rangle$ by a line segment with its initial point at the origin, in which case the terminal point must be the point with coordinates (a, b), as in Figure 41 for Example 1. We will call such a representation the **standard representation** or **standard position** for **v**. The **angle between two vectors** is the angle between standard representations of the vectors.

▶EXAMPLE 1 ***Representing a vector*** Points $O(0, 0)$, $P(3, 4)$, $R(4, 1)$, $S(7, 5)$, and $T(-2, -1)$ are shown in Figure 41. Suppose **v** is the vector $\langle 3, 4\rangle$, so $\mathbf{v} = \mathbf{OP}$. (**a**) Find the components and the magnitude of **v**. (**b**) Show that **RS** represents the same vector **v**. (**c**) Find the point U such that $\mathbf{TU} = \mathbf{v}$.

Solution

(a) The components of **v** are 3 and 4. Its magnitude is

$$|\mathbf{v}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

(b) From Equation (1), $\mathbf{RS} = \langle 7 - 4, 5 - 1 \rangle = \langle 3, 4 \rangle$, so $\mathbf{RS} = \mathbf{v}$.

(c) To have $\mathbf{TU} = \mathbf{v}$, the vector **TU** must have components $\langle 3, 4 \rangle$. If **U** has coordinates (u, w), this implies that

$$\mathbf{TU} = \langle u + 2, w + 1 \rangle = \langle 3, 4 \rangle,$$

so by the definition of equality of vectors, $u + 2 = 3$ and $w + 1 = 4$. Thus U has coordinates $(1, 3)$. The directed line segment **TU** is shown in Figure 41. ◀

Algebra of Vectors

The standard operations for vectors include **vector addition** and multiplication of a vector by a number (called **scalar multiplication,** to emphasize that we are not multiplying two vectors). Two other kinds of product can be defined for some vectors, the **dot product** and the **cross product,** but we will not discuss either of them here.

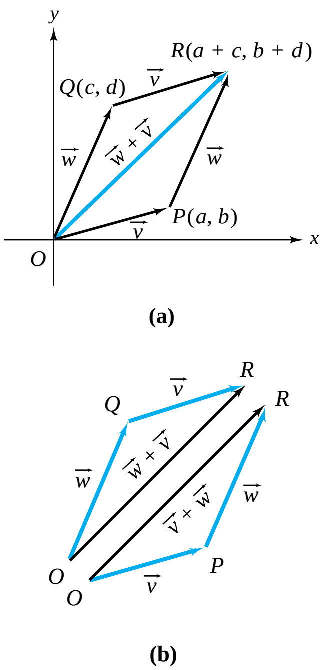


FIGURE 42

Definition: vector addition and multiplication by a scalar

Given vectors $\mathbf{v} = \langle a, b \rangle$ and $\mathbf{w} = \langle c, d \rangle$, the sum of **v** and **w** is a vector whose components are the sums of the corresponding components of **v** and **w**.

$$\mathbf{v} + \mathbf{w} = \langle a + c, b + d \rangle.$$

If k is any real number, the scalar product $k\mathbf{v}$ is defined by

$$k\mathbf{v} = \langle ka, kb \rangle.$$

Thus, to add two vectors, add components, and to multiply a vector by a number, multiply both components by the number. The geometric interpretation of vector addition and scalar multiplication is immediate. If $\mathbf{v} = \langle a, b \rangle$ and $\mathbf{w} = \langle c, d \rangle$, then their standard representations are directed line segments **OP** and **OQ** as shown in Figure 42. It follows by the definition that their sum $\mathbf{v} + \mathbf{w}$ is represented by the directed line segment **OR**, which is the diagonal of the parallelogram with adjacent edges OP and OQ. In the parallelogram $OPRQ$, the directed line segment **QR** is another representation of **v** and **PR** is another representation of **w**. Thus the vector sum $\mathbf{v} + \mathbf{w}$ is represented by the diagonal of the parallelogram. This is the **parallelogram method** of adding vectors.

Parallelogram method of adding vectors

For a representation of the vector sum $\mathbf{v} + \mathbf{w}$, take any directed line segment **PQ** to represent **v** and a directed line segment **QR** to represent **w** (with the tail of **w** at the tip of **v**). The directed line segment **PR** represents the sum $\mathbf{v} + \mathbf{w}$. It should be obvious that the vector sum satisfies $\mathbf{w} + \mathbf{v} = \mathbf{v} + \mathbf{w}$ and $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

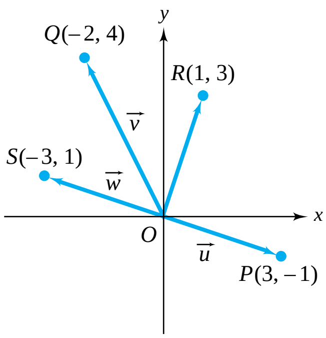


FIGURE 43

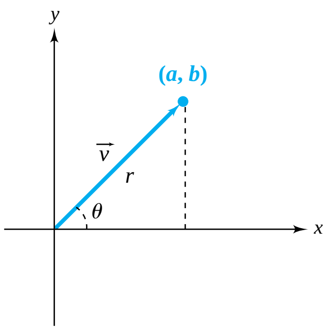


FIGURE 44

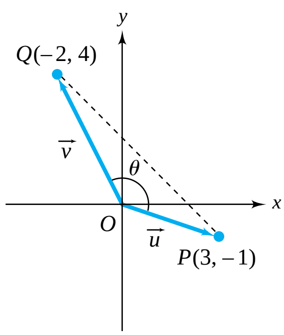


FIGURE 45

▶EXAMPLE 2 ***Vector addition*** Given vectors $\mathbf{u} = \langle 3, -1\rangle$, $\mathbf{v} = \langle -2, 4\rangle$, and $\mathbf{w} = \langle -3, 1\rangle$, find the vector sums $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} + \mathbf{w}$. Show the sums as directed line segments in standard position.

Solution

From the definition of vector addition, $\mathbf{u} + \mathbf{v} = \langle 3 - 2, -1 + 4\rangle = \langle 1, 3\rangle$ and $\mathbf{u} + \mathbf{w} = \langle 3 - 3, -1 + 1\rangle = \langle 0, 0\rangle = \mathbf{0}$. Figure 43 shows standard representations for $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. The parallelogram method tells us that **OR** represents $\mathbf{u} + \mathbf{v}$, but $\mathbf{u} + \mathbf{w}$ is the zero vector, and we cannot show a vector of length zero on the diagram. ◀

It is often convenient to use polar representation for a point in the plane to express a vector $\mathbf{v} = \langle a, b\rangle$. The terminal point of $\mathbf{v}$ in standard position is the point with coordinates (a, b). As Figure 44 shows, the coordinates (a, b) can be expressed in terms of the length of the vector, r, and the angle θ from the positive x-axis to the segment that represents $\mathbf{v}$ in standard position. Then

$$\mathbf{v} = \langle a, b\rangle = \langle r\cos\theta, r\sin\theta\rangle,$$

where $a^2 + b^2 = r^2$ and $\tan\theta = \dfrac{b}{a}$.

▶EXAMPLE 3 ***Angle between two vectors*** For the vectors $\mathbf{u} = \langle 3, -1\rangle$ and $\mathbf{v} = \langle -2, 4\rangle$, find the angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution

By the definition of the angle between vectors, we want the angle between standard representations, the angle $\theta = \angle QOP$ in Figure 45. The Law of Cosines can help if we have the lengths of the three sides of the triangle QOP. The lengths of two of the sides are the lengths of the vectors $\mathbf{u}$ and $\mathbf{v}$, and the distance formula can provide the length of $\overline{QP}$.

$$|\mathbf{u}| = \sqrt{3^2 + (-1)^2} = \sqrt{10} \qquad |\mathbf{v}| = \sqrt{(-2)^2 + 4^2} = \sqrt{20}$$
$$|\overline{QP}| = \sqrt{(3 + 2)^2 + (-1 - 4)^2} = \sqrt{50}.$$

From the Law of Cosines,

$$\cos\theta = \frac{10 + 20 - 50}{2\sqrt{10}\sqrt{20}} = \frac{-1}{\sqrt{2}} \quad \text{so} \quad \theta = \frac{3\pi}{4}, \quad \text{or} \quad 135°.$$

Alternative Solution Using the polar representations for $\mathbf{v}$ and $\mathbf{u}$ where $\mathbf{v} = \langle r_1\cos\theta_1, r_1\sin\theta_1\rangle$ and $\mathbf{u} = \langle r_2\cos\theta_2, r_2\sin\theta_2\rangle$, the angle θ is given by $\theta = \theta_1 - \theta_2$. See Figure 45, and so $\tan\theta_1 = -2$ and $\tan\theta_2 = -\frac{1}{3}$. Using identity (I-11),

$$\tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2}$$
$$= \frac{-2 + \dfrac{1}{3}}{1 + 2\left(\dfrac{1}{3}\right)} = \frac{-\dfrac{5}{3}}{\dfrac{5}{3}} = -1.$$

As above, $\theta = 135°$. ◀

If $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle -a, -b \rangle$, then $\mathbf{u} + \mathbf{v} = \langle 0, 0 \rangle = \mathbf{0}$; we say that $\mathbf{v}$ is the negative of $\mathbf{u}$ and write $\mathbf{v} = -\mathbf{u}$. It is clear that multiplying any vector by zero gives the zero vector. Multiplying by the number 1 doesn't change a vector, while multiplying by -1 changes a vector to its negative, that is, reverses its direction. We can also use the negative of a vector to define the operation of subtraction, much as we do for real numbers. We summarize these observations for convenient reference.

Additional properties of vectors

$$0 \cdot \mathbf{v} = 0 \qquad 1 \cdot \mathbf{v} = \mathbf{v}$$

$-1 \cdot \mathbf{v} = -\mathbf{v}$, where $-\mathbf{v}$ is the vector such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
For subtraction, $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$.

A **unit vector** is any vector of length 1. Given any nonzero vector $\mathbf{v}$, the unit vector in the direction of $\mathbf{v}$ is the vector $\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v}$; that is, multiply $\mathbf{v}$ by the number $\frac{1}{|\mathbf{v}|}$.

▶EXAMPLE 4 ***Vector arithmetic*** Let $\mathbf{v} = \langle 6, -8 \rangle$ and $\mathbf{w} = \langle -2, -3 \rangle$.
(a) Find the unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$. **(b)** Find $\mathbf{v} - 2\mathbf{w}$.

Solution

(a) The length of $\mathbf{v}$ is given by $|\mathbf{v}| = \sqrt{6^2 + (-8)^2} = 10$, so to find $\mathbf{u}$, multiply $\mathbf{v}$ by $\frac{1}{10}$: $\mathbf{u} = \langle \frac{6}{10}, -\frac{8}{10} \rangle = \langle 0.6, -0.8 \rangle$.

(b) $\mathbf{v} - 2\mathbf{w} = \langle 6, -8 \rangle - 2\langle -2, -3 \rangle = \langle 6, -8 \rangle + \langle 4, 6 \rangle = \langle 10, -2 \rangle$. ◀

Compass Directions for Vectors

We have used a coordinate system and directed line segments to identify vectors. It is often useful to describe the direction of a vector with compass directions, so a vector may have a given length in the direction 30° east of north, or 17.5° north of west. To represent such vectors in a coordinate plane, we usually take the positive y-axis as north and the positive x-axis as east.

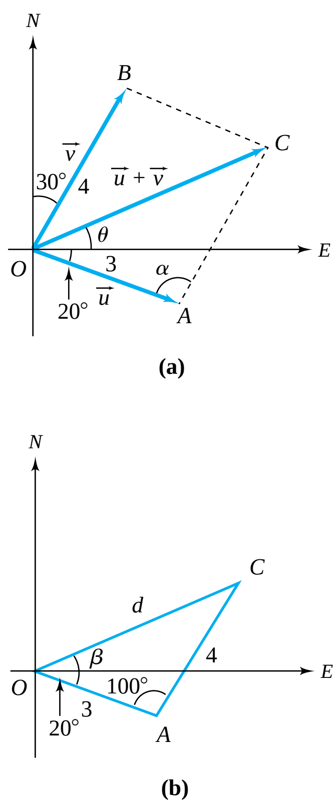


FIGURE 46

▶EXAMPLE 5 ***Adding with compass directions*** Vector $\mathbf{u}$ has length 3.0 and direction 20° south of east; $\mathbf{v}$ has length 4.0 and direction 30° east of north. Find the length and direction of $\mathbf{u} + \mathbf{v}$.

Solution

First draw a diagram (always an essential step) in the coordinate plane to show standard representations **OA** and **OB** for $\mathbf{u}$ and $\mathbf{v}$. See Figure 46. The diagram shows $\mathbf{u} + \mathbf{v}$ as the diagonal **OC** of the parallelogram $OACB$. Find the length and direction of **OC**. Both the length and direction can be described by solving the triangle OAC in the diagram, where $\alpha = \angle OAC = 100°$. (Why?) If we let $\beta = \angle COA$, then the angle of inclination of segment $\overline{OC}$ is given by $\theta = \beta - 20°$. Let $d = |\overline{OC}|$. (See Figure 46b.)

By the Law of Cosines,

$$d^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cdot \cos 100° \approx 29.16756, \quad d \approx 5.4.$$

Using either the Law of Sines or the Law of Cosines, $\beta \approx 47°$, from which $\theta \approx 27°$. Therefore $\mathbf{u} + \mathbf{v}$ is the vector of length 5.4, directed 27° north of east (or 63° east of north). ◀

▶EXAMPLE 6 ***Vector distances*** Mamie and Jody are approaching an intersection O. At noon, Mamie is a half-mile (2640 ft) from O and traveling south at 60 mph (88 ft/sec.) At the same time, Jody is one third-mile (1760 ft) from O and traveling west at 45 mph (66 ft/sec). See the diagram (Figure 47).

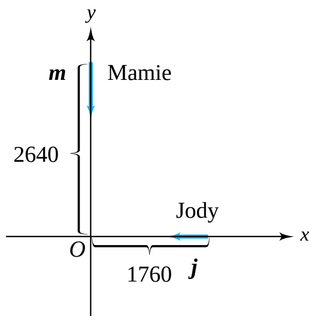


FIGURE 47

(a) Express the position of each at t seconds after noon as a vector function. When will each reach the intersection?
(b) Find their speed and location relative to O when $t = 20$, $t = 30$.
(c) Find a formula for the distance $d(t)$ between their cars at time t. Draw a graph of the distance and find how close they come to each other. At what time are they closest to each other?

Solution

(a) Using O as the origin with the usual orientation in Figure 47, their positions at time t are given by

$$\mathbf{m}(t) = \langle 0{,}2640 - 88t \rangle, \ \mathbf{j}(t) = \langle 1760 - 66t, 0 \rangle.$$

Mamie reaches the intersection when $2640 - 88t = 0$, or $t = 30$. Similarly, Jody gets to O when $1760 - 66t = 0$, or when $t = 26\frac{2}{3}$. Thus Jody passes through the intersection $3\frac{1}{3}$ seconds before Mamie.

(b)

$$\mathbf{m}(20) = \langle 0, 880 \rangle, \ \mathbf{j}(20) = \langle 440, 0 \rangle.$$
$$\mathbf{m}(30) = \langle 0, 0 \rangle, \ \mathbf{j}(30) = \langle -220, 0 \rangle.$$

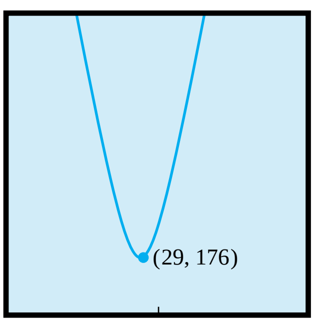
Thus at 20 seconds after noon, Mamie is twice as far from the intersection as Jody, and 10 seconds later, Jody is 220 feet beyond the intersection when Mamie arrives.

(c) The distance can be calculated either as the length (absolute value) of the vector $\mathbf{m} - \mathbf{j}$ or by using the distance formula between their locations at time t. In either case the distance is given by

$$d(t) = \sqrt{(66t - 1760)^2 + (2640 - 88t)^2}.$$

We can enter the function in this rather messy form, or we can simplify, collecting terms and factoring.

$$d(t) = 22\sqrt{25t^2 - 1440t + 20800}.$$



[20, 40] by [100, 500]

FIGURE 48

From the information in part (**b**) and checking some values of $d(t)$, we graph `Y = 22√(25X² − 1440X + 20800)` in $[20, 40] \times [100, 500]$ and get something like Figure 48. Tracing, we find that the minimum distance occurs at about 29 seconds, when they are about 176 feet apart. ◀

The simplification of the formula for $d(t)$ in part (**c**) above is mostly for our benefit, because the calculator will plot points just as quickly in either form. We, however, may be able to observe relationships from the simpler form that are not apparent in the more complicated form. In this instance for example, we may notice that we are taking the square root of a quadratic function. The minimum will obviously occur at the vertex of the parabola $y = 25t^2 - 1440t + 20800$, which we can locate without the use of technology. The vertex occurs when $t = \frac{-b}{2a} = \frac{1440}{50} = 28.8$, and $d(28.8) = 176$, information that we were able only to approximate from the calculator graph. Whether the greater precision of an exact form is valuable depends, of course, on the problem. In this instance, most of us would question whether part of a foot has any significance at all for Mamie and Jody. It isn't hard to imagine circumstances, though, in which a foot could make a vital difference. The point is that we must learn how to use and interpret a mathematical model in ways that are productive for *our* purposes, not for some arbitrary expectation set in some textbook.

►EXAMPLE 7 ***Air flight*** A direct air route from Denver to Chicago runs about 10° north of east. An airplane flies at a constant airspeed of 500 mph at a heading of 15° north of east while the jet stream is blowing at a constant speed of 100 mph due east. What is the velocity (speed and direction) of the plane relative to the ground?

Solution

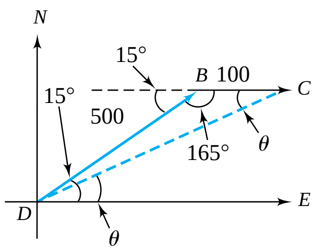


FIGURE 49

Represent the airspeed of the plane as vector **DB** of magnitude 500 in the direction 15° north of east. The jet stream contribution is vector **BC** of magnitude 100 directed due east. The ground speed of the plane is the vector sum **DB** + **BC**. The sum is represented by the directed line segment **DC** in Figure 49, at a heading of θ north of east. By the Law of Cosines, the length is

$$|\mathbf{DC}| = \sqrt{500^2 + 100^2 - 2 \cdot 500 \cdot 100 \cdot \cos 165^\circ} \approx 597.$$

Angle θ is also included in $\triangle BDC$, $\theta = \angle BCD$, and the Law of Sines is convenient:

$$\theta = \operatorname{Sin}^{-1}\left(\frac{500 \sin 165^\circ}{597}\right) \approx 12.5^\circ.$$

The ground speed of the plane is nearly 600 mph, but the heading of 12.5° north of east will take the plane too far north, and a correction will be needed to reach Chicago. ◄

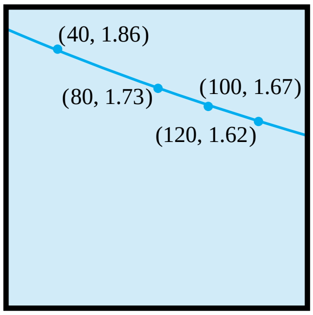
EXAMPLE 8 ***Variable jet stream*** Suppose that in Example 7 the jet stream is blowing due east but at a speed of x miles per hour. Then the flight time T is a function of x, equal to *distance/ground speed.*

(a) Find a formula for $T(x)$, assuming a distance of 1000 miles.
(b) From a graph of $T(x)$, find the flight time, and the number of minutes less than 2 hours, when the jet stream is blowing at 40 mph, at 80 mph, at 100 mph, at 120 mph.

Solution

(a) We can use the vector diagram in Figure 49, with a label of x for **BC**. As in Example 7, the ground speed is given by the length of **DC**. By the Law of Cosines,

$$\begin{aligned}|\mathbf{DC}| &= \sqrt{500^2 + x^2 - 2 \cdot 500 \cdot x \cos 165^\circ}\\ &\approx \sqrt{x^2 + 965.9x + 250{,}000}.\end{aligned}$$



[20, 140] by [1, 2]

FIGURE 50

Then the flight time, in hours, is given by

$$T(x) = 1000/\sqrt{x^2 + 965.9x + 250{,}000}.$$

(b) In $[20, 140] \times [1, 2]$ we get a graph like that in Figure 50, from which we can read approximate flight times as follows:

$$T(40) \approx 1.86 \text{ hr} \approx 111 \text{ min (about 9 minutes less than 2 hours)}$$
$$T(80) \approx 1.73 \text{ hr} \approx 104 \text{ min (about 16 minutes less than 2 hours)}$$
$$T(100) \approx 1.67 \text{ hr} \approx 100 \text{ min (about 20 minutes less than 2 hours)}$$
$$T(120) \approx 1.62 \text{ hr} \approx 97 \text{ min (about 23 minutes less than 2 hours)}$$ ◀

EXERCISES 7.5

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. Vectors $\langle 2, -3\rangle$ and $\langle 3, -2\rangle$ are directed in opposite directions.
2. Vectors $\langle 3, 4\rangle$ and $\langle -6, -8\rangle$ are directed in opposite directions.
3. The magnitude of vector $\langle 3, 4\rangle$ is greater than the magnitude of $\langle -2, -5\rangle$.
4. When a jet stream is blowing at 80 mi/hr in the direction 45° north of east, then for an airplane traveling east, the ground speed is greater than the speed of the instrument panel.
5. Answer Exercise 4 if the jet stream is blowing in the direction 45° south of west.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. If $\mathbf{v}$ is directed 45° north of east and $|\mathbf{v}| = 2$, then $\mathbf{v} =$ ______.
7. Vector $\mathbf{v}$ is directed 45° north of west. Vector $\mathbf{u}$ is in the opposite direction and $|\mathbf{u}| = 4$, then $\mathbf{u} =$ ______.
8. If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, $\mathbf{u}$ is directed east and $\mathbf{v}$ is directed south, then $\mathbf{u} + \mathbf{v} =$ ______.
9. If $\mathbf{u}$ is any nonzero vector, then the number of unit vectors perpendicular to $\mathbf{u}$ is ______.
10. If $|\mathbf{v}| = 16$ and $\mathbf{v}$ is directed 30° south of west, then $\mathbf{v} =$ ______.

Develop Mastery

For the first 26 exercises, vectors $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are given by* $\mathbf{u} = \mathbf{OP}$, $\mathbf{v} = \mathbf{OQ}$, *and* $\mathbf{w} = \mathbf{OR}$, *where O is the origin and the other points are* $P(1, 1)$, $Q(2, -5)$, *and* $R(-2, 1)$.

Exercises 1–8 ***Components*** *Find the components of each vector and draw a diagram to show the vector as a directed line segment in standard position.*

1. $4\mathbf{u}$
2. $-2\mathbf{v}$
3. $3\mathbf{w}$
4. $\mathbf{u} + \mathbf{v}$
5. $\mathbf{v} - \mathbf{u}$
6. $\mathbf{u} - 2\mathbf{v}$
7. $-2\mathbf{v} + 3\mathbf{w}$
8. $\mathbf{v} + \mathbf{u} + \mathbf{w}$
9. Find a vector $\mathbf{x}$ that, when added to $\mathbf{u}$, gives $\mathbf{w}$.
10. Find a vector $\mathbf{x}$ that, when subtracted from $2\mathbf{v}$, gives $\mathbf{w}$.
11. Find a vector $\mathbf{x}$ that, when added to $\mathbf{v} + \mathbf{w}$, gives $\mathbf{u}$.
12. Find a vector $\mathbf{x}$ that, when added to $\mathbf{u} - \mathbf{v} + \mathbf{w}$, gives $\mathbf{0}$.

Exercises 13–15 ***Angle Between Vectors*** *Find the angle (to the nearest degree) between the pair of vectors.*

13. $\mathbf{u}$ and $\mathbf{v}$
14. $\mathbf{u}$ and $\mathbf{w}$
15. $\mathbf{w}$ and $\mathbf{v}$
16. Find all unit vectors $\mathbf{OX}$ such that $\mathbf{OX}$ is perpendicular to $\mathbf{OP}$.
17. Find all unit vectors $\mathbf{OX}$ such that $\mathbf{OX}$ is perpendicular to $\mathbf{OR}$.
18. If $\mathbf{x} = 2\mathbf{u} + \mathbf{v} + \mathbf{w}$, show that $\mathbf{x}$ and $\mathbf{u}$ are perpendicular to each other.

Exercises 19–25 ***Length of Vector*** *Find the lengths of the vectors.*

19. $\mathbf{u}$ and $2\mathbf{u}$
20. $\mathbf{v}$ and $-\mathbf{v}$
21. $\mathbf{w}$ and $-3\mathbf{w}$
22. $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} + \mathbf{v}$
23. $\mathbf{u}$ and $k\mathbf{u}$, where $k > 0$.
24. $\mathbf{u}$ and $k\mathbf{u}$, where $k < 0$.
25. $\mathbf{w}$, $\mathbf{v}$ and $a\mathbf{w} + b\mathbf{v}$
26. Is there a positive integer n such that the vector $n\mathbf{u}$ has integer length? Explain your answer.

27. Find a vector $\mathbf{v}$ of length 4 whose first component is twice its second component.

28. Find all vectors $\mathbf{v}$ of length 8 whose components are equal in magnitude, but have opposite signs.

29. If $\mathbf{u} = \langle 3, -1 \rangle$, find all vectors $\mathbf{v} = \langle x, 2 \rangle$ for which $|\mathbf{v} - \mathbf{u}| = \sqrt{34}$.

30. If $\mathbf{u} = \langle 3, -1 \rangle$, find all vectors $\mathbf{v} = \langle x, 2 \rangle$ for which $|\mathbf{u} + \mathbf{v}| = \sqrt{17}$.

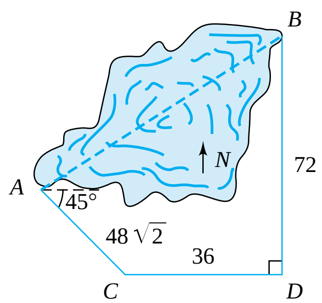
31. If $\mathbf{u} = \langle 3, -1 \rangle$, find all vectors $\mathbf{v} = \langle x, y \rangle$ for which $|\mathbf{u} + \mathbf{v}| = 2$.

32. Given point $P(-1, 2)$, find all points $Q(x, 2)$ for which the vector $\mathbf{PQ}$ has length 4.

33. Given point $P(-1, 2)$, find all points $Q(2, y)$ for which the vector $\mathbf{PQ}$ has length 5.

34. Given point $P(-1, 2)$, find all points $Q(x, y)$ for which the vector $\mathbf{PQ}$ has length 4.

35. ***Vector Sum*** Suppose A and B are points on opposite shores of a lake as shown in the diagram. A man starts at point A and reaches point B by walking from A to C ($48\sqrt{2}$ m 45° south of east), C to D (36 m east), D to B (72 m north). If he went directly by rowboat, how far and in what direction should he go? (*Hint:* Express each leg of his walk as a vector.)



36. A boat sails from port 72 km due east, then turns 60° toward the south and travels 48 km in the new direction. How far and in what direction is the boat now located, relative to port?

37. In playing golf Patty takes two putts to get the ball into the hole. The first putt takes her ball 8.0 feet 30° north of east. She then sinks the ball by putting 1.5 feet due north. To execute the putt in one stroke how far and in what direction should she have hit the ball?

38. Patty's golfing partner (from Exercise 37) ended up on the green exactly 20 feet due north of Patty's ball before her first putt. How far and in what direction should he aim his putt to hole out in one stroke? Is his putt easier or more difficult than Patty's (assuming that the green is level in all directions)?

39. A small motorboat has a speed of 5.0 mph in still water. It heads perpendicularly across a river whose current is 3.5 mph. Find the true heading and speed of the boat. (*Hint:* Draw a diagram to show both the heading and speed of the boat and the direction and speed of the current as vectors.)

40. Solve the problem in Example 6 if Mamie is $\frac{2}{3}$ mile (3520 ft) from 0 and Jody is $\frac{1}{3}$ mile (1760 ft) from 0 at noon.

41. Solve the problem in Example 6 if Mamie is driving at $96\,\dfrac{\text{ft}}{\text{sec}}\left(65.5\,\dfrac{\text{mi}}{\text{hr}}\right)$ and Jody is driving at $72\,\dfrac{\text{ft}}{\text{sec}}\left(49\,\dfrac{\text{mi}}{\text{hr}}\right)$.

42. ***Velocity of Airplane*** The instrument panel of a plane indicates a speed of 400 mph and a compass heading due north. If there is a cross wind of 80 mph from the southwest (blowing 45° east of north), what is the actual velocity (speed and direction) of the plane relative to the ground?

43. ***Variable Jet Stream*** In Exercise 42, suppose the cross wind is x mph in the direction 45° east of north. Find a formula for **(a)** the ground speed and **(b)** direction of the airplane as a function x. **(c)** If x is positive, the wind is from the southwest. If x is negative, in what direction is the wind blowing? **(d)** Use graphs to determine the speed and direction of the airplane when x is
(i) 40 **(ii)** 80 (Exercise 42)
(iii) 120 **(iv)** -50 (See Example 8).

44. In Exercise 43, let $T(x)$ be the time it takes for the plane to travel 1600 miles. Give a formula for $T(x)$. Use a graph to determine how long it takes when **(a)** $x = 50$, **(b)** $x = 100$, **(c)** $x = -60$.

45. An airplane has an airspeed of 500 mph in a northeasterly direction (45° north of east) and the jet stream is blowing at 100 mph due east. Find the velocity (speed and direction) of the plane relative to the ground.

46. ***Variable Jet Stream*** In Exercise 45, suppose the jet stream is blowing at x mph due east. Find formulas for the **(a)** ground speed and **(b)** direction of the airplane as a function of x. **(c)** Use graphs to determine the speed and direction of the airplane when x is
(i) 50 **(ii)** 100 (Exercise 45)
(iii) -80. Explain what $x = -80$ means.

47. In Exercise 46, determine a formula for the **(a)** time, $T(x)$, it takes for the airplane to travel 2400 miles. **(b)** Use a graph to find the time when x is
(i) 40 **(ii)** 100 **(iii)** -60.

48. Repeat Exercise 46 if the jet stream is x mph in the direction 45° south of west rather than due east.

49. ***Air Flight Time*** Suppose an airplane has an airspeed of 500 mph and the jet stream is blowing at 100 mph due east. If the plane has to travel from Seattle to Salt Lake City (900 miles at a heading 45° south of east), in what direction should the plane head? How long will the flight take?

50. Repeat Exercise 49 for the return flight, from Salt Lake City to Seattle, with the same airspeed and jet stream.

CHAPTER 7 REVIEW

Test Your Understanding

True or False. Give reasons.

1. In a right triangle suppose the hypotenuse c and angle β are given. Side b can be found by using $b = \dfrac{c}{\sin\beta}$.
2. In a right triangle the area A is given by $A = \frac{1}{2}c^2 \sin\alpha \sin\beta$.
3. In a right triangle the area A is given by $A = \frac{1}{4}c^2 \sin 2\alpha$.
4. Suppose angle β and the perimeter P of a right triangle are given. The hypotenuse c is given by
$$c = \frac{P}{1 + \sin\beta + \cos\beta}.$$
5. If $\alpha = 2\beta$ in a right triangle, then $a = 2b$.
6. There is exactly one triangle for which $a = 3.2$, $b = 4.1$, and $\gamma = 24°$.
7. Exactly one triangle is determined by $b = 7.3$, $c = 8.7$, and $\gamma = 120°$.
8. There is no triangle with $a = 3.6$, $b = 3.6$, and $\gamma = 135°$.
9. There is exactly one triangle ABC for which $b = 6$, $\alpha = 30°$, and $h = 3$, where h is the altitude from C to $\overline{AB}$.
10. There are two triangles for which $a = 6$, $b = 8$, and $\gamma = 40°$.
11. If $a = 5$, $b = 8$, and $c = 10$, then γ is an obtuse angle.
12. If $a = 4$, $b = 6$, and $c = 7$, then α is an acute angle.
13. If $a = 3$, $b = 5$, and $c = 6$, then $\alpha + \beta > 90°$.
14. In triangle ABC, if $\alpha = 50°$, $\beta = 60°$, then c must be the longest side.
15. A triangle with $a = 5$, $b = 12$, and $c = 13$ is a right triangle.
16. A triangle with $a = \sqrt{3}$, $b = \sqrt{5}$, and $c = 2\sqrt{2}$ is a right triangle.
17. If each side of a triangle is doubled, then its area is also doubled.
18. If $\alpha = 30°$ and $a = 16$, then $b = 32 \sin\beta$.
19. If $\beta = 45°$ and $b = 5\sqrt{2}$, then $c = 10 \sin\gamma$.
20. The area of a triangle with $a = \sqrt{3}$, $b = 8$, and $\gamma = 60°$ is a whole number.
21. If in a right triangle $a = \sqrt{3}$ and $\alpha = 30°$, then b is a whole number.
22. The area of a triangle with $a = 2$, $b = 4$, and $\gamma = 40°$ is twice that of a triangle with $a = 2$, $b = 4$, and $\gamma = 20°$.
23. The area of a triangle with $a = 4$, $b = 16$, and $\gamma = 43°$ is twice that of a triangle with $a = 2$, $b = 8$, and $\gamma = 43°$.
24. If the area of a triangle is 16 and $\gamma = 30°$, then the product $a \cdot b$ is equal to 32.
25. In a triangle with $a = 7.5$, $b = 5.3$, and $c = 5.6$, the largest angle is α.
26. The area of a triangle with $a = 4\sqrt{5}$, $b = \sqrt{15}$, and $\gamma = 60°$ is equal to 15.
27. If $c^2 > a^2 + b^2$, then $\cos\gamma$ is negative.
28. If $b^2 > a^2 + c^2$, then β is an obtuse angle.
29. If $a^2 = b^2 + c^2$, then γ is a right angle.
30. If γ is an obtuse angle, then $c > \sqrt{a^2 + b^2}$.
31. If $c > \sqrt{a^2 + b^2}$, then α must be an acute angle.
32. If b, β, and γ are given, then $c = \dfrac{b \sin\beta}{\sin\gamma}$.
33. If α, β, and c are given, then $b = \dfrac{c \sin\beta}{\sin(a + \beta)}$.
34. If $a = 2b$, then $\alpha = 2\beta$.
35. If $b = 2c$, then $\sin\beta = 2\sin\gamma$.
36. If $a = 2c$, then $\sin\alpha = \sin 2\gamma$.
37. If $\alpha = 2\beta$, then $a = 2b\cos\beta$.
38. In any triangle $\sin(\alpha + \beta) = \sin\gamma$.
39. If $z = -3 + 4i$, then $|z| = 5$.
40. A square root of $2i$ is $1 + i$.
41. A cube root of $8i$ is $\sqrt{3} + i$.
42. A polar form for $\sqrt{3} + i$ is $2(\cos 60° + i \sin 60°)$.
43. Vectors $\langle -1, 1\rangle$ and $\langle 1, -1\rangle$ are perpendicular to each other.
44. The angle between vectors $\langle 1, -1\rangle$ and $\langle 0, 1\rangle$ is 135°.
45. If $\mathbf{u} = \langle -1, 3\rangle$ and $\mathbf{v} = \langle -2, 1\rangle$ then $|\mathbf{u} + \mathbf{v}| = 5$.
46. If $\mathbf{u} = \langle 2, -1\rangle$ and $\mathbf{u} + \mathbf{v} = \langle -2, 1\rangle$ then $\mathbf{v} = \langle -4, 2\rangle$.
47. Vectors $\langle 2, -4\rangle$ and $\langle -4, 2\rangle$ are in opposite directions.
48. The vector $\langle 3, -3\rangle$ is in the direction of 45° north of west.
49. If the jet stream is blowing in the southeast direction, then the ground speed of an airplane traveling north is greater than its air speed.

50. If the jet stream is blowing in a southwesterly direction, then for an airplane traveling north, the ground speed is greater than the air speed.

Review for Mastery

In the following exercises assume standard labeling of parts of a triangle where a, b, and c denote the lengths of the sides and α, β, and γ are the angles opposite the respective sides. For right triangles $\gamma = 90°$.

1. In a right triangle $c = 37.4$ and $\beta = 25°20'$. Find α, a, and b.

2. Find the area of a right triangle in which $c = 2.56$ and $\alpha = 34°10'$.

3. The area of a right triangle is 0.924 m^2 and one of the angles is $37°20'$. Find the length of the hypotenuse.

4. The hypotenuse of a right triangle is 6.5 cm and its area is 8.4 cm^2. Find the two acute angles.

5. Determine the area of an isosceles triangle with equal sides of length 8.6 cm and opposite (base) angles of 36°.

6. Given $\alpha = 36°$, $\beta = 41°$, and $a = 7.6$, find c.

7. Given $a = 3.75$, $c = 5.76$, and $\beta = 137.4°$, find b.

8. If $a = 3.48$, $c = 5.63$, and $\gamma = 62.7°$, find b.

9. If $a = 3.7$, $b = 7.5$, and $c = 6.8$, determine angle α.

10. Given $a = 2.8$, $b = 3.7$, and $\beta = 54°$, find c and γ.

11. If $a = 1.53$, $b = 6.41$, and $\gamma = 37.4°$, find the area of the triangle.

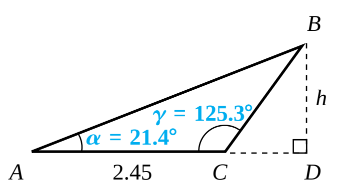
12. If $b = 3.7$, $c = 5.3$, and $\alpha = 115°$, find the perimeter of the triangle and length of the altitude to side c.

13. An equilateral triangle is inscribed in a circle of radius 12.4 inches. What is the area of the triangle?

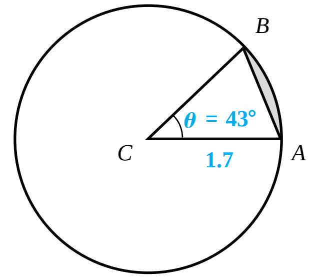
14. The area of triangle ABC is 427 m^2, $\gamma = 35.2°$ and $a = 23.5$ m. Find b.

15. The perimeter of a right triangle is 125 cm and one angle is 35.0°. Find the hypotenuse and the area of the triangle.

16. In the diagram, $|\overline{AC}| = 2.45$ km, $\alpha = 21.4°$, and $\gamma = 125.3°$. Find h, where $h = |\overline{BD}|$.

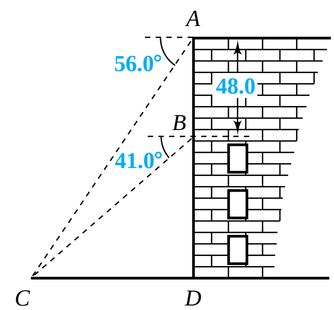


17. In the diagram the radius of the circle with center at C is 1.7 cm and the central angle $\theta = 43°$. Find the area of **(a)** triangle ABC, and **(b)** the shaded region.



18. The perimeter of triangle ABC is 145 cm, $\alpha = 36.0°$, and $\beta = 77.0°$. Find the length of side c.

19. ***Height of Building*** From the top of a building at point A, the angle of depression to point C on the ground is 56.0°, while from a point B, 48.0 feet directly below A, the angle of depression to point C is 41.0°. Find the height of the building. See the diagram.



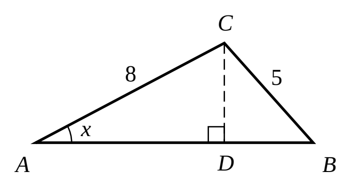
20. A circle is inscribed inside an equilateral triangle having sides of length 12. Find the area of the region that is inside the triangle and outside the circle.

21. ***Ambiguous Case*** In triangle ABC, $a = 5$, $b = 8$, x and y are as shown in the diagram.

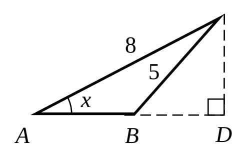
(a) Find a formula for y as a function f of x for the first diagram, and for y as a function g of x in the second diagram.

(b) Use graphs of f and g on the same screen to find y when x is 25°.

(c) For what values of x are there two solutions?

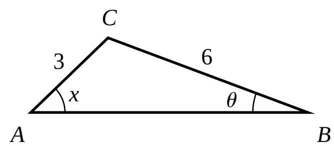


$y = |\overline{AD}| + |\overline{DB}|$



$y = |\overline{AD}| - |\overline{DB}|$

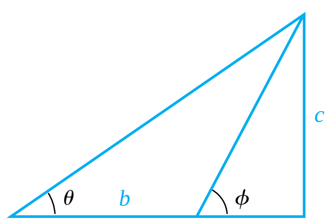
22. ***Maximum Angle*** In triangle ABC, $a = 6$, $b = 3$, x and θ are as shown in the diagram.
(a) Find a formula for θ as a function f of x.
(b) From a graph of f find θ when $x = 36°$.
(c) For what values of x is $\theta = 12°$?
(d) What is the maximum value θ can have?



23. ***Maximum Volume*** In triangle ABC, $b = 10$, $c = 5$. Let β be a variable angle, $\beta = x$. If $\triangle ABC$ is revolved about side $\overline{BC}$, a solid consisting of two cones joined together, is generated.
(a) Find a formula for the volume $V(x)$ of the solid.
(b) Use a graph to find the value of x that will give a maximum volume. What is the maximum volume?

24. Point E is the midpoint of side $\overline{BC}$ of square $ABCD$. Find $\sin\theta$, where θ is the angle formed by $\overline{AE}$ and the diagonal $\overline{AC}$. (*Hint:* First draw a diagram.)

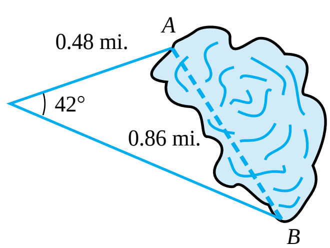
25. In the diagram show that $\cot\theta - \cot\phi = \dfrac{b}{c}$.



26. ***Height of CN Tower*** The CN Tower of Toronto is the world's tallest self-supporting structure. From a point on the ground 845.0 feet from the tower's base, the angle of elevation to the top is measured as 65.11°. What is the height of the tower?

27. ***Perimeter, Area of Octagon*** A regular octagon is inscribed in a circle of radius 24 feet.
(a) What is the perimeter of the octagon?
(b) What is the area of the region bounded by the octagon?

28. ***Inaccessible Distance*** Two cabins are located at points A and B on the shore of a lake (see the diagram). From the information in the diagram, find how far a boat would have to travel from A to B.



the information in the diagram, find how far a boat would have to travel from A to B.

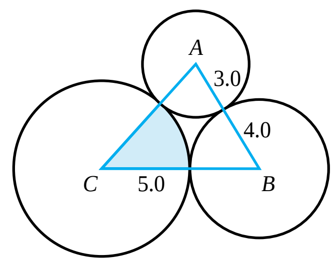
29. ***Vertices as Coordinates*** Points $A(12, 7)$, $B(7, 12)$, and $C(0, 0)$ are vertices of a triangles.
(a) Find the measure of angle $\theta = \angle ACB$ to the nearest tenth of a degree.
(b) What is the length of the altitude from C to $\overline{AB}$?
(c) What is the area of the region bounded by $\triangle ABC$?

30. For points $A(2, 1)$, $B(5, 5)$, and $C(6, 4)$ in the plane, find the measure of $\angle ABC$ **(a)** in degrees (to 1 decimal place) and **(b)** in radians (to 1 decimal place).

31. Find the measure to the nearest tenth of a degree of each angle of the parallelogram with vertices at $A(-2, 4)$, $B(2, 1)$, $C(5, 3)$, and $D(1, 6)$.

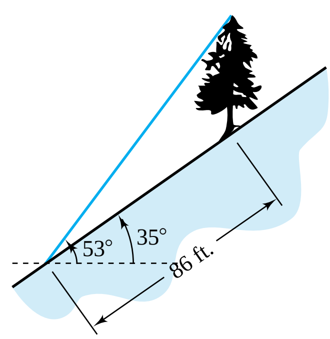
32. Find the largest angle to the nearest degree in the triangle with sides of lengths 39, 80, and 89.

33. Three mutually tangent circles with centers at A, B, and C have radii of 3.0, 4.0, and 5.0, respectively, as shown in the diagram. What is the area of the shaded region?



34. ***Height of a Building*** At a certain point A on the ground, the angle of elevation to the top of a building is 37.1°. At a point 64.2 feet farther away, the angle of elevation is 30.2°.
(a) What is the height of the building?
(b) How far from the base of the building is point A?

35. ***Height of a Tree*** A tree stands vertically on a hill that slopes 35° from the horizontal. At a point 86 feet downhill from the tree, the angle of elevation to the top of the tree is 53°. See the diagram. How tall is the tree?



36. The lengths of the diagonals of a parallelogram are 12 cm and 18 cm. If the smaller angle where the diagonals intersect is 32°, what are the lengths of the sides?

Exercises 37–42 How many triangles (if any) have the given parts?

37. $a = 3.6$, $b = 7.5$, $\gamma = 75°$

38. $a = 5.6$, $b = 4.5$, $\beta = 24°$

39. $\alpha = 40°$, $\beta = 115°$, $\gamma = 25°$

40. $a = 7.3$, $b = 2.8$, $c = 5.4$

41. $b = 8.2$, $c = 3.7$, $\beta = 43°$

42. $\alpha = 47°$, $\beta = 65°$, $a = 24$

Exercises 43–48 ***Powers of Complex Numbers*** *Evaluate the expression. Give the answer in rectangular form for complex numbers, a + bi, where a and b are real numbers. Whenever reasonable, give the result in exact form; otherwise give a and b rounded off to two decimal places.*

43. **(a)** $(1 + i)^5$ **(b)** $(3 - 2i)^6$

44. **(a)** $(1 + i)^{-2}$ **(b)** $(2 + 3i)^{-2}$

45. **(a)** $(\sqrt{3} + i)^6(1 + i)^{-4}$ **(b)** $(3 - 2i)^4(1 + 2i)^{-3}$

46. **(a)** $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{12}$ **(b)** $\left(\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}i\right)^{-8}$

47. **(a)** $\left[2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)\right]^6$

(b) $\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)^{-4}$

48. **(a)** $\left(\cos\frac{4\pi}{15} + i\sin\frac{4\pi}{15}\right)^{10}$

(b) $\left(\cos\frac{2\pi}{15} + i\sin\frac{2\pi}{15}\right)^{-10}$

49. Find the roots of **(a)** $z^2 - 2iz - 2 = 0$ and **(b)** $z^2 - (2 - i)z - i = 0$.

50. Find the cube roots of $\frac{\sqrt{3} - i}{2}$.

51. Find the fourth roots of $\frac{3 - 4i}{5}$.

52. Find the sixth roots of -729.

53. Find the fourth roots of $8\sqrt{23}(2 + i)$.

54. Find all complex number roots of the equation $x^5 + 1 = 0$.

Exercises 55–57 ***Vector Arithmetic*** *Vectors* **u** *and* **v** *are given by* $\mathbf{u} = \langle -1, 3\rangle$, $\mathbf{v} = \langle 0, -4\rangle$.

55. Find **(a)** $\mathbf{u} + \mathbf{v}$ and **(b)** $2\mathbf{u} - \mathbf{v}$

56. Find **(a)** $|\mathbf{u}|$ and **(b)** $|\mathbf{u} - \mathbf{v}|$

57. Find the angle between **u** and **v**.

58. A boat travels 64 km due west from port, then turns 60° toward the north and travels 48 km in the new direction. How far and in what direction is the boat's location relative to the port?

59. An airplane has an airspeed of 400 mph in the direction 30° east of north. If there is a wind of 80 mph blowing due east, **(a)** what are the speed and direction of the plane relative to the ground? **(b)** How long will it take for the plane to fly a distance of 1000 ground miles?

60. ***Air Flight*** The instrument panel of an airplane indicates a speed of 400 mph and a compass heading due south. **(a)** If there is a cross wind of x mph in the direction 45° east of north, find a formula for the ground speed of the airplane as a function of x.
(b) Use a graph to find the ground speed when $x = 40$; $x = 100$.

61. In Exercise 60, find a formula that gives the time it takes for the airplane to travel 1500 miles. Use a graph to find the time when $x = 20$; $x = 80$; $x = 120$.

62. ***Minimum Distance*** Two cars are traveling on highways that interesect at right angles. At noon car A is 2400 feet south of the intersection and is traveling north at a speed of 80 ft/sec, while car B is 2100 feet west of the intersection and is traveling east at a speed of 60 ft/sec.
- **(a)** What is the speed of each car in miles per hour?
- **(b)** Express the position of each car as a vector function of t, the number of seconds after noon.
- **(c)** Find a formula that gives the distance $d(t)$ between the cars. Use a graph to find t for which $d(t)$ is a minimum.

8 DISCRETE MATHEMATICS: FUNCTIONS ON THE SET OF NATURAL NUMBERS

THE UNIFYING THEME OF THIS chapter is the set of natural numbers. Functions defined on that set, called *sequences*, are discussed in the first three sections. Throughout this book we have invited you to explore and discover mathematics yourself. Number patterns encourage this curiosity, and they are a primary source of the fascination mathematics has always had for humans. Section 8.2 is a calculator-based exploration of limits and convergence. Section 8.4 is devoted to questions about patterns that illustrate where mathematical ideas come from and how theorems are discovered. Section 8.5 introduces mathematical induction, as both a productive way of thinking and a powerful tool to establish the validity of statements about the natural numbers. One such statement, the binomial theorem, is the focus of Section 8.6.

8.1 INTRODUCTION TO SEQUENCES; SUMMATION NOTATION

> In some cases such as the weather, the phenomenon always appears to be random but in other cases such as the dripping faucet, sometimes the dripping is periodic and other times each drip appears to be independent of the preceding one, thereby forming an irregular sequence.
>
> *B. J. West*

We live in a world largely ordered by numbers, most often by the natural numbers. Record books list many firsts; a runner in the Boston Marathon may be proud to come in 367th or 893rd. We sometimes use incredibly large numbers to identify things: nine digits suffice to provide unique Social Security numbers for 250 million Americans, but look at some of the numbers on insurance forms.

A familiar example from mathematics is the sequence of prime numbers, where 2 is the first, 3 is the second, 5 is the third, and so on. This order defines a function, with the rule of correspondence

$$f(n) = \text{the } n\text{th prime number.}$$

Euclid proved that there are infinitely many primes, which means that the domain of the prime number function is infinite, the set N of all natural numbers. There is *no* last prime. We have no simple formula to compute the millionth prime, $f(1{,}000{,}000)$, but we know that it exists.

Sequences and Notation

A **sequence** (sometimes, for emphasis, an **infinite sequence**) can be thought of as a **list,** with a term for every natural number, even though we sometimes use sequences like $a_0, a_1, a_2, \ldots$ or $b_4, b_5, b_6, \ldots$. Each has a first term, a second term, and so on. It is convenient to define a sequence in terms of its domain, but equivalent domains could be used, as well.

By the time I was well enough to go back to school I had missed more than two years. My parents arranged to have me tutored by a retired elementary school teacher. One day she told me that you could never carry the square root of 2 to a point where the decimal began to repeat. She knew that this fact had been proved, although she did not know how.

Julia Robinson

Definition: sequence

A **sequence** is a function whose domain is the set of natural numbers.

The rule that defines a sequence can be either given by a mathematical formula or stated in words. The terms of a sequence, the function values, can be listed as

$$f(1), f(2), f(3), \ldots, f(n), \ldots,$$

or we may use subscript notation, where $f(n)$, the nth term of the sequence, is denoted by a_n. We sometimes denote the whole list of sequence values by $\{a_n\}$:

$$a_1, a_2, a_3, \ldots, a_n, \ldots \text{ is equivalent to } \{a_n\}.$$

►EXAMPLE 1 **Evaluating terms** List the first four terms and the tenth term of the sequence **(a)** $f(n) = 2n - 1$, **(b)** $b_n = n^2 - 4$, and **(c)** $a_n = (-1)^n$.

Solution

(a) $f(1) = 2 \cdot 1 - 1 = 1$, $f(2) = 2 \cdot 2 - 1 = 3$, $f(3) = 2 \cdot 3 - 1 = 5$, $f(4) = 2 \cdot 4 - 1 = 7$, $f(10) = 2 \cdot 10 - 1 = 19$.

(b) $b_1 = 1^2 - 4 = -3$, $b_2 = 2^2 - 4 = 0$, $b_3 = 3^2 - 4 = 5$, $b_4 = 4^2 - 4 = 12$, $b_{10} = 10^2 - 4 = 96$. The sequence begins $-3, 0, 5, 12, \ldots$.

(c) $a_1 = (-1)^1 = -1$, $a_2 = (-1)^2 = 1$, $a_3 = (-1)^3 = -1$, and $a_4 = (-1)^4 = 1$. The sequence begins $-1, 1, -1, 1, \ldots$, and $a_{10} = (-1)^{10} = 1$. The same sequence is given by the rule

$$a_n = \begin{cases} -1 \text{ if } n \text{ is odd} \\ 1 \text{ if } n \text{ is even} \end{cases}. \quad ◄$$

►EXAMPLE 2 **Evaluating terms** List the first five terms.

(a) $a_n = 2n$

(b) $b_n = (n - 1)(n - 2)(n - 3) + 2n$

(c) $c_n = \begin{cases} 2n \text{ for } n = 1, 2, 3 \\ 0 \text{ for } n \geq 4 \end{cases}$

Solution

The sequences begin:

(a) $\{a_n\} = 2, 4, 6, 8, 10, \ldots$
(b) $\{b_n\} = 2, 4, 6, 14, 34, \ldots$
(c) $\{c_n\} = 2, 4, 6, 0, 0, \ldots$ ◀

Example 2 illustrates an important point. Giving a few terms does not define a sequence. Infinitely many sequences begin 2, 4, 6, ... Test makers sometimes ask for "the next term in the sequence 2, 4, 6, . . . ," expecting a response of "8," but as Example 2 shows, that question has no single correct response. Consider some less obvious sequences that begin 2, 4, 6, . . . :

Sequence Beginning	*Rule (Function)*
2, 4, 6, 10, 16, 26, . . .	Each term after the second is the sum of the two preceding terms.
2, 4, 6, 1, 3, 5, 0, . . .	The term a_n is the remainder when $9n$ is divided by 7.
2, 4, 6, 2, 3, 4, . . .	The sequence is taken from the decimal expansion for π starting in the 374th place, then listing every other digit.

Sequences Defined Recursively

We can define a sequence by stating the rule of correspondence, either by a formula or in words. Another useful method is to use **recursion,** continuing a sequence, based on known terms of the sequence, as illustrated in the following examples.

▶EXAMPLE 3 *Compare sequences* Evaluate several terms of the sequences

(a) $a_1 = 4$ and $a_n = a_{n-1} + 3$ for $n > 1$ **(b)** $b_n = 3n + 1$.

Solution

Strategy: **(a)** Evaluate the terms of a recursive sequence in order, using one term to get the next. In this case, given one term, we get the next one by adding 3.

(a) Start with $a_1 = 4$. The second part of the definition says that when n is greater than 1, each term is obtained by adding 3 to the preceding term, so $a_2 = a_1 + 3 = 4 + 3 = 7, a_3 = a_2 + 3 = 7 + 3 = 10$, and so on. The sequence begins 4, 7, 10, 13, 16, . . . , and continues by adding 3 each time.
(b) With a closed (explicit) formula, calculate the first few terms directly: 4, 7, 10, 13, 16. The sequence begins with exactly the same terms as those in part **a.** ◀

To be identical, two sequences must do more than agree for the first few terms; they must continue to agree. In Example 3, $\{a_n\}$ and $\{b_n\}$ have the same first five terms. Are they identical sequences? Notice how b_n is related to b_{n-1}. The formula gives $b_{n-1} = 3(n-1) + 1 = 3n - 2$ and $b_n = 3n + 1 = (3n - 2) + 3$, so $b_n = b_{n-1} + 3$. The two sequences satisfy the same recursive relation, so they are identical sequences.

Example 3 also shows that a sequence defined recursively, $\{a_n\}$, may also be given by a formula in closed form, $\{b_n\}$. Both methods are useful. Recursive relations often occur naturally and they lend themselves to computer programming, but an explicit formula gives any particular term directly, without the need to

HISTORICAL NOTE THE FIBONACCI SEQUENCE

Among the many sequences that mathematicians have studied, surely none is more fascinating than the Fibonacci sequence that begins

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, . . . ,

where each term after the second is the sum of the two preceding terms.

The sequence first appeared in print in the year 1202. Leonardo of Pisa, also known as Fibonacci, wrote a *Book of Counting (Liber Abaci)* that introduced the then modern mathematics of North Africa (including Arabic numerals) to Europe. The book was one of the most important sources of mathematical learning in Europe for several hundred years. One of the more trivial problems in the book dealt with the growth of a colony of rabbits and led to the sequence of numbers that now bears Fibonacci's name.

Once a person becomes aware of the numbers in the Fibonacci sequence, the numbers seem to pop up everywhere. All kinds of growth processes involve the sequence. As a simple example, the next time you pick up a pine cone or look at a sunflower head, count the number of spirals at some fixed angle. They will turn out to be Fibonacci numbers. The golden rectangle of Greek architecture, the rectangle considered to have the most pleasing proportions, has sides whose ratio is approached by ratios of successive Fibonacci numbers. The Fibonacci numbers also occur as sums of binomial coefficients taken along diagonals of Pascal's triangle. An entire journal, *The Fibonacci Quarterly,* is devoted to discoveries related to Fibonacci numbers and the Fibonacci sequence.

Fibonacci

The Fibonacci sequence is probably the only mathematical entity to have made the London stage. A recent play is based on the life of Alan Turing, the British mathematician who laid the theoretical foundations for modern computers and the study of artificial intelligence and who helped crack Germany's supersecret code in World War II. In the play Turing explains to the other characters the fascination of the Fibonacci sequence.

calculate earlier terms. For example, the first sequence in the table above is given recursively as:

$$a_1 = 2 \quad a_2 = 4 \quad \text{and} \quad a_n = a_{n-1} + a_{n-2} \quad \text{for } n > 2.$$

To get a_{10}, we would need a_9 and a_8, for which we would need a_7 and a_6, etc. The term a_{64} is defined, but it requires considerable work to determine that

$$a_{64} = 21{,}220{,}419{,}715{,}446.$$

The Fibonacci sequence

The Fibonacci sequence $\{f_n\}$ is defined recursively by

$$f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2} \quad \text{for } n > 2.$$

Check to see that the Fibonacci sequence begins 1, 1, 2, 3, 5 . . . .

Factorials

An important sequence is often defined recursively as

$$F_0 = 1 \quad \text{and} \quad F_n = nF_{n-1} \quad \text{for } n > 0.$$

The first few items are easily calculated.

$$\begin{aligned}
F_0 &= 1 & F_5 &= 5 \cdot F_4 = 120 \\
F_1 &= 1 \cdot F_0 = 1 & F_6 &= 6 \cdot F_5 = 720 \\
F_2 &= 2 \cdot F_1 = 2 & F_7 &= 7 \cdot F_6 = 5{,}040 \\
F_3 &= 3 \cdot F_2 = 6 & F_8 &= 8 \cdot F_7 = 40{,}320 \\
F_4 &= 4 \cdot F_3 = 24 & F_9 &= 9 \cdot F_8 = 362{,}880
\end{aligned}$$

The terms of the sequence grow very rapidly. F_{10} exceeds 3 million, and F_{13} is nearly 8 billion.

This sequence is the sequence of **factorials;** we reserve a special notation for factorials:

$$F_n = n! \qquad \text{(read “}n\text{ factorial”).}$$

The sequence has a recursive definition.

Factorial sequence (recursive form)

Suppose n is any nonnegative integer. The factorial sequence $\{n!\}$ is given by

$$\begin{aligned}
0! &= 1 \\
n! &= n(n-1)! \quad \text{for } n > 0.
\end{aligned}$$

A formula in closed form for $n!$ is suggested by the following examples.

$$2! = 2(1!) = 2 \cdot 1, \quad 3! = 3(2!) = 3 \cdot 2 \cdot 1, \quad 4! = 4(3!) = 4 \cdot 3 \cdot 2 \cdot 1.$$

It can be shown that for any positive integer n, $n!$ is the product of all the integers from 1 to n inclusive. This gives a closed-form definition.

Factorial sequence (closed form)

If n is any nonnegative integer, then the factorial sequence is given by

$$\begin{aligned}
0! &= 1 \\
n! &= n(n-1)\ldots 3 \cdot 2 \cdot 1 \text{ if } n \geq 1
\end{aligned}$$

►EXAMPLE 4 Factorials **(a)** Evaluate $\frac{8!}{4!\,4!}$. **(b)** Express the sum $\frac{7!}{2!\,5!} + \frac{7!}{3!\,4!}$ as a single fraction and then check by evaluating each expression.

Strategy: **(b)** To add fractions, find a common denominator that contains all factors of both given denominators. Since $5! = 5 \cdot 4!$ and $3! = 3 \cdot 2!$, the common denominator is $3!\,5!$; multiply the first term by $\frac{3}{3}$ and the second by $\frac{5}{5}$.

Solution

(a) Using the closed form for $n!$, write out each factorial as a product and then simplify.

$$\frac{8!}{4!\,4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} = 70$$

(b) Follow the strategy.

$$\begin{aligned}
\frac{7!}{2!\,5!} + \frac{7!}{3!\,4!} &= \frac{3 \cdot 7!}{(3 \cdot 2!)5!} + \frac{5 \cdot 7!}{3!\,(5 \cdot 4!)} = \frac{3 \cdot 7!}{3!\,5!} + \frac{5 \cdot 7!}{3!\,5!} \\
&= \frac{3 \cdot 7! + 5 \cdot 7!}{3!\,5!} = \frac{(3+5)7!}{3!\,5!} = \frac{8 \cdot 7!}{3!\,5!} = \frac{8!}{3!\,5!}
\end{aligned}$$

Thus

$$\frac{7!}{2!\,5!} + \frac{7!}{3!\,4!} = \frac{8!}{3!\,5!}.$$

As a check,

$$\frac{7!}{2!\,5!} + \frac{7!}{3!\,4!} = 21 + 35 = 56 \quad \text{and} \quad \frac{8!}{3!\,5!} = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56.$$ ◄

Partial Sums and Summation Notation

We will often be interested in the sum of certain terms of a given sequence. To illustrate, consider the sequence $\{b_n\}$ given by $b_n = 2n - 1$. This sequence begins $1, 3, 5, 7, \ldots$. There is a related sequence denoted by $\{S_n\}$, which we call the **sequence of partial sums,**

$$S_1 = b_1 = 1$$
$$S_2 = b_1 + b_2 = 1 + 3 = 4$$
$$S_3 = b_1 + b_2 + b_3 = 4 + 5 = 9$$
$$S_4 = b_1 + b_2 + b_3 + b_4 = 9 + 7 = 16.$$

The emerging pattern suggests that the general term is given by the formula $S_n = n^2$.

It is cumbersome to write out the partial sum for many terms of a sequence. For instance, S_{100} is the sum of 100 terms, $b_1 + b_2 + b_3 + \cdots + b_{100}$. We introduce some special notation to denote such sums, using the Greek letter sigma Σ:

$$S_{100} = \sum_{k=1}^{100} b_k = b_1 + b_2 + b_3 + \cdots + b_{100}$$

The summation notation is a convenient shorthand. We suggest frequent practice to become familiar with it.

Definition: sequence of partial sums

Suppose $\{a_n\}$ is any sequence of real numbers. The corresponding sequence of partial sums is $\{S_n\}$ where

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n.$$

The sigma notation $\Sigma_{k=1}^{m} a_k$ simply means the sum of the first m terms of the sequence $\{a_n\}$. The letter k does not appear in the expanded form and is sometimes called a **dummy variable.** Any other letter would do as well. For example, $\Sigma_{k=1}^{3} a_k = a_1 + a_2 + a_3$, and the same sum is given by $\Sigma_{j=1}^{3} a_j$, which also equals $a_1 + a_2 + a_3$. In $\Sigma_{k=1}^{m} a_k$ the integer 1 is called the **lower limit** for the sum, and m is the **upper limit.**

The sigma notation is useful in many other contexts, including differences of partial sums. Suppose we are interested in $S_6 - S_3$. In expanded form,

$$\begin{aligned} S_6 - S_3 &= (a_1 + a_2 + a_3 + a_4 + a_5 + a_6) - (a_1 + a_2 + a_3) \\ &= a_4 + a_5 + a_6 \\ &= \sum_{k=4}^{6} a_k. \end{aligned}$$

In general, for $m > n$,

$$S_m - S_n = \sum_{k=n+1}^{m} a_k$$

It should also be clear that the sequence of partial sums has a recursive form.

Sequence of partial sums (recursive form)

Suppose $\{a_n\}$ is any sequence of real numbers. The corresponding sequence of partial sums $\{S_n\}$ is

$$S_1 = a_1 \quad \text{and} \quad S_n = S_{n-1} + a_n \qquad \text{for } n > 1.$$

►EXAMPLE 5 ***Expanding sums*** Write out the sum in expanded form and evaluate.

(a) $\sum_{k=1}^{4} k(k+2)^2$

(b) $\sum_{i=3}^{6} (i-2)i^2$

Solution

(a) $\sum_{k=1}^{4} k(k+2)^2 = 1(1+2)^2 + 2(2+2)^2 + 3(3+2)^2 + 4(4+2)^2$
$= 9 + 32 + 75 + 144 = 260.$

(b) $\sum_{i=3}^{6} (i-2)i^2 = (3-2)3^2 + (4-2)4^2 + (5-2)5^2 + (6-2)6^2$
$= 9 + 32 + 75 + 144 = 260.$ ◄

Observe that the sums in Example 5 are identical even though they appear quite different in the compact sigma notation. We may write the same sum with any specified lower limit if we make appropriate adjustments in the upper limit and the defining formula. When rewriting a sum in sigma notation, always check at least the first and last terms to verify that your limits and formula give correct values.

►EXAMPLE 6 ***Converting to sigma notation*** Express the following in sigma notation.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6}$$

Solution

Sigma notation requires a formula that describes each term of the given sum. For this purpose it helps to identify what remains fixed and what changes from term to term. In each term the numerator is 1 and the denominator is the product of two consecutive integers, $k(k+1)$. In the first term, $k = 1$, and in the last, $k = 5$. The desired sum may be written as

$$\sum_{k=1}^{5} \frac{1}{k(k+1)}, \quad \text{or} \quad \sum_{n=2}^{6} \frac{1}{(n-1)n}. \quad ◄$$

EXERCISES 8.1

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. $\sum_{k=1}^{147}(-1)^k = 1.$
2. $\sum_{k=1}^{4} k^2 = \left(\sum_{k=1}^{4} k\right)^2.$
3. The sequence given by $b_n = 2n + 9 - n^2$ contains only positive integers.
4. For every positive integer n, $(n + 4)! = n! + 4!$.
5. $\sum_{k=1}^{4}\frac{1}{k} = \sum_{j=2}^{5}\frac{1}{j-1}.$

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. If $a_n = 2n + 1$, then $a_5 - a_2 =$ ______.
7. If $b_n = \dfrac{n! + 4}{n!}$, then $b_4 =$ ______.
8. If $a_n = 8 - 2n$, then $a_5 =$ ______.
9. $\sum_{k=1}^{4}(k-2)^2 =$ ______.
10. If $a_n = (-1)^n(2n - 1)$, then $\sum_{k=1}^{4} a_k =$ ______.

Develop Mastery

Exercises 1–12 ***Evaluate Terms*** *Find the first four terms and the eighth term.*

1. $f(n) = 3n + 1$
2. $f(n) = 10 - 2n$
3. $g(n) = 5^{-n}$
4. $g(n) = (-1)^n \cdot 2^{-n}$
5. $f(n) = n^2 + n + 41$
6. $f(n) = 2^n - 1$
7. $a_n = 1 - \dfrac{1}{2^n}$
8. $a_n = \dfrac{1}{1 - 2^n}$
9. $b_n = \dfrac{(2n)!}{n!}$
10. $b_n = \dfrac{n!}{2^n}$
11. $c_n = 4n^2 - 10n + 8$
12. $c_n = \left(1 + \dfrac{1}{n}\right)^n$

Exercises 13–24 ***Recursive Sequences*** *Find the first four terms of the sequence defined recursively.*

13. $a_1 = 3$ and $a_n = a_{n-1} - 4$ for $n \geq 2$
14. $a_1 = 5$ and $a_n = 3 - a_{n-1}$ for $n \geq 2$
15. $b_1 = 2$ and $b_n = 3 \cdot b_{n-1}$ for $n \geq 2$
16. $b_1 = -1$ and $b_n = -2 \cdot b_{n-1}$ for $n \geq 2$
17. $a_1 = 1, a_2 = 2$ and $a_n = a_{n-1} \cdot a_{n-2}$ for $n \geq 3$
18. $a_1 = 2, a_2 = 3$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$
19. $a_1 = 2, a_2 = 4$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$
20. $a_1 = 700$ and $a_n = \dfrac{a_{n-1}}{10}$ for $n \geq 2$
21. $a_1 = 1, a_2 = 2$ and $a_n = \dfrac{a_{n-1} + a_{n-2}}{2}$ for $n \geq 3$
22. $a_1 = \dfrac{1}{2}, a_n = a_{n-1} + \dfrac{1}{2^n}$ for $n \geq 2$
23. $a_1 = 1, a_n = \dfrac{a_{n-1}}{n!}$ for $n \geq 2$
24. $a_1 = 2, a_n = \sqrt{(a_{n-1})^2 + 1}$ for $n \geq 2$.

Exercises 25–30 ***Partial Sums*** *A formula for the kth term of a sequence is given. Find the first four terms of the corresponding partial sum sequence.*

25. $a_k = 2k + 1$
26. $a_k = 5 - 2k$
27. $a_k = \dfrac{1}{2^k}$
28. $a_k = \left(\dfrac{3}{2}\right)^k$
29. $a_k = \dfrac{1}{k}$
30. $a_k = \dfrac{1}{(k+1)(k+2)}$

Exercises 31–36 ***Expanded Form*** *Write out the terms for the given summation and then evaluate the sum.*

31. $\sum_{k=1}^{10}(k+1)$
32. $\sum_{j=1}^{5}\dfrac{1}{j+1}$
33. $\sum_{j=1}^{5}\left(\dfrac{1}{j+1} - \dfrac{1}{j}\right)$
34. $\sum_{i=1}^{6}\dfrac{i+1}{i}$
35. $\sum_{k=1}^{4}\left(1 - \dfrac{1}{2^k}\right)$
36. $\sum_{k=1}^{5} P_k$ where P_k is the kth prime number.

Exercises 37–42 ***nth Term Formula*** *The first few terms of a sequence are given. Determine a formula for the nth term. The answers are not unique, since many sequences could have the same starting terms. See Example 2.*

37. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
38. $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$
39. $\frac{1}{4}, -\frac{2}{9}, \frac{3}{16}, -\frac{4}{25}, \ldots$
40. 5, 8, 11, 14, . . .
41. 2, 3, 7, 25, 121, 721, . . .
(*Hint:* $1! = 1, 2! = 2, 3! = 6, 4! = 24$.)
42. $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots$

Exercises 43–46 ***Graph Sequence*** *A formula for a sequence is given. Draw a graph to show the values of function f for n = 1, 2, 3, 4, and 5. (Hint: The graph consists of isolated points.)*

43. $a_n = f(n) = (-1)^n$
44. $a_n = f(n) = \dfrac{n}{n+1}$

45. $a_n = f(n) = \dfrac{(-1)^n(n+1)}{n}$

46. $a_n = f(n) = \dfrac{(-1)^n}{2^{n-1}}$

Exercises 47–52 ***Sigma Notation*** *Express the sum in sigma notation.*

47. $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$

48. $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64$

49. $(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5})$

50. $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12}$

51. $\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6}$

52. $\ln 2 + \ln 3 + \ln 4 + \ln 5 + \ln 6$

Exercises 53–56 ***Factorials*** *Evaluate the expression.*

53. $\dfrac{6!}{2!\,4!}$ **54.** $\dfrac{12!}{8!}$ **55.** $\dfrac{50!}{48!}$ **56.** $\dfrac{15!}{5!\,10!}$

Exercises 57–58 ***Simplify Factorials*** *Express each sum as a single fraction involving factorials. Do not evaluate. (Hint: See Example 4b.)*

57. $\dfrac{8!}{3!\,5!} + \dfrac{8!}{4!\,4!}$ **58.** $\dfrac{10!}{3!\,7!} + \dfrac{10!}{4!\,6!}$

59. The decimal expansion for $\frac{5}{33}$ is given by $\frac{5}{33} = 0.1515151515\ldots.$ A sequence is described by $a_n =$ the nth decimal digit of $\frac{5}{33}$. For instance, $a_1 = 1$, $a_2 = 5$, $a_3 = 1$.
(a) Find a_{17} and a_{36}.
(b) Evaluate $\sum_{k=1}^{6} a_k$ and $\sum_{k=1}^{60} a_k$.

60. A representation of the number $\frac{1}{7}$ is the repeating decimal $0.\overline{142857}$. The sequence $\{a_n\}$ is described by $a_n =$ the nth decimal digit of $\frac{1}{7}$. For instance, $a_1 = 1$, $a_2 = 4$, and $a_3 = 2$.
(a) Find a_4, a_{17}, and a_{24}.
(b) Evaluate $\sum_{k=1}^{6} a_k$ and $\sum_{k=1}^{25} a_k$.

Exercises 61–62 ***Compare Sums*** *Write out the terms and compare the two sums. Are they identical?*

61. **(a)** $\displaystyle\sum_{k=1}^{4} \frac{1}{k(k+1)}$ **(b)** $\displaystyle\sum_{j=3}^{6} \frac{1}{(j-1)(j-2)}$

62. **(a)** $\displaystyle\sum_{k=1}^{6} k \cdot 2^{k+1}$ **(b)** $\displaystyle\sum_{j=2}^{7} (j-1) \cdot 2^j$

Exercises 63–66 ***Closed Form*** *Evaluate several terms of the sequence and look for a pattern that will help you guess a formula for a_n in closed form.*

63. $a_1 = 1, \quad a_n = 2 + a_{n-1} \quad \text{for } n \ge 2$

64. $a_1 = 2, \quad a_n = 2 \cdot a_{n-1} \quad \text{for } n \ge 2$

65. $a_n = \displaystyle\sum_{k=1}^{n} \frac{1}{2^k}$ **66.** $a_n = \displaystyle\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right).$

Exercises 67–70 ***Recursive Sequence*** *Write out the first five terms of the sequence $\{a_n\}$ defined recursively.*

67. $a_1 = 1, a_2 = 2 \quad a_{n+2} = (a_{n+1})^2 + (a_n)^2 \quad \text{for } n \ge 1$

68. $a_1 = 2 \quad a_{n+1} = \dfrac{n+1}{a_n} \quad \text{for } n \ge 1$

69. $a_1 = 1, a_2 = 2 \quad a_{n+2} = (a_{n+1})(a_n) \quad \text{for } n \ge 1$

70. $a_1 = 2 \quad a_{n+1} = a_n^2 \quad \text{for } n \ge 1$

71. Two sequences $\{a_n\}$ and $\{b_n\}$ are defined by $b_1 = 3$, $b_{n+1} = b_n + 2n$ and $a_n = n^2 - n + 3$ for $n \ge 1$.
(a) Evaluate the first five terms of each sequence.
(b) Are the sequences identical? Justify your answer.
(c) Compute b_{60}.

72. A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$, $a_2 = 4$, $a_{n+2} = a_{n+1} - a_n$ for $n \ge 1$.
(a) Write out the first ten terms of the sequence.
(b) What is the sum of the first 36 terms? Of the first 96 terms? Of the first 110 terms?

73. The increasing sequence 2, 3, 5, 6, 7, 8, 10, . . . consists of all positive integers that are not squares of integers, that is, all natural numbers not in the sequence 1, 4, 9, ... **(a)** What is the 60th term of the sequence? **(b)** How many terms of the sequence precede the number 124?

Exercises 74–76 ***The 3N + 1 Sequence*** *A curious sequence has been studied by many people; it is often called the* 3*N* + 1 *sequence and is defined recursively as:*

$$a_1 = \text{any positive integer}$$
$$a_{n+1} = \begin{cases} \dfrac{a_n}{2} & \text{if } a_n \text{ is an even integer} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd integer} \end{cases}$$

74. **(a)** Find several terms for each of the sequences starting with $a_1 = 4$, $a_1 = 5$, $a_1 = 24$, $a_1 = 27$. In each case, continue the sequence until you observe something interesting happening.
(b) Try different starting numbers of your own for a_1 and see if your observations in part (a) are still valid. Make a guess about what happens with any starting positive integer a_1.

75. It has been conjectured that starting with any positive integer a_1, the $3N + 1$ sequence eventually reaches 1, after which it repeats the loop 1, 4, 2, 1, 4, 2, ... Define a $3N - 1$ sequence in essentially the same way: begin with any positive integer a_1 and continue by defining $a_{n+1} = \frac{a_n}{2}$ if a_n is even, and $a_{n+1} = 3a_n - 1$ if a_n is odd.

(a) Take several odd, positive integers for a_1 and write out enough terms of the $3N - 1$ sequence to reach a repeating loop.

(b) Show that not every positive integer reaches the same loop (as appears to be the case for the $3N + 1$ sequence). How many different loops can you find?

76. Programs for computers or programmable calculators often use an IF . . . THEN . . . instruction that terminates the programs if certain conditions hold but branch to other instructions otherwise. Describe the kind of difficulties that we could conceivably encounter if we programmed a computer to run the $3N + 1$ sequence and print out the terms until reaching 1.

Exercises 77–78 ***Compare Sequences***

77. Evaluate the first five terms of $\{a_n\}$ and $\{b_n\}$ where

$$a_n = 2^{n-1},\ b_n = \frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}.$$

Are the sequences identical?

78. **(a)** Evaluate the first six terms of $\{a_n\}$ where

$$a_n = \sqrt{n} + \sqrt{n + 9 - 6\sqrt{n}}.$$

Use your calculator to simplify each term. Is $a_n = 3$ for every $n \geq 1$? Explain.

(b) If $b_n = \sqrt{n} + |\sqrt{n} - 3|$, are the sequences $\{a_n\}$ and $\{b_n\}$ identical? Explain.

Exercises 79–80 ***Large Numbers***

79. The factorial sequence $\{n!\}$ increases very rapidly. For instance,

$$10! \approx 3.6 \times 10^6 \qquad 20! \approx 2.4 \times 10^{18}$$
$$50! \approx 3.0 \times 10^{64} \qquad 70! \approx 1.2 \times 10^{100}$$

To get some idea of how large these numbers are, look at 20!. Simple computation gives 20! = 2,432,902,008,176,690,000. Now suppose a computer printer that operates at 100 characters per second were to print out a manuscript with 20! characters. How long would it take the printer to do the job?

80. For the manuscript described in Exercise 79, suppose each page contains about 4000 characters. How thick would the manuscript be? The thickness of a ream of paper (500 pages) is approximately 2 inches. For comparison, the distance from the earth to the sun is 93 million miles.

Exercises 81–82 ***Fibonacci Sequence*** *Use the definition on page 436.*

81. Show that $f_{n+1}/f_n - 1 = f_{n-1}/f_n$ for $n \geq 2$.

82. ***Explore*** Write out the first twelve terms of the Fibonacci sequence. Make a guess as to which terms are divisible by 2, by 3.

8.2 GRAPHS AND CONVERGENCE

> ...the vast scope of modern mathematics. I have in mind an expanse swarming with beaut[y],... worthy of being surveyed from one end to the other and studied even in its smallest details: its valleys, streams, rocks, woods and flowers.
>
> *Arthur Cayley*

> *It just came to me that I could use this technique, this theorem, in connection with these curves in Hilbert space that I was dealing with—and get the answer! It just came to me out of the blue one day. It has always struck me as so amazing. One half of me had been bouncing around with this theorem a lot and the other half had been doing this problem, and they had never gotten together.*
>
> Andrew Gleason

Calculus is based on the study of limits. At this point, we use only intuitive ideas of limits, but we can use graphs and their end behavior to get strong feelings about the existence or nonexistence of limits of certain sequences. Without making a precise definition, when a sequence $\{a_n\}$ has a limit L, we say that $\{a_n\}$ **converges** to L and we write $\lim_{n\to\infty} a_n = L$.

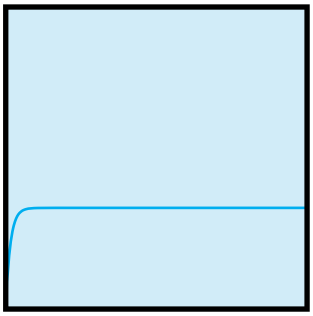
Since a sequence is a function, we can draw a graph; but we are only interested in those points for which the x-pixel values are positive integers. In general, our graphs will be drawn with an x-range of $[0, c]$, where c is the number of pixel columns on our calculator, and in almost all graphs we will want to use dot mode.

TECHNOLOGY TIP ◆ ***Pixel columns***

The number of pixel columns, c, for several calculators is as follows:

Model	*#cols*	*Model*	*#cols*
TI-81	95	HP-38,48	130
TI-82	94	Casio fx-7700	94
TI-85	126	Casio fx-9700	126

To illustrate convergence, we look at the end behavior for several sequences in the first example.



$[0, c]$ by $[0, 2]$

FIGURE 1

▶EXAMPLE 1 ***Convergence*** Use a graph to make a reasonable determination of the end behavior for the sequence, and then support your conclusion algebraically.

(a) $a_n = \dfrac{n-1}{n+2}$ **(b)** $b_n = 3n - 1$ **(c)** $c_n = (-1)^n \dfrac{n}{n+3}$

Solution

(a) ***Graphical*** Draw a graph of $y = \frac{x-1}{x+2}$ using $[0, c] \times [0, 2]$ (see Technology Tip above). See Figure 1. The right portion of the graph appears to be horizontal, but we know that the calculator has only so many pixels available. Tracing along the curve indicates that the y-coordinate is approaching 1 as x increases. That is, it appears that $\lim\limits_{n\to\infty} a_n = 1$; we say that the sequence $\{a_n\}$ converges to 1.

Algebraic If we divide the numerator and denominator of a_n by n, we get

$$a_n = \frac{n-1}{n+2} = \frac{1 - \dfrac{1}{n}}{1 + \dfrac{2}{n}}.$$

In this form, it should be clear that $a_n \to 1$ as $n \to \infty$.

(b) ***Graphical and Algebraic*** The graph of $y = 3x - 1$ is a line with slope 3. If we use an x-range of $[0, c]$, we need a correspondingly large y-range or the graph goes off scale almost immediately, but even without a graph for this particular function we know the end behavior of a line. The values of b_n continue to increase without bound and do not approach any number. We say that the sequence $\{b_n\}$ **diverges.** In this case, as in Chapter 3 when working with rational functions, we write $\lim\limits_{n\to\infty} b_n = \infty$.

(c) ***Graphical*** Graphing $y = (-1)^x x/(x + 3)$ in $[0, c] \times [-3, 3]$ gives very different looking graphs in connected or dot modes. Whichever you choose, make sure that each x-pixel coordinate is an integer and that you know how to interpret what the graph shows. Tracing (in either mode) shows that as x increases, the y-values jump back and forth, with the positive values approaching 1 and the negative values approaching -1. We conclude that $\{c_n\}$ diverges because the c_n values do not approach a single number as $n \to \infty$.

Algebraic Disregarding the $(-1)^n$, the expression $\frac{n}{n+3}$ does have a limit,

$$\lim_{n\to\infty} \frac{n}{n+3} = \lim_{n\to\infty} \frac{1}{1+\frac{3}{n}} = 1.$$

It follows that $c_n \to (-1)^n$, so that when n is even, $c_n \to 1$ and when n is odd, $c_n \to -1$. Again, $\{c_n\}$ diverges because the c_n values do not approach a single number as $n \to \infty$. ◀

Sequences Defined Recursively

For sequences defined recursively we cannot enter functions for graphing as we did in Example 1. Nevertheless, functionality built into our graphing calculators makes it possible to investigate limits of some such sequences quite easily. Consider the sequence $\{a_n\}$ defined by

$$a_1 = 2, \qquad a_{n+1} = 1 + \frac{3}{a_n} \quad \text{for } n \geq 1. \tag{1}$$

We want to calculate a number of terms of the sequence without having to go through all the steps of the recursive definition for each term. We describe some options in the following Technology Tip, and then look at additional examples.

TECHNOLOGY TIP ◆ ***Calculating recursively defined sequences***

For algebraic operation calculators (TI, Casio, and HP–38), essentially all of the steps for evaluating the sequence in Equation (1) can be handled on the home screen by making use of the machine capacity to store values.

Begin by storing the initial value 2 in the x-register: 2 → X , ENT .

Then compute and store the next value: 1 + 3/X→X , ENT .

The calculator displays the computed value, 2.5, which has been stored. When we press ENTER again, the same computation is repeated with the new x-value and displayed value, 2.2, is our a_3. As we ENTER repeatedly, the terms of the sequence are displayed. It soon becomes clear that the terms are approaching a limit, ≈2.3027756.

On the HP–48, we can write a simple program to accomplish the same thing.

Press ≪ ≫ (above the subtract key) to begin a program. Then type → X′1 + 3X′ → Num (above [EVAL]) and ENTER . What the program does is to take the number on the stack, call it x, compute $1 + 3/x$ symbolically, convert the symbolic computation to a number. The result is displayed on the stack so that the process can be repeated.

To use the program, we need to store it as a variable, so we type a name, say 'RECR' for "recursive". Then ENTER and [STO] . The name RECR should appear on your [VAR] menu. Now enter 2 on the stack, press the soft key beneath [RECR], and the new value appears. By repeatedly pressing [RECR], the value continues to change, approaching 2.3027756.

Strategy: For **(c)**, if the sequence converges to a number c, both a_n and a_{n+1} approach c, leading to the equation $c = \sqrt{3 + c}$. Then solve for c.

▶EXAMPLE 2 ***Nested square roots*** Sequence $\{a_n\}$ is defined by

$$a_1 = \sqrt{3}, \qquad a_{n+1} = \sqrt{3 + a_n} \quad \text{for } n \geq 1.$$

(a) Write out the first three terms in exact form.
(b) Use the Technology Tip (page 444) to approximate the first few terms of the sequence and find the apparent limit of the sequence (six decimal places).
(c) Justify your conclusion in **(b)** algebraically.

Solution

(a) $a_1 = \sqrt{3}, \qquad a_2 = \sqrt{3 + a_1} = \sqrt{3 + \sqrt{3}},$
$a_3 = \sqrt{3 + a_2} = \sqrt{3 + \sqrt{3 + \sqrt{3}}}.$

(b) Following the Technology Tip, for all machines except the HP-48, we store $\sqrt{3}$ in the x-register, and then enter √(3 + X) → X. Repeating the computation gives a sequence beginning 1.732051, 2.175328, 2.274935, 2.296723, ... After several more terms, the sequence settles on a number approximately equal to 2.302776. On the HP-48, we must change the recursive part of the definition in our program RECR by pressing the tick-mark key and the soft key under RECR . With 'RECR' on the stack, we press EDIT and go into the program, replacing '1 + 3/X' by the recursive part of our new sequence, '√(3 + X)'. With the new program, we enter $\sqrt{3}$ and then repeat the soft key under RECR , getting the same sequence of terms.

(c) Following the strategy, we want to solve the equation $c = \sqrt{3 + c}$ for c. Squaring both sides, we get the equation $c^2 = 3 + c$, or $c^2 - c - 3 = 0$. By the quadratic formula, taking the positive sign (why not $\pm$?), we get $c = (1 + \sqrt{13})/2 \approx 2.302775638$, obviously the number we were approximating in part **(b)**. ◀

▶EXAMPLE 3 ***Nested cube roots*** Repeat Example 2 for the sequence $\{c_n\}$ defined by

$$c_1 = \sqrt[3]{2}, \qquad c_{n+1} = \sqrt[3]{2 + c_n} \quad \text{for } n \geq 1.$$

Solution

(a) $c_1 = \sqrt[3]{2}, \quad c_2 = \sqrt[3]{2 + c_1} = \sqrt[3]{2 + \sqrt[3]{2}},$
$c_3 = \sqrt[3]{2 + c_2} = \sqrt[3]{2 + \sqrt[3]{2 + \sqrt[3]{2}}}.$

(b) On algebraic operation machines, we store 2^(1/3) in the x-register, and then enter (2 + X)^(1/3) → X. The sequence begins 1.259921, 1.482754, 1.515797, 1.520575, 1.521264, ... The sequence settles on a number approximately equal to 1.5213797. On the HP-48, we enter '(2 + X)^(1/3)' as the recursive part of the definition in RECR . After entering $\sqrt[3]{2}$ on the stack, repeating the soft key under RECR gives the same sequence of terms.

(c) Since it appears that the sequence converges to a number c, both c_n and c_{n+1} must approach the same number c, so the recursive portion of the definition gives an equation which can be cubed:

$$c = \sqrt[3]{2 + c}, \quad \text{or} \quad c^3 - c - 2 = 0.$$

The cubic equation is not one we can solve in exact form conveniently, but by graphical methods, or by using a solve routine, or by using Newton's Method from Chapter 3, the one real zero is approximately 1.5213797. ◀

Continued Fractions

Continued fractions is a topic studied in number theory courses that has applications in many areas, including the programming of routines for computers and graphing calculators. In the next example we illustrate the continued fraction, $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$, as a recursively defined sequence.

▶EXAMPLE 4 ***Continued fractions*** Sequence $\{a_n\}$ is defined by

$$a_1 = 1, a_{n+1} = 1 + 1/a_n \quad \text{for } n \geq 1.$$

(a) Write out the first four terms, first without simplifying, and then as a simple fraction.
(b) Approximate the first few terms of the sequence and find the apparent limit of the sequence.
(c) Justify your conclusion in **(b)** algebraically.

Solution

(a) $a_1 = 1, \quad a_2 = 1 + \frac{1}{1} = \frac{2}{1}, \quad a_3 = 1 + \frac{1}{a_2} = 1 + \frac{1}{2} = \frac{3}{2},$

$$a_4 = 1 + \frac{1}{a_3} = 1 + \frac{1}{1 + \frac{1}{2}} = 1 + \frac{1}{\frac{3}{2}} = 1 + \frac{2}{3} = \frac{5}{3}.$$

The numbers in the numerator and denominator of the fractions remind us of the Fibonacci sequence $\{f_n\}$ from page 436: 1, 1, 2, 3, 5, 8, 13, ... That is, $a_1 = f_2/f_1$, $a_2 = f_3/f_2$, $a_3 = f_4/f_3$, $a_4 = f_5/f_4$, and a reasonable guess is that $a_n = f_{n+1}/f_n$.

(b) For decimal approximations, we use the Technology Tip, beginning with 1 and using `1 + 1/X → X` for the recursion. The sequence appears to settle down on a number $c \approx 1.618034$.

(c) If both a_{n+1} and a_n approach c, then in the limit the recursion relation becomes

$$c = 1 + \frac{1}{c}.$$

Multiplying through by c leads to the quadratic equation $c^2 - c - 1 = 0$, whose positive root is given by $c = \frac{1 + \sqrt{5}}{2} \approx 1.618034$. ◀

The limit number of the sequence, $\frac{1 + \sqrt{5}}{2}$, is called the **Golden Ratio,** reflecting some aesthetic considerations of the ancient Greeks. It is a number that turns up in many diverse applications. See Exercise 27.

In the next example, we see another instance of a sequence that diverges even though parts of the sequence, called *subsequences,* converge. We had one such sequence in Example 1, given by $c_n = (-1)^n \frac{n}{n+3}$. From the graph in dot mode, we saw that the sequence consisting of the even-numbered terms $\{c_2, c_4, c_6, \ldots\}$ converges to 1; the odd-numbered terms form a subsequence that converges to -1. The same kind of behavior is possible with a recursively defined sequence.

▶EXAMPLE 5 **Subsequences** Sequence $\{a_n\}$ is defined by

$$a_1 = 3,\ a_{n+1} = \frac{2a_n}{(a_n - 2)} \quad \text{for} \quad n \geq 1.$$

Write out the first few terms. Does the sequence have a limit? Describe some convergent subsequences of the sequence.

Solution

Either by using the Technology Tip (page 444) or by direct computation, it is clear that the sequence begins 3, 6, 3, 6, 3, 6, 3, 6, ... The sequence has no limit because the terms are not getting close to any number as $n \to \infty$. The subsequence of odd-numbered terms contains only 3, $\{a_{2n-1}\} = \{3, 3, 3, 3, \ldots\}$, which obviously converges to 3. Similarly, the subsequence consisting of even-numbered terms $\{a_{2n}\} = \{6, 6, 6, \ldots\}$ converges to 6. ◀

TECHNOLOGY TIP ◆ ***Calculating with two-step recursions***

Any graphing calculator can be programmed to calculate more involved recursively defined sequences, but general programming is not our focus in this text. To learn about programming on your calculator, consult your instruction manual. The Texas Instrument TI-82 and TI-85, HP–38, and the Casio fx7700 and fx9700, allow us to handle two-step recursively defined sequences, such as the Fibonacci sequence, directly on the home screen, as described below.

The Fibonacci sequence (page 436) is defined by

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-2} + f_{n-1}, \quad n > 2.$$

On the home screen, we store the initial values and their sum:

1 → A: 1 → B: A + B → C and ENTER

(The colon is located above the decimal point on TI and HP–38, and on the PRGM menu on the Casio.) The display shows f_3 as 2. We must reassign values for the next step:

B → A:C → B: A + B → C .

Now ENTER and we see 3 as f_4, and we can repeat for as many terms as desired.

We revisit this problem in Matrix form in Exercise 70 of Section 9.6.

Equations of the Form $F(x) = x$

When looking for the roots of an equation it is sometimes possible to isolate an x, writing the equation in the form $F(x) = x$. Under certain circumstances it is possible to use an iterative process to approximate a root of such an equation to great accuracy. Basically, the solution is found as the limit of a recursive sequence. We begin with an initial approximation a_1 and define $a_{n+1} = F(a_n)$. Determining the conditions under which such an iteration converges requires calculus, but we illustrate the procedure in Example 6.

▶EXAMPLE 6 Solving an equation

(a) Approximate the root of the equation $2x - \cos x = 0$ from a graph.
(b) Write the equation in the form $f(x) = x$ and use the approximation from part **(a)** as a_1 and let $a_{n+1} = f(a_n)$. Iterate to approximate the limit L of the sequence to 8 decimal places and verify that L satisfies the original equation.

Solution

(a) From a graph, we can see that there is a root of the equation near 0.5.

(b) The equation is equivalent to $x = \frac{\cos x}{2}$, so we take $f(x) = \frac{\cos x}{2}$ and define the sequence by

$$a_1 = 0.5,\ a_{n+1} = \frac{\cos a_n}{2}.$$

Following the Technology Tip for recursive sequences, we store `0.5 → X`, `ENT`. Then follow with `(COS X)/2 → X`, `ENT` and iterate, getting a sequence beginning 0.43879128, 0.45263292, ... The sequence settles quickly on the number $L = 0.45018361$, and when we substitute L for x in the expression $2x - \cos x$, we get a number very near 0, as desired. ◀

EXERCISES 8.2

Check Your Understanding

It will be helpful to use the Technology Tip (page 444) to get the first several terms of $\{a_n\}$.

Exercises 1–5 True or False. Give reasons. Use sequence $\{a_n\}$ defined by $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{a_n}$.

1. Every term of $\{a_n\}$ is less than or equal to 2.
2. The sequence is decreasing; that is $a_{n+1} < a_n$ for every n.
3. The even-numbered terms are greater than the odd-numbered terms.
4. The subsequence consisting of the odd-numbered terms, $\{a_1, a_3, a_5, \dots\}$, is decreasing.
5. The subsequence consisting of the even-numbered terms, $\{a_2, a_4, a_6, \dots\}$, is increasing.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

Exercises 6–8 Sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = a_n + 4$ for $n > 1$.

6. The smallest integer greater than a_5 is ______.
7. The number of terms of $\{a_n\}$ between 8 and 20 is ______.
8. The sum of the first 5 terms is ______.

Exercises 9–10 Sequence $\{b_n\}$ is defined by $b_1 = 1$ and $b_{n+1} = \sqrt{1 + b_n^2}$ for $n \geq 1$.

9. $b_5 =$ ______.
10. The smallest prime number that is greater than b_5 is ______.

Develop Mastery

*Exercises 1–10 **Does it Converge?** Use a graph to help you determine whether or not the sequence appears to converge. Explain.*

1. $a_n = 2n - 5$
2. $a_n = \dfrac{(-1)^n(n + 1)}{n}$
3. $a_n = \dfrac{2n + 3}{n + 1}$
4. $a_n = \dfrac{n + 1}{n^2 + 1}$
5. $a_n = 3 - 2^{-n}$
6. $a_n = \left(1 + \dfrac{1}{n}\right)^n$
7. $a_n = 2 + \left(1 - \dfrac{1}{n}\right)^n$
8. $a_n = \left(1 + \dfrac{2}{n}\right)^n$
9. $a_n = \sqrt{n} + \sqrt{n + 64 - 16\sqrt{n}}$
10. $a_n = \sqrt{n} + |\sqrt{n} - 5|$

Exercises 11–16 ***Sequences Defined Recursively*** *Use the Technology Tip (page 444).* ***(a)*** *Give the first three terms of $\{a_n\}$.* ***(b)*** *The sequence converges to a number c. Use your calculator to get a six-decimal-place approximation for c.* ***(c)*** *Justify your answer algebraically. See Examples 2 and 3.*

11. $a_1 = \sqrt{3}, \quad a_{n+1} = \sqrt{3 - a_n} \quad$ for $n > 1$.

12. $a_1 = \sqrt{5}, \quad a_{n+1} = \sqrt{5 + a_n} \quad$ for $n > 1$.

13. $a_1 = \sqrt{7}, \quad a_{n+1} = \sqrt{7 - a_n} \quad$ for $n > 1$.

14. $a_1 = \sqrt[3]{5}, \quad a_{n+1} = \sqrt[3]{5 - a_n} \quad$ for $n > 1$.

15. $a_1 = \sqrt[3]{6}, \quad a_{n+1} = \sqrt[3]{6 + a_n} \quad$ for $n > 1$.

16. $a_1 = \sqrt[4]{5}, \quad a_{n+1} = \sqrt[4]{5 + a_n} \quad$ for $n > 1$.

Exercises 17–22 ***Continued Fractions*** ***(a)*** *Find the first three terms of $\{c_n\}$.* ***(b)*** *Find a 6-decimal-place approximation for the limit to which $\{c_n\}$ appears to converge.* ***(c)*** *Justify algebraically. See Example 4.*

17. $c_1 = 3, c_{n+1} = 3 + \dfrac{1}{c_n} \quad$ for $n > 1$.

18. $c_1 = 5, c_{n+1} = 5 + \dfrac{1}{c_n} \quad$ for $n > 1$.

19. $c_1 = 3, c_{n+1} = 3 - \dfrac{1}{c_n} \quad$ for $n > 1$.

20. $c_1 = 1, c_{n+1} = 1 + \dfrac{1}{c_n^2} \quad$ for $n > 1$.

21. $c_1 = 3, c_{n+1} = 3 + \dfrac{1}{c_n^2} \quad$ for $n > 1$.

22. $c_1 = 2, c_{n+1} = 2 - \dfrac{1}{c_n^2} \quad$ for $n > 1$.

Exercises 23–24 ***Repeating Terms*** *Sequence $\{a_n\}$ is defined recursively.* ***(a)*** *Give the first six terms. Does $\{a_n\}$ converge? You may wish to use the Technology Tip.* ***(b)*** *Determine a_{60} and the sum of the first sixty terms.*

23. $a_1 = 2, a_{n+1} = 1 - \dfrac{1}{a_n}$.

24. $a_1 = 2, a_{n+1} = 1 + \dfrac{2}{a_n}$.

Exercises 25–26 ***Recognizing a Pattern*** *Find the first four terms of $\{a_n\}$. Make a generalization and justify algebraically.*

25. **(a)** $a_1 = 1, \quad a_{n+1} = 0.5\left(a_n + \dfrac{1}{a_n}\right)$

(b) $a_1 = 2, \quad a_{n+1} = 0.5\left(a_n + \dfrac{4}{a_n}\right)$

(c) $a_1 = 3, \quad a_{n+1} = 0.5\left(a_n + \dfrac{9}{a_n}\right)$

(d) $a_1 = 4, \quad a_{n+1} = 0.5\left(a_n + \dfrac{16}{a_n}\right)$

26. **(a)** $a_1 = 1, \quad a_{n+1} = 2a_n - \dfrac{1}{a_n}$

(b) $a_1 = \dfrac{1}{2}, \quad a_{n+1} = 5a_n - \dfrac{1}{a_n}$

(c) $a_1 = \dfrac{1}{3}, \quad a_{n+1} = 10a_n - \dfrac{1}{a_n}$

(d) $a_1 = \dfrac{1}{4}, \quad a_{n+1} = 17a_n - \dfrac{1}{a_n}$

27. ***Fibonacci Related*** For $a_1 = 3, a_{n+1} = 3 - \frac{1}{a_n}$,

(a) Write the first six terms as simple fractions.

(b) Guess a relationship between $\{a_n\}$ and the Fibonacci sequence.

28. For $a_1 = \sqrt{\ln 2}, a_{n+1} = \sqrt{\ln 2 + a_n}$,

(a) use the Technology Tip to find a six-decimal-place approximation to the limit c to which $\{a_n\}$ converges.

(b) Show that c is a root of $e^{x^2-x} - 2 = 0$.

Exercises 29–30 ***Golden Ratio, $\frac{1+\sqrt{5}}{2}$*** *Given that $\{a_n\}$ converges to the number c, use an algebraic approach to verify that c is the number given. Then use the Technology Tip to get a calculator check.*

29. $a_1 = 2, a_{n+1} = \sqrt{1 + \dfrac{1}{a_n{}^2}}$; c is the square root of the golden ratio.

30. $a_1 = 3, a_{n+1} = \dfrac{1}{a_n} - 1$; c is the negative of the golden ratio.

Exercises 31–32 ***Explore*** *The recursive formula for a_{n+1} is given along with different values of a_1. In each case use the Technology Tip to get the first three terms and the limit (six decimal places) to which the sequence converges. Try other values of a_1. Describe the role that a_1 plays.*

31. $a_{n+1} = \sqrt{1 + \dfrac{1}{a_n^2}}$,

(a) $a_1 = 1$ **(b)** $a_1 = 8$ **(c)** $a_1 = 24$

32. $a_{n+1} = 1 + \dfrac{1}{a_n}$,

(a) $a_1 = 1$ **(b)** $a_1 = 3$

(c) $a_1 = -5$

Exercises 33–36 ***Subsequences*** ***(a)*** *Does $\{a_n\}$ converge?* ***(b)*** *Find subsequences of $\{a_n\}$ that converge. See Example 5.*

33. $a_n = \dfrac{(-1)^n n}{n + 1}$

34. $a_n = \dfrac{(-1)(2n)}{n + 1}$

35. $a_n = \sin\left(\dfrac{n\pi}{2}\right)$

36. $a_n = \cos(n\pi)$

Exercises 37–40 ***Your Choice*** *Give a sequence $\{a_n\}$ of your choice that meets the given conditions.*

37. $a_n > 0$ if n is odd, $a_n < 0$ if n is even.

38. The sequence $\{a_n\}$ does not converge, but the subsequence of odd-numbered terms (all of which are greater than 2) converges to 2, while the subsequence of even-numbered terms (all of which are less than -2) converges to -2.

39. For every n, $0 < a_n < 1$, and $\{a_n\}$ converges to **(a)** 1; **(b)** 0.5.

40. For all odd n, $0 < a_n < 1$, and for all even n, $1 < a_n < 2$, and $\{a_n\}$ converges to 1.

Exercises 41–43 ***Repeating Terms*** ***(a)*** *Give the first four terms of $\{a_n\}$.* ***(b)*** *What is a_{47}? a_{72}?* ***(c)*** *Find the sum of the first twenty terms.* ***(d)*** *Explain the repeating behavior of $\{a_n\}$. (Hint for Exercise 41: Consider $f(x) = 2x/(x - 2)$, and show that $f^{-1}(x) = f(x)$. What is $f^{-1}(f(x))$?)*

41. $a_1 = 6,\ a_{n+1} = \dfrac{2a_n}{a_n - 2}$

42. $a_1 = 1,\ a_{n+1} = \dfrac{-3a_n}{2a_n + 3}$

43. $a_1 = 4,\ a_{n+1} = \dfrac{3a_n}{2a_n - 3}$

Exercises 44–50 ***Roots of $f(x) = x$*** *Follow the instructions for Example 6 for the given equation.*

44. $\cos x = x$

45. $5x - 2\cos x = 0$

46. $x = e^{-x}$

47. $x = \cos(x/4)$

48. $x = \cos\left(\dfrac{4 + x}{4}\right)$

49. $x = \ln(4 + x)$

50. $x + 4 = e^x$

Exercises 51–52 ***Roots of $2^x = x^{10}$*** *Find the limit L of $\{a_n\}$ to eight decimal places. Show that L is a root of the equation $2^x = x^{10}$.*

51. $a_1 = 50,\ a_{n+1} = \dfrac{10 \ln a_n}{\ln 2}$

52. $a_1 = 1,\ a_{n+1} = 2^{0.1a_n}$

53. ***Explore*** In the recursive formula for Exercise 30, $a_{n+1} = \frac{1}{a_n} - 1$, many different initial values give sequences that converge to the same value. There are initial values that do not work, however. We obviously cannot use $a_1 = 0$ because a_2 would then be undefined, and we cannot use a number as an initial value that would lead to 0. For example, solving $\frac{1}{a_n} - 1 = 0$, we get $a_n = 1$. If we were to try $a_1 = 1$, we would get $a_2 = 0$ and then a_3 would be undefined. **(a)** Solve $\dfrac{1}{a_n} - 1 = 1$ and find another inadmissible value for a_1. **(b)** Find a sequence in exact form of inadmissible initial value numbers. **(c)** Show that $a_1 = 3/5$ is inadmissible by computing the first few terms in exact form. **(d)** Try $a_1 = 0.6$ and compute the first few terms by using the Technology Tip (page 444). Explain the difference in results from part **(c)**.

54. The sequence $\{a_n\}$ is given by $a_n = x^n + x^{-n}$, where $x = \dfrac{-1 + \sqrt{3}\, i}{2}$.

(a) Use DeMoivre's theorem to evaluate $x^n + x^{-n}$, then show that $a_n = 2\cos(n \cdot 120°)$.

(b) Write out the first six terms of the sequence and find their sum.

(c) What is the sum of the first 100 terms?

Exercises 55–60 *Sequence $\{a_n\}$ converges to a number L.* ***(a)*** *Use the Technology Tip to approximate L.* ***(b)*** *Use algebra to find the exact value of L.*

55. $a_1 = 1 \quad a_n = \dfrac{1}{2}\left(a_{n-1} + \dfrac{4}{a_{n-1}}\right) \quad$ for $n \geq 2$

56. $a_1 = 2 \quad a_n = \dfrac{1}{2}\left(a_{n-1} + \dfrac{3}{a_{n-1}}\right) \quad$ for $n \geq 2$

57. $a_1 = 1 \quad a_n = \dfrac{1}{2}\left(a_{n-1} + \dfrac{2}{a_{n-1}}\right) \quad$ for $n \geq 2$

58. $a_1 = \sqrt{2} \quad a_n = \sqrt{2 + a_{n-1}} \quad$ for $n \geq 2$

59. $a_1 = \sqrt{6} \quad a_n = \sqrt{6 + a_{n-1}} \quad$ for $n \geq 2$

60. $a_1 = 1 \quad a_n = \dfrac{4}{a_{n-1}} - \sqrt{2} \quad$ for $n \geq 2$

Exercises 61–62 *Sequences $\{a_n\}$ and $\{b_n\}$ converge; sequence $\{c_n\}$ diverges.* ***(a)*** *Find approximations (6 decimal places) for the limits of $\{a_n\}$ and $\{b_n\}$.* ***(b)*** *Find subsequences of $\{c_n\}$ that converge and approximate their limits.*

61. $a_1 = 1, \quad a_{n+1} = 4 + \dfrac{1}{a_n}; \quad b_1 = 1, \quad b_{n+1} = 4 - \dfrac{1}{b_n}$

$c_1 = 1, \quad c_{n+1} = 4 + \dfrac{(-1)^n}{c_n}$

62. $a_1 = 4, \quad a_{n+1} = 4 + \dfrac{3}{a_n}; \quad b_1 = 4, \quad b_{n+1} = 4 - \dfrac{3}{b_n}$

$c_1 = 4, \quad c_{n+1} = 4 + \dfrac{(-1)^n 3}{c_n}$

8.3 ARITHMETIC AND GEOMETRIC SEQUENCES

Whenever you tell me that mathematics is just a human invention like the game of chess I would like to believe you. But I keep returning to the same problem. Why does the mathematics we have discovered in the past so often turn out to describe the workings of the Universe?

John Barrow

I remember that when I was about twelve I learned from [my uncle] that by the distributive law −1 times −1 equals +1. I thought that was great.

Peter Lax

Two kinds of regular sequences occur so often that they have specific names, **arithmetic** and **geometric sequences.** We treat them together because some obvious parallels between these kinds of sequences lead to similar formulas. This also makes it easier to learn and work with the formulas. The greatest value in this association is understanding how the ideas are related and how to derive the formulas from fundamental concepts. Anyone learning the formulas this way can recover them whenever needed.

Both arithmetic and geometric sequences begin with an arbitrary first term, and the sequences are generated by regularly adding the same number (the **common difference** in an arithmetic sequence) or multiplying by the same number (the **common ratio** in a geometric sequence). Definitions emphasize the parallel features, which examples will clarify.

Definition: arithmetic and geometric sequences

Arithmetic Sequence

$$a_1 = a \quad \text{and} \quad a_n = a_{n-1} + d \quad \text{for } n > 1$$

The sequence $\{a_n\}$ is an arithmetic sequence with **first term** a and **common difference** d.

Geometric Sequence

$$a_1 = a \quad \text{and} \quad a_n = r \cdot a_{n-1} \quad \text{for } n > 1$$

The sequence $\{a_n\}$ is a geometric sequence with **first term** a and **common ratio** r.

The definitions imply convenient formulas for the nth term of both kinds of sequences. For an arithmetic sequence we get the nth term by adding d to the first term $n - 1$ times; for a geometric sequence, we multiply the first term by r, $n - 1$ times.

Formulas for the nth terms of arithmetic and geometric sequences

For an arithmetic sequence, a formula for the nth term of the sequence is

$$a_n = a + (n - 1)d. \tag{1}$$

For a geometric sequence, a formula for the nth term of the sequence is

$$a_n = a \cdot r^{n-1}. \tag{2}$$

The definitions allow us to recognize both arithmetic and geometric sequences. In an arithmetic sequence the difference between successive terms, $a_{n+1} - a_n$, is always the same, the constant d; in a geometric sequence the ratio of successive terms, $\dfrac{a_{n+1}}{a_n}$, is always the same.

▶EXAMPLE 1 ***Arithmetic or geometric?*** The first three terms of a sequence are given. Determine if the sequence could be arithmetic or geometric. If it is an arithmetic sequence, find d; for a geometric sequence, find r.

(a) 2, 4, 8, . . . **(b)** ln 2, ln 4, ln 8, . . . **(c)** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Strategy: Calculate the differences and/or ratios of successive terms.

Solution

(a) $a_2 - a_1 = 4 - 2 = 2$, and $a_3 - a_2 = 8 - 4 = 4$. Since the differences are not the same, the sequence cannot be arithmetic. Checking ratios, $\frac{a_2}{a_1} = \frac{4}{2} = 2$, and $\frac{a_3}{a_2} = \frac{8}{4} = 2$, so the sequence could be geometric, with a common ratio $r = 2$. Without a formula for the general term, we cannot say anything more about the sequence.

(b) $a_2 - a_1 = \ln 4 - \ln 2 = \ln(\frac{4}{2}) = \ln 2$, and $a_3 - a_2 = \ln 8 - \ln 4 = \ln(\frac{8}{4}) = \ln 2$, so the sequence could be arithmetic, with ln 2 as the common difference. As in part **(a)**, we cannot say more because no general term is given.

(c) $a_2 - a_1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$, and $a_3 - a_2 = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$. The differences are not the same, so the sequence is not arithmetic. $\frac{a_2}{a_1} = \frac{(\frac{1}{3})}{(\frac{1}{2})} = \frac{2}{3}$, and $\frac{a_3}{a_2} = \frac{(\frac{1}{4})}{(\frac{1}{3})} = \frac{3}{4}$, so the sequence is not geometric. Note that the sequence in part **(a)** *could be* geometric and the sequence in part **(b)** *could be* arithmetic, but in part **(c)** you can conclude unequivocally that the sequence cannot be either arithmetic or geometric. ◀

▶EXAMPLE 2 ***Arithmetic or geometric?*** Determine whether the sequence is arithmetic, geometric, or neither.

(a) $\{3 - 1.6n\}$ **(b)** $\{2^n\}$ **(c)** $a_n = \ln n$

Solution

(a) $a_2 - a_1 = (3 - 1.6 \cdot 2) - (3 - 1.6 \cdot 1) = (-0.2) - 1.4 = -1.6$, and $a_3 - a_2 = (3 - 1.6 \cdot 3) - (3 - 1.6 \cdot 2) = -1.6$. From the first three terms, this could be an arithmetic sequence with $d = -1.6$. Check the difference $a_{n+1} - a_n$.

$$a_{n+1} - a_n = [3 - 1.6(n + 1)] - [3 - 1.6n] = -1.6.$$

The sequence is arithmetic, with $d = -1.6$.

(b) $a_2 - a_1 = 4 - 2 = 2$, and $a_3 - a_2 = 8 - 4 = 4$, so the sequence is not arithmetic. Using the formula for the general term,

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n} = 2.$$

The sequence $\{2^n\}$ is geometric, with 2 as the common ratio.

(c) $a_{n+1} - a_n = \ln(n + 1) - \ln n = \ln \dfrac{n+1}{n}$. The difference depends on n, so the sequence is not arithmetic. Checking ratios, $\dfrac{a_{n+1}}{a_n} = \dfrac{\ln(n+1)}{\ln n}$, so the ratio also changes with n. The sequence is neither arithmetic nor geometric. ◀

▶EXAMPLE 3 ***Arithmetic sequences*** Show that the sequence is arithmetic; find the common difference and the twentieth term.

(a) $a_n = 2n - 1$ **(b)** $50, 45, 40, \ldots, 55 - 5n, \ldots$

Solution

(a) The first few terms of $\{a_n\}$ are $1, 3, 5, 7, \ldots$, from which it is apparent that each term is 2 more than the preceding term; this is an arithmetic sequence with first term and common difference $a = 1$ and $d = 2$. Check to see that $a_{n+1} - a_n = 2$. To find a_{20}, use either the defining formula for the sequence or Equation (1) for the nth term:

$$a_{20} = 2 \cdot 20 - 1 = 39 \quad \text{or} \quad a_{20} = a + 19d = 1 + 19 \cdot 2 = 39.$$

(b) If $b_n = 55 - 5n$, then $b_{n+1} - b_n = [55 - 5(n + 1)] - [55 - 5n] = -5$. This is an arithmetic sequence with $a = 50$, $d = -5$, and so $b_{20} = 55 - 5 \cdot 20 = -45$. ◀

Given the structure of arithmetic and geometric sequences, any two terms completely determine the sequence. Using Equation (1) or (2), two terms of the sequence give us a pair of equations from which we can find the first term and either the common difference or common ratio, as illustrated in the next example.

▶EXAMPLE 4 ***Arithmetic sequences*** Suppose $\{a_n\}$ is an arithmetic sequence with $a_8 = 6$ and $a_{12} = -4$. Find a, d, and the three terms between a_8 and a_{12}.

Solution

From Equation (1), $a_8 = a + 7d$, and $a_{12} = a + 11d$, from which the difference is given by $a_{12} - a_8 = 4d$. Use the given values for a_8 and a_{12} to get $-4 - 6 = 4d$, or $d = -\frac{5}{2}$. Substitute $-\frac{5}{2}$ for d in $6 = a + 7d$ and solve for a, $a = \frac{47}{2}$. Find the three terms between a_8 and a_{12} by successively adding $-\frac{5}{2}$:

$$a_9 = a_8 - \frac{5}{2} = \frac{7}{2}, \quad a_{10} = a_9 - \frac{5}{2} = 1, \quad a_{11} = a_{10} - \frac{5}{2} = -\frac{3}{2}.$$

Therefore, a_9 is $\frac{7}{2}$, a_{10} is 1, and a_{11} is $-\frac{3}{2}$. ◀

▶EXAMPLE 5 ***Geometric sequences*** Determine whether the sequence is geometric. If it is geometric, then find the common ratio and the terms a_1, a_3, and a_{10}.

(a) $\{2^n\}$ **(b)** $2, -\dfrac{2}{3}, \dfrac{2}{9}, \ldots, 2\left(-\dfrac{1}{3}\right)^{n-1}, \ldots$

Strategy: The property that identifies a geometric sequence is the common ratio: the values $\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \ldots$ must all be the same. For a geometric sequence, use Equation (2).

Solution

(a) The first few terms are 2, 4, 8, 16, . . . , each of which is twice the preceding term. This is a geometric sequence with first term $a = 2$, and common ratio given by $r = \frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n} = 2$. Using $a_n = 2^n$,

$$a_1 = 2 \quad a_3 = 2^3 = 8 \quad \text{and} \quad a_{10} = 2^{10} = 1024.$$

(b) Consider the ratio

$$\frac{a_{n+1}}{a_n} = \frac{2\left(-\frac{1}{3}\right)^n}{2\left(-\frac{1}{3}\right)^{n-1}} = -\frac{1}{3},$$

so the sequence is geometric with $a = 2$ and $r = -\frac{1}{3}$. Using $a_n = 2(-\frac{1}{3})^{n-1}$, we get $a_1 = 2$, $a_3 = ar^2 = \frac{2}{9}$, and $a_{10} = ar^9 = 2(-\frac{1}{3})^9 = -\frac{2}{19683}$. ◀

Partial Sums of Arithmetic Sequences

There is a charming story told about Carl Freidrich Gauss, one of the greatest mathematicians of all time. Early in Gauss' school career, the schoolmaster assigned the class the task of summing the first hundred positive integers, $1 + 2 + 3 + \cdots + 99 + 100$. That should have occupied a good portion of the morning, but while other class members busied themselves at their slates calculating $1 + 2 = 3$, $3 + 3 = 6$, $6 + 4 = 10$, and so on, Gauss sat quietly for a few moments, wrote a single number on his slate, and presented it to the teacher. Young Gauss observed that 1 and 100 add up to 101, as do the pair 2 and 99, 3 and 98, and so on up to 50 and 51. There are fifty such pairs, each with a sum of 101, for a total of $50 \cdot 101 = 5050$, the number he wrote on his slate.

This approach works for the partial sum of any arithmetic sequence, and we will use the method to derive some useful formulas. However, the ideas are more valuable than memorizing formulas. If you understand the idea, you can recreate the formula when needed.

To find a formula for the nth partial sum of an arithmetic sequence, that is, the sum of n consecutive terms, pair the first and last terms, the second and next-to-last, and so on; *each pair has the same sum.* In fact, it is easier to pair all terms twice, as illustrated with Gauss' sum:

$$\begin{aligned}
S_{100} &= 1 + 2 + \cdots + 99 + 100\\
S_{100} &= 100 + 99 + \cdots + 2 + 1\\
\hline
2S_{100} &= 101 + 101 + \cdots + 101 + 101
\end{aligned}$$

The sum on the right has 100 terms, so $2S_{100} = 100(101)$. Dividing by 2, $S_{100} = 50(101) = 5050$.

For the general case, pairing the terms in S_n and adding gives $2S_n = n(a_1 + a_n)$ because there are n pairs, each with the same sum. Dividing by 2 yields the desired formula.

Partial sums of an arithmetic sequence

Suppose $\{a_n\}$ is an arithmetic sequence. The sum S_n of the first n terms is given by

$$S_n = \frac{n(a_1 + a_n)}{2} \tag{3}$$

The formula is probably most easily remembered as n times the average of the first and last terms.

▶EXAMPLE 6 ***Partial sums*** For the sequence $\{a_n\} = \{2n - 1\}$,

(a) evaluate the sum $S_{25} = \Sigma_{k=1}^{25} (2k - 1)$ and
(b) find a formula for S_n.

Solution

Strategy: Let $a_n = 2n - 1$. To find S_{25} from Equation (3) requires a_1 and a_{25}, which the formula for a_n can provide. For **(b)**, substitute 1 for a_1 and $2n - 1$ for a_n in Equation (3) and simplify.

Follow the strategy.

(a) By Equation (3), $S_{25} = \dfrac{25(a_1 + a_{25})}{2}$. Now, find a_1 and a_{25}.

$$a_1 = 2 \cdot 1 - 1 = 1 \quad \text{and} \quad a_{25} = 2 \cdot 25 - 1 = 49$$

Thus, $S_{25} = \dfrac{25(1 + 49)}{2} = 625$.

(b) In general,

$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{n[1 + (2n - 1)]}{2} = \frac{n(2n)}{2} = n^2.$$

Hence, $S_n = n^2$. ◀

▶EXAMPLE 7 ***Arithmetic sequence*** The sum of the first eight terms of an arithmetic sequence $\{a_n\}$ is 24; the sixth term is 0. Find a formula for a_n.

Solution

For a_n, first find a and d. Since $a_6 = a + 5d$, $a + 5d = 0$. Express S_8 in terms of a and d,

$$S_8 = \frac{8[a + (a + 7d)]}{2} = 4(2a + 7d).$$

Since we are given $S_8 = 24$, Equation (3) states that $4(2a + 7d) = 24$. This gives a pair of equations to solve for a and d.

$$\begin{cases} a + 5d = 0 \\ 2a + 7d = 6 \end{cases}$$

We find $d = -2$ and $a = 10$. Therefore, the nth term is

$$a_n = a + (n - 1)d = 10 + (n - 1)(-2) = 12 - 2n. \quad ◀$$

Partial Sums of Geometric Sequences

The idea of pairing terms, which works so well for arithmetic sequences, does not help with a geometric sequence. Another idea does make the sum easy to calculate though. Multiply both sides by r and subtract:

$$\begin{array}{rl} S_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rS_n &= \phantom{a + {}} ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ \hline S_n - rS_n &= a - ar^n \end{array}$$

Thus,

$$S_n(1 - r) = a(1 - r^n).$$

If $r \neq 1$, dividing both sides by $(1 - r)$ yields a formula for S_n.

Partial sums of a geometric sequence

Suppose $\{a_n\}$ is a geometric sequence with $r \neq 1$. The sum of the first n terms is

$$S_n = \frac{a(1 - r^n)}{1 - r} \qquad \textbf{(4)}$$

In the special case where $r = 1$, the geometric sequence is also an arithmetic sequence, and $S_n = a + a + a + \cdots + a = na$.

►EXAMPLE 8 ***Partial sum*** Find a_n and S_n for the geometric sequence $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \ldots$.

Solution

Strategy: Since it is given that the sequence is geometric, find the common ratio $r = \dfrac{a_2}{a_1}$ and then use Equations (2) and (4).

Follow the strategy. We know that $a_1 = \frac{1}{3}$ and a_2 is $\frac{1}{6}$. The common ratio is $r = \dfrac{a_2}{a_1} = \dfrac{\left(\frac{1}{6}\right)}{\left(\frac{1}{3}\right)} = \dfrac{1}{2}$. From Equation (2),

$$a_n = ar^{n-1} = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^{n-1} = \frac{1}{3 \cdot 2^{n-1}}.$$

Since $r = \frac{1}{2}$, $1 - r = \frac{1}{2}$ and $1 - r^n = 1 - \left(\frac{1}{2}\right)^n$. Applying Equation (4) gives

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{\left(\frac{1}{3}\right)\left[1 - \left(\frac{1}{2}\right)^n\right]}{1 - \left(\frac{1}{2}\right)} = \frac{2}{3}\left(1 - \frac{1}{2^n}\right).$$

Therefore,

$$a_n = \frac{1}{3 \cdot 2^{n-1}} \quad \text{and} \quad S_n = \frac{2}{3}\left(1 - \frac{1}{2^n}\right) \quad ◄$$

►EXAMPLE 9 ***Limit of a sum*** **(a)** Find the sum of the first 5, 10, and 100 terms of the geometric sequence from Example 8. **(b)** Draw a graph of $S_n = \frac{2}{3}\left(1 - \frac{1}{2^n}\right)$ in $[0, c] \times [0, 2]$, where c is the number of pixel columns of your calculator (see inside front cover). Trace to find the smallest integer n for which the y-value is displayed as $\frac{2}{3}$.

Solution

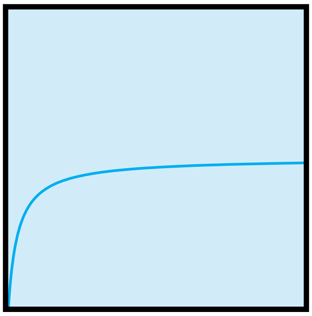
(a) In Example 8 we found a formula for the nth partial sum, $S_n = \frac{2}{3}(1 - \frac{1}{2^n})$. Substituting 5, 10, and 100 for n,

$$S_5 = \frac{2}{3}\left(1 - \frac{1}{2^5}\right) = \frac{31}{48} \approx 0.646, \qquad S_{10} = \frac{2}{3}\left(\frac{1023}{1024}\right) = \frac{1023}{1536} \approx 0.6660$$

$$S_{100} = \frac{2}{3}\left(1 - \frac{1}{2^{100}}\right).$$

The term $\frac{1}{2^{100}}$ has 30 zeros immediately following the decimal point. That means that S_{100} is so near 1 that a calculator cannot display the difference except as 1.

(b) In the window $[0, c] \times [0, 2]$ we see a graph something like Figure 2. Because calculators display trace coordinates differently, you may see something other than ours, but somewhere between 25 and 35, you should see the y-value displayed something like 0.6666666 . . . , the nearest your calculator can come to displaying $\frac{2}{3}$. ◀



$[0, c]$ by $[0, 2]$

FIGURE 2
$y = (2/3)(1 - 1/2\wedge x)$

Looking Ahead to Calculus: Infinite Series

As indicated above, each sequence $\{a_n\}$ is associated with a sequence of partial sums $\{S_n\}$, where $S_n = a_1 + a_2 + \cdots + a_n$. What happens to S_n as n gets larger and larger, that is, as we add more and more terms? We are considering an "infinite sum" written as $a_1 + a_2 + a_3 + \cdots$, or in summation notation,

$$\sum_{n=1}^{\infty} a_n.$$

This is called an **infinite series.**

Since we cannot add an infinite set of numbers, we need instead the notion of a limit. In one sense, calculus is the study of limits. It is beyond the scope of this book to deal with infinite series in general, but for a geometric sequence $\{a_n\}$, we can at least get an intuitive feeling for what happens to S_n as n becomes large.

In Examples 8 and 9, where $a_n = \frac{1}{3 \cdot 2^{n-1}}$ and $S_n = \frac{2}{3}\left(1 - \frac{1}{2^n}\right)$, it is reasonable to assume that $\frac{1}{2^n}$ gets close to 0 as n becomes large. In calculus notation

$$\lim_{n\to\infty} \frac{1}{2^n} = 0, \quad \text{from which} \quad \lim_{n\to\infty} S_n = \frac{2}{3}.$$

We say that the infinite series, $\sum_{n=1}^{\infty} \frac{1}{3 \cdot 2^{n-1}}$ **converges** to $\frac{2}{3}$, and we write

$$\sum_{n=1}^{\infty} \frac{1}{3 \cdot 2^{n-1}} = \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \cdots = \frac{2}{3}.$$

In general, we associate each geometric sequence $\{ar^{n-1}\}$ with an infinite geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots.$$

The only meaning we give to this infinite sum is the limit of the sequence of partial sums,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r},$$

which depends on $\lim_{n\to\infty} r^n$. Looking at different values of r, we conclude that if r is any number between -1 and 1, then $\lim_{n\to\infty} r^n = 0$, from which

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r}.$$

Infinite geometric series

Associated with every geometric sequence $\{ar^{n-1}\}$ is an infinite geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots.$$

If $-1 < r < 1$, then the series converges to $\frac{a}{1 - r}$, and we write

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \frac{a}{1 - r}. \qquad \textbf{(5)}$$

If $|r| \geq 1$, then the infinite series does not have a sum, and it diverges.

Repeating decimals. In Section 1.2 we said that the decimal representation of any rational number is a repeating decimal. The following example illustrates how we can use an infinite geometric series to express a repeating decimal as a fraction of integers.

▶***EXAMPLE 10*** ***Repeating decimal*** Write $1.2454545\cdots (= 1.2\overline{45})$ in terms of an infinite geometric series, then use Equation (5) to express $1.2\overline{45}$ in the form $\frac{p}{q}$, where p and q are integers.

Solution

$$\begin{aligned} 1.2454545\cdots &= 1.2 + 0.045 + 0.00045 + \cdots \\ &= \frac{6}{5} + \frac{45}{10^3} + \frac{45}{10^5} + \cdots \end{aligned}$$

The terms following $\frac{12}{10}$ form an infinite geometric series with $a = 0.045 = \frac{45}{1000}$ and $r = 0.01 = \frac{1}{100}$. Since r is between -1 and 1, we may use Equation (5) to express the sum as

$$\frac{6}{5} + \frac{0.045}{1 - 0.01} = \frac{6}{5} + \frac{45}{990} = \frac{137}{110}.$$

Therefore, $\frac{137}{110}$ and $1.2\overline{45}$ represent the same number. ◀

Functions represented by infinite series. The infinite series $1 + x + x^2 + \cdots$ is geometric (with $a = 1$ and $r = x$), so if x is any number between -1 and 1, the series converges:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}.$$

Hence the function $f(x) = \dfrac{1}{1 - x}$, where $-1 < x < 1$, can be represented by the infinite series $1 + x + x^2 + \cdots$.

An important topic arises in calculus when we represent functions by infinite series. For instance, it can be shown that the function $F(x) = \sin x$ is also given by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

The representation for $\sin x$ is not a geometric series, but it does converge for every real number x. It follows that $\sin x$ can be approximated by polynomial functions consisting of the first few terms of the infinite series. For example, if we let $p(x)$ be the sum of the first four terms,

$$p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}, \quad \text{then} \quad p(x) \approx \sin x.$$

Evaluating at $x = 0.5$,

$$\sin 0.5 \approx p(0.5) = 0.5 - \frac{(0.5)^3}{6} + \frac{(0.5)^5}{120} - \frac{(0.5)^7}{5040} \approx 0.4794255332.$$

To see how good this approximation is, use your calculator to evaluate sin 0.5 (in radian mode). In fact, your calculator is probably designed to use polynomial approximations to evaluate most of its built-in functions.

Following are series representations for some important functions we have studied in Chapters 4 and 5.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

EXERCISES 8.3

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. If $\{a_n\}$ is an arithmetic sequence, then $a_6 - a_3 = a_8 - a_5$.
2. The sequence beginning $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$ could be an arithmetic sequence.
3. If $\{c_n\}$ is a geometric sequence, then $\dfrac{c_5}{c_2} = r^3$.
4. The sequences $\{a_n\}$ and $\{b_n\}$ given by $a_n = 2n$ and $b_n = \log(100^n)$ are identical.
5. In a geometric sequence if the common ratio is negative, then after a certain point in the sequence, all the terms will be negative.
6. In an arithmetic sequence if the common difference is negative, then after a certain point in the sequence, all the terms will be negative.

Exercises 7–10 Fill in the blank so that the resulting statement is true.

7. $14 + \sum_{k=1}^{5} (-2)^k =$ ______.
8. $\sum_{k=1}^{15} (8 - k) =$ ______.
9. $0.999\ldots = 0.\overline{9} =$ ______.
10. $11(0.727272\ldots) = 11(0.\overline{72}) =$ ______.

Develop Mastery

Exercises 1–10 ***Arithmetic Sequences*** *The first three terms of an arithmetic sequence are given. Find* ***(a)*** *the common difference,* ***(b)*** *the sixth and tenth terms, and* ***(c)*** *the sum of the first ten terms.*

1. $3, 6, 9, \ldots$

2. $-18, -11, -4, \ldots$

3. $4, -1, -6, \ldots$

4. $\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \ldots$

5. $20, \frac{52}{3}, \frac{44}{3}, \ldots$

6. $0.24, 0.32, 0.40, \ldots$

7. $1 + \sqrt{2}, 1 - \sqrt{2}, 1 - 3\sqrt{2}, \ldots$

8. $1 + \sqrt{5}, 1, 1 - \sqrt{5}, \ldots$

9. $\ln 2, \ln 4, \ln 8, \ldots$

10. $\ln e, 4, 7, \ldots$

Exercises 11–20 ***Geometric Sequences*** *The first three terms of a geometric sequence are given. Find* ***(a)*** *the common ratio,* ***(b)*** *the sixth and eighth terms, and* ***(c)*** *the sum of the first five terms.*

11. $1, -\frac{1}{3}, \frac{1}{9}, \ldots$

12. $2, 1, \frac{1}{2}, \ldots$

13. $18, 6, 2, \ldots$

14. $1, -0.5, 0.25, \ldots$

15. $1, \sqrt{2}, 2, \ldots$

16. $1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \ldots$

17. $3, 1.5, 0.75, \ldots$

18. $\frac{4}{9}, -\frac{2}{3}, 1, \ldots$

19. $\sqrt{2} - 1, 1, \sqrt{2} + 1, \ldots$

20. $\frac{3}{2}, 1, \frac{2}{3}, \ldots$

Exercises 21–28 ***Arithmetic or Geometric?*** *The first three terms of a sequence are given. Determine whether the sequence could be arithmetic, geometric, or neither. If arithmetic, find the common difference; if geometric, give the common ratio.*

21. $3, -1, -4, \ldots$

22. $-2, -4, -8, \ldots$

23. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

24. $\frac{4}{9}, -\frac{2}{3}, 1, \ldots$

25. $\ln \sqrt{3}, \ln 3, \ln 3\sqrt{3}, \ldots$

26. $1, 4, 9, \ldots$

27. $0.21, 0.0021, 0.000021, \ldots$

28. $e^{-1}, e^{-2}, e^{-3}, \ldots$

Exercises 29–36 ***Arithmetic Sequences*** *Assume that the given information refers to an arithmetic sequence. Find the indicated quantities.*

29. $a_3 = 5, a_6 = 0; d, a_1$

30. $a_2 = 5, d = \frac{3}{2}; a_1, a_{10}$

31. $a_1 = 1, a_8 = 15; d, S_8$

32. $a_8 = 1, a_9 = 1; S_4, S_{16}$

33. $a_8 = 15, S_8 = 64; a_1, S_4$

34. $a_6 = -1, S_{16} = 8; a_1, S_6$

35. $a_5 = \dfrac{\pi}{3}, d = \dfrac{\pi}{3}, a_4, a_{16}, S_{16}$

36. $a_5 = \sqrt{2}, a_8 = 4\sqrt{2}; a_1, a_{12}, S_{12}$

Exercises 37–44 ***Geometric Sequences*** *Assume that the given information refers to a geometric sequence. Determine the indicated quantities.*

37. $a_1 = 4, a_2 = 6; r, a_6$

38. $a_2 = 3, a_3 = -\sqrt{3}; a_4, a_7$

39. $a_5 = \frac{1}{4}, r = \frac{3}{2}; a_1, S_5$

40. $a_4 = 6, a_7 = 48; r, a_{10}$

41. $a_1 = 18, S_2 = 24; a_5, S_5$

42. $a_4 = \frac{1}{3}, a_7 = -\frac{1}{81}; r, S_7$

43. $a_3 = -\frac{8}{5}, a_{10} = \frac{1}{80}; a_1, S_8$

44. $a_1 = \frac{1}{2}, S_2 = \frac{2}{3}; a_6, S_6$

Exercises 45–50 ***Find x*** *Determine the value(s) of x for which the given expressions will form the first three terms of the indicated type of sequence.*

45. $2, x, x^2 - 1$; arithmetic

46. $x - 2, x + 2, x + 6$; arithmetic

47. $2, x, x^2 - 1$; geometric

48. $x - 2, x + 2, x + 6$; geometric

49. $x + 1, 3x - 1, 3x + 3$; arithmetic

50. $2, 2^x, 2^{x-4}$; geometric

Exercises 51–56 ***Arithmetic or Geometric?*** *Three expressions are given. Determine whether, for every real number x, they are the first three terms of an arithmetic sequence or a geometric sequence.*

51. $x + 1, x + 3, x + 5$

52. $2 - x, 3 - 2x, 4 - 3x$

53. $2^x, 2^{x-1}, 2^{x-2}$

54. $2^{x-1}, 2^{2x-2}, 2^{3x-3}$

55. $(1 + x), (1 + x)^2, (1 + x)^3$

56. $\dfrac{1}{x^2 + 1}, \dfrac{2}{x^2 + 1}, \dfrac{4}{x^2 + 1}$

Exercises 57–60 ***Infinite Series*** *For the infinite series,* ***(a)*** *write out the first four terms, find the common ratio and a formula for S_n.* ***(b)*** *Find the sum of the series; that is, find the limit of S_n as n gets large.*

57. $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1}$

58. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2^n}$

59. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^{n+1}}$

60. $\displaystyle\sum_{n=1}^{\infty} 4\left(\frac{1}{5}\right)^{n-1}$

Exercises 61–62 ***Sum of an Infinite Series*** *Find the sum of the infinite geometric series*

61. $3 - 2 + \frac{4}{3} - \frac{8}{9} + \ldots.$

62. $-8 + 6 - \frac{9}{2} + \frac{27}{8} - \ldots.$

Exercises 63–64 ***Geometric Sequence, Partial Sums, Convergence*** *The first three terms of a geometric sequence are given.* ***(a)*** *Use Equation (4) to find a formula for S_n as a function of n, and draw a graph of S_n.* ***(b)*** *Using Equation (5) find the limit L of $\{S_n\}$.* ***(c)*** *Trace to find the smallest value of n for which $|S_n - L|$ is less than* 0.001, 0.00001. *See Example 9.*

63. $4, -2.4, 1.44, \ldots$

64. $2, 1.2, 0.96, \ldots$

Exercises 65–68 Express as a quotient of two integers in reduced form.

65. **(a)** 1.24, **(b)** $1.\overline{24}$ **66.** **(a)** 1.45, **(b)** $1.\overline{45}$

67. **(a)** 1.125, **(b)** $1.\overline{125}$ **68.** **(a)** 0.72, **(b)** $0.\overline{72}$

69. Evaluate the sum $\frac{\pi}{2} + \pi + \frac{3\pi}{2} + 2\pi + \frac{5\pi}{2} + \cdots + \frac{31\pi}{2} + 16\pi$.

70. **(a)** How many integers between 200 and 1000 are divisible by 11?
(b) What is their sum?

71. Find the sum of all odd positive integers less than 200.

72. Find the sum of all positive integers between 400 and 500 that are divisible by 3.

73. If $1 < a < b < c$ and a, b, and c are the first three terms of a geometric sequence, show that the numbers $\frac{1}{\log_a 4}$, $\frac{1}{\log_b 4}$, and $\frac{1}{\log_c 4}$ are three consecutive terms of an arithmetic sequence. (*Hint:* Use the change of the base formula from page 228.)

74. In a geometric sequence $\{a_n\}$ of positive terms, $a_2 - a_1 = 12$ and $a_5 - a_4 = 324$. Find the first five terms of the sequence.

75. If the sum of the first 60 odd positive integers is subtracted from the sum of the first 60 even positive integers, what is the result?

76. The measures of the four interior angles of a quadrilateral form four terms of an arithmetic sequence. If the smallest angle is 72°, what is the largest angle?

77. In a right triangle with legs a, b, and hypotenuse c, suppose that a, b, and c are three consecutive terms of a geometric sequence. Find the common ratio r.

78. The seats in a theater are arranged in 31 rows with 40 seats in the first row, 42 in the second, 44 in the third and so on.
(a) How many seats are in the thirty-first row? In the middle row?
(b) How many seats are in the theater?

79. A rubber ball is dropped from the top of the Washington Monument, which is 170 meters high. Suppose each time it hits the ground it rebounds $\frac{2}{3}$ of the distance of the preceding fall.
(a) What total distance does the ball travel up to the instant when it hits the ground for the third time?
(b) What total distance does it travel before it essentially comes to rest?

80. Suppose we wish to create a vacuum in a tank that contains 1000 cubic feet of air. Each stroke of the vacuum pump removes half of the air that remains in the tank.
(a) How much air remains in the tank after the fourth stroke?
(b) How much air was removed during the fourth stroke?
(c) How many strokes of the pump are required to remove at least 99 percent of the air?

81. From a helicopter hovering at 6400 feet above ground level an object is dropped. The distance s it falls in t seconds after being dropped is given by the formula $s = f(t) = 16t^2$.
(a) How far does the object fall during the first second?
(b) Let a_n denote the distance that the object falls during the nth second, that is, $a_n = f(n) - f(n-1)$. Find a formula for a_n. What kind of sequence is a_n?
(c) Evaluate the sum $a_1 + a_2 + \cdots + a_{12}$, and then find s when $t = 12$. Compare these two numbers.
(d) Clearly this is a finite sequence since the object cannot fall more than 6400 feet. How many terms are there in the sequence? What is the sum of these terms?

8.4 PATTERNS, GUESSES, AND FORMULAS

> What humans do with the language of mathematics is to describe patterns. Mathematics is an exploratory science that seeks to understand every kind of pattern—patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns.
>
> *Lynn Arthur Steen*

Arithmetic and geometric sequences are highly structured, and it is precisely because we can analyze the regularity of their patterns that we can do so much with them. The formulas developed in the preceding section are examples of what can be done when patterns are recognized and used appropriately.

One of the strongest urges of the human mind is to discover, seek out, or impose some kind of order in the world around us. If we can organize new information into some kind of recognizable pattern, we can learn more efficiently and remember more accurately.

Patterns and mathematical formulas to describe patterns permeate mathematics. Where did all these formulas come from? All too often people have the impression that mathematics just is, that it has always been around in precisely its current form. Students frequently get the feeling that their main responsibility is just to learn the wisdom that has passed down through the ages.

In the 1970s Penrose's lifelong passion for geometric puzzles yielded a bonus. He found that as few as two geometric shapes, put together in jigsaw-puzzle fashion, can cover a surface in patterns that never repeat themselves. "To a small extent I was thinking about how simple structures can force complicated arrangements, but mainly I was doing it for fun."
Roger Penrose

Mathematics should be seen as an experimental, growing, changing science. It has never been limited to professional mathematicians. Some very important discoveries have been made by amateurs, ordinary people who became involved in the fascinating questions that are always at the heart of mathematics. Mathematics has grown from discoveries that excited those who found answers in patterns they were investigating. It has been strengthened by vigorous disagreements and arguments between different investigators. It has grown in much the same way as other scientific disciplines and it continues to develop today as much as ever before.

Our goal in this section is not to make a mathematician of every reader, but we do want to *involve* you in the discovery process, to provide some opportunity to experience the feeling of creation that drives mathematics. Someone who sees what appears to be a relationship and then can work through to an understanding of why it is valid is truly doing mathematics, whether or not someone else may have made a similar discovery before.

As a first step, we will always need raw data, numbers we can look at to search for patterns or regularity. On the basis of our search, we will try to formulate a guess as to what is happening. A good guess will allow us to predict what should happen in the next case. Such a prediction allows us to test our guess. On the basis of checking a guess, we either strengthen our confidence that we have a good explanation, or we find out that some modification is necessary. We want to emphasize that there is no such thing as a bad guess if it explains something we have observed. Guesses may later turn out to be inadequate or incorrect, but any hope of finding new knowledge depends on a willingness to risk wrong guesses that can be corrected.

In many situations there may be several ways to write formulas, and there are almost always different ways to verify their correctness. In general, there is no single correct response. As you look for patterns, guess freely. Examine possible solutions. Try to understand what is happening. A guess remains just a guess until it is proven to be correct or shown to be incorrect. Proofs generally are much more difficult, and there is never any guarantee that a proof even exists. Some guesses that appear to be sound have not yet been established, even years after they were made. We illustrate some typical procedures in the following examples.

►EXAMPLE 1 ***Divisibility*** For what positive integers n is $2^n - 1$ divisible by 3?

Solution

Strategy: Evaluate $2^n - 1$ for the first few values of n and see which are divisible by 3. Use enough values of n to see a pattern.

Follow the strategy. Substituting $n = 1, 2, 3, \ldots$, calculate some numbers.

$$2^1 - 1 = 1 \quad 2^2 - 1 = 3 \quad 2^3 - 1 = 7 \quad 2^4 - 1 = 15$$

This indicates that $2^n - 1$ is divisible by 3 when n is 2 or 4. On the basis of this very small sample, we should probably hesitate to make a guess with much confidence, but it appears that every even value of n may give a number that is divisible by 3.

GUESS: $2^n - 1$ is divisible by 3 for every even positive integer n.

Now test the guess. The next even number for n is 6, and $2^6 - 1 = 63$, which does have a factor of 3, reinforcing confidence in the guess. Also check what happens when $n = 5$, to see if $2^5 - 1 (= 31)$ is not divisible by 3. The next even values for n yield $2^8 - 1 = 255$ and $2^{10} - 1 = 1023$, both of which are divisible by 3.

To prove that the guess is correct, we can use mathematical induction, which is discussed in the next section. ◀

Often one guess about a pattern leads to recognition of a related pattern. After evaluating $2^n - 1$ for several values of n, other patterns may emerge. For convenience, let us use function notation $f(n) = 2^n - 1$:

$$f(1) = 1 \quad f(2) = 3 \quad f(3) = 7 \quad f(4) = 15$$
$$f(5) = 31 \quad f(6) = 63 \quad f(7) = 127$$

A natural question is: For what n is $f(n)$ a prime number? If function g is given by $g(n) = 2^n + 1$, then ask when is $g(n)$ divisible by 3, or when is $g(n)$ a prime? For what values of n are $f(n)$ or $g(n)$ divisible by other numbers?

Pascal's Triangle

One marvelous source for pattern observation, called **Pascal's triangle,** is a triangular array of numbers named after Blaise Pascal (1623–1687). Pascal may be considered the father of modern probability theory, in which these numbers play an important role. The numbers in Pascal's triangle are also called **binomial coefficients,** a name we will justify in Section 8.6. Pascal was not the first, or only, discoverer of some of the properties of binomial coefficients. A beautiful representation of the triangle appeared in China as early as 1303, but Pascal did a great deal of work with the numbers we now associate with his name.

We shall examine binomial coefficients in greater detail in Section 8.6. At the moment, we are concerned primarily with the way the triangle is generated, one row at a time. Figure 3 shows only the first six rows, but the triangle can be continued as needed. The first and last entries on each row are always ones, and every other entry is obtained by adding the two adjacent entries immediately above.

Figure 3 shows the numbers themselves. In order to refer to specific entries in the triangle, we need to identify entries by location. The rows are numbered in obvious fashion; the columns are numbered diagonally, starting with column 0, not column 1. The entry in the nth row and the cth column is denoted by $\binom{n}{c}$. Figure 3b shows addresses of the corresponding entries in Figure 3a. For example, in the sixth row, the figure shows the following.

$$\binom{6}{0} = 1 \quad \binom{6}{1} = 6 \quad \binom{6}{2} = 15 \quad \binom{6}{3} = 20 \quad \binom{6}{4} = 15, \text{ etc.}$$

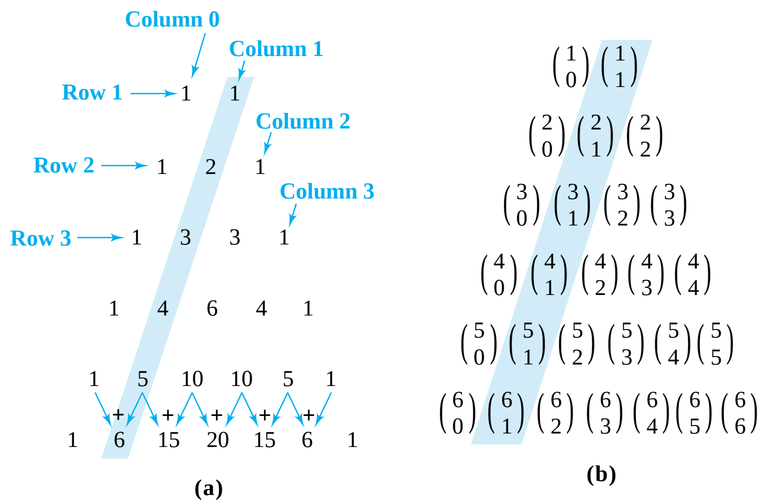


FIGURE 3

The rule for generating each row of Pascal's triangle from the one just above it is indicated by the arrows in Figure 3a, showing how the fifth row generates the sixth row. In address notation,

$$\binom{5}{0} + \binom{5}{1} = \binom{6}{1}, \quad \binom{5}{1} + \binom{5}{2} = \binom{6}{2}, \quad \binom{5}{2} + \binom{5}{3} = \binom{6}{3}, \quad \text{etc.}$$

This rule may be stated recursively.

Binomial coefficients (recursive form)

The symbol $\binom{n}{c}$ denotes the entry in the nth row and the cth column of Pascal's triangle. The end entries (where c is 0 or n) are 1 on each row. If $0 < c < n$, then adding adjacent entries in the nth row gives the entry between them in the next row.

$$\binom{n}{0} = \binom{n}{n} = 1 \quad \text{and} \quad \binom{n}{c} + \binom{n}{c+1} = \binom{n+1}{c+1}$$

►EXAMPLE 2 ***Sums in Pascal's triangle*** Guess a formula for the sum of the entries on the nth row of Pascal's triangle.

Solution

We first begin by looking at some specific cases that will help us understand the problem, and from which we may be able to recognize patterns. From Figure 3a, we add the numbers across each row and get the following sums.

Row	*Sum*
1	$1 + 1 = 2\ (= 2^1)$
2	$1 + 2 + 1 = 4\ (= 2^2)$
3	$1 + 3 + 3 + 1 = 8\ (= 2^3)$
4	$1 + 4 + 6 + 4 + 1 = 16\ (= 2^4)$

The sums appear to be doubling at each stage, suggesting an obvious guess.

HISTORICAL NOTE

COMPUTERS AND PATTERN RECOGNITION

Researchers in the area of artificial intelligence marvel at the capacity of the human mind to see patterns and discover relationships. As Douglas Hofstadter has said, "An inherent property of intelligence [is] that it is always looking for, and often finding, patterns."

How can a machine be instructed to recognize that a set of data points lie along some line, that they are essentially linear? Such judgments require the ability to ignore exceptional cases and to consider others in clusters. Recent applications of pattern recognition routines are as diverse as space probes that can make midcourse corrections using star patterns for guidance, computer programs in medicine that can make probable diagnoses from patient responses to programmed questions, and programs to identify insect pests by analyzing the patterns in which leaves are chewed.

Pulitzer Prize-winning computer scientist Douglas Hofstadter

Now investigators are exploring ways to connect large numbers of computer chips into neural networks to more closely simulate the way they think the brain may work. Some results are very promising. Neural nets are proving to be remarkably adept at predicting the biological activity of comparatively short fragments of DNA. They can find patterns in sequences that appear to be random. While we do not clearly understand just how these networks operate, they seem to take functions that describe the given numbers and combine the functions to predict the next terms. Scientists still cannot approach the capabilities of the human mind with machines, but, in some instances, the neural nets can now recognize patterns more complicated than human brains can handle.

GUESS: The sum of the entries on the nth row of Pascal's triangle is 2^n.

The sum of the entries on the fifth row is 32, which is 2^5, and for the sixth row, the sum is 64, which is 2^6. The guess still looks good.

The key to understanding why the guess is correct is the way each row is derived from the row above it. Look at the arrows from the fifth to the sixth row in Figure 3a. The first 6 comes from adding the 1 and 5 above, and the same 5 with the 10 gives 15. Similarly, the same 10 is used again for the next entry. Thus each entry on the fifth row is added twice to get the sixth row, including the outside 1s to get the outside 1s on the sixth row. It follows that the sum of the entries on Row 6 is twice the sum of the entries on Row 5. Since 2^5 is the sum for Row 5, the sum of Row 6 must equal $2(2^5) = 2^6$. ◀

This argument is essentially a proof by mathematical induction, which we will discuss formally in the next section.

►EXAMPLE 3 ***Using Pascal's triangle*** Find a formula for the sum of the first n positive integers in terms of the entries in Pascal's triangle.

Solution

Let $f(n)$ denote the sum of the first n positive integers,

$$f(n) = 1 + 2 + 3 + \cdots + n.$$

Express $f(n)$ in terms of entries in Pascal's triangle. First get some data:

$$f(1) = 1, \qquad f(2) = 1 + 2 = 3, \qquad f(3) = 1 + 2 + 3 = 6,$$
$$f(4) = 1 + 2 + 3 + 4 = 10.$$

The numbers 1, 3, 6, and 10 are successive entries in Column 2 of Pascal's triangle in Figure 8.1a. In address notation,

$$f(1) = \binom{2}{2}, \qquad f(2) = \binom{3}{2}, \qquad f(3) = \binom{4}{2}, \qquad f(4) = \binom{5}{2}.$$

Based on the data we gathered, we arrive at the following guess.

$$\text{GUESS: } f(n) = \binom{n+1}{2} \text{ for every positive integer } n.$$

If we look at the way each entry is obtained from the two immediately above it, we may see that we are adding precisely what is needed for the pattern to continue. Therefore, our guess must be correct. ◄

►EXAMPLE 4 ***Dividing a circle*** Given n points on a circle, consider two functions:

$C(n)$ is the number of chords determined by connecting each pair of these points.

$R(n)$ is the number of regions into which the chords divide the interior of the circle, where no three chords have a common point of intersection inside the circle.

Guess a formula for **(a)** $C(n)$ and **(b)** $R(n)$.

Solution

Strategy: From a table showing the first few values of $C(n)$ and $R(n)$, look for numbers that may be related to obvious powers of numbers or to entries in Pascal's triangle.

For each value of n, make a sketch from which we can get the information needed for the table (see Figure 4).

n	$C(n)$	$R(n)$
1	0	1
2	1	2
3	3	4
4	6	8
5	10	16

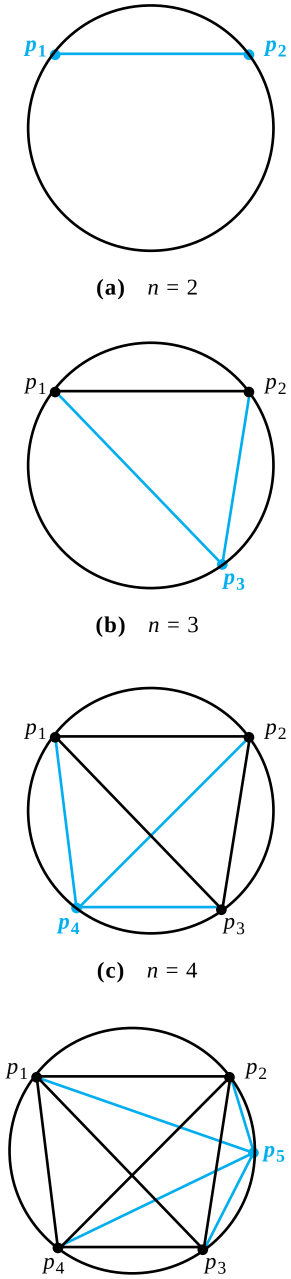


FIGURE 4

(a) For $n \geq 2$, the values of $C(n)$ are numbers in Column 2 of Pascal's triangle. In address notation

$$C(2) = \binom{2}{2}, \qquad C(3) = \binom{3}{2}, \qquad C(4) = \binom{4}{2}, \qquad C(5) = \binom{5}{2}$$

A pattern emerges on which to base a reasonable guess.

$$\text{GUESS: } C(n) = \binom{n}{2} \text{ for } n \geq 2.$$

To see why the guess for $C(n)$ continues to give the correct values, see what happens when you add one more point (going from n to $n + 1$). Check to see that you get n new chords. Compare the resulting values with the values predicted by the formulas.

(b) From the table of values of $R(n)$, the number of pieces appears to be a power of 2, doubling with each new point:

$$R(1) = 1 = 2^0, \qquad R(2) = 2 = 2^1, \qquad R(3) = 4 = 2^2,$$
$$R(4) = 8 = 2^3, \quad \text{and} \quad R(5) = 16 = 2^4.$$

Make the obvious guess: $R(n) = 2^{n-1}$ for every positive integer n.

Draw circles with 6 and 7 points and carefully count the number of regions. According to the guess $R(n) = 2^{n-1}$, we should get $R(6) = 32$ and $R(7) = 64$. What numbers do you get?

This is an excellent guess based on a beautiful pattern that simply happens to be wrong. Sometimes people speak of a pattern "breaking down." The pattern does not break down; we have failed to find the right pattern. The correct formula is more complicated and may be expressed in terms of the entries in Pascal's triangle.

$$R(n) = \binom{n}{0} + \binom{n}{2} + \binom{n}{4}.$$

For instance, when n is 5, from Figure 8.1

$$\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 1 + 10 + 5 = 16$$

which is the value of $R(5)$. Evaluate

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4}$$

when n is 6 and when n is 7. (Extend Pascal's triangle to include row seven.) Compare your results with the actual count of the number of regions, $R(6)$ and $R(7)$.

In Section 8.6 we will see how to evaluate binomial coefficients to get

$$R(n) = \frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}. \quad ◀$$

EXERCISES 8.4

Check Your Understanding

Exercises 1–10. True or False. Give reasons.

1. When n is 1, 2, 3, 4, or 5, the sum of the first n odd positive integers is equal to n^2.
2. When n is 1, 2, 3, 4, or 5, the sum of the first n even positive integers is equal to $n(n + 1)$.
3. If $f(n) = n^2 - n + 17$, then $f(n)$ is a prime number for $n = 1, 2, 4, 8$, and 17.
4. If $f(n) = n^2 + n$, then $f(n)$ is an even number for every positive integer n.
5. Evaluating the expressions $(n + 1)^2$ and 2^n for $n = 1, 2, 3, 4, 5$, and 6, it is reasonable to conclude that $(n + 1)^2 > 2^n$ for every positive integer.
6. For every positive integer n, $3^n + 1$ is an even number.
7. For every positive integer n, the units digit of $5^n - 1$ is 4.
8. When n is 1, 2, 3, or 4, $5^n + 1$ is not divisible by 4.
9. For every positive integer n, the units digit of 2^n is 2, 4, or 8.
10. For every positive integer n, the units digit of $4^n - 1$ is 3 or 5.

Develop Mastery

In these exercises, n always denotes a natural number.

Exercises 1–5 ***Recognize Patterns*** *As a first step for each exercise, complete the following table by entering the values of f(n) for the given function. In order to see patterns, it is very important that computations be correct. As a check, one of the values of f(n) is given.*

n	1	2	3	4	5	6
$f(n)$						

1. $f(n) = 5^n - 1$. *Check:* $f(4) = 5^4 - 1 = 624$.
   (a) For what values of n in your table is $f(n)$ a multiple of 3? Of 4? Of 12?
   (b) Based on these observations, make a guess that describes all natural numbers n for which $f(n)$ is a multiple of 3, of 4, and of 12. State your guess in complete sentences.
2. $f(n) = 5^n + 1$. *Check:* $f(5) = 5^5 + 1 = 3126$.
   (a) For what values of n in your table is $f(n)$ a multiple of 6? Of 7?
   (b) Make a guess that describes all natural numbers n for which $f(n)$ is a multiple of 6. Express your guess in complete sentences.
   (c) For additional information: Is $f(9)$ a multiple of 7? Is $f(15)$ a multiple of 7? Is $f(18)$ a multiple of 7? For which n is $f(n)$ a multiple of 7?
3. $f(n) = 9 + 9^2 + 9^3 + \cdots + 9^n$.
   Check: $f(3) = 9 + 9^2 + 9^3 = 819$.
   Based on the data in your table, make a guess that describes all natural numbers n for which the units digit of $f(n)$ is a zero, a one, a nine. Convince your teacher that your guess is correct.
4. $f(n) = n^2 - n + 11$.
   Check: $f(5) = 5^2 - 5 + 11 = 31$.
   (a) For what values of n in your table is $f(n)$ a prime number? Is $f(n)$ a prime number for every natural number n?
   (b) Make a guess concerning the units digit of $f(n)$.
   (c) To lead to a recursive formula, enter appropriate numbers in each of the blank spaces:
   $$f(2) = f(1) + \underline{\qquad}$$
   $$f(3) = f(2) + \underline{\qquad}$$
   $$f(4) = f(3) + \underline{\qquad} \ldots$$
   Now guess the quantity that should be entered in the general case:
   $$f(n + 1) = f(n) + \underline{\qquad}.$$
   Prove that your guess is valid (or not valid) by actually evaluating $f(n + 1)$ and $f(n) + \underline{\qquad}$ to see if they are equal.
   (d) Evaluate $f(11)$, $f(22)$, and $f(33)$. Now check your conclusion in **(a).**
5. $f(n) = n^2 - n + 41$.
   Check: $f(5) = 5^2 - 5 + 41 = 61$.
   (a)–(c) Same as in Exercise 4.
   (d) Evaluate $f(41)$ and $f(82)$. Are these primes?

Exercises 6–14 ***Guess a Formula*** *Function f is defined as a sum.* ***(a)*** *Make a table that shows the values of f(n) for n = 1, 2, 3, 4, 5, and 6.* ***(b)*** *Based on the data in the table, guess a simpler formula for f. The value of f(4) is also given, which should serve as a check on your computations and also possibly as a hint to help you recognize patterns.*

6. $f(n) = 1 + 3 + 5 + \cdots + (2n - 1)$
   $$= \sum_{k=1}^{n} (2k - 1); \qquad f(4) = 16.$$
7. $f(n) = 2 + 4 + 6 + \cdots + 2n = \sum_{j=1}^{n} 2j$;
   $f(4) = 4 \cdot 5$.
8. $$f(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)}$$
   $$= \sum_{i=1}^{n} \frac{1}{i(i + 1)}; \qquad f(4) = \frac{4}{5}.$$

9. $f(n) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$
$= \sum_{k=1}^{n} \frac{1}{2^k}; \qquad f(4) = \frac{16-1}{16}.$

10. $f(n) = 2\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n}\right)$
$= 2 \cdot \left(\sum_{k=1}^{n} \frac{1}{3^k}\right); \qquad f(4) = \frac{80}{81}.$

11. $f(n) = 1 + 2(3^0 + 3^1 + 3^2 + \cdots + 3^{n-1})$
$= 1 + 2\sum_{k=1}^{n} 3^{k-1}; \qquad f(4) = 3^4$

(c) Use the result in **(b)** to get a formula for $\Sigma_{k=1}^{n} 3^{k-1}$.

12. $f(n) = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}$
$= \sum_{k=1}^{n} \frac{k}{(k+1)!}; \qquad f(4) = \frac{119}{120}.$

(*Hint:* Keep in mind the values of factorials, such as $6 = 3!$, $24 = 4!$, $120 = 5!, \cdots$.)

13. $f(n) = 1 + (1 + 2 + 4 + \cdots + 2^{n-1})$
$= 1 + \sum_{j=1}^{n} 2^{j-1}; \qquad f(4) = 2^4.$

(c) Use your results in **(b)** to get a simpler formula for $g(n) = 1 + 2 + 4 + \cdots + 2^{n-1}$.

14. $f(n) = 1 + 2 + 3 + \cdots + (n-1) + n + (n-1) + \cdots + 3 + 2 + 1$
$f(4) = 1 + 2 + 3 + 4 + 3 + 2 + 1 = 16.$

15. For $f(n) = 2\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{(n+1)^2}\right)$, evaluate $f(n)$ for $n = 1, 2, 3, 4, 5,$ and 6. For instance, $f(4) = \frac{6}{5} = \frac{4+2}{4+1}$. Based on your data, guess a simpler formula for $f(n)$.

Exercises 16–18 ***Recursive Functions*** *For the functions defined recursively, **(a)** Make a table that shows the values of $f(n)$ for $n = 1, 2, 3, 4, 5,$ and 6. **(b)** Using the data from this table, guess a closed form formula for f.*

16. $f(1) = 2$ and $f(n) = 2f(n-1)$ for $n \geq 2$; $f(5) = 2^5$.

17. $f(1) = 2$ and $f(n) = f(n-1) + 2n$ for $n \geq 2$; $f(5) = 5 \cdot 6$.

18. $f(1) = 3$ and $f(n) = f(n-1) + (2n+1)$ for $n \geq 2$; $f(4) = 4 \cdot 6$.

19. ***Number of Handshakes*** Suppose that each of the n people at a party shakes hands with every other person exactly once. Let $f(n)$ denote the total number of handshakes, so that $f(1) = 0$ (one person, no handshakes), $f(2) = 1$ (two people, one handshake), $f(3) = f(2) + 2$ (adding a person adds two more handshakes). A newcomer shakes hands with each of the k people present and so $f(k+1) = f(k) + k$. Find a formula for f.

20. In Example 4 let $D(n)$ be the number of diagonals of the polygon obtained by connecting the points on the circle. For example, $D(1) = D(2) = D(3) = 0$, $D(4) = 2$, and $D(5) = 5$. Guess a formula for $D(n)$. (*Hint:* Count the number of new diagonals when adding a point.)

Exercises 21–24 ***Pascal's Triangle*** *Extend Pascal's triangle, shown in Figure 3, for a few more rows. When you are asked to find a number in the triangle, express it in address notation, $\binom{n}{c}$.*

21. How many entries appear on row n?

22. On what rows of the triangle are the two middle entries the same?

23. Each row in Figure 3 is symmetrical. (Each reads the same forward and backward.) Explain how you can know that Row 7 is symmetrical without computing any entries in Row 7? How about Row 8? Row n?

24. In Example 2, we showed that the sum of all the entries on Row n is 2^n $\left(\text{symbolically, } \sum_{c=0}^{n}\binom{n}{c} = 2^n\right)$. Let $f(n)$ be the sum of the entries in the even-numbered columns of Row n. That is,

$$f(n) = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{m}$$

where m is the last even-numbered column on Row n. For instance,

$$f(4) = \binom{4}{0} + \binom{4}{2} + \binom{4}{4} = 1 + 6 + 1 = 8$$

$$f(5) = \binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 1 + 10 + 5 = 16.$$

Evaluate $f(n)$ for several other values of n and then use the information to help you guess a simpler formula for f.

25. Follow the instructions for Exercise 24, but let $f(n)$ denote the sum of all the entries in odd-numbered columns on Row n, that is

$$f(n) = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots + \binom{n}{m}$$

where m is the last odd-numbered column in Row n.

Exercises 26–29 ***Sums in Pascal's Triangle*** *Let $f(n)$ be the sum of the first n entries in the given column.* ***(a)*** *Evaluate $f(n)$ for several values of n ($f(4)$ is given as a check).* ***(b)*** *Locate these sums in the triangle, and then guess a simpler formula for f in terms of address notation.*

26. *Column 0*

$$f(n) = \binom{1}{0} + \binom{2}{0} + \binom{3}{0} + \cdots + \binom{n}{0}$$

$$f(4) = \binom{4}{1}.$$

27. *Column 1*

$$f(n) = \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1}$$

$$f(4) = \binom{5}{2}.$$

28. *Column 2*

$$f(n) = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n+1}{2}$$

$$f(4) = \binom{6}{3}.$$

29. *Column 3*

$$f(n) = \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \cdots + \binom{n+2}{3}$$

$$f(4) = \binom{7}{4}.$$

30. Let $f(n)$ denote the sum of the squares of the first n natural numbers:

$$f(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

(a) Evaluate $f(n)$ for $n = 1, 2, 3, 4, 5$, and 6. For instance, $f(4) = 30$.

(b) Now look in Column 3 of Pascal's triangle and find two consecutive entries whose sum is $f(n)$. For instance,

$$f(4) = 30 = 10 + 20 = \binom{5}{3} + \binom{6}{3}.$$

Use this information to help you guess a formula that gives $f(n)$ as the sum of two consecutive entries in Column 3. Use address notation in your answer.

31. Let $P(n)$ denote the number of pieces (regions) into which n lines divide the plane. Assume that no two lines are parallel, and that no three lines contain a common point. Draw figures to illustrate the cases for $n = 1, 2, 3, 4$, and 5. By actually counting the pieces in each case, evaluate $P(n)$ for $n = 1, 2, 3, 4$, and 5. Guess a formula for $P(n)$. (*Hint:* Look for $P(n) - 1$ in Pascal's triangle.)

32. ***Related Functions*** Function f is defined recursively by

$$f(1) = 1 \quad f(2) = 5 \quad \text{and}$$
$$f(n) = f(n-1) + 2f(n-2) \quad \text{for } n \geq 3.$$

Functions g and h are given in closed form by

$$g(n) = 2^n + 1 \qquad h(n) = 2^n - 1.$$

(a) Complete the following table:

n	1	2	3	4	5	6
$f(n)$						
$g(n)$						
$h(n)$						

As a check, you should have $f(4) = 17$, $g(4) = 17$, $h(4) = 15$.

(b) Based on the data in the table, make a guess about the values of n for which $f(n) = g(n)$ and for which $f(n) = h(n)$.

(c) Using your guess in **(b)**, is $f(n) = 2^n + (-1)^n$ for every natural number n?

33. At the end of Example 4, we indicated that $R(n)$ is given by the formula:

$$R(n) = \frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}.$$

Evaluate $R(n)$ for $n = 4, 5, 6$, and 7. For each value of n draw an appropriate diagram and actually count the number of regions to see if there is agreement with the formula prediction.

34. Sequence $\{b_n\}$ is defined recursively by $b_1 = 3, b_2 = 5$, $b_n = b_{n-1} - b_{n-2}$, $n > 2$. That is, after the second term, each term is the difference of the two preceding terms.

(a) List the first ten terms.

(b) Based on the results in part **(a)**, describe a general pattern for the sequence.

(c) What is the sum of the first 1996 terms? The first 2000 terms?

35. Repeat Exercise 34 for $b_1 = x$, $b_2 = y$.

8.5 MATHEMATICAL INDUCTION

> The idea of mathematical induction is simply that if something is true at the beginning of the series, and if this is "inherited" as we proceed from one number to the next, then it is also true for all natural numbers. This has given us a method to prove something for all natural numbers, whereas to try out all such numbers is impossible with our finite brains. We need only prove two things, both conceivable by means of our finite brains: that the statement in question is true for 1, and that it is the kind that is "inherited."
>
> *Rózsa Péter*

In Section 1.4 we discussed statements (sentences that are either true or false) and open sentences, whose truth value depends on replacing a variable or placeholder with a number. In this section we consider sentences of the type

$$n! < 8n \text{ for every positive integer } n. \qquad \textbf{(1)}$$

Such a sentence involves a quantifier : for every positive integer. Looking at the first few statements, we get

$$\begin{array}{lll} 1! < 8 \cdot 1 & 4! < 8 \cdot 4 & 6! < 8 \cdot 6 \\ 1 < 8 \text{ (True)} & 24 < 32 \text{ (True)} & 720 < 48 \text{ (False)} \end{array}$$

Since $n! < 8n$ does not yield a true statement for every positive integer n, Sentence (1) is false. To show that a sentence of the type (1) is false, all we need is one value of n ($n = 6$ in this case) that yields a false statement; any such value of n is a counterexample.

A statement of the form

$$P(n) \text{ for every positive integer } n, \qquad \textbf{(2)}$$

means that the infinite set of statements $P(1)$, $P(2)$, $P(3)$, . . . are all true. Establishing the truth of such a statement requires a special method of proof called **mathematical induction.** Consider an example of the type given in Statement (2). An appeal to intuition leads us to the formal statement of the Principle of Mathematical Induction.

Suppose $P(n)$ is given by

$$P(n)\text{: The sum of the first } n \text{ odd positive integers is } n^2.$$

> *The real inspiration . . . I got from my father. He showed me a few mathematical things when I came back from Berlin. He showed me mathematical induction, for instance. Maybe I was twelve.*
> Lipman Bers

It is difficult to work mathematically with a statement given verbally. We can restate $P(n)$ in mathematical terms.

$$P(n)\text{: } 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

In sigma notation

$$P(n)\text{: } \sum_{m=1}^{n} (2m - 1) = n^2.$$

If we claim that $P(n)$ is true for every positive integer n, then we must somehow show that each of the statements $P(1), P(2), P(3), \ldots$ is true. We list a few of these:

$$P(1)\text{: } \sum_{m=1}^{1} (2m - 1) = 1^2 \quad \text{or} \quad 1 = 1^2 \qquad \text{True}$$

$$P(2)\text{: } \sum_{m=1}^{2} (2m - 1) = 2^2 \quad \text{or} \quad 1 + 3 = 2^2 \qquad \text{True}$$

$$P(3)\text{: } \sum_{m=1}^{3} (2m - 1) = 3^2 \quad \text{or} \quad 1 + 3 + 5 = 3^2 \qquad \text{True}$$

So far we have verified that $P(1)$, $P(2)$, and $P(3)$ are all true. Clearly we cannot continue with the direct verification for the remaining positive integers n, but suppose we can accomplish two things:

(a) Verify that $P(1)$ is true.
(b) For any arbitrary positive integer k, show that the truth of $P(k + 1)$ follows from the truth of $P(k)$.

If **(a)** and **(b)** can be done, then we reason as follows: $P(1)$ is true by **(a)**; since $P(1)$ is true, then by **(b)** it follows that $P(1 + 1)$, or $P(2)$, must be true; now since $P(2)$ is true then by **(b)** it follows that $P(3)$ is true; and so on. Therefore, it is intuitively reasonable to conclude that $P(n)$ is true *for every positive integer n*. This type of reasoning is the basis for the idea of mathematical induction.

Having done **(a)** for the example, let us see if we can accomplish **(b)**. We can state **(b)** in terms of a hypothesis and a conclusion and argue that the conclusion follows from the induction hypothesis:

$$\text{Hypothesis } P(k)\text{:} \qquad 1 + 3 + 5 + \cdots + (2k - 1) = k^2 \tag{3}$$

$$\text{Conclusion } P(k+1)\text{:} \qquad 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2 \tag{4}$$

Now use Equation (3) and argue that Equation (4) follows from it. To get to Equation (4) from Equation (3), add $2k + 1$ to both sides.

$$[1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1) = k^2 + (2k + 1) \tag{5}$$

The right side of Equation (5) can be written as

$$k^2 + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2.$$

Therefore, Equation (5) is equivalent to

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2.$$

This is precisely Equation (4), so we have accomplished **(b)**. Hence we can conclude that the statement

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \text{ for every positive integer } n$$

is true.

The mathematical basis for our conclusion follows from the Principle of Mathematical Induction.

Principle of mathematical induction

Suppose $P(n)$ is an open sentence that gives statements $P(1)$, $P(2)$, $P(3)$, ... If we can accomplish the following two things:

(a) verify that $P(1)$ is true,
(b) for any arbitrary positive integer k, show that the truth of $P(k + 1)$ follows from the truth of $P(k)$,

then $P(n)$ is true *for every positive integer n.*

►EXAMPLE 1 ***Truth values*** For the open sentence, write out $P(1)$, $P(2)$, and $P(5)$. Determine the truth value of each.

(a) $P(n)$: $n^3 + 11n = 6(n^2 + 1)$.
(b) $P(n)$: $5^n - 1$ is divisible by 4.

Solution

(a) $P(1)$: $1^3 + 11 \cdot 1 = 6(1^2 + 1)$ or $1 + 11 = 6(1 + 1)$ True
$P(2)$: $2^3 + 11 \cdot 2 = 6(2^2 + 1)$ or $8 + 22 = 6(5)$ True
$P(5)$: $5^3 + 11 \cdot 5 = 6(5^2 + 1)$ or $125 + 55 = 6(26)$ False
(b) $P(1)$: $5^1 - 1$ is divisible by 4, $5^1 - 1 = 4$ True
$P(2)$: $5^2 - 1$ is divisible by 4, $5^2 - 1 = 24$ True
$P(5)$: $5^5 - 1$ is divisible by 4, $5^5 - 1 = 3124$ True ◄

►EXAMPLE 2 ***Proof by mathematical induction*** Let $P(n)$ be the open sentence

$P(n)$: $5^n - 1$ is divisible by 4.

Prove that $P(n)$ is true for every positive integer n.

Solution

Strategy: After verifying that $P(1)$ is true, for mathematical induction, show that $P(k)$ implies $P(k + 1)$; relate $5^{k+1} - 1$ to $5^k - 1$.

Proof will follow if we can accomplish **(a)** and **(b)** of the Principle of Mathematical Induction. For **(a)** we must show that $P(1)$ is true. This has already been done in Example 1b.

For **(b)**, state the induction hypothesis and conclusion.

Hypothesis $P(k)$: $5^k - 1$ is divisible by 4. **(6)**
Conclusion: $P(k + 1)$: $5^{k+1} - 1$ is divisible by 4. **(7)**

Since by hypothesis, $5^k - 1$ is divisible by 4, there is an integer m such that

$$5^k - 1 = 4m \quad \text{or} \quad 5^k = 4m + 1.$$

Therefore,

$$\begin{aligned} 5^{k+1} - 1 &= 5 \cdot 5^k - 1 = 5(4m + 1) - 1 \\ &= 20m + 4 = 4(5m + 1). \end{aligned}$$

Hence if $5^k - 1$ is divisible by 4, then $5^{k+1} - 1$ is also divisible by 4. This establishes **(b)**, and proves that $5^n - 1$ is divisible by 4 for every positive integer n. ◄

▶EXAMPLE 3 ***Proof by mathematical induction*** Show that $2^{n+1} > n + 2$ for every positive integer n.

Solution

(a) When n is 1, $2^{1+1} > 1 + 2$, or $4 > 3$, which is true.
(b) Hypothesis $P(k)$: $2^{k+1} > k + 2$
Conclusion $P(k + 1)$: $2^{k+2} > k + 3$

Begin with the hypothesis and multiply both sides of the inequality by 2.

$$2 \cdot 2^{k+1} > 2k + 4 \quad \text{or} \quad 2^{k+2} > 2k + 4$$

We would like $k + 3$ on the right-hand side, but $2k + 4 = (k + 3) + (k + 1)$, and since k is a positive integer, $k + 1 > 0$, so $2k + 4 > k + 3$. Therefore,

$$2^{k+2} > 2k + 4 > k + 3.$$

By the Principle of Mathematical Induction, we conclude that $2^{n+1} > n + 2$ for every positive integer n. ◀

EXERCISES 8.5

Check Your Understanding

Exercises 1–10 True or False. Give reasons.

1. There is no positive integer n such that $(n + 1)! = n! + 1!$.
2. There is no positive integer n such that $n^2 + n = 6$.
3. For every positive integer n, $(n + 1)^2 \geq 2^n$.
4. For every positive integer n, $\sin n\pi = 0$.
5. For every positive integer n, $(2n - 1)(2n + 1)$ is an odd number.
6. For every positive integer n, $n^2 + n$ is an even number.
7. For every positive integer n, $n^2 + 1 \geq 2n$.
8. For every positive integer n, $(n + 1)^3 - n^3 - 1$ is divisible by 6.
9. For every positive integer n, $n^2 - n + 17$ is a prime number.
10. For every integer n greater than 1, $\log_2 n \geq \log_n 2$.

Develop Mastery

Exercises 1–8 ***Truth Values*** *Denote the given open sentence as $P(n)$. Write out $P(1)$, $P(2)$, and $P(5)$, and determine the truth value of each.*

1. $n^2 - n + 11$ is a prime number.
2. $4n^2 - 4n + 1$ is a perfect square.
3. $n^2 < 2n + 1$
4. $3^n > n^2$
5. $n! \leq n^2$
6. $4^n - 1$ is a multiple of 3.
7. $1^3 + 2^3 + 3^3 + \cdots + n^3$ is a perfect square.
8. The sum of the first n even positive integers is equal to $n(n + 1)$.

Exercises 9–12 ***When is $P(n)$ False?*** *Find the smallest positive integer n for which $P(n)$ is false.*

9. $P(n)$: $n! \leq n^3$
10. $P(n)$: $n^2 - n + 5$ is a prime number.
11. $P(n)$: $n^3 < 3^n$
12. $P(n)$: $n! < 3^n$

Exercises 13–18 ***Hypothesis and Conclusion*** *Step* **(b)** *of the Principle of Mathematical Induction involves an induction hypothesis and a conclusion. Write out the hypothesis and the conclusion.*

13. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n + 1)^2}{4}$
14. $2 + 5 + 8 + \cdots + (3n - 1) = \dfrac{n(3n + 1)}{2}$
15. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \dfrac{n(n + 1)(n + 2)}{3}$
16. $3^n > n^2$
17. $4^n - 1$ is a multiple of 3.
18. The sum of the first n even positive integers equals $n(n + 1)$.

Exercises 19–32 **Give a Proof** *Use mathematical induction to prove that the given formula is valid for every positive integer n.*

19. $2 + 5 + 8 + \cdots + (3n - 1) = \dfrac{n(3n + 1)}{2}$

20. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \dfrac{n(n + 1)(n + 2)}{3}$

21. The sum of the first n positive integers is equal to $\dfrac{n(n + 1)}{2}$.

22. The sum of the first n even positive integers is equal to $n(n + 1)$.

23. $2 \cdot 1 + 2 \cdot 4 + 2 \cdot 7 + \cdots + 2(3n - 2) = 3n^2 - n$.

24. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$.

25. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n + 1)^2}{4}$.

26. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$.

27. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$.

28. $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n + 1)! - 1$

29. $\displaystyle\sum_{m=1}^{n} (3m^2 + m) = n(n + 1)^2$

30. $\displaystyle\sum_{m=1}^{n} (2m - 3) = n(n - 2)$

31. $\displaystyle\sum_{m=1}^{n} (2^{m-1} - 1) = 2^n - n - 1$

32. $\displaystyle\sum_{m=1}^{n} \ln m = \ln(n!)$

Exercises 33–40 **Give a Proof** *Prove that P(n) yields a true statement when n is replaced by any positive integer.*

33. $P(n)$: $4^n - 1$ is divisible by 3.

34. $P(n)$: $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

35. $P(n)$: $n^3 + 2n$ is divisible by 3.

36. $P(n)$: $2^{2n-1} + 1$ is divisible by 3.

37. $P(n)$: $2^n \le (n + 1)!$

38. $P(n)$: $3^n \le (n + 2)!$

39. $P(n)$: $2^n \ge n + 1$

40. $P(n)$: $2^{n+1} > 2n + 1$

Exercises 41–49 **Is it True?** *Let P(n) denote the open sentence. Either find the smallest positive integer n for which P(n) is false, or prove P(n) is true for every positive integer n.*

41. $n(n^2 - 1)$ is divisible by 6.

42. $n^2 + n$ is an even number.

43. $n^2 - n + 41$ is an odd number.

44. $5n^2 + 1$ is divisible by 3.

45. $5n^2 + 1$ is not a perfect square.

46. $2^n < (n + 1)^2$.

47. $n^2 - n + 41$ is a prime number.

48. $n^4 + 35n^2 + 24 = 10n(n^2 + 5)$.

49. $\left(1 + \dfrac{1}{1}\right)\left(1 + \dfrac{1}{2}\right)\left(1 + \dfrac{1}{3}\right) \cdots \left(1 + \dfrac{1}{n}\right) = n + 1$.

Exercises 50–51 **Explore** ***(a)*** *Evaluate the sum when n is 1, 2, 3, and 4.* ***(b)*** *Guess a formula for the sum.* ***(c)*** *Prove that your formula is valid for every positive integer n.*

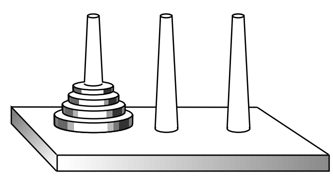
50. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n + 1)}$

51. $1 + 2 + 3 + \cdots + n + \cdots + 3 + 2 + 1$

52. Let $f(n) = 5^n - 4$.
- **(a)** Evaluate f when n is 1, 2, 3, 4, and 5.
- **(b)** For what positive integers n is $f(n)$ divisible by 3? Prove that your guess is correct.
- **(c)** For what positive integers n is $f(n)$ divisible by 21? Prove that your guess is correct.

53. ***Explore*** Sequence $\{a_n\}$ is defined recursively by $a_1 = 6$, $a_{n+1} = 5a_n/(a_n - 5)$ for $n \ge 1$ and $\{b_n\}$ is defined by $b_n = a_n a_{n+1}$. **(a)** Find the first four terms of $\{b_n\}$. **(b)** Give a simpler formula for b_n. **(c)** Is mathematical induction necessary to prove that your formula for b_n is correct? Explain.

54. ***Towers of Hanoi*** There are three pegs on a board. Start with n disks on one peg, as suggested in the drawing. Move all disks from the starting peg onto another peg, one at a time, placing no disk atop a smaller one. (You can experiment yourself without pegs; use coins of different sizes, for example, a dime on top of a penny, on top of a nickle, on top of a quarter. It may take some patience to find the minimum number of moves required.)



- **(a)** What is the minimum number of moves required if you start with 1 disk? 2 disks? 3 disks? 4?

(b) Based on your results for **(a)**, guess the minimum number of moves required if you start with an arbitrary number of n disks. (*Hint:* To help see a pattern, add 1 to the number of moves for $n = 1, 2, 3, 4$.)

(c) A legend claims that monks in a remote monastery are working to move a set of 64 disks, and that the world will end when they complete their sacred task. Moving one disk per second without error, 24 hours a day, 365 days a year, how long would it take to move all 64 disks?

55. ***Number of Handshakes*** Suppose there are n people at a party and that each person shakes hands with every other person exactly once. Let $f(n)$ denote the total number of handshakes. See Exercise 19, page 469. Show

$$f(n) = \frac{n(n-1)}{2} \quad \text{for every positive integer } n.$$

56. Suppose n is an odd positive integer not divisible by 3. Show that $n^2 - 1$ is divisible by 24. (*Hint:* Consider the three consecutive integers $n - 1$, n, $n + 1$. Explain why the product $(n - 1)(n + 1)$ must be divisible by 3 and by 8.)

57. ***Finding Patterns*** If $a_n = \sqrt{24n + 1}$, **(a)** write out the first five terms of the sequence $\{a_n\}$. **(b)** What odd integers occur in $\{a_n\}$? **(c)** Explain why $\{a_n\}$ contains all primes greater than 3. (*Hint:* Use Exercise 56.)

8.6 THE BINOMIAL THEOREM

We remake nature by the act of discovery, in the poem or in the theorem. And the great poem and the great theorem are new to every reader, and yet are his own experiences, because he himself recreates them. [And] in the instant when the mind seizes this for itself, in art or in science, the heart misses a beat.

J. Bronowski

In this section we derive a general formula to calculate an expansion for $(a + b)^n$ for any positive integer power n, or to find any particular term in such an expansion. We begin by calculating the first few powers directly and then look for significant patterns. To go from one power of $(a + b)$ to the next, we simply multiply by $(a + b)$:

$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\begin{array}{lr} (\times) & a + b \\ \hline & a^4 + 3a^3b + 3a^2b^2 + ab^3 \qquad\qquad\qquad \\ & a^3b + 3a^2b^2 + 3ab^3 + b^4 \\ \hline \end{array} \quad \text{Add like terms}$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$\begin{array}{lr} (\times) & a + b \\ \hline & a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4 \qquad\qquad\qquad \\ & a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5 \\ \hline \end{array} \quad \text{Add like terms}$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

The thing that started it all was this silly newspaper puzzle that asked you to count up the total number of ways you could spell the words "Pyramid of Values" from a triangular array of letters. This led my friend and me to discover Pascal's triangle. This happened in grade 10 or 11.

Bill Gosper

When we look at these expansions of $(a + b)^n$ for $n = 1, 2, 3, 4$, and 5, several patterns become apparent.

1. There are $n + 1$ terms, from a^n to b^n.
2. Every term has essentially the same form: some coefficient times the product of a power of a times a power of b.
3. In each term the sum of the exponents on a and b is always n.
4. The powers (exponents) on a decrease, term by term, from n down to 0 where the last term is given by $b^n = a^0b^n$, and the exponents on b increase from 0 to n.

Knowing the form of the terms in the expansion and that the sum of the powers is always n, we will have the entire expansion when we know how to calculate the coefficients of the terms. If we display the coefficients from the computations above, we find precisely the numbers in the first few rows of Pascal's triangle:

$$\begin{array}{c} 1\quad 1 \\ 1\quad 2\quad 1 \\ 1\quad 3\quad 3\quad 1 \\ 1\quad 4\quad 6\quad 4\quad 1 \\ 1\quad 5\quad 10\quad 10\quad 5\quad 1 \end{array}$$

Using the address notation for Pascal's triangle that we introduced in Section 8.4, the last row of coefficients in the triangle is $\binom{5}{0}, \binom{5}{1}, \binom{5}{2}, \binom{5}{3}, \binom{5}{4}, \binom{5}{5}$, and so we may write the expansion for $(a + b)^5$:

$$(a + b)^5 = \binom{5}{0}a^5b^0 + \binom{5}{1}a^4b^1 + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}a^1b^4 + \binom{5}{5}a^0b^5$$

Each term exhibits the same form. For $n = 5$, each coefficient has the form $\binom{5}{r}$, where r is also the exponent on b. For each term the sum of the exponents on a and b is always 5, so that when we have b^r, we must also have a^{5-r}. Finally, since the first term has $r = 0$, the second term has $r = 1$, etc., the $(r + 1)$st term involves r.

This leads to a general conjecture for the expansion of $(a + b)^n$ which we state as a theorem that can be proved using mathematical induction. (See the end of this section.)

Binomial theorem

Suppose n is any positive integer. The expansion of $(a + b)^n$ is given by

$$(a + b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}a^0b^n \qquad \textbf{(1)}$$

where the $(r + 1)$st term is $\binom{n}{r}a^{n-r}b^r$, $0 \le r \le n$. In summation notation,

$$(a + b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r. \qquad \textbf{(2)}$$

At this point, we have established only that the form of our conjecture is valid for the first five values of n, and we have not completely justified our use of the name Pascal's triangle of **binomial coefficients.** Nonetheless, the multiplication of $(a + b)^4$ by $(a + b)$ to get the expansion for $(a + b)^5$ contains all the essential ideas of the proof.

We still lack a closed-form formula for the binomial coefficients. We know, for example, that the fourth term of the expansion of $(x + 2y)^{20}$ is $\binom{20}{3}x^{17}(2y)^3$, but we cannot complete the calculation without the binomial coefficient $\binom{20}{3}$. This would require writing at least the first few terms of 20 rows of Pascal's triangle.

Pascal himself posed and solved the problem of computing the entry at any given address within the triangle. He observed that to find $\binom{n}{r}$, we can take the product of all the numbers from 1 through r, and divide it into the product of the same number of integers, from n downward. This leads to the following formula.

Pascal's formula for binomial coefficients

Suppose n is a positive integer and r is an integer that satisfies $0 < r \leq n$. The binomial coefficient $\binom{n}{r}$ is given by

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r} \tag{3}$$

We leave it to the reader to verify that the last factor in the numerator, $(n - r + 1)$, is the rth number counting down from n. This gives the same number of factors in the numerator as in the denominator.

►EXAMPLE 1 ***Using Pascal's formula*** Find the first five binomial coefficients on the tenth row of Pascal's triangle, and then give the first five terms of the expansion of $(a + b)^{10}$.

Solution

Strategy: We know $\binom{10}{0} = 1$. Use Equation (3) to get the remaining coefficients.

Follow the strategy.

$$\binom{10}{1} = \frac{10}{1} = 10,\ \binom{10}{2} = \frac{10\cdot 9}{1\cdot 2} = 45,\ \binom{10}{3} = \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} = 120, \text{ and}$$
$$\binom{10}{4} = \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} = 210.$$

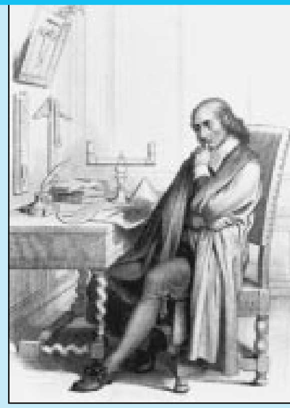
Therefore the first five terms in the expansions of $(a + b)^{10}$ are

$a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4$. ◄

There is another very common formula for binomial coefficients that uses factorials. Equation (3) has a factorial in the denominator, and we can get a factorial in the numerator if we multiply numerator and denominator by the product of the rest of the integers from $n - r$ down to 1:

$$\binom{n}{r} = \frac{n(n-1)\ldots(n-r+1)}{1\cdot 2\cdot\ldots r}$$
$$= \frac{n(n-1)\ldots(n-r+1)}{r!}\cdot\frac{(n-r)\ldots 2\cdot 1}{(n-r)\ldots 2\cdot 1} = \frac{n!}{r!\,(n-r)!}.$$

HISTORICAL NOTE BLAISE PASCAL



Blaise Pascal made significant contributions to the study of mathematics before deciding to devote his life to religion.

Pascal's triangle is named after Blaise Pascal, born in France in 1623. Pascal was an individual of incredible talent and breadth who made basic contributions in many areas of mathematics, but who died early after spending much of life embroiled in bitter philosophical and religious wrangling.

For some reason, Pascal's father decided that his son should not be exposed to any mathematics. All mathematics books in the home were locked up and the subject was banned from discussion. We do not know if the appeal of the forbidden was at work, but young Pascal approached his father directly and asked what geometry was. His father's answer so fascinated the 12-year-old boy that be began exploring geometric relationships on his own. He apparently rediscovered much of Euclid completely on his own. When Pascal was introduced to conic sections (see Chapter 10) he quickly absorbed everything available; he submitted a paper on conic sections to the French academy when he was only 16 years of age.

At the age of 29, Pascal had a conversion experience that led to a vow to renounced mathematics for a life of religious contemplation. Before that time, however, in addition to his foundational work in geometry, he built a mechanical computing machine (in honor of which the structured computer language Pascal is named), explored relations among binomial coefficients so thoroughly that we call the array of binomial coefficients Pascal's triangle even though the array had been known, at least in part, several hundred years earlier, proved the binomial theorem, gave the first published proof by mathematical induction, and invented (with Fermat) the science of combinatorial analysis, probability, and mathematical statistics.

Before his death ten years later, Pascal spent only a few days on mathematics. During a night made sleepless by a toothache, he concentrated on some problems about the cycloid curve that had attracted many mathematicians of the period. The pain subsided, and, in gratitude, Pascal wrote up his work for posterity.

While Equation (3) does not give a formula for $\binom{n}{0}$, the formulation in terms of factorials does apply.

$$\binom{n}{0} = \frac{n!}{0!\,(n-0)!} = \frac{n!}{1 \cdot n!} = \frac{n!}{n!} = 1.$$

This gives an alternative to Pascal's formula.

Alternative formula for binomial coefficients

Suppose n is a positive integer and r an integer that satisfies $0 \le r \le n$. The binomial coefficient $\binom{n}{r}$ is given by

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} \tag{4}$$

▶EXAMPLE 2 ***Symmetry in binomial coefficients*** Show that

$$\textbf{(a)} \binom{6}{2} = \binom{6}{4} \quad \textbf{(b)} \binom{n}{r} = \binom{n}{n-r}.$$

Solution

Follow the strategy.

Strategy: Use Equation (4) to evaluate both sides of the given equations to show that the two sides of each equation are equal.

$$\textbf{(a)} \binom{6}{2} = \frac{6!}{2!\,(6-2)!} = \frac{6!}{2!\,4!} \quad \text{and} \quad \binom{6}{4} = \frac{6!}{4!\,(6-4)!} = \frac{6!}{4!\,2!}.$$

$$\textbf{(b)} \binom{n}{r} = \frac{n!}{r!\,(n-r)!} \quad \text{and}$$

$$\binom{n}{n-r} = \frac{n!}{(n-r)!\,[n-(n-r)]!} = \frac{n!}{(n-r)!\,r!}$$

Thus

$$\binom{n}{r} = \binom{n}{n-r}. \quad ◀$$

▶EXAMPLE 3 ***Adding binomial coefficients*** Show that $\binom{8}{3} + \binom{8}{4} = \binom{9}{4}$. Get a common denominator and add fractions, but do not evaluate any of the factorials or binomial coefficients.

Solution

Use Equation (3) to get $\binom{8}{3}$ and $\binom{8}{4}$, get common denominators, then add.

$$\binom{8}{3} + \binom{8}{4} = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} + \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{(8 \cdot 7 \cdot 6) \cdot 4}{(1 \cdot 2 \cdot 3) \cdot 4} + \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$= \frac{8 \cdot 7 \cdot 6(4+5)}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = \binom{9}{4}.$$

Thus

$$\binom{8}{3} + \binom{8}{4} = \binom{9}{4}. \quad ◀$$

Example 3 focuses more on the process than the particular result, hence the instruction to add fractions without evaluating. When we write out the binomial coefficients as fractions, we can identify the extra factors we need to get a common denominator and then add. In Example 2, we proved that $\binom{n}{r} = \binom{n}{n-r}$ giving a symmetry property for the nth row of Pascal's triangle. Example 3 illustrates the essential steps to prove the following additivity property (see Exercise 61):

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Symmetry and additivity properties

Binomial coefficients have the following properties:

Symmetry $\binom{n}{r} = \binom{n}{n-r}$ **(5)**

Additivity $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ **(6)**

Notice that we used the additivity property from Equation (6) in Section 8.4 to get the $(n + 1)$st row from the nth row in Pascal's triangle. This justifies our claim that the entries in Pascal's triangle are binomial coefficients.

TECHNOLOGY TIP ◆ **Evaluating binomial coefficients**

Most graphing calculators have the capacity to evaluate binomial coefficients directly, but we need to know where to look for the needed key. Most calculators use the notation nCr (meaning "number of combinations taken r at a time," language from probability), and the key is located in a probability (PRB, PROB) submenu under the MATH menu. To evaluate, say $\binom{20}{4}$, the process is as follows.

All TI-calculators: Having entered 20 on your screen, press MATH PRB nCr, which puts nCr on the screen. Then type 4 so you have 20nCr4. When you enter, the display should read 4845.

All Casio calculators: Having entered 20 on your screen, press MATH PRB nCr, which puts C on the screen. Then type 4 so you have 20C4. When you execute, the display should read 4845.

HP-38: Press MATH, go down to PROB, highlight COMB, OK. Then, on the command line you want COMB(20,4). Enter to evaluate.

HP-48: Put 20 and 4 on the stack. As with many HP-48 operations, COMB (for "combinations") works with two numbers. MTH NXT PROB COMB returns 4845.

▶EXAMPLE 4 Binomial Theorem Use the binomial theorem to write out the first five terms of the binomial expansion of $(x + 2y^2)^{20}$ and simplify.

Solution

Use Equation (1) with $a = x$, $b = 2y^2$, and $n = 20$. The first five terms of $(x + 2y^2)^{20}$ are

$$x^{20} + \binom{20}{1}x^{19}(2y^2) + \binom{20}{2}x^{18}(2y^2)^2 + \binom{20}{3}x^{17}(2y^2)^3 + \binom{20}{4}x^{16}(2y^2)^4.$$

Before simplifying, find the binomial coefficients, using either Equation (3) or the Technology Tip.

$$\binom{20}{1} = \frac{20}{1} = 20 \qquad \binom{20}{2} = \frac{20 \cdot 19}{1 \cdot 2} = 190$$

$$\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3} = 1140 \qquad \binom{20}{4} = \frac{20 \cdot 19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 4} = 4845$$

Therefore, the first five terms of $(x + 2y^2)^{20}$ are

$$x^{20} + 20 \cdot 2x^{19}y^2 + 190 \cdot 4x^{18}y^4 + 1140 \cdot 8x^{17}y^6 + 4845 \cdot 16x^{16}y^8, \text{ or}$$

$$x^{20} + 40x^{19}y^2 + 760x^{18}y^4 + 9120x^{17}y^6 + 77520x^{16}y^8. \quad ◀$$

▶EXAMPLE 5 ***Finding a middle term*** In the expansion of $(2x^2 - \frac{1}{x})^{10}$, find the middle term.

Solution

There are $10 + 1$ or 11 terms in the expansion of a tenth power, so the middle term is the sixth (five before and five after). The sixth term is given by $r = 5$.

$$\binom{10}{5}(2x^2)^5\left(-\frac{1}{x}\right)^5 = 252(32x^{10})\left(-\frac{1}{x}\right)^5 = -8064x^5$$

The middle term is $-8064x^5$. ◀

▶EXAMPLE 6 ***Finding a specified term*** In the expansion of $(2x^2 - \frac{1}{x})^{10}$, find the term whose simplified form involves $\frac{1}{x}$.

Solution

Strategy: First find the general term, then simplify. Finally, find the value of r that gives -1 as the exponent of x.

Follow the strategy. The general term given in Equation (2) is

$$\binom{10}{r}(2x^2)^{10-r}\left(-\frac{1}{x}\right)^r = \binom{10}{r}2^{10-r}x^{20-2r}(-1)^r x^{-r}$$
$$= \binom{10}{r}(-1)^r 2^{10-r}x^{20-3r}.$$

For the term that involves $\frac{1}{x}$ or x^{-1}, find the value of r for which the exponent on x is -1: $20 - 3r = -1$, or $r = 7$. The desired term is given by

$$\binom{10}{7}(2x^2)^3\left(-\frac{1}{x}\right)^7 = -\frac{120(8)x^6}{x^7} = -\frac{960}{x}.$$ ◀

Proof of the Binomial Theorem

We can use mathematical induction to prove that Equation (1) holds for every positive integer n.

(a) For $n = 1$, Equation (1) is $(a + b)^1 = \binom{1}{0}a^1b^0 + \binom{1}{1}a^0b^1 = a + b$, so Equation (1) is valid when n is 1.

(b) Hypothesis:
$$(a + b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \cdots + \binom{k}{r}a^{k-r}b^r + \cdots + \binom{k}{k}b^k \qquad \textbf{(7)}$$

Conclusion:
$$(a + b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^kb + \cdots + \binom{k+1}{r}a^{k+1-r}b^r + \cdots + \binom{k+1}{k+1}b^{k+1} \qquad \textbf{(8)}$$

Since $(a+b)^{k+1} = (a+b)^k(a+b) = (a+b)^k a + (a+b)^k b$, multiply the right side of Equation (7) by a, then by b, and add, combining like terms. It is also helpful to replace $\binom{k}{0}$ by $\binom{k+1}{0}$ and $\binom{k}{k}$ by $\binom{k+1}{k+1}$, since all are equal to 1.

$$\begin{aligned}(a+b)^k(a+b) = \binom{k+1}{0}a^{k+1} &+ \left[\binom{k}{0} + \binom{k}{1}\right]a^k b \\ &+ \left[\binom{k}{1} + \binom{k}{2}\right] a^{k-1}b^2 + \cdots \\ &+ \left[\binom{k}{r-1} + \binom{k}{r}\right]a^{k+1-r}b^r + \cdots \\ &+ \binom{k+1}{k+1}b^{k+1}.\end{aligned}$$

Apply the additive property given in Equation (6) to the expressions in brackets to get Equation (8), as desired. Therefore, by the Principle of Mathematical Induction, Equation (1) is valid for every position integer n.

EXERCISES 8.6

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. For every positive integer n, $(3n)! = (3!)(n!)$.
2. There are ten terms in the expression of $(1+x)^{10}$.
3. The middle term of the expansion of $(x + \frac{1}{x})^8$ is 70.
4. The expansion of $(x^2 + 2x + 1)^8$ is the same as the expansion of $(x+1)^{16}$.
5. $\binom{8}{1} + \binom{8}{2} - \binom{9}{2} = 0$.
6. For every positive integer x, $\left(\sqrt{x} + \dfrac{1}{x}\right)^4 = x^2 + \dfrac{1}{x^4}$.

Exercises 7–10 Fill in the blank so that the resulting statement is true.

7. After simplifying the expansion of $(x^2 - \frac{1}{x})^5$, the coefficient of x^4 is ______.
8. In the expansion of $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^6$, the middle term is ______.
9. $\binom{8}{3} - \binom{8}{2} =$ ______.
10. The number of terms in the expansion of $(x^2 + 4x + 4)^{12}$ is ______.

Develop Mastery

Exercises 1–14 Evaluate and simplify. Use Equations (3)–(6). Then verify by calculator.

1. **(a)** $\binom{9}{3}$ **(b)** $\binom{9}{6}$
2. **(a)** $\binom{14}{3}$ **(b)** $\binom{14}{11}$
3. **(a)** $\binom{8}{5}$ **(b)** $\binom{8}{3}$
4. **(a)** $\binom{100}{98}$ **(b)** $\binom{100}{2}$
5. **(a)** $\binom{20}{2} + \binom{20}{3}$ **(b)** $\binom{21}{3}$
6. **(a)** $\binom{7}{3} + \binom{7}{4}$ **(b)** $\binom{8}{4}$
7. **(a)** $\frac{5}{6} \cdot \binom{10}{5}$ **(b)** $\binom{10}{6}$
8. **(a)** $\frac{9}{4} \cdot \binom{12}{3}$ **(b)** $\binom{12}{4}$
9. **(a)** $\binom{10}{6} \cdot \binom{6}{3}$ **(b)** $\binom{10}{7} \cdot \binom{7}{3}$
10. **(a)** $\binom{12}{10} \cdot \binom{10}{4}$ **(b)** $\binom{8}{5} \cdot \binom{5}{3}$
11. **(a)** $\dfrac{10!}{7!}$ **(b)** $\dfrac{10!}{7!\,3!}$
12. **(a)** $8! + 2!$ **(b)** $10!$
13. **(a)** $\dfrac{6! + 4!}{3!}$ **(b)** $\dfrac{8! - 5!}{3!}$
14. **(a)** $6! - 3!$ **(b)** $(6-3)!$

Exercises 15–18 ***Calculator Evaluation*** *Use the Technology Tip to evaluate the expression.*

15. **(a)** $\binom{24}{8}$ **(b)** $\binom{37}{3} + \binom{37}{5}$
16. **(a)** $\binom{31}{5}$ **(b)** $\binom{16}{4} - \binom{12}{10}$
17. **(a)** $\binom{12}{8} \cdot \binom{20}{3}$ **(b)** $\binom{25}{7} \div \binom{25}{4}$
18. **(a)** $\binom{32}{3} \cdot \binom{31}{29}$ **(b)** $\binom{31}{8} \div \binom{31}{3}$

Exercises 19–24 Evaluate and simplify.

19. $\binom{n}{n-1}$ **20.** $\binom{n}{n-2}$ **21.** $\binom{n+1}{n-1}$

22. $\dfrac{(n+1)!}{(n-1)!}$ **23.** $\dfrac{\binom{n}{k+1}}{\binom{n}{k}}$ **24.** $\dfrac{\binom{n+1}{r}}{\binom{n}{r-1}}$

Exercises 25–30 ***Binomial Theorem*** *Use the binomial theorem formula to expand the expression, then simplify your result.*

25. $(x-1)^5$ **26.** $(x-3y)^4$

27. $\left(\dfrac{1}{x}-2y^2\right)^4$ **28.** $\left(x^2+\dfrac{2}{x}\right)^6$

29. $\left(3x+\dfrac{1}{x^2}\right)^5$ **30.** $(x-1)^7$

Exercises 31–34 ***Expansion*** *Use the formula in Equation (2).* ***(a)*** *Write the expansion in sigma form.* ***(b)*** *Expand and simplify.*

31. $(2-x)^5$ **32.** $\left(2x+\dfrac{y}{2}\right)^5$

33. $\left(x^2+\dfrac{2}{x}\right)^5$ **34.** $(x^2-2)^6$

Exercises 35–38 ***Number of Terms*** ***(a)*** *How many terms are there in the expansion of the given expression?* ***(b)*** *If the answer in* ***(a)*** *is odd, then find the middle term. If it is even, find the two middle terms.*

35. $(x^2-3)^8$ **36.** $\left(x^2-\dfrac{1}{x}\right)^{15}$

37. $(1+\sqrt{x})^5$ **38.** $(x+2\sqrt{x})^{10}$

Exercises 39–40 Find the first three terms in the expansion of

39. $\left(x+\dfrac{1}{x}\right)^{20}$ **40.** $\left(x-\dfrac{3}{x}\right)^{25}$

Exercises 41–44 ***Find Specified Term*** *If the expression is expanded using Equation (1), find the indicated term and simplify.*

41. $\left(x^3-\dfrac{2}{x}\right)^5$; third term

42. $\left(\dfrac{x}{2}-2y\right)^{12}$; tenth term

43. $\left(2x-\dfrac{y}{2}\right)^{10}$; fourth term

44. $(x^{-1}+2x)^8$; fourth term

Exercises 45–52 ***Specified Term*** *If the expression is expanded and each term is simplified, find the coefficient of the term that contains the given power of x. See Example 6.*

45. $\left(x^3-\dfrac{2}{x}\right)^4$; x^4 **46.** $\left(2x-\dfrac{1}{3}\right)^{10}$; x^7

47. $(x^2+2)^{11}$; x^8 **48.** $\left(x^2-\dfrac{2}{x}\right)^{10}$; x^8

49. $\left(x^3-\dfrac{1}{x}\right)^{15}$; x^{25} **50.** $\left(x^2-\dfrac{3}{x}\right)^{12}$; x^9

51. $(x^2-2x+1)^3$; x^4 **52.** $(x^2+4x+4)^3$; x^2

Exercises 53–60 ***Solve Equation*** *Find all positive integers n that satisfy the equation.*

53. $(2n)! = 2(n!)$ **54.** $(3n)! = (3!)(n!)$

55. $2(n-2)! = n!$ **56.** $(3n)! = 3(n+1)!$

57. $\binom{n}{3} = \binom{n}{5}$ **58.** $\binom{n}{3} + \binom{n}{4} = \binom{8}{4}$

59. $\binom{n}{2} = 15$ **60.** $\binom{n}{2} = 28$

61. **(a)** Show that $\binom{10}{6} + \binom{10}{7} = \binom{11}{7}$ by carrying out the following steps. Using Equation (3), express each term of $\binom{10}{6} + \binom{10}{7}$ as a fraction with factorials; then, without expanding, get a common denominator and express the result as a fraction involving factorials. By Equation (3), show that the result is equal to $\binom{11}{7}$. See Example 3.
(b) Following a pattern similar to that described in part **(a)**, prove the additivity property for the binomial coefficients

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}.$$

62. By expanding the left- and right-hand sides, verify that

$$\binom{n}{k+1} = \frac{n-k}{k+1}\cdot\binom{n}{k}.$$

63. ***Explore***
Let $S_n = 1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! + 1$.
$S_1 = 1\cdot 1! + 1 = 2 = 2!$ and
$S_2 = 1\cdot 1! + 2\cdot 2! + 1 = 6 = 3!$
(a) Evaluate S_3, S_4, and S_5 and look for a pattern. On the basis of your data, guess the value of S_8. Verify your guess by evaluating S_8 directly.
(b) Guess a formula for S_n and use mathematical induction to prove that your formula is correct.

64. ***Explore*** Suppose $f_n(x) = (x+\frac{1}{x})^{2n}$ and let a_n be the middle term of the expansion of $f_n(x)$.
(a) Find a_1, a_2, a_3, and a_4.
(b) Guess a formula for the general term a_n. Is $a_n = \dfrac{(2n)!}{n!\,n!}$?

65. Observe that 5! (= 120) ends with one zero (meaning that 5! is a multiple of 10). Find the smallest positive integer n such that $n!$ is a multiple of **(a)** 100, **(b)** 1000, and **(c)** 10^6.

66. Find the smallest positive integer n such that $n!$ exceeds **(a)** 1 billion, **(b)** 1 trillion, and **(c)** 10^{15}.

67. Show that $\binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5}$ is equal to 0. (*Hint:* Expand $[1 + (-1)]^5$ using Equation (1).

68. Show that $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$ is equal to 2^5. (*Hint:* Expand $(1 + 1)^5$ using Equation (1).)

CHAPTER 8 REVIEW

Test Your Understanding

Determine the truth value. Give reasons.

1. If $a_n = n^2 - n + 17$, then all terms of the sequence $\{a_n\}$ are prime numbers.

2. In an arithmetic sequence the common difference d equals $a_8 - a_7$.

3. In an arithmetic sequence, if d is negative, then all terms of the sequence must be negative from some point on.

4. The numbers $5, -\frac{5}{2}, \frac{5}{4}, -\frac{5}{6}$, are four consecutive terms of a geometric sequence.

Exercises 5–8 Assume that $\{a_n\}$ is an arithmetic sequence and that each term is a positive integer.

5. The common difference d cannot be a negative number.

6. If a_1 is even and d is even, then every a_n is even.

7. If a_1 is odd and d is odd, then every a_n is odd.

8. If a_1 is odd and d is even, then every a_n is odd.

9. The terms $x + 1$, $x - 2$, and $x - 3$ constitute three consecutive terms of an arithmetic sequence for every real number x.

10. There is no real number x for which x, $x + 1$, $x + 2$ will be three consecutive terms of a geometric sequence.

11. There is no sequence that is both arithmetic and geometric.

12. The sequence whose nth term is $\ln(2n)$ is neither arithmetic nor geometric.

13. The sequence whose nth term is $n \ln 2$ is arithmetic.

14. The sequence whose nth term is $\ln 2^n$ is geometric.

15. If $a_{n+1} = a_n - 3$ for $n = 1, 2, 3, \ldots$, then $\{a_n\}$ is an arithmetic sequence.

16. If $a_1 < 0$ and $a_{n+1} = -a_n$, then $\{a_n\}$ is a geometric sequence.

17. In a geometric sequence if a_1 is an irrational number, then every term of the sequence must be an irrational number.

18. If $a_n = 2n - 1$ and $b_m = 3m + 1$, then no number is in both sequences $\{a_n\}$ and $\{b_m\}$.

19. The eleventh term of the sequence $\{2n + 3\}$ is the same as the ninth term of $\{3n - 2\}$.

20. The numbers $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$, are the first three terms of an arithmetic sequence.

21. $\sum_{k=1}^{15} (2k - 1) = 15^2$.

22. $\sum_{k=1}^{4} \left(-\frac{1}{2}\right)^k = -\frac{5}{16}$

23. The numbers e, e^2, e^3 are the first three terms of a geometric sequence.

24. If $a_n = \frac{1}{n^2}$, then $\{a_n\}$ is a geometric sequence.

25. If $a_n = \sin[(2n - 1)\frac{\pi}{2}]$, then $\{a_n\}$ is a geometric sequence.

26. If $a_n = \cos n\pi$, then $\{a_n\}$ is a geometric sequence.

27. If $a_k = \cos k\pi$, then $\Sigma_{k=1}^{n} a_k$ equals 0 whenever n is even.

28. $1.\overline{21} = \frac{40}{33}$.

29. $n^2 - n + 3$ is an odd number for every positive integer n.

30. $n^2 - 2n + 4$ is an even number for every positive integer n.

31. $\frac{(n + 1)!}{(n - 1)!} = n^2 + n$ for every positive integer n.

32. $\binom{8}{5} = \binom{8}{3}$

33. $\binom{7}{4} + \binom{7}{5} = \binom{8}{5}$

34. $\frac{16!}{2!\,14!} = 240$

35. $\frac{3! + 6!}{3!} = 1 + 2!$

36. There are eight terms in the expansion of $\left(x - \frac{1}{x}\right)^8$.

37. In the expansion of $\left(x + \frac{1}{x}\right)^6$ the fourth term is a constant.

38. $\binom{n}{n - 1} = n$ for every positive integer n.

39. $\binom{2n}{n} = \frac{2n!}{n!\,n!}$ for every positive integer n.

40. $2\binom{n}{2} + \binom{n}{1} = n^2$ for every positive integer n.

41. The smallest prime number greater than the limit of the sequence defined by $a_n = 8n/(0.5n + 1)$ is 17.

42. The sequence defined by $a_n = (-1)^n n/(2n + 1)$ converges.

43. The sequence defined by $a_1 = 4$, $a_{n+1} = 3a_n/(a_n - 3)$ converges.

44. The sequence defined by $a_1 = 1$, $a_{n+1} = (\cos a_n)/4$ converges to a number between 0.24 and 0.25.

45. The subsequence of even-numbered terms of the sequence defined by $a_1 = 6$, $a_{n+1} = 5a_n/(a_n - 5)$ converges to 30.

Review for Mastery

Exercises 1–4 For the sequence whose nth term is given, ***(a)*** *find the first four terms, and* ***(b)*** *evaluate $\sum_{k=1}^{4} a_k$.*

1. $a_n = 1 - \dfrac{1}{2^n}$

2. $a_n = 1 - \dfrac{1}{2^{n-1}}$

3. $a_n = 3n - 1$

4. $a_n = \dfrac{1}{n(n+2)}$

5. If $a_1 = 3$ and $a_n = 2a_{n-1}$ for $n \geq 2$, find the first five terms of the sequence $\{a_n\}$.

6. If $a_1 = 1$, $a_2 = 2$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$, find the first five terms of the sequence $\{a_n\}$.

Exercises 7–12 ***Guess a Formula*** *The first four terms of a sequence $\{a_n\}$ are given. Guess a formula for a_n that could generate the sequence. There is no unique correct answer. (Why?)*

7. 3, 8, 13, 18, . . .

8. $\sqrt{2}, 2, 2\sqrt{2}, 4, \ldots$

9. 1, 3, 7, 15, . . .

10. 3, 5, 9, 17, . . .

11. $-1, 1, -1, 1, \ldots$

12. $\frac{1}{2}, -\frac{1}{6}, \frac{1}{12}, -\frac{1}{20}, \ldots$

13. The first three terms of an arithmetic sequence are 3, 8, 13. Find **(a)** the twenty-fourth term, and **(b)** the sum of the first 24 terms.

14. In an arithmetic sequence $a_4 = 16$ and $a_{13} = -2$. Find **(a)** a_{20}, **(b)** $\sum_{k=1}^{20} a_k$, and **(c)** the number n of terms such that $\sum_{k=1}^{n} a_k = -140$.

15. Find all values of x such that $x^2, x, -3$ are three consecutive terms of arithmetic sequence.

16. The first three terms of a geometric sequence are $3, \frac{3}{2}, \frac{3}{4}$. Find **(a)** the fifth term, and **(b)** the sum of the first five terms.

17. Suppose a sequence $\{a_n\}$ is given by $a_n = 1 + \dfrac{1}{2^n}$.

(a) Write out the first four terms.
(b) Is this a geometric sequence?
(c) Find the sum of the first four terms.

18. In a geometric sequence $a_1 = \frac{2}{3}$ and $r = \frac{1}{3}$. Find the number of terms n such that the sum S_n equals $\frac{6560}{6561}$.

19. Find the repeating decimal expansion for **(a)** $\frac{14}{15}$, **(b)** $\frac{18}{11}$, and **(c)** $\frac{3}{14}$.

20. Express the repeating decimal 0.727272 . . . (that is, $0.\overline{72}$) as a quotient of two integers.

Exercises 21–27 Evaluate the sum.

21. $\displaystyle\sum_{k=1}^{15} (2k - 1)$

22. $\displaystyle\sum_{k=1}^{50} (3k + 2)$

23. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k$

24. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k$

25. $\displaystyle\sum_{k=1}^{5} (2^k - k)$

26. $\displaystyle\sum_{k=1}^{10} (-1)^k(2k - 1)$

27. $\displaystyle\sum_{k=1}^{10} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

Exercises 28–31 ***Specified Term*** *The sequence $\{a_n\}$ is either arithmetic or geometric. Find the indicated term.*

28. $5, 8, 11, 14, \cdots;\quad a_{16}$

29. $3, \frac{5}{2}, 2, \frac{3}{2}, \cdots;\quad a_{24}$

30. $2, 3, \frac{9}{2}, \frac{27}{4}, \cdots;\quad a_8$

31. $4, -2, 1, -\frac{1}{2}, \cdots;\quad a_{10}$

Exercises 32–33 ***Geometric Series*** *Find the sum of the infinite geometric series.*

32. $\frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \cdots$

33. $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots$

Exercises 34–36 ***Mathematical Induction*** *Use the Principle of Mathematical Induction to prove that $P(n)$ yields a true statement for every positive integer n.*

34. $P(n)$: $3 + 9 + 15 + \cdots + (6n - 3) = 3n^2$

35. $P(n)$: $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

36. $P(n)$: $7^n - 1$ is divisible by **(a)** 2 **(b)** 3 **(c)** 6.

37. Is it true that $3n^3 + 6n$ is divisible by 9 for every positive integer n? Give reasons for your answer.

38. Is this a true statement: $3^n \leq (n + 3)^2$ for every positive integer n? If so, give a proof; if not, give a counterexample.

39. Evaluate **(a)** $\dfrac{6!}{2!\,4!}$ **(b)** $\dbinom{15}{3}$ **(c)** $\dbinom{8}{2} + \dbinom{8}{3}$.

Exercises 40–45 ***Binomial Theorem*** *Use the binomial theorem to expand the expression.*

40. $\left(2x - \dfrac{1}{x}\right)^5$

41. $(3 + 2x)^4$

42. $(4 - 3x)^5$

43. $(1 - \sqrt{2})^6$

44. $\left(1 - \dfrac{1}{x}\right)^5$

45. $(x^{-1} + \sqrt{y})^6$

46. Find the fourth term in the expansion of $(x + 2y)^8$.

47. In the expansion of $(2 - \frac{x}{3})^{10}$, write out the term that contains x^8.

48. In the expansion of $(1 - \sqrt{x})^8$, write out the term that contains x^3.

49. Suppose $(x^2 + \frac{1}{x})^{15}$ is expanded and the resulting terms are simplified. Find the term that involves x^6.

50. Find the sum of all positive integers less than 400 that are divisible by both 2 and 3.

51. Suppose a sequence $\{a_n\}$ is given by $a_n = \left(1 - \dfrac{1}{n}\right)^n$.

Find the term and give results rounded off to five decimal places.
(a) a_{10} **(b)** a_{100} **(c)** $a_{1,000}$ **(d)** $a_{10,000}$
(e) Evaluate e^{-1} and compare with a_n for large n.

52. Expand $(1 + 2\sqrt{x} + x)^3$. [*Hint:* First show that $1 + 2\sqrt{x} + x = (1 + \sqrt{x})^2$.]

53. ***Explore*** Using Pascal's Triangle, **(a)** evaluate the sum of the squares of all entries in Row 1, Row 2, Row 3 and then find the sum as an entry in the triangle. **(b)** On the basis of your results in part **(a)**, guess a formula for the sum of the squares of all entries in Row n. **(c)** Test your formula for $n = 4$, $n = 5$.

Exercises 54–56 ***Limit of a Sequence*** ***(a)*** *Use the Technology Tip (page 444) to find the limit L(6 decimal places) of the sequence.* ***(b)*** *Use algebra to find the exact value of L.*

54. $a_1 = 4,\ a_{n+1} = 4 - 1/a_n$

55. $a_1 = 2,\ a_{n+1} = 0.5(a_n + 5/a_n)$

56. $a_1 = \sqrt{3},\ a_{n+1} = \sqrt{5 + 2a_n}$

Exercises 57–58 ***Subsequences*** *The sequence $\{a_n\}$ diverges. Find a subsequence that converges.*

57. $a_1 = 5,\ a_{n+1} = 4a_n/(a_n - 4)$

58. $a_1 = 4,\ a_{n+1} = 3a_n/(a_n - 3)$

Exercises 59–60 ***Calculator Evaluation*** *Evaluate.*

59. **(a)** $\binom{20}{17}$ **(b)** $\binom{16}{3} + \binom{16}{14}$

60. **(a)** $\binom{30}{28} \cdot \binom{30}{3}$ **(b)** $\binom{17}{5} \div \binom{17}{15}$

9 SYSTEMS OF EQUATIONS AND INEQUALITIES

A CONSISTENT THEME THROUGHOUT THIS book is **solving equations.** So far, our equations have involved a single variable or unknown. In this chapter we explore methods for solving *systems* of equations, where the equations involve more than one variable.

When we look at a given problem, it is often natural to label more than one variable. Then we try to relate the variables. For example, suppose we know that the hypotenuse of a right triangle is 17, the perimeter is 40, and we want to find the lengths of the two legs. If x and y represent the lengths of the legs as shown in Figure 1, then equations relating x and y are

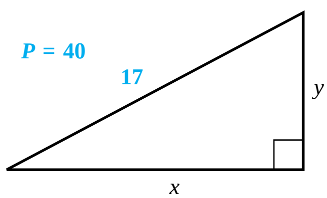
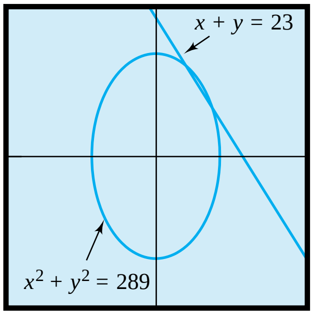


FIGURE 1

$$
\begin{aligned}
x + y + 17 &= 40 \quad \text{or} \quad x + y = 23 \\
x^2 + y^2 &= 17^2 \quad \text{or} \quad x^2 + y^2 = 289.
\end{aligned} \tag{1}
$$

The two equations in (1) form a **system of equations** in two variables. It is simple to verify that the solutions are given by $x = 8$, $y = 15$, or $x = 15$, $y = 8$. In either case, the lengths of the legs are 8 and 15.

How do we find the number pairs (8, 15) and (15, 8) that satisfy system (1)? This type of question will occupy much of this chapter. We first consider the role of technology and then, in the first section, develop some systematic tools to help us find solutions to systems of linear equations. Linear systems are the focus of the first two sections of the chapter. We turn to systems of nonlinear equations in Section 9.3. In Section 9.4 we consider systems of inequalities and the vital application of linear programming. In the last two sections we give a brief introduction to determinants and matrix algebra.



[−40, 40] by [−25, 25]

FIGURE 2

The Role of Technology

Perhaps one of the most important lessons we can learn in this course, where technology is always part of our concern, is twofold:

technology is an incredibly powerful tool, and
technology has unavoidable limitations.

We already have considerable experience in using graphs to find solutions. If we look at system (1) again, we recognize a line ($x + y = 23$) and a circle ($x^2 + y^2 = 289$), and the solution to system (1) must consist of the coordinates of any points of intersection of the line and circle. To see intersections, we need a fairly good-sized window, say $[-40, 40] \times [-25, 25]$, which shows us something like Figure 2. It appears that the line may meet the circle twice (although that isn't entirely clear in this window). Tracing and zooming several times only allows us to conclude that there are indeed two intersections, somewhere near (8, 15) and (15, 8). Unless we are unreasonably lucky, we are unlikely to find either set of coordinates exactly. If we use more sophisticated programs, such as the SOLVE routine of the TI-85 or the HP-48, we don't get reliably better information. Solving $23 - x = \sqrt{289 - x^2}$ with starting guesses near 8, our calculators return answers such as 8.00000000005, 7.9999999999998, 8, and 8.0000000000002. Which is closest, and how do we tell?

Returning to system (1), it is easy to get an exact solution. From the first equation, $y = 23 - x$. Substituting this expression into the second equation, we get a quadratic equation x, which factors:

$$\begin{aligned} x^2 + (23 - x)^2 &= 289, \\ 2x^2 - 46x + 240 &= 0, \\ 2(x - 8)(x - 15) &= 0. \end{aligned}$$

Thus $x = 8$ or 15, and the corresponding y-values are 15 and 8. Here is an example where algebraic methods easily give us exact solutions.

While it is easy to be enthusiastic about the superiority of exact answers, we must also keep in mind that exact answers aren't always easy to get or even meaningful. Suppose the problem leading to system (1) had a perimeter of **(a)** 42 or **(b)** 38. The corresponding systems are

$$\textbf{(a)} \begin{cases} x + y = 25 \\ x^2 + y^2 = 289 \end{cases} \qquad \textbf{(b)} \begin{cases} x + y = 21 \\ x^2 + y^2 = 289 \end{cases}$$

System (a) has no solution, and system (b) has two solutions given by $x = \dfrac{21 \pm \sqrt{137}}{2}$ (see Exercises 49–50). Graphs show that for system **(a)** the line does not meet the circle. For system **(b)**, in most practical problems, we would almost certainly be happy with decimal approximations for x and y.

What do we conclude? Both algebra and technology are needed. Neither is as powerful without the other. Algebraic methods give meaning to, and an understanding of limitations of, answers provided by technology; we cannot intelligently make use of technology without a thorough understanding of the concepts that underlie the technology. And technology allows us to go beyond algebra to get approximate answers that may be entirely beyond the capacity of analytic methods. The interplay of these two phases of our learning is nowhere better illustrated, nor more important, than in systems of equations and inequalities.

TECHNOLOGY TIP ◆ ***Limitations and power***

Always remember, when using technology to solve systems of equations or inequalities, because of inherent, unavoidable complexities,

technology will sometimes give misleading, incorrect answers.

Nonetheless,

technology will enable us to get answers that would be impossible to obtain in any other way.

9.1 SYSTEMS OF LINEAR EQUATIONS; GAUSSIAN ELIMINATION

Mathematics is effective precisely because a relatively compact mathematical scheme can be used to predict over a relatively long period of time the future behavior of some physical system to a certain level of accuracy, and thereby generate more information about the system than is contained in the mathematical scheme to begin with.

P. W. C. Davies

I also engaged in wild mathematical discussions, formulating vast and new projects, new problems, theories and methods bordering on the fantastic. . . .

Stan Ulam

Our focus in this section is **linear equations** in several variables, such as

$$3x - 4y + 2z + w = 5 \quad \text{and} \quad -3s + 2t = 1.$$

The following equations are not linear:

$$x^2 - y = 4 \qquad \text{Not linear in } x$$
$$x + 3|y| - z = 7 \qquad \text{Not linear in } y$$
$$uv + \ln w = 0 \qquad \text{Not linear in } u, v, \text{ or } w$$

For a *system* of linear equations, we indicate both the number of equations and the number of variables. A 2×2 system consists of two equations in two variables, and a 3×3 system has three equations in three variables:

$$\begin{cases} -3x + 4y = 11 \\ 2x - 3y = -8 \end{cases} \tag{2}$$

$$\begin{cases} 2a - 5b + 3c = 8 \\ a + 5b - c = 4 \\ 3a \qquad + 2c = 12 \end{cases} \tag{3}$$

A **solution** to a system of linear equations consists of a value for each variable such that when we substitute these values, every equation becomes a true statement. For system (2) above, the values $x = -1$, and $y = 2$ satisfy both equations in the system. A solution to system (3) can be written $(a, b, c) = (6, -1, -3)$, which means that $a = 6$, $b = -1$, and $c = -3$. The ordered pair of numbers $(-1, 2)$ is the only solution to system (2), but $(8, -2, -6)$ is one of many solutions to system (3).

Equivalent Systems

We need a systematic procedure to find all solutions to a system of equations. There are several methods, some of which you may have seen in previous courses. We will describe a technique that replaces a system of equations in turn by other, simpler systems with the same solutions until we get a system simple enough that we can read off the solution. For example, consider these 3×3 systems:

$$\begin{cases} 2x - 5y + 3z = -4 \\ x - 2y - 3z = 3 \\ -3x + 4y + 2z = -4 \end{cases} \tag{4}$$

$$\begin{cases} 2x - 5y + 3z = -4 \\ y - 9z = 10 \\ -z = 1 \end{cases} \tag{5}$$

It is simple to solve system (5) by starting with the last equation to get $z = -1$. Substitute into the second equation and find $y = 1$, and then substitute both y and z values into the first equation to get $x = 2$. In fact, it is easy to see that $(x, y, z) = (2, 1, -1)$ is the only solution for system (5). In Example 1 we will show that the two systems have the same solution, and hence that our solution for system (5) is the solution for system (4). Two systems of linear equations are **equivalent** if they have identical solutions.

In the process of going from system (4) to system (5), we successively eliminate variables. So x has been eliminated from the second equation in system (5), and both x and y have been eliminated in the third equation. System (5) is called an **echelon,** or **upper triangular,** form of system (4).

Definition: echelon (upper triangular) form

A system of three linear equations in variables x, y, z is said to be in echelon form if it can be written as

$$\begin{aligned} a_1x + a_2y + a_3z &= d_1 \\ b_2y + b_3z &= d_2 \\ c_3z &= d_3 \end{aligned}$$

where the coefficients a, b, c, and d are given numbers, some of which may be zero.

Elementary Operations and Gaussian Elimination

The systematic elimination of variables to change a system of linear equations into an equivalent system in echelon form from which we can read the solution is called **Gaussian elimination** in honor of Carl Friedrich Gauss, one of the most brilliant mathematicians of all time.

The key to Gaussian elimination (which can be done efficiently on computers) is the idea of **elementary operation,** the replacement of one equation in a system by another in a way that leaves the solution unchanged. Each of the following operations gives an equivalent system, that has the same solution set. E_k denotes the kth equation of the system and $-2E_1 + E_2$ is what we get when we multiply both sides of equation E_1 by -2 and add the result to equation E_2.

Elementary operations and equivalent systems

Operation	*Notation and Meaning*
1. Interchange two equations	$E_2 \leftrightarrow E_3$ means interchange equations E_2 and E_3.
2. Multiply by a nonzero constant	$4E_3 \to E_3$ means replace equation E_3 with $4E_3$.
3. Add a multiple of one equation to another equation	$4E_2 + E_3 \to E_3$ means replace E_3 with $4E_2 + E_3$.

Performing any of the elementary operations on a system of linear equations gives an equivalent system.

Follow the next example closely, performing each operation as indicated, to be certain that you understand both the process by which we reduce the original system to echelon form and the notation by which we keep track of and check each step.

▶EXAMPLE 1 ***Echelon form*** Reduce the following system to echelon form and then find the solution.

$$\begin{array}{lr} E_1 & 2x - 5y + 3z = -4 \\ E_2 & x - 2y - 3z = 3 \\ E_3 & -3x + 4y + 2z = -4 \end{array}$$

Solution

Strategy: Since the coefficient of x in E_2 is 1, first interchange E_1 and E_2, then eliminate x from the other two equations without involving fractions.

Follow the strategy. We will not repeatedly write the equation numbers, simply assuming in each system that the equations are numbered E_1, E_2, and E_3, from top to bottom. Beginning with the given system, we perform elementary operations as indicated:

$$E_1 \leftrightarrow E_2 \quad \begin{cases} x - 2y - 3z = 3 \\ 2x - 5y + 3z = -4 \\ -3x + 4y + 2z = -4 \end{cases}$$

$$\begin{array}{r} -2E_1 + E_2 \to E_2 \\ 3E_1 + E_3 \to E_3 \end{array} \quad \begin{cases} x - 2y - 3z = 3 \\ -y + 9z = -10 \\ -2y - 7z = 5 \end{cases}$$

$$(-2)E_2 + E_3 \to E_3 \quad \begin{cases} x - 2y - 3z = 3 \\ -y + 9z = -10 \\ -25z = 25 \end{cases}$$

We now have a system in echelon form that is equivalent to the given system.

To solve the echelon-form system, start with the last equation and solve for z: $z = \frac{25}{-25} = -1$. Substitute -1 for z into E_2 and solve for y: $-y + 9(-1) = -10$, or $y = 1$. Substitute -1 for z and 1 for y into E_1 and solve for x: $x - 2(1) - 3(-1) = 3$, or $x = 2$. The solution is given by $x = 2$, $y = 1$, $z = -1$. ◀

The process of solving a system of equations in echelon form has the name **back-substitution.** This suggests the procedure of starting at the bottom and working toward the top, substituting into each successive equation.

▶EXAMPLE 2 ***Eliminate x*** Use elementary operations to get an equivalent system, eliminating the x-variable from E_2 and E_3.

$$\begin{aligned} 2x - 3y + z &= -1 \\ -3x + 4y - z &= 2 \\ 2x - y + 2z &= -3 \end{aligned}$$

Strategy: We can easily use E_1 to eliminate x in E_3, but to avoid fractions for E_2, first multiply E_2 by 2, then add $3E_1$ to eliminate x.

Solution

Carry out the elementary operations suggested in the strategy:

$$(-1)E_1 + E_3 \to E_3 \quad \begin{cases} 2x - 3y + z = -1 \\ -3x + 4y - z = 2 \\ 2y + z = -2 \end{cases}$$

$$2E_2 \to E_2 \quad \begin{cases} 2x - 3y + z = -1 \\ -6x + 8y - 2z = 4 \\ 2y + z = -2 \end{cases}$$

$$3E_1 + E_2 \to E_2 \quad \begin{cases} 2x - 3y + z = -1 \\ -y + z = 1 \\ 2y + z = -2 \end{cases}$$

Complete the solution and verify that $z = 0$, $y = -1$, and $x = -2$. ◀

▶EXAMPLE 3 ***Gaussian elimination*** Solve the system by using Gaussian elimination.

(a) $\begin{cases} x + 2y - 2z = 3 \\ 2x + 3y - 3z = 1 \\ -4x - 5y + 5z = 3 \end{cases}$

(b) $\begin{cases} x + 2y - 2z = 3 \\ 2x + 3y - 3z = 1 \\ -4x - 5y + 5z = 5 \end{cases}$

Solution

(a) The following elementary operations lead to an echelon form, from which we find x, y, and z.

$$(-2)E_1 + E_2 \to E_2 \quad \begin{cases} x + 2y - 2z = 3 \\ -y + z = -5 \\ -4x - 5y + 5z = 3 \end{cases}$$

$$4E_1 + E_3 \to E_3 \quad \begin{cases} x + 2y - 2z = 3 \\ -y + z = -5 \\ 3y - 3z = 15 \end{cases}$$

$$3E_2 + E_3 \to E_3 \quad \begin{cases} x + 2y - 2z = -3 \\ -y + z = -5 \\ 0 \cdot z = 0 \end{cases}$$

We now have an echelon form system in which E_3, $0 \cdot z = 0$, is satisfied by any number z. Therefore, we have infinitely many solutions. Let $z = t$, where t is any number. E_2 implies $y = z + 5 = t + 5$. Finally, we get x from E_1.

$$x = 3 - 2y + 2z = 3 - 2(t + 5) + 2t = 3 - 2t - 10 + 2t = -7.$$

Infinitely many solutions are given by

$$x = -7,\ y = t + 5,\ z = t,$$

where t is any number. For instance,

$$t = 0 \text{ gives } x = -7,\ y = 5,\ z = 0$$
$$t = -3 \text{ gives } x = -7,\ y = 2,\ z = -3.$$

(b) Note that the system of equations given here is the same as that in part **(a)** except for the right side of E_3. The same elementary operations performed in the solution to Example 3a yield the following echelon form for the system.

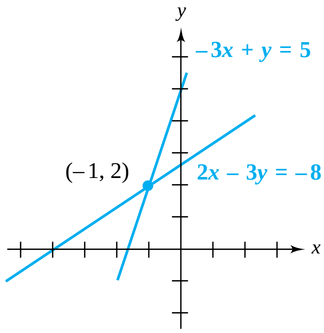
$$\begin{aligned} x + 2y - 2z &= 3 \\ -y + z &= -5 \\ 0 \cdot z &= 2 \end{aligned}$$

Since no number z satisfies the equation $0 \cdot z = 2$, the system has no solution. ◀

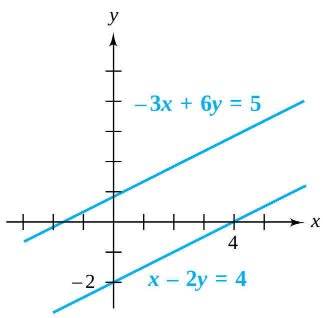
A system of linear equations that has infinitely many solutions is said to be **dependent,** while a system with no solutions is called **inconsistent.** The system in Example 3a is dependent and that in 3b is inconsistent. Another advantage of echelon form is that the last equation tells us the nature of the solutions, which must be one of the following possibilities.

Nature of solutions for a system of linear equations

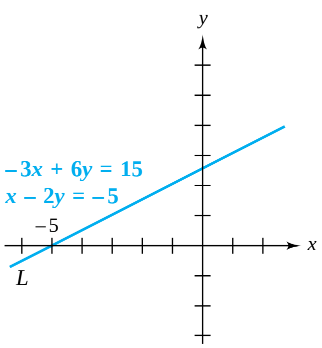
1. There is exactly one solution; the solution is unique.
2. There are no solutions; the system is inconsistent.
3. There are infinitely many solutions; the system is dependent.



(a) Unique solution



(b) Inconsistent system



(c) Dependent system

FIGURE 3

The next example illustrates the three possibilities for 2×2 systems. It shows geometrically a unique solution, a dependent system, and an inconsistent system.

▶***EXAMPLE 4*** ***Solutions and graphs*** Graph the pair of equations on the same coordinate system, then solve the system.

(a) $\begin{cases} -3x + y = 5 \\ 2x - 3y = -8 \end{cases}$

(b) $\begin{cases} -3x + 6y = 5 \\ x - 2y = 4 \end{cases}$

(c) $\begin{cases} -3x + 6y = 15 \\ x - 2y = -5 \end{cases}$

Solution

The graphs are shown in Figure 3. Use Gaussian elimination to verify the following solutions.

(a) Unique solution; $x = -1$, $y = 2$. The two lines intersect at $(-1, 2)$.
(b) No solution; the system is inconsistent. The two lines are parallel; they have no intersection.
(c) Infinitely many solutions; the system is dependent. Both equations determine the same line; every point of the line satisfies both equations. ◀

A system of any number of linear equations must have either a unique solution, no solution, or be dependent, just as the 2×2 systems in Example 4. Unfortunately, we cannot see the geometry as easily with larger systems as we can with 2×2 systems. In the next example we illustrate how linear systems occur in applications.

▶EXAMPLE 5 ***Mixture problem*** Dessert consists of chocolate pudding and whipped cream. We are interested in the energy (calories) and vitamin A content. The necessary information in the table is taken from a handbook on nutrition.

Food	*Energy (calories)*	*Vitamin A (units)*
Pudding (1 cup)	385	390
Cream (1 tablespoon)	26	220

How much pudding (in cups) and cream (in tablespoons) will give a dessert with 283 calories and 674 units of vitamin A?

Strategy: To find the numbers of cups and tablespoons, assign variables and write equations for the number of calories (= 283) and units of vitamin A (= 674).

Solution

Follow the strategy. Let x be the number of cups of pudding and y be the number of tablespoons of cream.

Since each cup of pudding contains 385 calories (see the table), x cups must contain $385x$ calories. Similarly, y tablespoons of cream contain $26y$ calories. Set the sum of these two equal to 283 calories: $385x + 26y = 283$. In a similar manner, to get 674 units of vitamin A, $390x + 220y = 674$. Therefore, solve the following system of equations.

$$\begin{aligned} E_1&: \quad 385x + 26y = 283 \qquad \text{Calories} \\ E_2&: \quad 390x + 220y = 674 \qquad \text{Vitamin A} \end{aligned}$$

To eliminate x from E_2, first multiply E_1 by 390 ($390E_1 \rightarrow E_1$) and E_2 by -385 ($-385E_2 \rightarrow E_2$). Then add the resulting equations ($E_1 + E_2 \rightarrow E_2$). This gives for the last equation

$$-74{,}560y = -149{,}120 \quad \text{or} \quad y = 2. \qquad \text{Check!}$$

Substitute 2 for y in one of the original equations to get $x = 0.6$. Hence $\frac{3}{5}$ cup of pudding with 2 tablespoons of cream will give the desired proportions of calories and vitamin A. ◀

Technology Support for 2 × 2 Systems (Cramer's Rule)

In one sense, all 2×2 systems of linear equations are the same; all can be solved with exactly the same steps. The results can be summarized in a form that lends itself to convenient implementation on a graphing calculator. A 2×2 system can be written in the form

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

We can solve the system by eliminating either x or y. To eliminate x, replace E_2 by $aE_2 - cE_1$, getting $(ad - bc)y = af - ce$. If we choose to eliminate y, we replace E_1 by $dE_1 - bE_2$, getting $(ad - bc)x = de - bf$. In both cases the coefficient of the variable is identical, $ad - bc$, and *the system has a solution if* $ad - bc \neq 0$. If $ad - bc = 0$, then we do not use Cramer's Rule; the system is either dependent or inconsistent.

Furthermore, when $ad - bc$ is nonzero, we can write down the solution:

$$x = \frac{de - bf}{ad - bc}, \quad y = \frac{af - ce}{ad - bc}. \tag{6}$$

A simple way to remember the form of this solution comes from determinants, which we will introduce more formally in Section 9.5. At this point, however, since we have solved the system, we only want a convenient way to keep the result in mind.

The denominator and both numerators have the same form in solution (6). Each can be written as a number associated with a 2 by 2 array, called a *determinant*. The denominator is called the *coefficient determinant of the system:*

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc; \text{ the product } \begin{matrix} a & \\ & \searrow \\ & d \end{matrix} \text{ minus the product } \begin{matrix} & b \\ \swarrow & \\ c & \end{matrix}.$$

With this notation, the numerator for each variable is also a determinant, where we replace the column of coefficients of each variable in D by the column of constants on the right side:

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{D} = \frac{ed - bf}{D}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{D} = \frac{af - ec}{D}.$$

The entire process is known as Cramer's Rule for 2 by 2 linear systems.

Cramer's rule for 2 by 2 linear systems

Given a system of two linear equations of the form

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

there is a solution if and only if the number $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ is nonzero, in which case the solution is given by

$$x = \frac{ed - bf}{D}, \quad y = \frac{af - ec}{D},$$

where the numerator in each case is the determinant obtained from D by replacing the coefficients of the variable by the column of constants.

We illustrate in the next example by using Cramer's Rule for two systems we have already solved.

▶EXAMPLE 6 ***Cramer's rule*** Use Cramer's Rule for the systems

(a) $\begin{cases} -3x + 6y = 5 \\ x - 2y = 4 \end{cases}$ **(b)** $\begin{cases} 385x + 26y = 283 \\ 390x + 220y = 674 \end{cases}$

Solution

(a) We begin by computing the coefficient determinant D:

$$D = \begin{vmatrix} -3 & 6 \\ 1 & -2 \end{vmatrix} = (-3)(-2) - (6)(1) = 0.$$

Since $D = 0$, Cramer's Rule does not apply. The system has no solution as we already saw in Example 4b.

(b) With a calculator we don't even have to write anything down, simply reading the values for a, b, c, d from the system. We start with the determinant: $D = 385 \cdot 220 - 26 \cdot 390$, and store 74560, say in memory D. For x, we replace the column $\begin{bmatrix} 385 \\ 390 \end{bmatrix}$ by the constant column, $\begin{bmatrix} 283 \\ 674 \end{bmatrix}$, so $x = (283 \cdot 220 - 26 \cdot 674)/D = 0.6$. Similarly, replacing the y-coefficients by the constant column, $y = (385 \cdot 674 - 283 \cdot 390)/D = 2$. The solution, as we found in Example 5, is given by $x = 0.6$, $y = 2$. ◀

EXERCISES 9.1

Check Your Understanding

Exercises 1–7 True or False. Give reasons.

1. The equation $3x - \sqrt{2}y = 5$ is linear in x and y.
2. The equation $3\sqrt{x^2} + 4y = 7$ is linear in x and y.
3. The graphs of $2x - 3y = 3$ and $x + y = 3$ intersect in the first quadrant.
4. Both $(0, 0, 0)$ and $(-3, 2, 1)$ are solutions to the system
$$\begin{aligned} x + y + z &= 0 \\ y - 2z &= 0 \\ x - 2y - z &= 0 \end{aligned}$$
5. The solution to the system
$$\begin{aligned} 2x + y &= 5 \\ x + 3y &= -4 \end{aligned}$$
consists of a pair of positive integers.
6. The system
$$\begin{aligned} 2x + y &= 0 \\ x - 3y &= 5 \end{aligned}$$
is dependent.
7. In the solution to the following system, x and y are negative and z is positive.
$$\begin{aligned} x + y - z &= 4 \\ y + 2z &= 0 \\ 3x + y &= 5 \end{aligned}$$

Exercises 8–10 Fill in the blank so that the resulting statement is true. Lines L_1, L_2, and L_3 are given by L_1: $x - 3y = 0$, L_2: $x + 3y = 6$, L_3: $x - 9y = 6$.

8. Lines L_1 and L_2 intersect at ______.
9. Lines L_1 and L_3 intersect at ______.
10. Lines L_3 and L_2 intersect at ______.

Develop Mastery

Exercises 1–6 ***Pairs of Lines*** *Solve the system of equations and graph the pair of lines on the same system of coordinates. (See Example 4.)*

1. $\begin{aligned} x + y &= 4 \\ 3x - 2y &= -3 \end{aligned}$
2. $\begin{aligned} 3x + y &= -5 \\ -x + 2y &= 4 \end{aligned}$
3. $\begin{aligned} 3x + 4y &= -1 \\ -3x + 5y &= -2 \end{aligned}$
4. $\begin{aligned} 3x - 2y &= 4 \\ -5x + 2y &= 8 \end{aligned}$
5. $\begin{aligned} 4x - 2y &= 3 \\ -2x + y &= 5 \end{aligned}$
6. $\begin{aligned} 2x + 4y &= 3 \\ x + 2y &= 1.5 \end{aligned}$

Exercises 7–36 ***Linear Systems*** *Solve the system of equations.*

7. $\begin{aligned} 2x - y + z &= 6 \\ 3y + 2z &= 3 \\ -z &= 3 \end{aligned}$
8. $\begin{aligned} x + 3y - z &= 4 \\ 2y - 3z &= 8 \\ 3z &= -6 \end{aligned}$
9. $\begin{aligned} x + y + z &= 1 \\ 2x - y - z &= 5 \\ -x + 2y - 3z &= -4 \end{aligned}$
10. $\begin{aligned} 2x - 3y + z &= 6 \\ x + 2y + 2z &= -5 \\ -3x - y - z &= 6 \end{aligned}$

11. $2x - y + 3z = 1$
$x + y - 5z = 2$
$3x - 2z = 3$

12. $x + 3y - z = 1$
$-2x + y + 3z = 0$
$-4x + 9y + 7z = 3$

13. $3.1x - 2.5y = 13.7$
$1.7x + 2.4y = -3.8$

14. $\frac{3}{4}x + \frac{1}{3}y = \frac{1}{12}$
$2x - y = 5$

15. $371x + 258y = 2710$
$137x + 125y = 971$

16. $325x - 175y = -625$
$173x - 276y = 33$

17. $2x - y = -7$
$3x - 4z = -1$
$3x + y - 4z = 0$

18. $x - y = -4$
$x + z = 1$
$3x + y + 2z = 4$

19. $x + y + z = 0$
$x + 2y + z = 0$
$2x + 3y + 2z = 0$

20. $2x + y + z = 0$
$3x + 2y + 4z = 0$
$x - 2y - 3z = 0$

21. $x - y + 2z = 4$
$2x + 3y - z = 5$
$3x + 2y + z = 8$

22. $x + 5y + 3z = -3$
$4x + 3y + 2z = 2$
$3x + y + z = 3$

23. $x + 3y + z = 0$
$-2x + y = -4$
$8x + 3y + 2z = 12$

24. $x - 3y - 3z = -5$
$5x - 7y - 3z = 15$
$4x - 4y - 3z = 8$

25. $2x - 4y + 3z = 0$
$x - y - 2z = -6$
$6x - 4y + z = -8$

26. $5x + 6y + 3z = -1$
$x + 4y - 2z = 8$
$x + 3y + 2z = 2$

27. $-x + 6y + 2z = 1$
$2x - 7y + z = 13$
$-5x + 7y + 3z = 0$

28. $-x - 4y + z = -14$
$3x - y + 3z = 2$
$-x + 4y - z = 12$

29. $x - 2y + 2z = -3$
$x - 2y + 7z = -13$
$3x - 2y + 7z = -3$

30. $x + 3y + 2z = 0$
$6x + 3y + 2z = 10$
$3x + y + 3z = 17$

31. $x + y - 2z = 9$
$2x - y = 0$
$3x + z = 0$

32. $2x - y + z = 4$
$x - y = 0$
$2x + z = 0$

33. $6x - 4y + z = -24$
$7x - 4y + z = -26$
$6x - 3y + z = -20$

34. $x - 2y = 0$
$-3x - 4y + z = 0$
$2y + z = 0$

35. $2x - 3y + z = 11$
$3x - y + 2z = 10$
$5x + 4y - z = 1$

36. $-2x + y - 3z = 14$
$3x - 2y - z = -5$
$2x + 2y - 3z = 7$

Exercises 37–42 ***Cramer's Rule*** *Use Cramer's Rule to solve the system. Then find a window in which you can see the intersection of the graphs.*

37. $15x + 37y = 19$
$17x + 14y = 245$

38. $192x - 135y = 2709$
$64x + 83y = 519$

39. $72x + 43y = 141$
$129x - 22y = -1233$

40. $429x - 362y = -5285$
$611x + 243y = -1306.8$

41. $17x + 43y = -118$
$12x - 28y = -200$

42. $42x - 36y = -113.4$
$61x - 24y = 72.9$

Exercises 43–46 ***Substitution*** *Solve for x and y. (Hint: First let $\frac{1}{x} = u$ and $\frac{1}{y} = v$.)*

43. $\frac{1}{x} + \frac{1}{y} = 4$
$\frac{3}{x} - \frac{2}{y} = -3$

44. $\frac{3}{x} + \frac{1}{y} = -5$
$\frac{1}{x} - \frac{2}{y} = -4$

45. $\frac{3}{x} - \frac{2}{y} = 4$
$\frac{-5}{x} + \frac{2}{y} = 8$

46. $\frac{1}{x} - \frac{3}{y} = 0$
$\frac{4}{x} + \frac{1}{y} = 6$

47. Find the point of intersection of the two lines given by $2x - 3y = 4$ and $3x + y = -5$.

48. Find the point of intersection of the two lines given by $y = 2x - 5$ and $2y = 3x - 8$.

Exercises 49–50 ***Nonlinear Systems*** *Follow the procedure in the introductory section to solve the system; then draw graphs of both equations on the same screen.*

49. $x + y = 25$
$x^2 + y^2 = 289$

50. $x + y = 21$
$x^2 + y^2 = 289$

Exercises 51–54 ***Perimeter and Area*** *One vertex of a triangle is the point of intersection of lines L_1 and L_2, and the other two vertices are the x-intercept points of L_1 and L_2. Find **(a)** the perimeter of the triangle and **(b)** the area of the triangular region.*

51. L_1: $x + y = 6$
L_2: $x - 3y = -2$

52. L_1: $x + 2y = 4$
L_2: $3x - y = -9$

53. L_1: $y = -0.5x + 2.5$
L_2: $y = -3x$

54. L_1: $y = x - 2$
L_2: $y = 0.5x + 0.5$

Exercises 55–60 ***Systems*** *Solve the system of equations.*

55. $\frac{xy}{x + y} = 3, \frac{xz}{x + z} = 4, \frac{yz}{z + y} = 6$
$\left(\textit{Hint: } \text{If } \frac{xy}{x + y} = 3, \text{ then } \frac{x + y}{xy} = \frac{1}{y} + \frac{1}{x} = \frac{1}{3}.\right)$

56. $\log(xyz) = 2, \log\left(\frac{xy}{z}\right) = 0, \log\left(\frac{yz}{x}\right) = 0$
(*Hint:* $\log(xyz) = \log x + \log y + \log z$.)

57. $\ln(xyz) = 0.5, \ln(x^2y) = 1, \ln\left(\frac{yz}{x}\right) = -1.5$
(*Hint:* See Exercise 56.)

58. $2^{2x+2y} = 4^z \quad 4 \cdot 2^{x-y} = 8^z \quad 32 \cdot 2^{y+z} = 4^x$
(*Hint:* Use properties of exponents.)

59. $4^x = 8 \cdot 2^{x+2y}$
$9^{x-6y} = 9 \cdot 3^{-4y}$

60. $\log(2x - y) + \log 5 = 1$
$\log x - \log y = 0$

61. ***Triangle*** Suppose lines L_1, L_2, L_3 are given by the equations:

$$L_1: -x + 2y = 1$$
$$L_2: x + 2y = 3 \qquad L_3: 3x + 2y = 13.$$

(a) Draw a graph to show lines L_1, L_2, and L_3.
(b) Find the points of intersection for each pair of the three lines.
(c) For the triangle formed by the three lines in **(a)**, find the largest angle to the nearest degree.

62. ***Rectangle*** The area of a rectangle remains unchanged if its width is increased by 2 and its length is decreased by 2, or if its width is decreased by 2 and its length is increased by 3. What is the perimeter of the rectangle?

63. ***Rectangle*** The perimeter of a rectangle is 24 cm. If its length is 2 cm greater than its width, what is the area of the rectangular region?

64. ***Gardening*** A gardener wants to buy two kinds of flowers to plant a border. Ajugas are \$1.10 each, and Lilliput Zinnias are \$0.85 each. The gardener wants to spend exactly \$200 to purchase exactly 200 plants. Can some combination of ajugas and zinnias meet this need? If so, how many of each should be bought?

65. ***Investing*** A total of \$2500 is invested at simple interest in two accounts. The first pays 8 percent interest and the second pays 10 percent interest per year. The total interest earned from the two accounts after one year is \$234. How much is invested in each account?

66. ***Mixture Problem*** A mixture of 36 pounds of peanuts and cashews costs a total of \$33. If peanuts cost \$0.80 per pound and cashews cost \$1.10 per pound, how many pounds of each does the mixture contain?

67. ***Two Numbers*** The sum of two numbers is 63 and the first is twice the second. What is the product of the two numbers?

68. ***Fencing*** A rectangular lot has a length-to-width ratio of 4 to 3. If 168 meters of fence will enclose it, what are the dimensions of the lot?

69. ***Mixture Problem*** Suppose x grams of food A and y grams of food B are mixed and the total weight is 2000 grams. Food A contains 0.25 units of vitamin D per gram, and food B contains 0.50 units of vitamin D per gram. Suppose the final mixture contains 900 units of vitamin D. How many grams of each type of food does the mixture contain?

70. ***Filling a Tank*** Two pipelines A and B are used to fill a tank with water. The tank can be filled by running A for three hours and B for six hours, or it can be filled by having both of the supply lines open for four hours. How long would it take for A to fill the tank alone? How long would it take for B to fill the tank alone? (*Hint:* If x is the number of hours it takes A to fill the tank alone, then in one hour, A will fill $\frac{1}{x}$ of the total capacity of the tank.)

71. ***Airspeed*** When flying with the wind, it takes a plane 1 hour and 15 minutes to travel 600 kilometers; when flying against the wind it takes 1 hour 40 minutes to travel 600 kilometers. What is the airspeed of the plane and the speed of the wind?

72. ***Mixture Problem*** One cup of half-and-half cream contains 28 g of fat and 7 g of protein, while one cup of low-fat milk contains 5 g of fat and 8 g of protein. How many cups of half-and-half and how many cups of low-fat milk should be combined to get a mixture that contains 71 g of fat and 38 g of protein?

73. ***Finding Costs*** The cost of a sandwich, a drink, and a piece of pie is \$2.50. The sandwich costs a dollar more than the pie, and the pie costs twice as much as the drink. What is the cost of each?

74. ***Investing*** A total of \$3600 is invested in three different accounts. The first account earns interest at a rate of 8 percent, the second at 10 percent, and the third at 12 percent. The amount invested in the first account is twice as much as that in the second account. If the total amount of simple interest earned in one year is \$388, how much is invested in each account?

75. ***Mixture Problem*** Suppose x grams of food A, y grams of food B, and z grams of food C are mixed together for a total weight of 2400 grams. The vitamin D and calorie content of each food is given in the table.

Food	*Units of Vitamin D per Gram*	*Calories per Gram*
A	0.75	1.4
B	0.50	1.6
C	1.00	1.5

The 2400-gram mixture contains a total of 1725 units of vitamin D and 3690 calories. How many grams of each type of food does it contain?

76. ***Finding a Quadratic***

(a) Find an equation for the quadratic function whose graph passes through the three points $(-1, 8)$, $(0, 5)$, and $(1, -4)$. (*Hint:* Let the parabola have equation $y = Ax^2 + Bx + C$, substitute coordinates of the given points, and solve for A, B, and C.)
(b) What is the distance between the x-intercept points of the parabola?

77. ***Filling a Tank*** A large tank full of water has three outlet pipes, A, B, and C. If only A and B are opened, the tank empties in three hours. If only A and C are open, the tank drains in four hours. If only pipes B and C are open, the tank drains in six hours. How long does it take to empty the tank if all three pipes are open? (*Hint:* If outlet A can empty the tank in x hours, how much drains through A in one hour?)

9.2 SYSTEMS OF LINEAR EQUATIONS AS MATRICES

> If you have a thousand equations in a thousand unknowns, you know there exists a solution, but how do you compute it?
>
> *Garrett Birkhoff*

A close inspection of Gaussian elimination shows that the method combines application of elementary operations together with careful bookkeeping. One way to streamline the entire process is to note that it makes no difference what letters we use for the variables. For instance, if we used u, v, and w, in place of x, y, and z, the problem and technique for solving the system would be the same. However, if we altered any of the coefficients or constants in the equations, the problem would change. The coefficients of the variables and the numbers on the right side completely describe the system.

> *I applied the matrix tower idea to the Ramanujan series for pi. You are evaluating the series exactly. In other words if you evaluate a million terms of this series and that's worth say eight million digits of pi, what you actually have is the exact rational fraction which is the sum of those million terms, which is something massively larger than eight million digits. I must have a hundred million digits of stuff in there.*
>
> Bill Gosper

Therefore, we may consider a system of linear equations in terms of a rectangular array of numbers consisting of the coefficients and constants on the right side, arranged in the same order as they appear in the equations. Then Gaussian elimination becomes a matter of operating on rows of numbers. We refer to the rectangular array of numbers as a **matrix.**

To illustrate the notion of a matrix let us consider the system of equations given in Example 1 of the preceding section.

$$\begin{array}{ll} E_1 & 2x - 5y + 3z = -4 \\ E_2 & x - 2y - 3z = 3 \\ E_3 & -3x + 4y + 2z = -4 \end{array} \tag{1}$$

System (1) can be described by the matrix M:

$$M = \begin{array}{c} \begin{bmatrix} 2 & -5 & 3 & -4 \\ 1 & -2 & -3 & 3 \\ -3 & 4 & 2 & -4 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array} \tag{2}$$

Array (2) consists of three rows and four columns of numbers; it is a 3×4 matrix. We refer to the rows as R_1, R_2, and R_3.

Gaussian elimination can now be described as a process of elementary operations on rows of a matrix to obtain a sequence of matrices that correspond to equivalent systems of equations, until we get one in echelon form. This process is referred to as **row reduction to echelon form.**

When we apply any elementary row operation to a matrix we get an **equivalent matrix.** To describe row operations, we shall use notation analogous to that in the preceding section. For instance, $R_1 \leftrightarrow R_2$ means interchange rows one and two; $3R_1 + R_2 \rightarrow R_2$ means multiply R_1 by 3 and add to R_2 to get the new R_2.

Elementary Row Operations

The elementary operations on equations of a system of linear equations listed in the preceding section translate into corresponding elementary row operations on matrices as follows.

Elementary row operations

Operation	Notation and Meaning
1. Interchange two rows	$R_1 \leftrightarrow R_3$: interchange rows R_1 and R_3.
2. Multiply by a nonzero constant	$2R_2 \rightarrow R_2$: replace R_2 by $2R_2$.
3. Add a multiple of one row to another row.	$R_3 + 2R_2 \rightarrow R_3$: replace R_3 by $R_3 + 2R_2$; that is, add $2R_2$ to R_3.

We illustrate elementary row operations and related notation in the following example which uses matrices to solve the system given in Example 1 of the preceding section.

Strategy: Since the coefficient of x in E_2 is 1, first interchange R_1 and R_2 in order to avoid working with fractions.

▶EXAMPLE 1 ***Solution by matrices*** Express the system of equations in terms of a matrix, and then get an equivalent matrix in echelon form that can be used to get the solution.

$$\begin{aligned} 2x - 5y + 3z &= -4 \\ x - 2y - 3z &= 3 \\ -3x + 4y + 2z &= -4 \end{aligned}$$

Solution

The matrix M that corresponds to the system of equations is

$$M = \begin{bmatrix} 2 & -5 & 3 & -4 \\ 1 & -2 & -3 & 3 \\ -3 & 4 & 2 & -4 \end{bmatrix}$$

The following steps are analogous to those in the solution of Example 1 in Section 9.1.

$$\begin{bmatrix} 2 & -5 & 3 & -4 \\ 1 & -2 & -3 & 3 \\ -3 & 4 & 2 & -4 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & -2 & -3 & 3 \\ 2 & -5 & 3 & -4 \\ -3 & 4 & 2 & -4 \end{bmatrix} \quad \begin{matrix} -2R_1 + R_2 \rightarrow R_2 \\ 3R_1 + R_3 \rightarrow R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & -3 & 3 \\ 0 & -1 & 9 & -10 \\ 0 & -2 & -7 & 5 \end{bmatrix} \quad -2R_2 + R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & -2 & -3 & 3 \\ 0 & -1 & 9 & -10 \\ 0 & 0 & -25 & 25 \end{bmatrix}$$

The final matrix corresponds to a system of equations in echelon form. Use it for back-substitution to get the desired solution. The last row represents the equation $-25z = 25$ and so $z = -1$. Similarly from rows 2 and 1, y and x are given by $y = 1$, $x = 2$. ◀

It isn't always necessary to reduce a matrix to echelon form, even though it is often a good practice. The next example illustrates the use of a matrix to solve a system, always keeping in mind what each row represents.

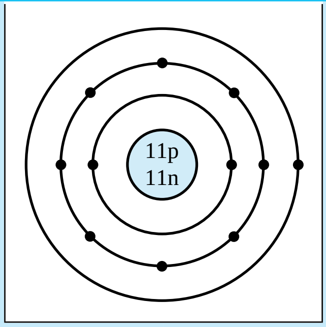
Strategy: A circle has an equation of the form $x^2 + y^2 + ax + by + c = 0$. Substitute coordinates of the given points into the equation and solve the resulting system.

▶EXAMPLE 2 ***Finding an equation*** Any three noncollinear points (not on any line) determine a unique circle. Find the center and radius of the circle containing the points $P(6, 8)$, $Q(7, 1)$, $R(-2, 4)$.

Solution

Follow the strategy. Substituting the coordinates of each point into the equation $x^2 + y^2 + ax + by + c = 0$, we get the following equations:

HISTORICAL NOTE MATRICES



The atomic structure of sodium

In this section we introduce a matrix as a single array that carries all the significant information about a system of linear equations. As one mathematician described the process, "strip the linear functions of every piece of clothing and there remain the matrices."

Matrices have been around in one form or another for a long time. Cauchy (France, 1789–1857) seems to have been the first to use such arrays, but the British mathematician Arthur Cayley was the first to study them systematically, considering sums and products of matrices. He announced to the Royal Society of London in 1858 a "peculiarity" of matrix multiplication: if A and B are square matrices of the same size, then the products AB and BA are also square, but, in general, *they are not equal;* $AB \neq BA$.

In 1925 Werner Heisenberg, trying to keep track of the characteristic states of orbiting electrons in the atomic nucleus, entered the data in square arrays. He then worked out ways to combine these arrays to describe subatomic interactions and developed the first successful quantum mechanics. Heisenberg appears to have been embarrassed by the discovery that his multiplication of arrays is not generally commutative ($AB \neq BA$). He mentions it casually in one sentence and immediately gives an example without the noncommutativity. Heisenberg's teacher and collaborator, Max Born, was apparently one of the few European physicists who knew anything about matrix analysis. He recognized Heisenberg's matrices for what they were and explored all kinds of applications of the new physics. Interestingly, both Heisenberg and Born were later awarded the Nobel prize in physics.

Today matrices are indispensable throughout mathematics and physics. Most calculus sequences are followed by a linear algebra course in which matrices are a powerful and indispensable tool for analysis of all kinds of linear systems.

$$\begin{aligned}
P(6, 8):&\quad 36 + 64 + 6a + 8b + c = 0, \text{ or } 6a + 8b + c = -100.\\
Q(7, 1):&\quad 49 + 1 + 7a + b + c = 0, \text{ or } \quad 7a + b + c = -50.\\
R(-2, 4):&\quad 4 + 16 - 2a + 4b + c = 0, \text{ or } -2a + 4b + c = -20.
\end{aligned}$$

The matrix representation for this system is $\begin{bmatrix} 6 & 8 & 1 & -100 \\ 7 & 1 & 1 & -50 \\ -2 & 4 & 1 & -20 \end{bmatrix}$. Rather than reducing this to echelon form, we will simply use the matrix to keep track of the operations as we eliminate variables. It is simple to eliminate two entries in the third column (coefficients of c):

$$\begin{matrix} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{matrix} \begin{bmatrix} 6 & 8 & 1 & -100 \\ 1 & -7 & 0 & 50 \\ -8 & -4 & 0 & 80 \end{bmatrix}. \quad \tfrac{1}{4}R_3 + 2R_2 \to R_3 \begin{bmatrix} 6 & 8 & 1 & -100 \\ 1 & -7 & 0 & 50 \\ 0 & -15 & 0 & 120 \end{bmatrix}.$$

The last row represents the equation $-15b = 120$, from which $b = -8$. Substituting -8 for b into the second equation, we have $a - 7(-8) = 50$ or $a = -6$. Finally, substituting both of these values into the first equation, we find that $c = 0$.

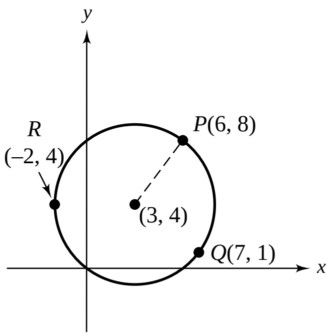


FIGURE 4

Therefore, an equation for the circle containing points P, Q, and R is $x^2 + y^2 - 6x - 8y = 0$.

To identify the center and radius of the circle, we complete the squares on the x- and y-terms:

$$(x^2 - 6x + 3^2) + (y^2 - 8y + 4^2) = 3^2 + 4^2, \text{ or}$$
$$(x - 3)^2 + (y - 4)^2 = 25.$$

The center of the circle is the point (3, 4) and the radius is 5. The circle is shown in Figure 4. ◀

Echelon form is especially useful when a system of linear equations is dependent (has infinitely many solutions) or inconsistent (no solutions).

▶EXAMPLE 3 ***Echelon form*** Solve the system of equations using matrix notation and row reduction to echelon form.

$$\begin{aligned} x + 3y - z &= 0 \\ -2x + y &= -4 \\ 8x + 3y - 2z &= 12 \end{aligned}$$

Solution

The matrix that corresponds to the system is

$$M = \begin{bmatrix} 1 & 3 & -1 & 0 \\ -2 & 1 & 0 & -4 \\ 8 & 3 & -2 & 12 \end{bmatrix}$$

Find a sequence of equivalent matrices:

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ -2 & 1 & 0 & -4 \\ 8 & 3 & -2 & 12 \end{bmatrix} \quad \begin{matrix} 2R_1 + R_2 \to R_2 \\ -8R_1 + R_3 \to R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 7 & -2 & -4 \\ 0 & -21 & 6 & 12 \end{bmatrix} \quad 3R_2 + R_3 \to R_3 \quad \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 7 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation that corresponds to R_3 of the final matrix is $0 \cdot x + 0 \cdot y + 0 \cdot z = 0$, which is satisfied by any numbers for x, y, and z. We want x, y, and z that satisfy the equations for R_1 and R_2. Thus the system reduces to

$$\begin{aligned} x + 3y - z &= 0 \\ 7y - 2z &= -4 \end{aligned}$$

If $z = k$ (any number), then from the second equation $y = \dfrac{2k - 4}{7}$. Substitute $\dfrac{2k - 4}{7}$ for y and k for z in the first equation and solve for x.

$$x + \frac{3(2k - 4)}{7} - k = 0 \quad \text{or} \quad x = \frac{k + 12}{7}$$

The system of equations is dependent and has infinitely many solutions given by

$$x = \frac{k + 12}{7} \qquad y = \frac{2k - 4}{7} \qquad z = k,$$

where k is any number. ◀

Partial Fractions

Elementary algebra courses devote considerable time to learning to add and subtract fractions to get a single fraction. Here we consider the problem of going in the opposite direction. Suppose we have a given rational expression in which the denominator can be expressed in factored form with linear or quadratic factors. What fractions could have been added or subtracted to get the given rational expression? The following examples illustrate a method to answer this question. The technique demonstrated here is called the **method of partial fractions.**

▶EXAMPLE 4 ***Partial fractions*** Express $\dfrac{6x^2 + 3x + 1}{x^3 - x}$ as a sum of fractions.

Strategy: To add fractions, we need a common denominator. To reverse the process, identify the factors that make up the denominator. Begin by factoring, $x^3 - x = x(x + 1)(x - 1)$. Look for fractions with denominators of x, $x + 1$, and $x - 1$ that can be added to get the given fraction.

Solution

Follow the strategy. The given fraction can be written as

$$\frac{6x^2 + 3x + 1}{x^3 - x} = \frac{6x^2 + 3x + 1}{x(x + 1)(x - 1)}.$$

It seems reasonable to expect that the given rational expression must have come from adding three fractions whose denominators are x, $x + 1$, and $x - 1$. Find three numbers a, b, and c (the unknowns) for which the following is an identity:

$$\frac{6x^2 + 3x + 1}{x(x + 1)(x - 1)} = \frac{a}{x} + \frac{b}{x + 1} + \frac{c}{x - 1} \tag{3}$$

Add the fractions on the right side and collect like terms in the numerator to get:

$$\frac{6x^2 + 3x + 1}{x(x + 1)(x - 1)} = \frac{(a + b + c)x^2 + (-b + c)x - a}{x(x + 1)(x - 1)}$$ Check this.

The two fractions in the last equation will be identically equal if we choose a, b, and c so that the corresponding coefficients in the numerators are the same; that is, if

$$\begin{aligned} a + b + c &= 6 \\ -b + c &= 3 \\ -a &= 1 \end{aligned}$$

We solve this system of equations and get $a = -1$, $b = 2$, and $c = 5$. Replacing a, b, and c by -1, 2, and 5, respectively, in Equation (3) gives the desired result.

$$\frac{6x^2 + 3x + 1}{x(x + 1)(x - 1)} = -\frac{1}{x} + \frac{2}{x + 1} + \frac{5}{x - 1}$$ ◀

In the next example we consider a denominator that contains a quadratic factor.

►EXAMPLE 5 ***Another partial fraction*** Express $\dfrac{3x^2 - 2x + 5}{x^3 - 1}$ as a sum of fractions with simpler denominators.

Strategy: Begin by factoring the denominator as a difference of cubes.

Solution

Follow the strategy. Note that $x^3 - 1 = (x - 1)(x^2 + x + 1)$. It is reasonable to assume that the given fraction can be written as a sum of fractions of the form

$$\frac{3x^2 - 2x + 5}{(x - 1)(x^2 + x + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}$$

Add the two fractions on the right side and collect like terms in the numerator to get (*Check!*):

$$\frac{3x^2 - 2x + 5}{(x - 1)(x^2 + x + 1)} = \frac{(A + B)x^2 + (A - B + C)x + (A - C)}{(x - 1)(x^2 + x + 1)}$$

Equating corresponding coefficients in the numerator gives a system of linear equations.

$$\begin{aligned} A + B &= 3 \\ A - B + C &= -2 \\ A - C &= 5 \end{aligned}$$

Solving the system, $A = 2$, $B = 1$, and $C = -3$. Therefore

$$\frac{3x^2 - 2x + 5}{(x - 1)(x^2 + x + 1)} = \frac{2}{x - 1} + \frac{x - 3}{x^2 + x + 1}$$ ◄

Up to this point we have not had any repeated linear factors in a denominator. Check this addition:

$$\frac{2}{x - 2} + \frac{3}{(x - 2)^2} = \frac{2x - 1}{(x - 2)^2}$$

This sets the pattern for the next example.

►EXAMPLE 6 ***Repeated factor*** Express $\dfrac{5x^2 - 8x + 2}{x^3 - 2x^2 + x}$ as a sum of fractions with simpler denominators.

Strategy: The denominator factors as $x(x - 1)^2$, so try x, $x - 1$, $(x - 1)^2$ as denominators.

Solution

Follow the strategy. The given fraction may be expressible as a sum of fractions as follows.

$$\frac{5x^2 - 8x + 2}{x(x - 1)^2} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}$$

Add the fractions on the right side and collect like terms in the numerator to get

$$\frac{5x^2 - 8x + 2}{x(x - 1)^2} = \frac{(A + B)x^2 + (-2A - B + C)x + A}{x(x - 1)^2}$$

Equating corresponding coefficients in the numerator gives us the following system of linear equations:

$$\begin{aligned} A + B &= 5 \\ -2A - B + C &= -8 \\ A &= 2 \end{aligned}$$

Solving the system, we find that $A = 2$, $B = 3$, and $C = -1$. Therefore

$$\frac{5x^2 - 8x + 2}{x(x-1)^2} = \frac{2}{x} + \frac{3}{x-1} - \frac{1}{(x-1)^2}. \blacktriangleleft$$

EXERCISES 9.2

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. The matrix for the system

$$\begin{aligned} 2x - y &= 5 \\ x + 2y &= 3 \end{aligned}$$

is

$$\begin{bmatrix} 2 & -1 & 5 \\ 1 & 2 & 3 \end{bmatrix}.$$

2. The system of linear equations that correspond to the matrix

$$\begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is

$$\begin{aligned} 3x - y &= 0 \\ y &= 0. \end{aligned}$$

3. For the system of equations in x and y that correspond to the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 6 \end{bmatrix},$$

the solution is given by $x = 1$, $y = 2$.

4. The systems of linear equations that correspond to the following matrices are equivalent.

$$\begin{bmatrix} 1 & -2 & -3 & 3 \\ 0 & 1 & -9 & 10 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

5. The triangle formed by the three lines $2x - 3y = 1$, $2x + 3y = 4$, and $3x + 2y = 3$ is a right triangle. (*Hint:* Consider the slopes of the lines.)

6. The triangle formed by the three lines $x + 2y = 3$, $2x - 2y = 5$, and $x - 2y = 4$ is a right triangle. (*Hint:* Consider the slopes of the lines.)

Exercises 7–10 Fill in the blank so that the resulting statement is true. Solve the system of equations that corresponds to the matrix.

7. $\begin{bmatrix} 1 & -1 & 3 \\ 0 & -1 & 2 \end{bmatrix}$; solution is ______.

8. $\begin{bmatrix} 2 & 0 & -4 \\ 1 & 2 & 6 \end{bmatrix}$; solution is ______.

9. $\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -3 & 6 \end{bmatrix}$; solution is ______.

10. $\begin{bmatrix} 1 & 0 & -1 & 4 \\ -1 & 0 & 0 & -3 \\ 0 & 2 & -1 & 5 \end{bmatrix}$; solution is ______.

Develop Mastery

Exercises 1–4 ***Matrix to System*** *For the given matrix, write the corresponding system of linear equations.*

1. $\begin{bmatrix} 1 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix}$

2. $\begin{bmatrix} -2 & 0 & 3 \\ 1 & -4 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 3 & 5 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 & 3 & 2 \\ -1 & -2 & 0 & 0 \\ 3 & 2 & 1 & -1 \end{bmatrix}$

Exercises 5–8 ***System to Matrix*** *Write the matrix that corresponds to the system of equations.*

5. $\begin{aligned} 2x - y &= 3 \\ x + 2y &= -1 \end{aligned}$

6. $\begin{aligned} -3x + 2y &= 1 \\ 5x - y &= -3 \end{aligned}$

7. $\begin{aligned} x + y - z &= 1 \\ 2x - y &= 3 \\ -x + 2y - z &= 0 \end{aligned}$

8. $\begin{aligned} 3x - y + z &= -4 \\ x + y &= 3 \\ y - z &= 5 \end{aligned}$

Exercises 9–12 ***Matrix Systems*** *Solve the system of linear equations given in matrix form. Use x, y, and z as the variables.*

9. $\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -1 & 3 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 3 & -6 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 0 & 2 & -4 \\ 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 2 & 2 & 5 \\ 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$

Exercises 13–24 ***Linear Systems*** *Solve the system of equations by expressing it in terms of a matrix, and then complete row reduction to achieve echelon form.*

13. $\begin{aligned} 3x + y &= -1 \\ x - y &= 3 \end{aligned}$

14. $\begin{aligned} x - 3y &= 5 \\ 3x + y &= 5 \end{aligned}$

15. $\begin{aligned} 0.4x - 0.5y &= 2.8 \\ -1.5x + 0.6y &= -5.4 \end{aligned}$

16. $\begin{aligned} 6x - 12y &= 7 \\ 4x - 8y &= -5 \end{aligned}$

17. $\begin{aligned} 4x - 8y &= -5 \\ -2x + 4y &= 2.5 \end{aligned}$

18. $\begin{aligned} \frac{x}{3} - \frac{y}{2} &= 4 \\ \frac{x}{2} - y &= 7 \end{aligned}$

19. $\begin{aligned} x + 2y + z &= 3 \\ -3x + 4z &= 5 \\ -3y + 2z &= 1 \end{aligned}$

20. $\begin{aligned} x + 2y + z &= 1 \\ -2x + y - 2z &= -2 \\ -x + 8y - z &= 2 \end{aligned}$

21. $\begin{aligned} x + y + 3z &= -1 \\ 3x - 4z &= -4 \\ -x + 2y + 2z &= 2 \end{aligned}$

22. $\begin{aligned} 3x + 4y - 4z &= -1 \\ 6x - 2y - 2z &= -2 \\ y - 3z &= -3 \end{aligned}$

23. $\begin{aligned} -x + y &= 2 \\ 3x + 4z &= 5 \\ 4x - y + 4z &= 3 \end{aligned}$

24. $\begin{aligned} 2x - y - 3z &= 1 \\ x + y + 5z &= 2 \\ 3x + 2z &= 3 \end{aligned}$

Exercises 25–28 ***Fitting Points*** *Find an equation for the geometric figure that contains the given points. (Hint: Assume an equation for a parabola of the form $y = ax^2 + bx + c$.)*

25. Circle; $P(0, 2)$, $Q(6, -1)$, $R(0, -7)$

26. Circle; $P(3, 2)$, $Q(0, -1)$, $R(-2, 7)$

27. Parabola; $P(2, 1)$, $Q(-1, -5)$, $R(-3, 7)$

28. Parabola; $P(-1, 0)$, $Q(1, 4)$, $R(2, 3)$

Exercises 29–36 ***Partial Fractions*** *Use the method of partial fractions to express the rational expression as a sum or difference of fractions with simpler denominations.*

29. $\dfrac{-8}{3x^2 - 4x - 4}$

30. $\dfrac{14x}{3x^2 + 5x - 2}$

31. $\dfrac{-10x - 4}{x^3 - 4x}$

32. $\dfrac{5x^2 + x}{2x^3 + x^2 - 2x - 1}$

33. $\dfrac{5x^2 - 3x + 2}{x^3 - x^2 + x - 1}$

34. $\dfrac{3x^2 - 3x + 5}{(x - 1)(x^2 + 2x + 2)}$

35. $\dfrac{x^2 + 5x - 12}{x^3 - 4x^2 + 4x}$

36. $\dfrac{x^2 - 6x - 13}{(x - 1)(x + 2)^2}$

37. ***Finding Numbers*** The average of three numbers is 8. The first is 3 greater than twice the second, and the third is the sum of the first two. What are the numbers?

38. ***Mixture Problem*** A grocery store sells two kinds of candy, *A* and *B*, each at a certain price per pound. When these are combined in a ratio of 3 to 1 (by weight) of *A* to *B*, then the price per pound of the mixture is \$1.10. However, if the corresponding ratio is 3 to 2, then the price per pound is \$1.04. If a ratio of 4 to 1 were made, what should be the price per pound of the resulting mixture?

39. ***Mixture Problem*** A breakfast menu is to consist of oatmeal, whole milk, and fresh orange juice. We are interested in the protein, calcium, and vitamin C content. The following table gives the pertinent information.

Food	*Protein (grams)*	*Calcium (milligrams)*	*Vitamin C (milligrams)*
Oatmeal (1 cup; 245 g)	5	22	0
Milk (1 cup; 244 g)	8	291	2
Orange juice (1 cup; 248 g)	2	27	124

How many cups of each (oatmeal, milk, and orange juice) are required to get a breakfast with 9 grams of protein, 185.7 milligrams of calcium, and 125 milligrams of vitamin C?

40. ***Mixture Problem*** A mixture of 50 pounds of peanuts, cashews, and walnuts costs a total of \$49. If peanuts cost \$0.80 per pound, cashews cost \$1.10 per pound, and walnuts cost \$1.20 per pound, and if the mixture contains twice as many pounds of peanuts as walnuts, how many pounds of each does the mixture contain?

9.3 SYSTEMS OF NONLINEAR EQUATIONS

In economics and psychology, linear least squares or constant input–output matrices are often used, and indeed, sometimes used automatically by means of canned programs, when fundamental force interactions are nonlinear. Unfortunately, linear models may be very poor approximations for nonlinear models.

Donald Greenspan

I would read about the history of the subject I was taking . . . so I had a more comprehensive view of the subject than my fellow students. Many times I would walk up to the exam with some classmate and say, "I will review you for the exam, I'll ask you some questions," and he would give me the answers he had studied. I'd go in, take the exam, and get 20 percent more than he did. He'd be so full of the subject, he couldn't see the woods for the trees.

I. I. Rabi

Gaussian elimination and matrix methods are well-suited for systems of linear equations. However, we often must deal with systems that include nonlinear equations. Such systems can sometimes be difficult to solve, even with the aid of technology. Several of the ideas we have used to solve systems of linear equations have applicability to nonlinear systems. In particular, we often use the **method of substitution,** which is a special case of **eliminating a variable.** If we can graph the equations, we can make use of technology as well.

▶EXAMPLE 1 ***A nonlinear system*** For the system $\begin{cases} x^2 + y^2 = 5 \\ x = y^2 - 3 \end{cases}$, **(a)** describe three different strategies for solving, and **(b)** show that two of your strategies in part **(a)** yield the same solution set.

Solution

(a) **(i)** Since the second equation is already solved for x, we can *substitute* $y^2 - 3$ for x in the first equation and solve the resulting equation for y.

(ii) Writing the system in the form $\begin{cases} x^2 + y^2 = 5 \\ x - y^2 = -3 \end{cases}$, we can *eliminate* y^2 by adding equations (replace E_2 by $E_2 + E_1$).

(iii) We can solve both equations for y and graph four equations: $y = \pm\sqrt{5 - x^2}$, $y = \pm\sqrt{x + 3}$. Then use graphical methods.

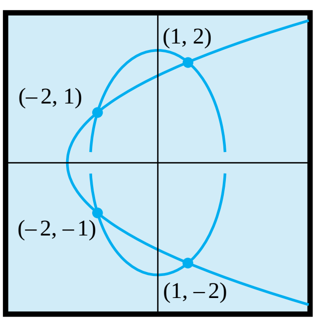
(b) **(i)** Substituting for x, the first equation becomes

$$(y^2 - 3)^2 + y^2 = 5, \quad \text{or} \quad y^4 - 5y^2 + 4 = 0.$$

Factoring, $(y^2 - 1)(y^2 - 4) = 0$, so $y = \pm 1$ or $y = \pm 2$. The corresponding x-values are -2 and 1. The solution set consists of the four points $\{(-2, 1), (-2, -1), (1, 2), (1, -2)\}$.

(ii) Adding equations gives $x^2 + x = 2$, with solutions $x = -2, 1$. Substituting each x-value into either of the original equations and solving for y gives the same four points as the solution set.

(iii) Graphing the four equations $y = \pm\sqrt{5 - x^2}$, $y = \pm\sqrt{x + 3}$ on the same screen gives something like Figure 5. In a decimal window we can read the coordinates of the same four points exactly, but in general we would have to settle for approximations. ◀



[−5, 5] by [−3, 3]

FIGURE 5

▶EXAMPLE 2 ***Solving with graphs*** Solve the system of equations and show the solutions graphically.

$$x + 2y = -4$$
$$y = x^2 - 2x - 3$$

Strategy: Eliminate either x or y by substituting from the first equation into the second, and then solving a quadratic.

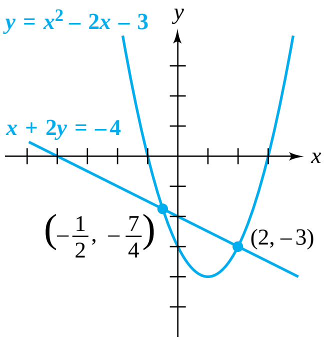


FIGURE 6

Solution

Follow the strategy. Solving the first equation for x gives $x = -2y - 4$. Substitute into the second equation and solve for y.

$$\begin{aligned} y &= (-2y - 4)^2 - 2(-2y - 4) - 3 \\ y &= 4y^2 + 16y + 16 + 4y + 8 - 3 \\ 4y^2 + 19y + 21 &= 0 \\ (4y + 7)(y + 3) &= 0 \end{aligned}$$

Therefore, $y = -\frac{7}{4}$ or $y = -3$. Now use $x = -2y - 4$ to get the corresponding values of x. For $y = -\frac{7}{4}$, $x = -\frac{1}{2}$; for $y = -3$, $x = 2$.

The graph of the first equation is a line and that of the second is a parabola that opens upward with vertex at $(1, -4)$. See Figure 6. The points of intersection are $(-\frac{1}{2}, -\frac{7}{4})$ and $(2, -3)$, which correspond to the two solutions. ◀

▶EXAMPLE 3 ***Complex solution*** Solve the system of equations and interpret the solution graphically.

$$\begin{aligned} 2x + y &= 3 \\ x^2 + y &= 1 \end{aligned}$$

Solution

Solve the second equation for y to get $y = 1 - x^2$. Substitute into the first equation and solve for x.

$$\begin{aligned} 2x + (1 - x^2) &= 3 \\ x^2 - 2x + 2 &= 0 \end{aligned}$$

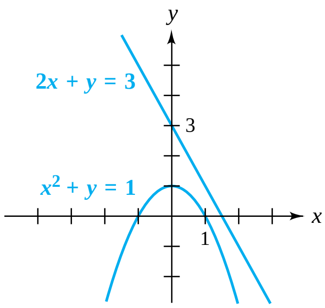


FIGURE 7

To solve the quadratic equation, apply the quadratic formula.

$$x = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

We find the coordinates of the points of intersection, if any, of the parabola and the line by solving the system for real number solutions only. Since we have imaginary number solutions, there are no points of intersection, as we see from the graphs in Figure 7. ◀

▶EXAMPLE 4 ***Eliminate variable*** Find the solution set for the system of equations

$$\begin{aligned} x^2 - y^2 &= 3 \\ 2x^2 + y^2 &= 9. \end{aligned}$$

Solution

Eliminate y by adding the two equations and then solve for x.

$$3x^2 = 12 \qquad x^2 = 4 \qquad x = \pm 2.$$

To get the corresponding values of y, substitute 2 or -2 into either of the given equations, say the first.

$$4 - y^2 = 3 \qquad y^2 = 1 \qquad y = \pm 1.$$

This gives four solutions. The solution set is $\{(2, 1), (2, -1), (-2, 1), (-2, -1)\}$. ◀

Strategy: The domain of $y = 2 \ln x$ is $(0, \infty)$ while that of $y = \ln(4 - x)$ is $(-\infty, 4)$. Thus any solution will be such that x is in $(0, 4)$. To draw graphs, use properties of the ln function from Chapter 4.

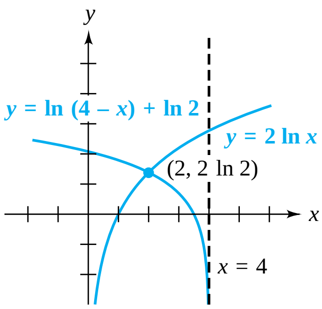


FIGURE 8

▶EXAMPLE 5 ***Graph intersections*** Find the points of intersection of the graphs of $y = 2 \ln x$ and $y = \ln(4 - x) + \ln 2$. Draw the graphs.

Solution

Follow the strategy. Eliminate y from the equations.

$$2 \ln x = \ln(4 - x) + \ln 2$$

Now use properties of logarithms from Chapter 4 and then solve for x.

$$\begin{aligned} \ln x^2 &= \ln[2(4 - x)] \\ x^2 &= 2(4 - x) \\ x^2 + 2x - 8 &= 0 \\ (x - 2)(x + 4) &= 0 \end{aligned}$$

Therefore, we get two possible solutions for x: $x = 2$ or $x = -4$, but -4 is not a solution. See the Strategy. To find the corresponding value of y use either of the given equations, say $y = 2 \ln x$. For $x = 2$, $y = 2 \ln 2 = \ln 4$. The graphs of the two equations intersect at only one point $(2, \ln 4) \approx (2, 1.39)$. See Figure 8. ◀

EXERCISES 9.3

Check Your Understanding

Exercise 1–6 True or False. Give reasons.

1. The system of equations

$$\begin{aligned} \sqrt{2}x - \sqrt{5}y &= 4 \\ \sqrt{3}x + \sqrt{7}y &= 3 \end{aligned}$$

is a nonlinear system.

2. The graphs of $y = x^2$ and $y = -x - 1$ intersect at two points.

3. The graphs of $y = |x|$ and $y = x$ have in common only one point, $(0, 0)$.

4. The system of equations

$$\begin{aligned} x - y &= 0 \\ x^2 + y^2 &= 8 \end{aligned}$$

has exactly two solutions.

5. The system

$$\begin{aligned} x - 2y - 1 &= 0 \\ x^2 + y^2 + 1 &= 0 \end{aligned}$$

has no real solutions.

6. The system

$$\begin{aligned} x^2 + y &= 0 \\ x^2 - y &= 0 \end{aligned}$$

has no real solutions.

Exercises 7–10 Fill in the blank so that the resulting statement is true.

7. The graphs of $y = x^2 - 1$ and $y = 1 - x^2$ intersect at ______.

8. The graphs of $x^2 + y^2 = 25$ and $4x + 3y = 0$ intersect at ______.

9. The graphs of $|x| + y = 0$ and $y + 2 = 0$ intersect at ______.

10. The graphs of $y = |x| - 1$ and $y = 1 - |x|$ intersect at ______.

Develop Mastery

Exercises 1–16 ***Solve, Draw Graphs*** *Find all pairs of real numbers x, y that satisfy the system of equations. Draw graphs and show points of intersection (if any).*

1. $y = 3x + 4$, $y = x^2$

2. $2x - y + 2 = 0$, $x^2 + y^2 = 169$

3. $3x + y = 0$, $2x^2 + 4x + y = 0$

4. $2x - y = -2$, $xy = 4$

5. $2x + 3y = -3$, $xy = -3$

6. $5x - y = 10$, $x^2 + x - y = 6$

7. $2x - y = 0$, $x^2 - y = -3$

8. $x + y = 2$, $x^2 + y^2 = 2$

9. $3x - y = 5$, $x^2 + y^2 = 25$

10. $y = x^2 - 4x + 4$, $y = -2x^2 + x + 16$

11. $x - y = 2$
$x^2 + y = 2$

12. $y = \sqrt{x}$
$y = 2x - 6$

13. $x - y = 2$
$\sqrt{x} - y = 0$

14. $2x - y = 0$
$xy - y = 2$

15. $x^2 - y^2 = 0$
$x^2 + y^2 = 8$

16. $x - y = 0$
$x^3 - 3x + y = 0$

Exercises 17–30 ***Nonlinear Systems*** *Solve the system of equations. If results involve irrational numbers, give approximations rounded off to two decimal places.*

17. $y = \ln x$
$y = \ln(2 - x)$

18. $y = e^x$
$x + \ln y = 0$

19. $y = 2 \ln x$
$y = \ln(3 - x) + \ln 4$

20. $y = \ln x^2$
$y = \ln(3 - x) + \ln 4$

21. $x - \ln y = 2$
$x - \ln(y - 3) = 3$

22. $x + \ln(y + 1) = 2$
$x - \ln y = 1$

23. $2^x + y = 16$
$2^{x+1} - y = 8$

24. $3^x + 3y = 10$
$3^{x-1} - y = 8$

25. $x^2y = 2$
$y = 2x^2$

26. $xy = 2$
$y = \sqrt{x} + 1$

27. $x^2 + 2y^2 = 6$
$xy = 2$

28. $x^2y = 1$
$y = -x^2 + 2$

29. $x^2 + y^2 - xy = 3$
$x^2 + y^2 = 5$

30. $2x^2 + 5xy + 3y^2 = 4$
$xy = -2$

Exercises 31–34 ***Trigonometric Functions*** *Solve the system of equations. Assume that* $0 \le x \le 2\pi$; *for Exercises* 33 *and* 34, $0 \le y \le 2\pi$.

31. $\sin x - y = 0$
$\cos x - y = 0$

32. $\sin x + y = 0$
$\sin 2x - y = 0$

33. $2 \sin x + \cos y = 2$
$\sin x - \cos y = -0.5$

34. $\sin x + \cos y = 0$
$2 \sin x - 4 \cos y = 3\sqrt{2}$

Exercises 35–36 ***Absolute Values*** *Solve the system of equations.* (*Hint: How could you graph an equation involving* $|y|$?)

35. $3|x| - 2|y| = -2$
$|x| + 3|y| = 14$

36. $2|x| - 3|y| = 0$
$4|x| + 3|y| = 18$

Exercises 37–42 ***Nonlinear Systems***

37. $6e^x - e^y = 1$
$3e^x + e^y = 8$

38. $e^x + 2e^y = 8$
$2e^x - e^y = 1$

39. $\ln x + \ln y = 0$
$2 \ln x + \ln y = 1$

40. $\ln x + \ln y = 0$
$3 \ln x + 4 \ln y = 2$

41. $x + y + |x| = 9$
$x - y + |y| = 12$
(*Hint:* What are the possible values of $x + |x|$?)

42. $x + y + \sqrt{x^2} = 6$
$x + \sqrt{y^2} - y = 8$

Exercises 43–46 ***Rectangles***

43. The perimeter of a rectangle is 40 cm and the area is 96 cm^2. Find the dimensions of the rectangle.

44. Find the dimensions of a rectangle that has a diagonal of length 13 cm and a perimeter of 34 cm.

45. One side of a rectangle is 3 cm longer than twice the shorter side, and the area is 230 cm^2. Find the perimeter of the rectangle.

46. A rectangle is inscribed in a circle of radius $\sqrt{10}$. If the area of the rectangle is 16, find its dimensions.

Exercises 47–48 ***Line Through Intersections***

47. Find an equation for the line that passes through the points of intersection of the graphs of $y = x^2 + 2x$ and $y = -x^2$.

48. Find an equation for the line that passes through the points of intersection of the graphs of $y = x^2 - 4x - 5$ and $y = -x^2 + 2x + 3$.

49. An altitude of a triangle is twice as long as the corresponding base and the area of the triangle is 36 cm^2. Find the altitude and the base. Does the given information determine a unique triangle? Suppose the problem states that one of the other sides is $4\sqrt{10}$. What is the perimeter of the triangle?

50. ***Explore*** Find all pairs of real numbers (if any) such that
(a) their difference is 1 and their product is 1.
(b) their sum is 1 and their product is 1.
(c) their difference is 1 and their quotient is 1.
(d) their sum is 1 and their quotient is 1.

9.4 SYSTEMS OF LINEAR INEQUALITIES; LINEAR PROGRAMMING

My earliest recollection of feeling that mathematics might some day be something special was perhaps in the fourth grade when I showed the arithmetic teachers that the squares always end in—well, whatever it is that they end in.

Irving Kaplansky

Consider the problem of assigning 70 men to 70 jobs. Unfortunately there are 70 factorial permutations, or ways to make the assignments. The problem is to compare 70 factorial ways and to select the one which is optimal, or "best" by some criterion. Even if the Earth were filled with nano-speed computers, all programmed in parallel from the time of the Big Bang until the sun grows cold, it would [be] impossible to examine all the possible solutions. The remarkable thing is that the simplex method with the aid of a modern computer can solve this problem in a split second.

George P. Dantzig

In earlier chapters we solved inequalities that involved single variables. We noted that the solution sets could be shown on a number line. In this section we are interested in solving inequalities in which two variables are involved. We shall see that the solution set may be shown as a region of the plane.

Linear Inequalities

In Section 9.1 we studied linear equations that can be written in the form $ax + by = c$. If we replace the equal sign by one of the inequality symbols, $\leq$, $<$, $\geq$, or $>$, we have a **linear inequality.** The example that follows illustrates a technique for representing the solution set for a linear inequality.

▶EXAMPLE 1 ***A line and Inequalities*** Show all points in the plane that satisfy **(a)** $-x + 2y = 4$, **(b)** $-x + 2y < 4$, and **(c)** $-x + 2y > 4$.

Solution

(a) The points (x, y) that satisfy the equation are on line L whose equation may also be written $y = \frac{1}{2}x + 2$. This appears in Figure 9, which also shows some typical points M, P, and Q, where M is on the line, P is below M, and Q is above M. Since $y_2 < y_1$ and $y_1 = \frac{1}{2}x_1 + 2$, then $y_2 < \frac{1}{2}x_1 + 2$. Similarly, $y_3 > \frac{1}{2}x_1 + 2$.

(b) The inequality can be written $y < \frac{1}{2}x + 2$. The diagram in Figure 10 shows that the coordinates of any point below the line L, such as $P(x_1, y_2)$, will satisfy the given inequality. Any point on or above the line will not. Therefore, the set of points (x, y) that satisfy $-x + 2y < 4$ consists of all points below L. This

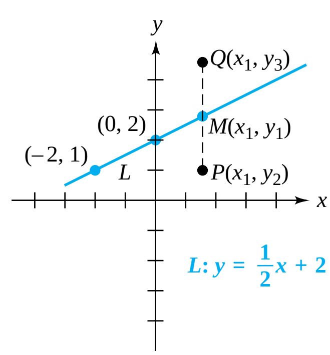


FIGURE 9

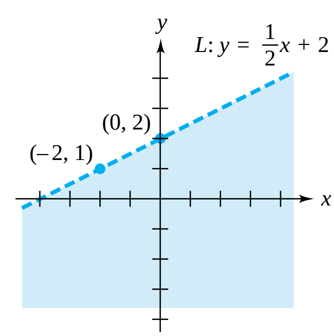


FIGURE 10

is the shaded region (or half-plane) in Figure 10, where L is shown as a broken line to indicate that the points on L are not included in the solution set. Your graphing calculator may be able to graph the kinds of shaded regions in Figures 10 and 11. Look for a DRAW menu.

(c) In a similar manner, the given inequality is equivalent to $y > \frac{1}{2}x + 2$, and the solution set consists of all points in the half-plane above L. See Figure 11. ◀

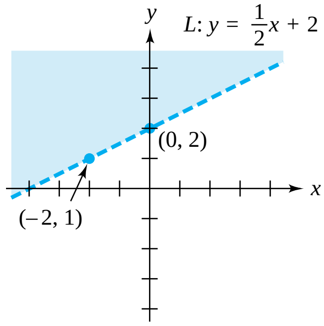


FIGURE 11

Parts **(b)** and **(c)** of Example 1 suggest the following definition.

Definition: half-plane

The solution set for a linear inequality, such as $ax + by < c$, consists of all points on one side of the defining line, $ax + by = c$. The graph of the linear inequality is a **half-plane.**

▶EXAMPLE 2 ***A linear inequality*** Graph the inequality $3x - 2y \le 6$.

Strategy: Begin with the boundary line L, $3x - 2y = 6$, and choose a test point that is not on the line. If the coordinates satisfy the desired inequality, the solution is the half-plane that contains the test point; if not, choose the other half-plane.

Solution

We want all points (x, y) that satisfy $3x - 2y < 6$ and all those that satisfy $3x - 2y = 6$. The graph will consist of all points in a half-plane together with the points *on* the boundary line.

Follow the strategy, referring to Figure 12. We must decide which half-plane (above or below L) satisfies the inequality. To do this, take a test point not on L, say (0, 0), and see if it satisfies the inequality.

$$3 \cdot 0 - 2 \cdot 0 \le 6 \quad \text{or} \quad 0 \le 6$$

Since $0 \le 6$ is a true statement, the half-plane that contains (0, 0) is the one we want, the portion of the plane above and to the left of L. The shaded region in Figure 12 including the line L (drawn solid) is the graph of the inequality. ◀

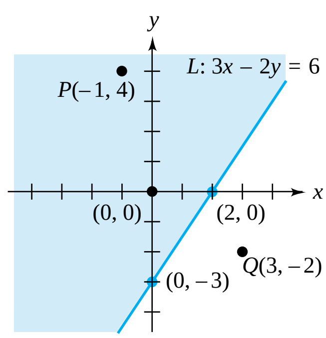


FIGURE 12

The technique for determining the solution set by drawing a graph of a linear inequality, as illustrated in the above example, can be expressed in algorithmic form.

Algorithm for solving a linear inequality

1. Replace the inequality symbol by an equal sign and graph the corresponding line L (broken, for a strict inequality, solid otherwise).
2. Take a test point P not on line L and see if it satisfies the inequality. If it does, then the desired solution set includes all points in the half-plane that contains P; if not, then the solution set consists of the half-plane on the other side of L.

Systems of Inequalities

A **system of linear inequalities** consists of two or more linear inequalities that must be satisfied simultaneously. The following two examples illustrate techniques for determining the solution set or the graph of such a system.

▶EXAMPLE 3 ***System of linear inequalities*** Solve the system of inequalities and show the solution set as a graph in the plane.

$$\begin{aligned} x + 2y &\leq 3 \\ -3x + y &< 5 \\ -3x + 8y &\geq -23 \end{aligned}$$

Strategy: Each inequality defines a half-plane, so the solution set for the system is the intersection of three half-planes. Draw each boundary line, find the coordinates of the intersections, and identify the correct half-planes by taking test points.

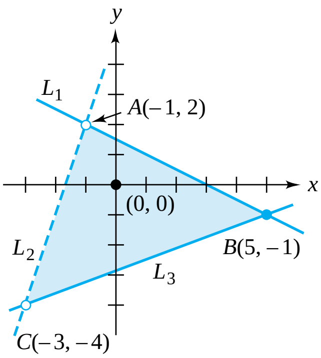


FIGURE 13

Solution

Follow the strategy. First draw graphs of the three lines L_1, L_2, and L_3:

$$L_1\colon x + 2y = 3 \qquad L_2\colon -3x + y = 5 \qquad L_3\colon -3x + 8y = -23.$$

The points of intersection of these three lines, called **corner points,** are obtained by solving the equations in pairs.

$$A\begin{cases} x + 2y = 3 \\ -3x + y = 5 \end{cases} \qquad B\begin{cases} x + 2y = 3 \\ -3x + 8y = -23 \end{cases} \qquad C\begin{cases} -3x + y = 5 \\ -3x + 8y = -23 \end{cases}$$

The three corner points are $A(-1, 2)$, $B(5, -1)$, and $C(-3, -4)$. In Figure 13 L_2 is shown as a broken line, and points A and C are indicated by open circles, since the points on L_2 are not in the solution set.

Returning to the inequalities, identify the points that belong to all three half-planes. Using (0, 0) as a test point, the desired half-planes are below L_1, below L_2, and above L_3. The intersection of the three half-planes, the solution set, is shown as the shaded region in the figure. Any other test point not on any of the three lines would serve as well to identify the three half-planes and their intersection. ◀

▶*EXAMPLE 4* ***Mixture problem*** A dietitian wishes to combine two foods, *A* and *B*, to make a mixture that contains at least 50 g of protein, at least 130 mg of calcium, and not more than 550 calories. The nutrient values of foods *A* and *B* are given in the table.

Food	*Protein (g/cup)*	*Calcium (mg/cup)*	*Calories (cup)*
A	20	20	100
B	10	50	150

How many cups of each of the foods should the dietitian use?

Strategy: We want numbers of cups of *A* and *B*, so assign letters (variables). Write inequalities for grams of protein (≥ 50), milligrams of calcium (≥ 130), and number of calories (≤ 550), then draw a graph to show the solution set.

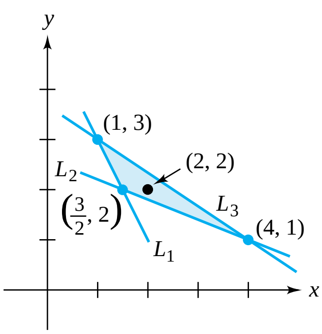


FIGURE 14

Solution

Follow the strategy. Let x be the number of cups of food A and y be the number of cups of food B. The three conditions to be met can be written as inequalities:

$$\begin{aligned} &\text{Protein:} && 20x + 10y \geq 50 \\ &\text{Calcium:} && 20x + 50y \geq 130 \\ &\text{Calories:} && 100x + 150y \leq 550. \end{aligned}$$

Simplify the inequalities by dividing each of the first two by 10 and the third by 50, and then graph the three lines L_1, L_2, and L_3,

$$L_1\colon 2x + y = 5 \qquad L_2\colon 2x + 5y = 13 \qquad L_3\colon 2x + 3y = 11.$$

Find the points of intersection of L_1, L_2, and L_3 and draw the lines, as shown in Figure 14. The solution set for the system of inequalities is the region shown. Therefore, any point in the region will give a combination of foods A and B that will satisfy the given constraints. For instance, point (2, 2) is in the region. Taking two cups of each type of food will provide 60 g of protein, 140 mg of calcium, and 500 calories. ◀

HISTORICAL NOTE

SIMPLEX AND KARMARKAR ALGORITHMS FOR LINEAR PROGRAMMING

Example 5 illustrates a kind of problem that modern industry and government face all the time—that of maximizing or minimizing some function subject to constraints or restrictions. An oil refinery, for example, may produce a dozen products (grades of engine oil, gasoline, diesel, and so on), each of which requires different crude oil purchases, refining processes, and storage. Transportation costs and customer demand vary. Refinery and storage capacity and raw material availability also affect what can be produced and the profitability of the whole operation.

The constraints can usually be described by a set of linear inequalities such as those in Example 5. The set of points satisfying the system of inequalities forms some kind of polyhedral region in a high dimensional space like the regions pictured in Figure 15. It turns out that the desired maximum or minimum always occurs at a corner point of the graph. Many industrial or economic applications may present dozens or even hundreds of variables, and locating and testing corner points becomes a staggering problem.

Algorithms for linear programming are used to solve complex problems that face oil companies and firms in other industries

In 1947 an American mathematician, George B. Dantzig, developed a new method for dealing with such problems called the *simplex algorithm for linear programming.* The algorithm uses computers to manipulate matrices in a way that essentially moves from one corner to the next, improving the result at each step. The simplex algorithm has saved untold billions of dollars for industries and consumers worldwide.

Now a new algorithm under investigation promises to deal with even larger problems in less time. This new algorithm, named for its developer, Narendra Karmarkar of Bell Laboratories, intuitively takes shortcuts through the polyhedron, instead of moving along the edges. Scientists, engineers, and economists are working and experimenting to see if computer utilization of the Karmarkar algorithm can significantly improve on the simplex algorithm.

Linear Programming

The Historical Note in this section describes some applications of linear programming. For most such problems we want to maximize or minimize a function, called the **objective function,** subject to conditions (linear inequalities) called **constraints.** The constraints define a set (the set satisfying the system of inequalities) referred to as the **feasible set.** The remarkable fact that makes it possible to solve such optimization problems effectively is the following theorem.

Linear programming theorem

If the objective function of a linear programming problem has a maximum or minimum value on the feasible set, then the extreme value must occur at a corner point of the feasible set.

Some of the problems that linear programming helps solve can include dozens of variables and even more constraints. Such complex problems require sophisticated computer techniques, but we can illustrate all of the key ideas with much simpler problems. We begin by outlining the basic ideas for solving a linear programming problem.

Solving a linear programming problem

1. Name the variables; express the constraints and the objective function in terms of the variables.
2. Sketch the boundaries of the feasible set (one boundary for each constraint).
3. Find the corners of the feasible set.
4. Evaluate the objective function at each corner point to identify maximum and minimum values.

►EXAMPLE 5 ***Linear programming*** A farmer planning spring planting has decided to plant up to a total of 120 acres in corn and soybeans. An estimate of the investment required and the expected return per acre for each appears in the table.

Crop	*Investment*	*Return*
Corn	\$20	\$50
Soybeans	\$35	\$80

Because corn is needed for feed purposes on the farm, the farmer needs at least 38 acres of corn, and the budget can cover at most \$3000 for both corn and soybeans. How many acres of corn and how many acres of soybeans should be planted to maximize the return from these two crops?

Solution

Let x be the number of acres to be planted in soybeans and y the number of acres of corn. Then we must have $x \geq 0$ and the need for corn as feed implies $y \geq 38$. The total allowable acreage for the two crops is 120 acres, so $x + y \leq 120$. The investment required by x acres of soybeans and y acres of corn is $35x + 20y$, so $35x + 20y \leq 3000$. Finally, the objective function is the expected return, which is $R(x, y) = 80x + 50y$.

We want to maximize $R(x, y)$ on the feasible set, which is defined by the inequalities

$$x \geq 0 \qquad y \geq 38 \qquad x + y \leq 120 \qquad 35x + 20y \leq 3000.$$

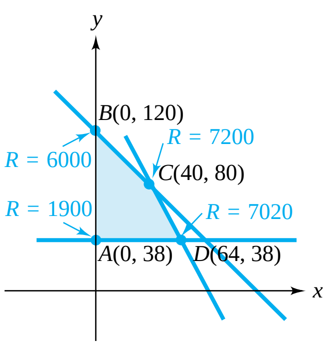


FIGURE 15

Draw a diagram and shade the feasible set. See Figure 15. To find coordinates of the corner points, find the intersections of the boundary lines. The corner points are: $A(0, 38)$, $B(0, 120)$, $C(40, 80)$, and $D(64, 38)$. Finally, determine the estimated return for each choice, that is, evaluate $R(x, y) = 80x + 50y$ at each corner point:

$$R(0, 38) = 1900 \qquad R(0, 120) = 6000$$
$$R(40, 80) = 7200 \qquad R(64, 38) = 7020.$$

The farmer will get the greatest return, subject to the given constraints, by planting 40 acres of soybeans and 80 acres of corn, for a return of \$7200. ◄

EXERCISES 9.4

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. The point $(-1, 2)$ is in the solution set for $2x + 3y < 4$.
2. The solution set for the system $x < 0$, $y > 0$, $x + y > 1$ contains only points in the second quadrant.
3. The solution set for the system $x < 0$, $y > 0$, $x + y > 1$ is the empty set.
4. The solution set for the system $x < 0, x - y > 1$ contains points in the third quadrant only.
5. The solution set for the system $x < 1$, $y < x$ contains no points in the fourth quadrant.
6. Point $(2, -3)$ is a corner point for the system of inequalities, $2x + 3y \leq -5$, $3x - y \geq 9$, $x - y \leq 1$.

Exercises 7–10 Fill in the blank with the quadrant(s) that make the resulting statement true.

7. The solution set for $y > x + 2$ contains no points in ______.
8. The solution set for $y \geq 2x + 1$ contains points in ______.
9. The solution set for the system $y \geq x$, $y \leq -x$ contains points in ______.
10. The expression $\sqrt{x - y - 2}$ is a real number for some points (x, y) in ______.

Develop Mastery

Exercises 1–4 ***Locating Points*** *Determine whether or not the given pair of numbers (x, y) belongs to the solution set of the system of inequalities.*

1. $x - 3y < 4$, $2x + y < 3$ **(a)** $(1, 1)$ **(b)** $(\sqrt{2}, -0.5)$
2. $-2x + y > -3$, $5x + 2y < 1$ **(a)** $(-1, 2)$ **(b)** $(1, -5)$
3. $x - 3y \geq 1$, $4x - y \leq \pi$ **(a)** $(1, -1)$ **(b)** $(\sqrt{2}, \pi)$
4. $y \leq 2x$, $3x + y > 0$ **(a)** $(0, 0)$ **(b)** $(-1, 3)$

Exercises 5–12 ***Graphing Inequalities*** ***(a)*** *Draw a graph showing all points (x, y) in the solution set of the given inequality.* ***(b)*** *Give coordinates of any two specific pairs (x, y) that satisfy the inequality.*

5. $x + 2y < 4$
6. $-x + y > 3$
7. $2x - 3y \geq 6$
8. $4x - 2y \leq 9$
9. $x + y + 4 < 0$
10. $2x > y - 4$
11. $y \geq 2x$
12. $2y < 3x - 4$

Exercises 13–24 ***Solving Inequalities with Graphs*** *Draw a graph showing the solution set for the system of inequalities. Determine the coordinates of any corner points and show them on your diagram. Indicate which boundary curves and corner points belong and which do not belong to the solution set.*

13. $x + y < 4$, $2x - y < -1$
14. $3x - 2y > 5$, $-x - y < -5$
15. $x - 2y \geq 4$, $|x| > 2$
16. $3x - 4y < 6$, $|x| < 2$, $|y| < 3$
17. $-x + 2y < 5$, $2x + y > 0$, $3x - y < 5$
18. $4x + 3y \leq 16$, $-x + y > -4$, $6x + y \geq 10$
19. $y > 0$, $x + y > 1$
20. $x < 0$, $x + y > 1$
21. $x < 0$, $y > 0$, $x + y > 1$
22. $-1 < x - y \leq 2$, $-2 < x + y \leq 2$
23. $x > 2$, $y > -1$, $x + y < 3$
24. $|x - y| \leq 2$, $|x + y| \leq 2$

Exercises 25–28 ***Which Quadrants?*** *For the system of inequalities, determine which quadrants contain points in the solution set.*

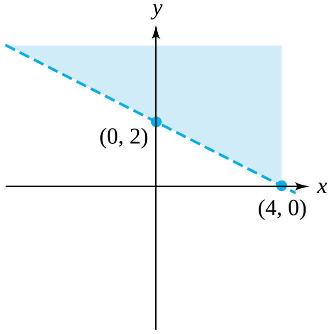
25. $y > 2x$, $y > 4 - x$
26. $x > 1$, $y > x$
27. $x + y \leq 1$, $x - y \leq -1$
28. $x - y \geq 2$, $2x + y \geq 4$

Exercises 29–36 ***Domains*** *Show on a graph all points (x, y) for which the expression will be a real number.*

29. $\sqrt{2x - y - 4}$
30. $\sqrt{x - y + 1}$
31. $\ln(2x + y - 2)$
32. $\log(x - 2y - 4)$
33. $\text{Arcsin}(y - x)$
34. $\text{Arcsin}(x + y + 1)$
35. $\ln x + \ln(y - x)$
36. $\log(x + y) - \log(2x - y)$

Exercises 37–39 ***Write an Inequality*** *Write a linear inequality whose solution set is the shaded region in the diagram.*

37.



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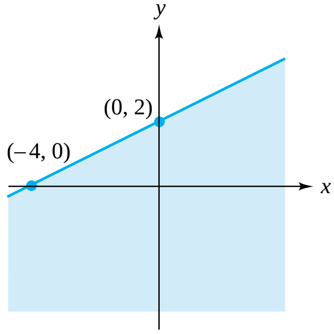

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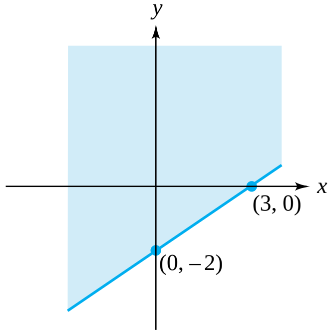
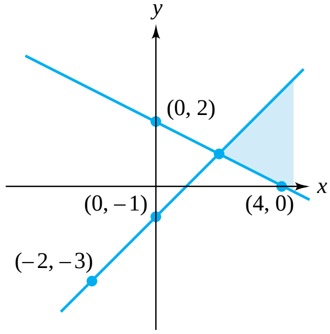


Exercises 40–42 ***Write a System*** *Find a system of inequalities whose solution set is the shaded region in the diagram and give the coordinates of the corner points.*

40.



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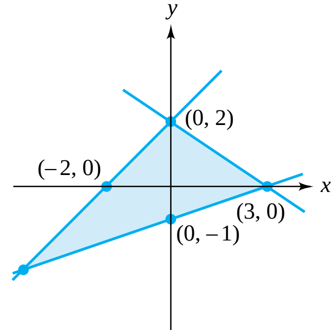

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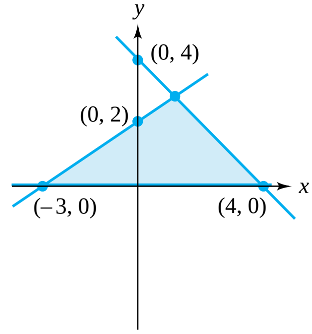


Exercises 43–46 ***System Defining Triangular Region*** ***(a)*** *Draw a diagram showing the set of all points inside the triangle whose vertices are the points A, B, and C.* ***(b)*** *Find a system of inequalities whose solution set consists of all points inside the triangle.*

43. $A(-2, 0)$ $B(0, 4)$ $C(4, -2)$

44. $A(-3, 2)$ $B(3, -2)$ $C(5, 2)$

45. $A(-3, 0)$ $B(0, 4)$ $C(2, 0)$

46. $A(0, 0)$ $B(-2, 2)$ $C(4, 2)$

Exercises 47–48 ***Verbal to System*** *Sketch a graph for the set described and find a system of inequalities for which the set described is the solution set.*

47. All points above the line $2x - y = 1$ and below the line $x + 2y = 4$.

48. All points above the line $y = 2x$ and below the line $x + 2y = 5$.

Exercises 49–54 ***Linear Programming*** *Find the minimum and maximum values of the objective function subject to the given constraints. (Hint: First draw a diagram showing the feasible set and use the linear programming theorem.)*

49. Objective function: $T = 48x + 56y + 120$
$x + y \geq 4,\ y \leq 2x + 1,\ 4x + y \leq 13$

50. Objective function: $T = 36x + 73y - 16$
$x \geq 1,\ y \leq x,\ y \geq 3x - 8$

51. Objective function: $T = 67x + 35y$
$y \le 2,\ y \le 2x,\ y \ge x - 4$

52. Objective function: $T = 65x + 124y - 200$
$x + y \ge 3,\ y \le 2x,\ 4x + y \le 12$

53. Objective function: $T = 84x + 73y - 78$
$x \ge 0,\ y \ge 0,\ x - 3y + 14 \ge 0,\ 5x + 2y \le 32,$
$4x + 5y \ge 12$

54. Objective function: $T = 47x + 56y - 24$
$x - 3y + 11 \ge 0,\ 4x + y \le 21,\ 3x + 4y \ge 6$

Exercises 55–58 ***Applied Inequalities***

55. A concert is to be presented in an auditorium that has a seating capacity of 800. The price per ticket for 200 of the seats is $6, and $3 each for the remaining 600 seats. The total cost for putting on the concert will be $2100. Draw a graph to show the various possible pairs of numbers of $6 and $3 tickets that must be sold for the concert to avoid financial loss.

56. A rancher wants to purchase some lambs and goats—at least five lambs and at least four goats—but cannot spend more than $800. Each lamb costs $80, and goats cost $50 each. How many of each can the rancher buy? Draw a graph to list all possible pairs, keeping in mind that lambs and goats come in whole numbers.

57. A sheep rancher raises two different kinds of sheep for market, Rambis and Eustis, with only enough summer range to support 3000 animals for sale each year. To satisfy loyal customers, the rancher must have at least 750 of each breed available, and because of different range demands, at least a third of the herd should be Rambis. The average profit for the Rambis breed is $8 per animal, while each Eustis should yield an average of $10. How many of each breed should the rancher raise to maximize the profit? (*Hint:* If x is the number of Rambis sheep and y is the number of Eustis, the condition that at least a third should be Rambis can be expressed as $x \ge \frac{(x + y)}{3}$ or $y \le 2x$.)

58. A fish cannery packs tuna in two ways, chunk style and solid pack. Limits on storage space and customer demand lead to these constraints:

> The total number of cases produced per day must not exceed 3000.
>
> The number of cases of chunk style must be at least twice the number of cases of solid pack.
>
> At least 600 cases of solid pack must be produced each day.

How many cases of each type can be produced per day if all constraints are to be satisfied? Draw a graph of the solution set and show the coordinates of the corner points.

Exercises 59–61 ***Mixture Problems*** *Use the information from the following table, which gives nutrient values for four foods, A , B, C, and D. Each unit is 100 grams.*

Food	*Energy (calories/unit)*	*Vitamin C (mg/unit)*	*Iron (mg/unit)*
A	200	2	0.5
B	100	3	1.5
C	300	0	2.0
D	400	1	0.0

Food	*Calcium (mg/unit)*	*Protein (g/unit)*	*Carbohydrate (g/unit)*
A	10	2	15
B	4	3	30
C	20	9	10
D	5	3	10

59. In preparing a menu, determine how many units of *A* and of *B* can be included so that the combined nutrient values will satisfy the following constraints:

> At least 8 milligrams of vitamin C
>
> At least 18 milligrams of calcium
>
> Not more than 800 calories

60. How many units of *A* and *C* can be included in a menu to contribute:

> At least 3 milligrams of vitamin C
>
> At least 40 milligrams of calcium
>
> Not more than 60 grams of carbohydrates

61. How many units of *C* and *D* will give a combined total that satisfies these constraints:

> At least 2 milligrams of vitamin C
>
> At least 15 grams of protein
>
> Not more than 6 milligrams of iron
>
> Not more than 2100 calories

Exercises 62–66 ***Acreage and Fertilizer Choices***

62. Would the farmer's decision in Example 5 be different if there were no minimum acreage to be alotted to corn?

63. What would be the optimal planting scheme for Example 5 if the expected return on soybeans were **(a)** $100 per acre? **(b)** $110 per acre?

64. In Example 5 how many acres of corn should be planted and how many of soybeans if the return of corn were to drop to $25 per acre?

65. A commercial gardener wants to feed plants a very specific mix of nitrates and phosphates. Two kinds of fertilizer, Brand A and Brand B, are available, each sold in 50 pound bags, with the following quantities of each mineral per bag:

	Phosphate	*Nitrate*
Brand A	2.5 lbs	10 lbs
Brand B	5.0	5

The gardener wants to put at least 30 lbs of nitrates and 15 lbs of phosphates on the gardens and not more than 250 lbs of fertilizer altogether. If Brand A costs \$8.50 a bag and Brand B costs \$3.50 a bag, how many bags of each would minimize fertilizer costs?

66. Repeat Exercise 65 if the cost of Brand B fertilizer increases to \$6.00 a bag.

9.5 DETERMINANTS

. . . A staggering paradox hits us in the teeth. For abstract mathematics happens to work. It is the tool that physicists employ in working with the nuts and bolts of the universe! There are many examples from the history of science of a branch of pure mathematics which, decades after its invention, suddenly finds a use in physics.

F. David Peat

From childhood on, Shannon was fascinated by both the particulars of hardware and the generalities of mathematics. (He) tinkered with erector sets and radios given him by his father . . . and solved mathematical puzzles supplied by his older sister, Catherine, who became a professor of mathematics.

Claude Shannon

In Section 9.2 we introduced matrices as convenient tools for keeping track of coefficients and handling the arithmetic required to solve systems of linear equations. Matrices are being used today in more and more applications. A matrix presents a great deal of information in compact, readable form. Finding optimal solutions to large linear programming problems requires extensive use of matrices. The properties and applications of matrices are studied in *linear algebra,* a discipline that includes much of the material of this chapter. In this section we introduce the determinant of a square matrix as another tool to help solve systems of linear equations.

Dimension (Size) of a Matrix and Matrix Notation

A matrix is a rectangular array arranged in horizontal **rows** and vertical **columns.** The number of rows and columns give the dimension, or size, of the matrix. A matrix with m rows and n columns is called an $\boldsymbol{m}$ **by** $\boldsymbol{n}$ $(\boldsymbol{m \times n})$ **matrix.** Double subscripts provide a convenient system of notation for labeling or locating matrix entries.

Here are some matrices of various sizes:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{33} \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix A is 3×3, B is 3×1, and C is 2×2. A and B show the use of double subscripts: a_{ij} is the entry in the ith row and the jth column. The first subscript identifies the row, the second tells the column; virtually all references to matrices are given in the same order, row first and then column. A matrix with the same number of rows and columns is a **square matrix.**

Determinants

Every square matrix A has an associated number called its **determinant,** denoted by $\det(A)$ or $|A|$. To evaluate determinants, we begin by giving a recursive definition, starting with the determinant of a 2×2 matrix, the definition we gave informally in Section 9.1.

Determinant of a 2 × 2 matrix. For 2×2 matrix A, we obtain $|A|$ by multiplying the entries along each diagonal and subtracting.

Definition: determinant of a 2 × 2 matrix

For the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the **determinant of A** is given by

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

As in Section 9.1, the easiest way to remember the formula is by visualizing products taken in the direction of two arrows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Thus, for example,

$$\begin{vmatrix} 3 & 2 \\ -4 & 1 \end{vmatrix} = (3)(1) - (2)(-4) = 3 + 8 = 11, \quad \text{and}$$

$$\begin{vmatrix} 9 & 0 \\ 2 & -5 \end{vmatrix} = (9)(-5) - (0)(2) = -45 - 0 = -45.$$

For larger square matrices, the determinant definition uses determinants of smaller matrices within the given matrix. The determinant of a 3×3 matrix uses 2×2 determinants, the determinant of a 4×4 matrix uses 3×3 determinants, and so on.

Minors and cofactors. We associate with each entry a_{ij} of square matrix A a **minor determinant** M_{ij} and a **cofactor** C_{ij}. The minor determinant, more commonly called simply the **minor,** of an entry is the determinant obtained by deleting the row and column of the entry, so M_{ij} is the determinant we get by crossing out the ith row and the jth column. The cofactor C_{ij} is the signed minor given by

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

In Example 1, to make it easier to visualize the minor determinant for a given element, we shade the row and column containing that element. When you practice evaluating 3×3 (or larger) determinants, it may help to have a mental picture of a similar shading.

Strategy: The elements of the first row are a_{11}, a_{12}, a_{13}. Apply the definition of cofactor for each set of subscripts.

$$\begin{bmatrix} 1 & -3 & -2 \\ 3 & 2 & -1 \\ -1 & 5 & 0 \end{bmatrix}$$

(a) Minor M_{11} of a_{11} (unshaded).

$$\begin{bmatrix} 1 & -3 & -2 \\ 3 & 2 & -1 \\ -1 & 5 & 0 \end{bmatrix}$$

(b) Minor M_{12} of a_{12} (unshaded).

$$\begin{bmatrix} 1 & -3 & -2 \\ 3 & 2 & -1 \\ -1 & 5 & 0 \end{bmatrix}$$

(c) Minor M_{13} of a_{13} (unshaded).

►EXAMPLE 1 ***Finding cofactors*** Find the cofactor for each element in the first row of the matrix.

$$A = \begin{bmatrix} 1 & -3 & -2 \\ 3 & 2 & -1 \\ -1 & 5 & 0 \end{bmatrix}$$

Solution

Follow the strategy. In the first row, $a_{11} = 1$, $a_{12} = -3$, and $a_{13} = -2$. For the minor M_{11}, we delete the shaded row and column in the first margin matrix, leaving the (unshaded) minor $\begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix}$ and then use $C_{11} = (-1)^{1+1}M_{11}$.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix} = (-1)^2[0 - (-5)] = 5.$$

To obtain M_{12}, delete row 1 and column 2 (see the second margin in matrix) and then use $C_{12} = (-1)^{1+2}M_{12}$.

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -1 \\ -1 & 0 \end{vmatrix} = -[0 - (1)] = 1$$

In a similar manner (third margin matrix) C_{13} is given by

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ -1 & 5 \end{vmatrix} = [15 - (-2)] = 17 \quad ◄$$

Determinant of a 3 × 3 matrix. The determinant of a 3×3 matrix can be obtained using the elements of the first row.

Definition: cofactor expansion by the first row

Let A be a 3×3 matrix with entires a_{ij}. If C_{ij} and M_{ij} are the cofactor and minor, respectively, of a_{ij} as defined above, then the determinant of A is given by

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}. \qquad \textbf{(1)}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3 × 3 Sign Matrix

It is helpful to remember that the cofactors have signs, so that each term of the cofactor expansion of a determinant is a product of three factors: an *entry* a_{ij}, a *sign factor* $(-1)^{i+j}$, and a *minor* M_{ij}. Because the sign factor is either 1 or -1 and depends only on the address (location) of a_{ij}, many people like to use a "sign matrix," that gives the pattern of signs. The sign matrix in the margin may be extended as needed, following the same pattern. Then the above expansion of the determinant has the form

$$|A| = \underbrace{a_{11}}_{\text{entry}}\underbrace{(+1)}_{\text{sign}}\underbrace{M_{11}}_{\text{minor}} + a_{12}(-1)\,M_{12} + a_{13}(+1)\,M_{13}.$$

Determinants of any size have a remarkable property. We get the same number using the entries and cofactors of *any* row or column. For example, each of the following gives the same value for $|A|$ as Equation (1).

Expansion by second row $\quad |A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

Expansion by third column $\quad |A| = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

To illustrate that the cofactor expansion is independent of the row or column chosen, we return to the matrix from Example 1, for which we already have some cofactors.

EXAMPLE 2 ***Cofactor expansion*** Evaluate the determinant of matrix A by **(a)** the first row **(b)** the second column.

$$A = \begin{bmatrix} 1 & -3 & -2 \\ 3 & 2 & -1 \\ -1 & 5 & 0 \end{bmatrix}$$

Strategy: **(a)** Since matrix A is the same as the matrix in Example 1, we already have the cofactors for expansion by the first row. Multiply each cofactor by its entry, and add.

Solution

Follow the strategy.

(a) Using $C_{11} = 5$, $C_{12} = 1$, and $C_{13} = 17$ from Example 1, then by Equation (1),

$$|A| = 1 \cdot 5 + (-3) \cdot 1 + (-2) \cdot 17 = 5 - 3 - 34 = -32.$$

(b) Expansion by the second column gives

$$\begin{aligned} |A| &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= (-3)(-1)M_{12} + (2)(+1)M_{22} + (5)(-1)M_{32} \\ &= 3\begin{vmatrix} 3 & -1 \\ -1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} - 5\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\ &= 3(-1) + 2(-2) - 5 \cdot 5 = -32, \end{aligned}$$

the same value as for the first-row expansion. ◀

Determinant of an $n \times n$ matrix. Since we know how to evaluate 3×3 determinants, we can use a similar cofactor expansion for a 4×4 determinant. Choose any row or column and take the sum of the products of each entry with the corresponding cofactor. The determinant of a 4×4 matrix involves four 3×3 determinants, one for each of the four entries in the chosen row or column. Similarly, the determinant of a 5×5 matrix uses five 4×4 determinants. We give no formal definition of the procedure to evaluate the determinant of an $n \times n$ matrix, but it should be clear from the form of Equation (1). It should also be clear that the number of arithmetic operations required to evaluate a determinant grows staggeringly large as the size of the matrix increases.

Elementary row (column) operations and determinants. One way to simplify the evaluation of determinants is to recognize that certain elementary matrix operations leave the determinant unchanged.

Elementary operation property

Given a square matrix A, if the entries of one row (column) are multiplied by a constant and added to the corresponding entries of another row (column), then the determinant of the resulting matrix is still equal to $|A|$.

Applying the Elementary Operation Property (EOP) may give some zero entries that make the evaluation of a determinant much easier, as illustrated in the next example.

▶EXAMPLE 3 ***Elementary operations*** Evaluate the determinant of the matrix

$$A = \begin{bmatrix} -2 & 2 & 0 & 1 \\ 2 & -1 & 3 & 0 \\ -1 & 0 & 2 & -4 \\ 0 & -3 & 5 & 3 \end{bmatrix}$$

Strategy: Use the EOP to get a matrix with three zeros in a row or column and use that row or column for the cofactor expansion.

Solution

Follow the Strategy. Several choices seem reasonable, including using the last 1 in the first row to get three zeros in the first row, or using the -1 in the first column to get zeros in the first column or in the third row. To get zeros in the first column, perform the following elementary row operations: $-2R_3 + R_1 \to R_1$ and $2R_3 + R_2 \to R_2$. The result is matrix B. Evaluate its determinant by the first column expansion.

$$|B| = \begin{vmatrix} 0 & 2 & -4 & 9 \\ 0 & -1 & 7 & -8 \\ -1 & 0 & 2 & -4 \\ 0 & -3 & 5 & 3 \end{vmatrix} = 0 \cdot C_{11} + 0 \cdot C_{21} + (-1)C_{31} + 0 \cdot C_{41}.$$

Thus

$$|A| = |B| = (-1)(+1)\begin{vmatrix} 2 & -4 & 9 \\ -1 & 7 & -8 \\ -3 & 5 & 3 \end{vmatrix}$$

Apply elementary row operations $2R_2 + R_1 \to R_1$ and $-3R_2 + R_3 \to R_3$ to get a matrix with two zeros in the first column:

$$|B| = (-1)\begin{vmatrix} 0 & 10 & -7 \\ -1 & 7 & -8 \\ 0 & -16 & 27 \end{vmatrix} = (-1)\begin{vmatrix} 10 & -7 \\ -16 & 27 \end{vmatrix}$$

$$= -(270 - 112) = -158.$$

Since $|A| = |B|$, $|A| = -158$. ◀

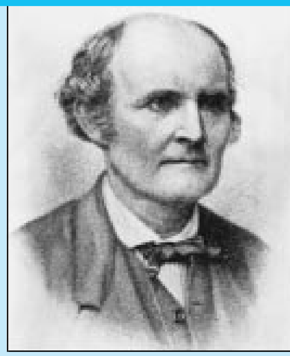
Technology and Larger Determinants

The arithmetic of determinant evaluation grows so rapidly that computers and calculators must use approximation techniques. Most graphing calculators will give excellent approximations for determinants (look for operations in the Matrix menu). To use the power of this technology well, we must understand something about determinants ourselves while at the same time being alert to computational limitations.

As a simple example, we know from the definition that a determinant is a sum of signed products of entries of a matrix. It follows that *if all the entries in a matrix are integers, then its determinant must be an integer*. For

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

HISTORICAL NOTE DETERMINANTS



English mathematician Arthur Cayley

Most students of mathematics today learn about determinants only in connection with matrices. Historically, though, determinants had a lively role of their own long before matrices were recognized. Matrices as such have been studied only for a little more than one hundred years, and were not widely known even into the first third of this century (see "Matrices" in Section 9.2). Determinants are numbers rather than arrays, and it probably should not be surprising that they have been recognized more than twice as long as matrices.

At least three important mathematicians independently developed and used some properties of determinants. Leibnitz, best known for his part in the invention of calculus, wrote letters in 1693 that described how to determine whether a given system of homogeneous equations is consistent by calculating a single number, which we now call a determinant. Maclaurin probably used Cramer's rule twenty years before Cramer published it in 1750.

We would probably not recognize Cramer's rule in its original form. It used none of the special notation we use today. There were also formulas for the solution of three by three systems, but it is likely that neither Maclaurin nor Cramer extended the rule to larger systems—with good reason. A formula for quotients of two 24-term expressions is too complicated to be worth much.

By 1773 Lagrange was using essentially modern notation for certain problems. He is responsible for the formula given in this section for the area of a triangle as a determinant. Cauchy applied the name *determinant* to a class of functions including those that we now call determinants, and Jacobi broadened Cauchy's usage to a determinant consisting of derivatives. Cayley finally related determinants and matrices in 1858, when he used them to describe points and lines in higher-dimensional geometry.

several calculators (including the TI-81 and TI-82) give $|A| = 0$, but the TI-85 returns a value of -2.4E-12, or $|A| = -0.0000000000024$. Obviously the TI-85 is programmed in a way that gives an approximation that is (very slightly) in error. This is not a criticism of the TI-85; every calculator will fail on some relatively simple similar example. What we need to recognize is the meaning of the result. When we see such a ridiculously small number, we should understand that the calculator is telling us (see Exercise 12) that the determinant of matrix A is equal to zero.

If you keep such calculator limitations in mind, you should not hesitate to use your calculator to check all determinant computations. The chances are very good that your calculator makes fewer arithmetic errors than you do, and the greatest source of error is probably entering numbers incorrectly or pressing a wrong key.

Why learn cofactor expansion? With all of the power and convenience of calculator computation, why shouldn't we rely entirely on technology? In addition to the fact that we cannot use technology wisely without having some feeling for what a machine is doing for us ("garbage in, garbage out"), it turns out that a number of the most important applications of determinants require the evaluation of highly symbolic determinants, where the result is not a number at all. In vector calculus and linear algebra and differential equations, it is necessary to know how to calculate and manipulate determinants; it is not enough to know what buttons to push to get a number.

In the next example we illustrate the use of a determinant involving unit vectors, $\boldsymbol{i}, \boldsymbol{j},$ and $\boldsymbol{k}$ that are used in physics and engineering. This particular example computes the cross product of two vectors, an operation that we do not discuss but that is used in calculus. Example 5 comes directly from linear algebra.

►EXAMPLE 4 ***A vector product*** *Looking Ahead to Calculus* Suppose $\boldsymbol{u} = \boldsymbol{i} + 2\boldsymbol{k}$, $\boldsymbol{v} = 3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ are vectors in 3-space. Then the cross product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 2 \\ 3 & -1 & 1 \end{vmatrix},$$

where the second and third rows are the components of $\boldsymbol{u}$ and $\boldsymbol{v}$. Use cofactor expansion by the first row to obtain the cross product in standard form.

Solution

Using the definition,

$$\begin{aligned} \boldsymbol{u} \times \boldsymbol{v} &= \boldsymbol{i}\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} \\ &= \boldsymbol{i}(0 + 2) - \boldsymbol{j}(1 - 6) + \boldsymbol{k}(-1 - 0) \\ &= 2\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}. \end{aligned}$$

This last expression describes a 3-dimensional vector that is perpendicular to the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. ◄

►EXAMPLE 5 ***A determinant equation*** **(a)** Expand the determinant and **(b)** solve the equation for x.

$$\begin{vmatrix} x - 1 & -4 & 2 \\ -3 & x - 2 & -4 \\ 0 & 0 & x + 1 \end{vmatrix} = 0$$

Solution

(a) Using the cofactor expansion by the last row (since there are two zeros), the determinant equals

$$\begin{aligned} 0 - 0 + (x + 1)\begin{vmatrix} x - 1 & -4 \\ -3 & x - 2 \end{vmatrix} &= (x + 1)((x - 1)(x - 2) - 12) \\ &= (x + 1)(x^2 - 3x - 10) = (x + 1)(x - 5)(x + 2). \end{aligned}$$

(b) The equation reduces to $(x + 1)(x - 5)(x + 2) = 0$, whose solutions are given by $x = -2, -1, 5$. We suggest that you check by substituting each x-value into the original determinant. ◄

Applications of Determinants

As suggested in the previous examples, applications of determinants abound in different areas of mathematics. We will see another in Section 9.6 when we use inverses of matrices for solving systems of linear equations. Determinants also provide a convenient way to do some things we have previously considered in this text, among them a way of writing an equation for a line through two given points and another way to compute the area of a triangle from the coordinates of its vertices. We are not interested here in deriving Equations (2) and (3), but are merely illustrating uses of determinants. Examples and exercises support the validity of these formulas.

Equation of a line

Given two points $P(a, b)$ and $Q(c, d)$, an equation for the line PQ may be written as

$$\begin{vmatrix} 1 & x & y \\ 1 & a & b \\ 1 & c & d \end{vmatrix} = 0. \tag{2}$$

Area of a triangle

Given ΔPQR with vertices $P(a, b)$, $Q(c, d)$, and $R(e, f)$ going around the triangle *counterclockwise,* then the area K of the triangle is given by

$$K = \frac{1}{2}\begin{vmatrix} 1 & a & b \\ 1 & c & d \\ 1 & e & f \end{vmatrix}. \tag{3}$$

If we disregard the order of vertices, then we must take the absolute value of the determinant.

►EXAMPLE 6 ***Determinant applications*** Given points $A(-1, 1)$, $B(0, -2)$, $C(5, 3)$.

(a) Verify that Equation (2) gives an equation for the line AC.
(b) Show that $\triangle ABC$ is a right triangle and verify that the area K of the triangle is given by Equation (3).

Solution

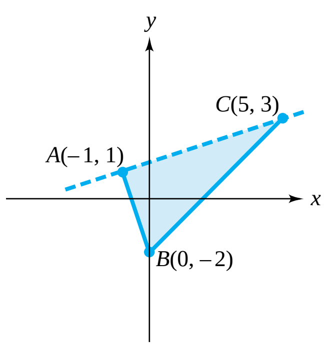


FIGURE 16

(a) Figure 16 shows $\triangle ABC$ and line AC. Substituting the coordinates of points A and C into Equation (2) and expanding by the first row gives us

$$\begin{vmatrix} 1 & x & y \\ 1 & -1 & 1 \\ 1 & 5 & 3 \end{vmatrix} = 1(-3 - 5) - x(3 - 1) + y(5 + 1) = 0,$$

or $-x + 3y = 4$, which is obviously an equation of a line. It is a simple task to verify that the coordinates of both A and C satisfy the equation, so Equation (2) is an equation for the line containing the points A and C.

(b) From the diagram in Figure 16 we see that the slope of line AC is $\frac{1}{3}$ and the slope of line AB is -3. Thus the lines are perpendicular and ΔABC is a right triangle. Using Equation (3), we can go around the triangle counterclockwise in order ABC (or, if we prefer, BCA or CAB). We have, using the first row for cofactor expansion,

$$K = \frac{1}{2}\begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 5 & 3 \end{vmatrix} = \frac{1}{2}(1(0 + 10) - (-1)(3 + 2) + 1(5 - 0)) = 10.$$

Because we have a right triangle with legs b and c, we can compute the area as $\frac{1}{2}bc$ as soon as we have those lengths.

$$b = |\overline{AC}| = \sqrt{6^2 + 2^2} = 2\sqrt{10}, \quad c = |\overline{AB}| = \sqrt{1^2 + 3^2} = \sqrt{10}.$$

Thus $K = \frac{1}{2}bc = \frac{1}{2}(2\sqrt{10})(\sqrt{10}) = 10$, in agreement with Equation (3). ◀

It is interesting to observe that Equation (3) does not depend on whether or not the triangle has a right angle. Equation (3) can be used with any triangle in the coordinate plane. To find the area of a general triangle without the use of a determinant would require considerably more work.

Cramer's Rule

We conclude this section by revisiting a topic we introduced in Section 9.1. There is a technique, known as Cramer's Rule, for solving systems of linear equations using determinants. In Section 9.1 we solved a 2×2 linear system directly and observed that the solution could be expressed in terms of what we now know are determinants. The same process works for any $n \times n$ linear system. For completeness, we state the theorem here in its more general form, but we do not recommend its use for larger systems. Computationally it is too inefficient. In the next section we will get a matrix approach that is very easy to implement with technology.

Cramer's rule

Given a system of n linear equations in variables $x_1, x_2, \ldots, x_n$, where A is the coefficient matrix and B is the column of constants, let $D = |A|$ and let D_i be the determinant of the matrix obtained by replacing the ith column of A by column B. If $D \neq 0$, the system has a unique solution given by

$$x_1 = \frac{D_1}{D}, \quad ex_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D}.$$

EXERCISES 9.5

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. The determinant of

$$\begin{bmatrix} 2 & -1 \\ 3 & -5 \end{bmatrix}$$

is equal to -7.

2. The only solution of the equation

$$\begin{vmatrix} x & -2 \\ 4 & 2 \end{vmatrix} = 6$$

is given by $x = -1$.

3. $\begin{vmatrix} 1 & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}.$

4. The solution of the equation

$$\begin{vmatrix} x & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 1 & 1 \end{vmatrix} = 3$$

is given by $x = -1$.

5. The solution set for the equation

$$\begin{vmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{vmatrix} = 1$$

is the empty set.

6. If every element of 2×2 matrix A is a positive number, then the determinant of A is a positive number.

Exercises 7–10 Fill in the blank so that the resulting statement is true. All questions refer to the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}.$$

7. The determinant of A is equal to ______.

8. The minor M_{31} is equal to ______.

9. The cofactor C_{11} is equal to ______.

10. The cofactor C_{12} is equal to ______.

Develop Mastery

Exercises 1–4 ***Cofactor Evaluation*** *Evaluate the indicated cofactors.*

1. $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -5 \\ 1 & 0 & -2 \end{vmatrix}$ Find C_{12}, C_{31}.

2. $\begin{vmatrix} -1 & 0 & 0 \\ 2 & 5 & 3 \\ 2 & -1 & 4 \end{vmatrix}$ Find C_{23}, C_{32}.

3. $\begin{vmatrix} 0 & -2 & \sqrt{3} \\ 5 & \sqrt{3} & 2 \\ -2 & 0 & 1 \end{vmatrix}$ Find C_{22}, C_{33}.

4. $\begin{vmatrix} 2 & -1 & e \\ 3 & e & -1 \\ 5 & 2 & -3 \end{vmatrix}$ Find C_{11}, C_{13}.

Exercises 5–12 ***Determinants by Cofactors*** *Evaluate the determinant of the given matrix. Use cofactors for the 3×3 matrices.*

5. $A = \begin{bmatrix} 3 & -5 \\ 2 & 5 \end{bmatrix}$

6. $A = \begin{bmatrix} 0 & 4 \\ -3 & 2 \end{bmatrix}$

7. $B = \begin{bmatrix} 6 & -4 & 1 \\ 7 & -4 & 1 \\ 6 & -3 & 1 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 0 & 1 & 5 \end{bmatrix}$

9. $B = \begin{bmatrix} 2 & 5 & 2 \\ 6 & 2 & -1 \\ 2 & 2 & 1 \end{bmatrix}$

10. $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

11. $C = \begin{bmatrix} 3 & -2 & 2 \\ 4 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

12. $D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Exercises 13–20 ***Determinants by Technology*** *Use a calculator to evaluate the determinant of the matrix.*

13. $A = \begin{bmatrix} 2 & -1 & 0 \\ \sqrt{3} & \sqrt{12} & \sqrt{27} \\ \sqrt{75} & \sqrt{48} & -\sqrt{3} \end{bmatrix}$

14. $B = \begin{bmatrix} 1 & 3 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 2 & 0 & 4 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

15. $C = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 3 & -1 & 2 & 1 \\ -2 & 1 & 4 & -1 \\ -1 & -2 & 3 & 1 \end{bmatrix}$

16. $A = \begin{bmatrix} -1 & 0 & -2 & 0 \\ -3 & 2 & 1 & -3 \\ 1 & -2 & -1 & 3 \\ 2 & 1 & 4 & -2 \end{bmatrix}$

17. $A = \begin{bmatrix} 0.3 & 0.7 & 1.2 \\ -0.8 & -1.3 & 0.4 \\ 0.0 & 1.0 & 2.1 \end{bmatrix}$

18. $D = \begin{bmatrix} 4 & 12 & 28 \\ 2 & -1 & 0 \\ 30 & 20 & 70 \end{bmatrix}$

19. $M = \begin{bmatrix} 1001 & 101 & 11 \\ 2001 & 201 & 21 \\ 4001 & 401 & 41 \end{bmatrix}$

20. $D = \begin{bmatrix} 17 & 0 & 0 & 0 \\ -83 & 20 & 0 & 0 \\ 25 & 100 & 500 & 0 \\ -6 & -8 & -10 & 5000 \end{bmatrix}$

Exercises 21–34 ***Solving for x*** *The equation involves the variable x.* ***(a)*** *Expand the determinant and* ***(b)*** *solve for x.*

21. $\begin{vmatrix} 2x & -4 \\ 3x & 2 \end{vmatrix} = 3$

22. $\begin{vmatrix} 3 & -4x \\ x & -5 \end{vmatrix} = 6$

23. $\begin{vmatrix} e^x & e \\ e & 1 \end{vmatrix} = 0$

24. $\begin{vmatrix} 1 & 0 & x \\ 3 & -1 & 2 \\ -5 & 3 & 0 \end{vmatrix} = -2$

25. $\begin{vmatrix} -2x & 1 & 0 \\ 0 & 3 & 2 \\ -x & 1 & 5 \end{vmatrix} = 4$

26. $\begin{vmatrix} x & 4 & 0 \\ 2 & 2 & -x \\ 1 & 1 & 1 \end{vmatrix} = 0$

27. $\begin{vmatrix} x & -2x & 1 \\ 3x & 1 & -2 \\ 2x & 2x+1 & -3 \end{vmatrix} = 0$

28. $\begin{vmatrix} e^x & 0 & 1 \\ -1 & 2 & -3 \\ 0 & 2 & -1 \end{vmatrix} = 4$

29. $\begin{vmatrix} 4\sin x & 1 & 0 \\ -1 & 1 & 3 \\ 2 & -1 & -1 \end{vmatrix} = -3$

30. $\begin{vmatrix} x & 0 & 1 \\ x & 2 & -1 \\ 1 & x & 1 \end{vmatrix} = 0$

31. $\begin{vmatrix} x-2 & 1 & 0 \\ 0 & x-3 & -1 \\ 3 & 1 & x+1 \end{vmatrix} = 0$

32. $\begin{vmatrix} x-1 & 3 & 2 \\ -2 & x-4 & -4 \\ 0 & -1 & x-2 \end{vmatrix} = 0$

33. $\begin{vmatrix} x & 0 & 0 \\ x & x+1 & 5 \\ 2 & 1 & x-1 \end{vmatrix} = 0$

34. $\begin{vmatrix} x-1 & 2 & 2 \\ 2 & x-2 & -2 \\ 1 & 0 & x \end{vmatrix} = 0$

Exercises 35–40 ***Cramer's Rule*** *Use Cramer's Rule to solve the system of equations. If the determinant of the coefficient matrix is zero, use Gaussian elimination.*

35. $6.3x + 2.1y = 18.9$
$1.5x + 3.4y = -4.2$

36. $2.4x - 5.2y = -8.0$
$1.6x + 2.4y = 6.4$

37. $371x + 285y = 2726$
$137x + 125y = 977$

38. $325x - 175y = -625$
$173x - 276y = 33$

39. $x + y - 2z = 0$
$3x - 2y - z = 0$
$-x + 4y - 3z = 0$

40. $x + 2y - z = 5$
$2x + y + 2z = 3$
$x - y + 3z = 0$

Exercises 41–48 ***Areas of Polygons*** *Find the area enclosed by the polygon with the given vertices. (Hint: If there are more than three vertices, break up the figure into triangles.)*

41. $A(1, 0), B(6, 4), C(8, 0)$

42. $A(1, 0), B(5, -2), C(7, 2)$

43. $A(2, 0), B(1, -2), C(4, 3)$

44. $A(5, 5), B(5, -5), C(0, -1)$

45. $A(1, 0), B(4, 6), C(8, 0), D(7, -3)$

46. $A(0, 0), B(4, 6), C(3, 0), D(5, -2)$

47. $A(0, 4), B(2, 4), C(0, 2), D(-2, 4)$

48. $A(0, 0), B(7, -3), C(8, 0), D(8, 6), E(4, 6)$

Exercises 49–52 ***Explore*** *Evaluate the three determinants. State a theorem about such determinants and explain why you think your theorem is true.*

49. (a) $\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}$ **(b)** $\begin{vmatrix} 5 & -3 & 2 \\ 0 & 0 & 0 \\ 4 & 1 & -3 \end{vmatrix}$

(c) $\begin{vmatrix} 5 & 3 & 0 & -1 \\ 0 & 2 & 0 & 5 \\ 4 & -6 & 0 & 8 \\ 5 & 0 & 0 & -2 \end{vmatrix}$

50. (a) $\begin{vmatrix} 2a & a \\ 3 & 4 \end{vmatrix}$ **(b)** $\begin{vmatrix} k & 5 \\ -2k & 3 \end{vmatrix}$ **(c)** $\begin{vmatrix} 2 & 1 \\ c & -c \end{vmatrix}$

51. (a) $\begin{vmatrix} 1 & a & 0 \\ 2 & -a & 3 \\ -1 & 2a & 1 \end{vmatrix}$ **(b)** $\begin{vmatrix} k & -k & k \\ 2 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 0 & -2 \\ 5 & 1 & 0 \\ c & 2c & c \end{vmatrix}$

52. (a) $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & -5 & 1 \end{vmatrix}$ **(b)** $\begin{vmatrix} 2 & 6 & -3 \\ 6 & 18 & -9 \\ 1 & 0 & 5 \end{vmatrix}$

(c) $\begin{vmatrix} 40 & -25 & 0 \\ 8 & -5 & 0 \\ 5 & 1 & -4 \end{vmatrix}$

(*Hint*: Consider the first two rows.)

Exercises 53–54 ***Lines Through Two Points*** *Use Equation (2) to find an equation for the line that passes through points P and Q.*

53. $P(-1, 2)\ Q(3, 4)$

54. $P(2, -3)\ Q(-3, 5)$

Exercises 55–56 ***Cross Product*** *Find the cross product of the vectors **u** and **v**. See Example 4.*

55. $\boldsymbol{u} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}, \quad \mathbf{v} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

56. $\boldsymbol{u} = \boldsymbol{i} - \boldsymbol{j}, \quad \mathbf{v} = 3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$

9.6 MATRIX ALGEBRA

> A genuine discovery should do more than merely conform to the facts: it should feel right, it should be beautiful. Aesthetic qualities are important in science, and necessary, I think, for great science.
>
> *Roger Penrose*

In Sections 9.2 and 9.5 we introduced some ideas related to matrices, but we did not discuss the algebra of matrices themselves. In this section we present a small portion of matrix algebra for solving systems of linear equations. We limit most of our discussion to 2×2 or 3×3 systems, but all of the essential ideas can be applied to larger systems, as well.

Matrix Equality

Since matrices have many entries, we need to know when two matrices are equal. Equality requires not only that the matrices are the same size, but that all corresponding entries be the same.

Definition: equality of matrices

Matrices A and B are equal, written $A = B$, if and only if

1. A and B have the same size, and
2. each entry in A is equal to the corresponding entry in B: $a_{ij} = b_{ij}$.

Matrix Product

The product of two matrices is probably most easily introduced with an example.

[In college] there were no women teaching mathematics but I remember women teaching biology and psychology. Naturally I elected to major in mathematics. [Most math] students [were] planning to be engineers. There were also girls who were going to be teachers. At that time I had no idea that such a thing as a mathematician (as opposed to math teacher) existed.
Julia Robinson

►EXAMPLE 1 ***Sales by matrix multiplication*** A bicycle dealer has three outlets, one downtown, one in a mall, and one at a nearby resort. A special mountain bike sale features three brands of bikes with these sale prices: Hoppit (\$375), Runner (\$425), Climber (\$315). The numbers of bikes sold at the three outlets during the special promotion are displayed in a matrix:

$$\begin{array}{l c} & \begin{array}{ccc} H & R & C \end{array} \\ \begin{array}{l} \text{Downtown} \\ \text{Mall} \\ \text{Resort} \end{array} & \begin{bmatrix} 8 & 7 & 12 \\ 4 & 14 & 9 \\ 5 & 8 & 16 \end{bmatrix} \end{array}$$

Find the sales total in dollars at each outlet.

Solution

We could find the desired information without using matrices. The dollar total from the downtown store is $8(\$375) + 7(\$425) + 12(\$315) = \$9{,}755$, and the same operations will give us the gross sales figures for the mall store (\$10,285) and the resort store (\$10,315). Matrix multiplication is defined to do precisely these operations. Let A and B be matrices.

$$A = \begin{bmatrix} 8 & 7 & 12 \\ 4 & 14 & 9 \\ 5 & 8 & 16 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 375 \\ 425 \\ 315 \end{bmatrix}$$

The product AB is a 3×1 matrix C:

$$AB = \begin{bmatrix} 8 & 7 & 12 \\ 4 & 14 & 9 \\ 5 & 8 & 16 \end{bmatrix} \cdot \begin{bmatrix} 375 \\ 425 \\ 315 \end{bmatrix}$$
$$= \begin{bmatrix} 8 \cdot 375 + 7 \cdot 425 + 12 \cdot 315 \\ 4 \cdot 375 + 14 \cdot 425 + 9 \cdot 315 \\ 5 \cdot 375 + 8 \cdot 425 + 16 \cdot 315 \end{bmatrix} = \begin{bmatrix} 9{,}755 \\ 10{,}285 \\ 10{,}315 \end{bmatrix} = C$$

From the matrix C, read off sales totals: $c_{11} = \$9{,}755$ (downtown), $c_{21} = \$10{,}285$ (mall), and $c_{31} = \$10{,}315$ (resort). ◀

$$\overrightarrow{[a \quad b \quad c]} \times \begin{bmatrix} r \\ s \\ t \end{bmatrix}\Big\downarrow = ar + bs + ct$$

Row-by-column product is a number.

The matrix product in Example 1 is sometimes called a *row-by-column product*. Each entry in product AB is obtained by multiplying the entries of a row of A by the entries of a column of B, and each entry c_{ij} of the product is the sum of the products of the entries in the ith row of A with the corresponding entries of the jth column of B. More specifically, c_{11} is given by $c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$. Similarly, c_{21} comes from the second row of A and the first column of B: $c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$, and $c_{31} = a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}$. See the illustration in the margin.

The row-by-column idea defines the product of two matrices in general. The product AB requires that the number of entries in each row of A matches the number of entries in each column of B. It is easy to see in a particular example whether or not A and B allow multiplication, but we can also read the information from the dimensions of A and B.

Definition: product of two matrices

Let A be an $m \times k$ matrix and B be a $k \times n$ matrix. The product AB is an $m \times n$ matrix C, where the entry c_{ij} is obtained by multiplying the entries of the ith row of A by the corresponding entries of the jth column of B and then adding the resulting products:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ik}b_{kj}.$$

Strategy: **(a)** Using the row-by-column definition, if $AB = C$, then $c_{11} = 1 \cdot 4 + (-2)0 + 0(-2) = 4$, and so on.

▶EXAMPLE 2 ***AB* ≠ *BA*** Find the products AB and BA if matrices A and B are given by

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ 2 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 1 & 3 \\ -2 & 1 & -1 \end{bmatrix}.$$

Solution
Follow the strategy.

$$AB = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ 2 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & 2 \\ 0 & 1 & 3 \\ -2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 4 + (-2)0 + 0(-2) & 1(-1) + (-2)1 + 0 \cdot 1 & 1 \cdot 2 + (-2)3 + 0(-1) \\ 3 \cdot 4 + 2 \cdot 0 + (-1)(-2) & 3(-1) + 2 \cdot 1 + (-1)1 & 3 \cdot 2 + 2 \cdot 3 + (-1)(-1) \\ 2 \cdot 4 + 0 \cdot 0 + (-1)(-2) & 2(-1) + 0 \cdot 1 + (-1)1 & 2 \cdot 2 + 0 \cdot 3 + (-1)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & -4 \\ 14 & -2 & 13 \\ 10 & -3 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 1 & 3 \\ -2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -10 & -1 \\ 9 & 2 & -4 \\ -1 & 6 & 0 \end{bmatrix} \blacktriangleleft$$

In the solution to Example 2 note that $AB \neq BA$, which implies that *matrix multiplication is not necessarily commutative.*

▶EXAMPLE 3 Associativity Matrices A, B, and C are

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}.$$

Find the matrix products **(a)** $(AB)C$ and **(b)** $A(BC)$.

Strategy: **(a)** First find AB, then multiply the result by C (with C on the right) to get $(AB)C$.

Solution
Follow the strategy.

(a) $$(AB)C = \left(\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix}\right)\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 \\ 8 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -4 & 6 \end{bmatrix}$$

(b) $$A(BC) = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}\left(\begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 7 & 12 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -4 & 6 \end{bmatrix}. \blacktriangleleft$$

The solution to Example 3 illustrates a general property of matrix multiplication: *matrix multiplication is associative.* Whenever the products are defined, $(AB)C = A(BC)$.

▶EXAMPLE 4 Identity Matrices A and B are given by

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find the matrix products **(a)** AB and **(b)** BA.

Solution

(a) $AB = \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}$ ◀

The solution to Example 4 shows that $AB = BA = A$, so the matrix B acts much like the number 1 in ordinary arithmetic ($a \cdot 1 = 1 \cdot a = a$). For any 2×2 matrix, C, $CB = BC = C$, and we call B the **identity matrix** for the set of 2×2 matrices. It is customary to denote the identity matrix by the letter I. There is an identity matrix of size $n \times n$ for every dimension n. The 3×3 identity is the matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The same letter I can denote the identity matrix of any size under discussion, but the context should make it clear which size identity is intended.

▶EXAMPLE 5 Inverses Find matrix products AB and BA, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -5 \\ -2 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & -1 \\ 5 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$$

Solution

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -5 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 1 & -1 \\ 5 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 1 & -1 \\ 5 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -5 \\ -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ◀

The product of the two matrices in Example 5 (in either order) is the identity matrix. In the set of real numbers two numbers whose product is 1 are called *reciprocals* or *multiplicative inverses* of each other. We use the same terms in matrix algebra. If $AB = BA = I$, then A and B are inverses of each other, $B = A^{-1}$. In general, $AA^{-1} = A^{-1}A = I$. Not all matrices have inverses, but every square matrix with a nonzero determinant does have an inverse.

We sum up our discussion so far in a list of some properties of matrix algebra.

Properties of matrix algebra

1. In general, matrix multiplication is *not* commutative: $AB \neq BA$.
2. Matrix multiplication is associative: $(AB)C = A(BC)$.
3. The square matrix I with 1s on the main diagonal and 0s everywhere else is an identity matrix: $AI = IA = A$.
4. Any square matrix A with a nonzero determinant has an inverse: $AA^{-1} = A^{-1}A = I$.
5. The matrix kA is obtained by multiplying every entry of A by the number k.

Finding the Inverse of a Square Matrix without Technology

Matrix inverses have several important applications. Among them is another technique for solving systems of linear equations. To use the technique we need a method for finding the inverse of a matrix. The following algorithm is simple and relatively efficient.

Algorithm to find the inverse of a square matrix

Suppose A is a square matrix with a nonzero determinant.

1. Adjoin the identity matrix to the right of A, getting a matrix with the structure $[A \mid I]$.
2. Use elementary row operations on $[A \mid I]$ to get a matrix of the form $[I \mid B]$.
3. The inverse of A is the matrix B.

We illustrate the algorithm with matrix A of Example 5.

$$[A \mid I] = \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ -5 & 1 & -5 & 0 & 1 & 0 \\ -2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} 5R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$(-1)R_2 + R_3 \to R_3 \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right]$$

$$(-1)R_3 + R_1 \to R_1 \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 4 & 1 & -1 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right]$$

The last matrix has the form $[I \mid B]$, so

$$A^{-1} = B = \begin{bmatrix} 4 & 1 & -1 \\ 5 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix},$$

as we found in Example 5, which showed that $AB = I$.

Solving Systems of Linear Equations

We stated that a goal of this section was to develop the matrix algebra needed to express an $n \times n$ system of linear equations as a matrix equation and then to use matrix algebra to solve the system. Two examples illustrate this process.

▶EXAMPLE 6 ***Matrix form of linear system*** For the matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -5 \\ -2 & 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix},$$

(a) write the matrix product AX, and
(b) write the system of linear equations that results if $AX = C$.

Solution

(a)

$$AX = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -5 \\ -2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + \quad z \\ -5x + y - 5z \\ -2x + y - z \end{bmatrix}$$

(b) If $AX = C$, then

$$\begin{bmatrix} x \quad + z \\ -5x + y - 5z \\ -2x + y - z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \quad \text{so} \quad \begin{aligned} x \quad + z &= 3 \\ -5x + y - 5z &= -2 \\ -2x + y - z &= 4 \end{aligned}$$

◀

Using Technology to Solve Matrix Equations

Any system of linear equations can be expressed as a matrix equation $AX = B$ as in Example 6. If the coefficient matrix A is square and has an inverse, then we can solve the system, at least symbolically. We simply multiply both sides of the equation *on the left* by A^{-1} and use the associative property of matrix multiplication.

$$A^{-1}(AX) = A^{-1}B, \qquad (A^{-1}A)X = A^{-1}B, \qquad IX = A^{-1}B, \quad \text{or} \quad X = A^{-1}B.$$

That is, given the matrix equation $AX = B$, as long as A has an inverse, we can solve the system by premultiplying B by A^{-1}. This is a tremendous boon if we can use technology to find the inverse and perform the matrix multiplication. Therein, of course, lies the rub. Finding the inverse of a large matrix can tax the most sophisticated computer software and finding better ways to manipulate linear systems to improve methods of solution continues as an area of active mathematical research.

If we recognize the limitations, though, the technology we have available to us allows us to solve a great many linear system problems efficiently and easily. Determinants continue to play a role in solving a system. It turns out that the inverse of a matrix involves division by the determinant of the matrix. When the determinant of a matrix is zero, there is no inverse, and if the determinant is near zero, there is a possibility of substantial error in approximations.

A warning example. Consider the following system:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

We mentioned in Section 9.5 that the TI-85 evaluates the determinant of the coefficient matrix A as −2.4E-12. We should recognize that such a number indicates that the determinant is equal to 0 and that hence that A has no inverse. If, however, we disregard our warning, enter the matrix A and the column matrix B and go ahead and compute $A^{-1}B$, the calculator immediately returns an answer. We read that $x =$ 6.25E12, $y =$ − 1.25E12, and $z =$ 6.25E12, a highly suspicious result to say the least. If we want to check by multiplying the result by A, we should get B again, but the calculator indicates that the entries are 1.1, −3, −6, nowhere close to the numbers 2, −1, 1 in B. If we look at A^{-1}, we see *huge* numbers (suspicious again), and if we calculate AA^{-1} or $A^{-1}A$, we get reasonable looking numbers but *not* the identity matrix.

We emphasize again that this problem is part of *any* computing technology, not a peculiarity of the TI-85. With a little work we can find such an example for any calculator. What we must do is work within the limitations of our technology. With care, we can take advantage of the power we have available, as outlined below.

Solving a matrix system $AX = B$

1. Enter the coefficient matrix A in your calculator.
2. Compute the determinant of A. If $\det A = 0$ or if the calculator shows $\det A$ as a very small number, **stop.** Use Gaussian elimination to solve the system.
3. If you are confident that $\det A \neq 0$, then enter the column matrix B.
4. On your home screen evaluate $A^{-1}B$. The result is the solution matrix.
5. Check by premultiplying your result by A. The product should be B.

TECHNOLOGY TIP ◆ Calculating $A^{-1}B$

Note that to solve the system $AX = B$, we need not necessarily even compute A^{-1}, although some calculators do compute and display A^{-1} in the process.

TI-calculators Having entered the matrix A, on the home screen, we just enter A (or [A]) and the x^{-1} key, followed by B, and ENTER.

Casio fx-7700 only handles three matrices. After entering matrix A, press F4 to evaluate A^{-1}, which is displayed in the C register. We want it in the A register, so F1 performs the interchange. Then after putting in B, press PRE. Then F5 does the multiplication and puts the product in C.

Casio fx-9700 allows us to store several matrices, so after entering A and B, we EXIT twice and simply press F1 ALPHA A X^{-1} F1 ALPHA B to display Mat A^{-1} × Mat B and EXE.

HP-38 Having entered matrix A as $M1$ and B as $M2$, return to the home screen. On the command line, type M1^−1*M2 (use the A...Z key for M, the
d inverse key for ^−1), and ENTER.

HP-48 Having entered A on the stack, 1/X computes the matrix A^{-1} and enters it in place of A. Then put B on the stack in Level 1 and press the amultiplication key (the order of matrix multiplication on the HP-48 is Level 2 × Level 1).

We strongly suggest that you check your ability to handle matrices by doing all of the calculations in Example 7.

►EXAMPLE 7 *Matrix solution* Use a graphing calculator to find the inverse of the coefficient matrix, and solve the system.

$$\begin{aligned} x \phantom{{}+y} + z &= 3 \\ -5x + y - 5z &= -2 \\ -2x + y - z &= 4 \end{aligned}$$

Solution

For this system, $A = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -5 \\ -2 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$. The calculator shows the inverse of A as $A^{-1} = \begin{bmatrix} 4 & 1 & -1 \\ 5 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$. As a check, see Example 5. The solution is given by $A^{-1}B = \begin{bmatrix} 6 \\ 13 \\ -3 \end{bmatrix}$, so $x = 6$, $y = 13$, $z = -3$. ◀

EXERCISES 9.6

Check Your Understanding

Exercises 1–10 True or False. Give reasons.

1. If $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
2. The inverse of $\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}$ is $\begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}$.
3. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $A^{-1} = A$.
4. If $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, then $BA = AB$.
5. If $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, then $BA = AB$.

Exercises 6–10 Let $A = [-1, 3]$

$B = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$.

6. The only entry in AB is positive.
7. AC is a square matrix.
8. All entries in BA are negative.
9. $(BA)C = B(AC)$
10. $A(BC)$ is undefined.

Develop Mastery

Exercises 1–4 ***Matrix Notation*** *(a) Give the dimension of matrix A and* **(b)** *find a_{12} and a_{21} when possible. If this is not possible, explain why.*

1. $A = \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix}$
2. $A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$
3. $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 1 & 2 & 4 \end{bmatrix}$
4. $A = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$

Exercises 5–12 ***Matrix Products*** *Evaluate the matrix product when possible; if the product is not defined, explain why. Use the matrices*

$$A = \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 & 2 \\ 3 & -1 & 4 \\ -2 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} 0 & -1 & 2 \\ 3 & -2 & 1 \\ 4 & 0 & 2 \end{bmatrix}$$

5. AB
6. BA
7. CD
8. AE
9. EA
10. CF
11. FC
12. $A(EA)$

Exercises 13–20 ***Matrix Inverse*** *Use the algorithm of this section to find the inverse of the matrix if it has an inverse; if it has no inverse, explain how you know. Check by technology.*

13. $A = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$
14. $B = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$
15. $C = \begin{bmatrix} 1 & 0 \\ 6 & 2 \end{bmatrix}$
16. $A = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$
17. $B = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$
18. $A = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$
19. $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 3 \\ 1 & 2 & 3 \end{bmatrix}$
20. $B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ -2 & 6 & -3 \end{bmatrix}$

Exercises 21–28 ***Inverses by Calculator*** *Use your calculator to find A^{-1} If A has an inverse; if not, explain why. (Hint: It may help to evaluate (det A) A^{-1}.)*

21. $A = \begin{bmatrix} -4 & 2 & -3 \\ 10 & -5 & 8 \\ -1 & 1 & -1 \end{bmatrix}$
22. $A = \begin{bmatrix} 4 & 1 & -1 \\ 5 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$

23. $A = \begin{bmatrix} -3 & -1 & 1 \\ 2 & 1 & 2 \\ 5 & 2 & 0 \end{bmatrix}$

24. $A = \begin{bmatrix} 1 & 7 & -3 \\ 2 & 4 & 1 \\ -4 & -8 & -9 \end{bmatrix}$

25. $A = \begin{bmatrix} 2 & 0 & 2 \\ -4 & 1 & -4 \\ -2 & 1 & -1 \end{bmatrix}$

26. $A = \begin{bmatrix} 57 & 93 & 179 \\ 412 & -611 & 84 \\ -4255 & 13589 & 5995 \end{bmatrix}$

27. $A = \begin{bmatrix} 4 & 1 & -1 & 3 \\ 5 & 1 & 0 & 5 \\ 3 & -1 & 1 & -2 \\ 2 & 0 & 1 & 5 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 5 & 1 & -1 & 1 \\ 0 & 4 & 1 & 1 \\ -4 & 7 & 5 & 3 \end{bmatrix}$

Exercises 29–40 ***Matrix Systems*** *(**a**) Evaluate $|A|$. (**b**) If A has an inverse, find A^{-1} and solve the system of equations by solving the matrix equation $AX = C$.*

29. $\begin{aligned} 3x + 4y &= 2 \\ -7x - 9y &= 3 \end{aligned}$

30. $\begin{aligned} x + 3y &= 4 \\ 3x + 5y &= -2 \end{aligned}$

31. $\begin{aligned} -3x + 2y &= 4 \\ 5x - 3y &= -1 \end{aligned}$

32. $\begin{aligned} -2x + y &= 3 \\ -5x + 3y &= 1 \end{aligned}$

33. $\begin{aligned} x - y &= 0 \\ -y + z &= 4 \\ -2x + 6y - 3z &= 1 \end{aligned}$

34. $\begin{aligned} x + 2y + 2z &= -1 \\ x + 3y + 2z &= -2 \\ 2x + 6y + 5z &= 3 \end{aligned}$

35. $\begin{aligned} x + 2y + 4z &= -1 \\ x + 3y + 3z &= 2 \\ x + 2y + 3z &= -4 \end{aligned}$

36. $\begin{aligned} x + 7y - 3z &= 0 \\ 2x + 4y + z &= -4 \\ -4x - 8y - 9z &= 22 \end{aligned}$

37. $\begin{aligned} -3x - y + z &= 2 \\ 2x + y + 2z &= -1 \\ -x + 3z &= 0 \end{aligned}$

38. $\begin{aligned} 2x - 2y + 3z &= 15 \\ x + y - z + 2w &= -6 \\ x + y - z + w &= -4 \\ 2x + y - z + 3w &= -6 \end{aligned}$

39. $\begin{aligned} x - y + 4z - w &= 4 \\ 2x + y - 3z + 5w &= -1 \\ 4x + 3z - 2w &= 13 \\ -2x + 4y - 3z &= 5 \end{aligned}$

40. $\begin{aligned} x + y + z + w + v &= 4 \\ 2x - y + 3z + w - v &= -2 \\ 5x + y + w + 2v &= 11 \\ x - 2y - z - 2w &= -7 \\ -x + 3y + 2z - 6w + 4v &= 3 \end{aligned}$

Exercises 41–43 ***Find a Circle*** *For the circle that passes through the three points, (**a**) write an equation in the form $x^2 + y^2 + bx + cy = d$ and (**b**) find the radius and the coordinates of the center.*

41. $(-1, -2), (5, 6), (6, 5)$

42. $(-1, -1), (0, 2), (2, 2)$

43. $(0, 2), (7, 1), (8, -2)$

Exercises 44–46 ***Find a Parabola*** *For the parabola that passes through the three points, (**a**) write an equation in the form $y = ax^2 + bx + c$ and (**b**) find the coordinates of the x-intercept points and the vertex.*

44. $(0, 1), (1, -2), (2, -3)$

45. $(-1, -1), (0, 2), (2, 2)$

46. $(0, 7), (1, 1), (2, -1)$

47. The height from ground level of an object is given by an equation of the form $h(t) = at^2 + bt + c$, where t is the time in seconds and h is measured in feet.
(a) Find a, b, and c, if $h(1) = 240$, $h(2) = 246$, and $h(3) = 248$.
(b) At what time will the object be at ground level?

Exercises 48–51 ***Inverse of a Product*** *For matrices A and B, find (**a**) AB, (**b**) $(AB)^{-1}$, (**c**) $A^{-1}B^{-1}$* and *$B^{-1}A^{-1}$.*

48. $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

49. $A = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

50. $A = \begin{bmatrix} -4 & 2 & -3 \\ 10 & -5 & 8 \\ -1 & 1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

51. $A = \begin{bmatrix} 3 & 2 & -2 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -3 & -1 & 1 \\ 2 & 1 & 2 \\ 5 & 2 & 0 \end{bmatrix}$

Exercises 52–63 ***Powers of a Matrix*** *For matrix A, find (**a**) A^2 $(= A \cdot A)$, (**b**) A^3, (**c**) A^{16}, and (**d**) A^{47}.*

52. $A = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$

53. $A = \begin{bmatrix} \frac{1}{4} & \frac{5}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$

54. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

55. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & 5 & -4 \end{bmatrix}$

56. $A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 5 & 3 \\ -6 & -12 & -7 \end{bmatrix}$

57. $A = \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix}$

58. $A = \begin{bmatrix} 6 & 10 \\ -3 & -5 \end{bmatrix}$

59. $A = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$

60. $A = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

61. $A = \begin{bmatrix} -2 & 1 & -3 \\ 6 & -3 & 9 \\ 4 & -2 & 6 \end{bmatrix}$

62. $A = \begin{bmatrix} -6 & 8 & 2 \\ -3 & 4 & 1 \\ -9 & 12 & 3 \end{bmatrix}$

63. $A = \begin{bmatrix} 2 & 6 & 2 \\ -1 & -3 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

Exercises 64–65 ***Multiple of I*** *For the given matrix, show that the given expression is equal to some multiple of the identity matrix.*

64. $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}; 13A - A^3$

65. $B = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}; 3B - B^2$

Exercises 66–69 ***Your Choice*** *Find two* 2×2 *matrices A and B such that the first row of A is* $1, -1$ *and such that A and B satisfy the given condition.*

66. $AB = I$

67. $AB = BA$

68. $AB \neq BA$

69. A and B have no zero entries but AB is the zero matrix.

70. ***Explore: Generating the Fibonacci Sequence*** Let F be the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and let $A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) Compute the first few powers of F by entering F and then iterating $F*ANS$.

(b) Guess a formula for F^n in terms of the Fibonacci sequence.

(c) Enter A and iterate $F*ANS$. How would you describe the entries you observe?

(d) Describe how to generate the Lucas sequence, which is defined recursively by $L_0 = 1$, $L_1 = 3$, and $L_{n+1} = L_n + L_{n-1}$, $n \geq 1$.

CHAPTER 9 REVIEW

Test Your Understanding

True or False. Give reasons.

1. The equation $3x - 4y + z = 7$ is linear in x, y, and z.

2. The equation $\sqrt{3}x - \sqrt{5}y = \sqrt{6}$ is not a linear equation in x and y.

3. The system

$$\begin{aligned} 2x - 3y &= 5 \\ -4x + 6y &= 7 \end{aligned}$$

has infinitely many solutions. It is dependent.

4. The solution for the system

$$\begin{aligned} x - 2y - 3z &= 4 \\ y - 2z &= 6 \\ 3z &= -9 \end{aligned}$$

is given by $x = -5$, $y = 0$, $z = -3$.

Exercises 5–8 *Refer to the system of inequalities:*

$$\begin{aligned} 2x - y &\leq 0 \\ 2x + y &\geq 4 \end{aligned}$$

5. Point $(0, 1)$ is in the solution set.

6. Point $(2, 4)$ is not in the solution set.

7. Point $(1, 2)$ is a corner point.

8. The solution set contains no points in Quadrants III or IV.

Exercises 9–12 *Lines* L_1 *and* L_2 *are given by*

$$L_1:\quad x + 2y = 0 \qquad L_2:\quad 3x - 4y = -5$$

9. Point $(-2, 1)$ is on both L_1 and L_2.

10. Point $(-1, \frac{1}{2})$ is on both L_1 and L_2.

11. Point $(1, 2)$ is on L_2, but not on L_1.

12. Point $(0, 1)$ is above L_1 and below L_2.

Exercises 13–18 *Let G be the set of all points* (x, y) *that satisfy the system*

$$\begin{aligned} x - 2y &\geq -6 \\ x + y &\geq -3 \\ 7x - 2y &\leq 6 \end{aligned}$$

13. Point $(0, 3)$ is in G.

14. Point $(0, 0)$ is in G.

15. Point $(0, -3)$ is a corner point of G.

16. Point $(-4, 1)$ is not a corner point of G.

17. Point $(2, 4)$ is not in G.

18. There is no point on the line $2x + y = 0$ that is also in G.

Exercises 19–23 *Line L and parabola P are given by*

$$L:\quad x - 2y = -1 \qquad P:\quad y = x^2 - 1$$

19. There is exactly one point that is on both L and P.

20. There are exactly two points that are on both L and P.

21. Point (1, 1) is on both L and P.

22. Point $(-1, 0)$ is not on both L and P.

23. Point $(0, -1)$ is on L, but not on P.

24. If $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

25. If $A = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}$, then $AB = I$.

Exercises 26–31 *Let G be the graph of the equation given by*

$$\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

26. G is a line.

27. Point (2, 3) is on G.

28. Point $(-1, -1)$ is on G.

29. G is a line with slope $\frac{4}{3}$.

30. The x-intercept point of G is $(\frac{1}{2}, 0)$.

31. The y-intercept point of G is $(0, -\frac{2}{3})$.

32. The equation $\begin{vmatrix} x^2 & 3 \\ -1 & 1 \end{vmatrix} = 4$ has two real solutions.

33. The equation $3 + \begin{vmatrix} x & 1 \\ -1 & 1 \end{vmatrix} = 4$ has no real solutions.

Exercises 34–36 *Assume that* $A = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$.

34. $|A| = -11$

35. $A^{-1} = \begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix}$

36. The solution the matrix equation $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ is $\begin{bmatrix} 11 \\ 30 \end{bmatrix}$.

Exercises 37–40 *Assume that* $A = \begin{bmatrix} 3 & 2 & 2 \\ 3 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}$.

37. $|A|$ is a positive number.

38. Matrix A has an inverse all of whose entries are positive.

39. The solution to the matrix equation $AX = C$, where $C = \begin{bmatrix} 0 \\ -3 \\ 4 \end{bmatrix}$, is given by $x = -8$, $y = -27$, $z = -15$.

40. Not all entries of A^2 are positive.

Review for Mastery

Exercises 1–9 ***Linear Systems*** *Solve the system of equations. If it is dependent (has infinitely many solutions), describe all solutions and then give two specific ones.*

1. $\begin{aligned} 3x - 2y &= 5 \\ x - y &= -1 \end{aligned}$

2. $\begin{aligned} -2x + y &= 3 \\ 5x - 3y &= -4 \end{aligned}$

3. $\begin{aligned} \frac{x}{2} - \frac{y}{3} &= 4 \\ \frac{x}{4} + \frac{y}{2} &= -2 \end{aligned}$

4. $\begin{aligned} 0.4x + 0.6y &= 0 \\ 0.8x - 1.2y &= 2 \end{aligned}$

5. $\begin{aligned} x - 2y + z &= 3 \\ -2x + y - z &= 0 \\ 4x - 3y + 2z &= 1 \end{aligned}$

6. $\begin{aligned} x + 2y &= 2 \\ 3x - 4y + z &= -2 \\ x + 3z &= -8 \end{aligned}$

7. $\begin{aligned} x + 2y - 5z &= 1 \\ 3x + 2y + z &= -2 \\ 3x - 2y + 17z &= -7 \end{aligned}$

8. $\begin{aligned} x - y + z &= 3 \\ 5x - 4y + 3z &= 2 \\ x - 2y + 3z &= 16 \end{aligned}$

9. $\begin{aligned} \frac{1}{x} - \frac{2}{y} &= \frac{1}{3} \\ \frac{2}{x} - \frac{5}{y} &= -\frac{2}{5} \end{aligned}$

Exercises 10–15 ***Solve and Graph*** *Solve the system of equations and draw graphs to illustrate the solution graphically.*

10. $\begin{aligned} y &= -3x + 4 \\ y &= x^2 \end{aligned}$

11. $\begin{aligned} 2x - 3y &= -26 \\ x^2 + y^2 &= 169 \end{aligned}$

12. $\begin{aligned} y &= -2x \\ y &= -x^2 - 3x \end{aligned}$

13. $\begin{aligned} 2y &= x + 2 \\ xy &= 4 \end{aligned}$

14. $\begin{aligned} x + y &= 4 \\ x^2 + y^2 &= 4 \end{aligned}$

15. $\begin{aligned} y &= x^2 - 4x + 4 \\ y &= -2x^2 + 5x + 4 \end{aligned}$

Exercises 16–19 ***Graphical Inequalities*** *Draw a graph of the set of points (x, y) that satisfy the inequality or inequalities.*

16. $2x - y < 1$

17. $x + y > 1$ and $2x - y < 5$

18. $y \leq x$ and $x - y < 2$

19. $2x + y < 4$ and $x - 2y \geq 1$

Exercises 20–22 ***System of Inequalities*** *Draw a graph of the region described by the system of inequalities, identifying all corner points.*

20. $\begin{aligned} x - y &\leq 4 \\ 2x + y &\geq 2 \\ x + 2y &\leq 4 \end{aligned}$

21. $\begin{aligned} 2x - y &\geq 8 \\ 2x + y &\leq 4 \\ x - y &\leq 8 \end{aligned}$

22. $\begin{aligned} y &\geq x - 1 \\ x + 2y &\leq 10 \end{aligned}$

Exercises 23–25 ***Solution Set with Determinants*** *Find the solution set.*

23. $\begin{vmatrix} 2^x & 1 \\ 1 & 2 \end{vmatrix} = 7$

24. $\begin{vmatrix} x & 0 & 1 \\ 0 & x & 0 \\ 1 & 1 & x \end{vmatrix} = 0$

25. $\begin{vmatrix} x-1 & x & 0 \\ 1 & 2 & x \\ 0 & 1 & x+1 \end{vmatrix} = 5x$

Exercises 26–31 ***Matrix Products and Equations*** *Use the matrices:*

$$A = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

26. Find **(a)** AB **(b)** BA.

27. Find **(a)** A^{-1} **(b)** B^{-1}.

28. Find **(a)** $(AB)^{-1}$ **(b)** $B^{-1}A^{-1}$.

29. Find **(a)** $(BA)^{-1}$ **(b)** $A^{-1}B^{-1}$.

30. **(a)** Express the matrix equation $AX = C$ as a system of equations

(b) Solve the system using $X = A^{-1}C$.

31. Solve the matrix equation $BX = C$ for X.

32. Find the inverse of the matrix A where

$$A = \begin{bmatrix} 4 & 5 & -3 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

33. Use the result in Exercise 32 to solve the system of equations

$$\begin{aligned} 4x + 5y - 3z &= 1 \\ x + y - z &= 3 \\ -x + z &= -4 \end{aligned}$$

Exercises 34–37 ***Linear Programming*** *A system of linear constraints is given.* ***(a)*** *Draw a graph to show the feasible set F and determine the corner points.* ***(b)*** *For the given objective function, determine the point in F that will give the indicated optimal solution.*

34. Constraints: $x - y \geq 0$, $4x + 2y \leq 5$, $y \geq 0$. Objective function: $z = 5x - 2y$; maximum.

35. Constraints: $x - y \geq 0$, $4x + 2y \leq 5$, $y \geq 0$. Objective function: $z = 5x + 4y$; maximum.

36. Constraints: $y \leq x + 2$, $y \geq 2x - 1$, $y \geq -x + 2$. Objective function: $z = 2x + 3y$; maximum.

37. Constraints: $y \leq x + 2$, $y \geq 2x - 1$, $y \geq -x + 2$. Objective function: $z = 3x + 4y$; minimum.

Exercises 38–39 ***Area*** *Find the area enclosed by the polygon with the given vertices.*

38. $A(5, 4)$, $B(2, -1)$, $C(0, 1)$

39. $A(5, 4)$, $B(2, -1)$, $C(0, 1)$, $D(2, 1)$

40. ***Mixture Problem*** If 1 cup of oatmeal contains 5 grams of protein and 20 milligrams of calcium, and 1 cup of milk contains 8 grams of protein and 300 milligrams of calcium, determine the amount (in cups) of oatmeal and milk that will give a serving that contains 12 grams of protein and 383 milligrams of calcium.

41. ***Ticket Sales*** A musical sponsored by the student association is to be held in the school auditorium, which seats 1500. Ticket prices are \$5 each for the 500 reserved seats and \$3 each for the remaining 1000 general admission seats. The cost to present the musical will be \$3700. How many reserved seat tickets and how many general admission tickets must be sold to cover the cost of the production?

42. ***Filling a Reservoir*** Two pipelines, A and B, supply water to a reservoir, while pipeline C (located at the bottom) drains the reservoir. When all three pipelines are open it takes 18 hours to fill the reservoir. If A and B are open and C is closed, it takes 12 hours to fill the reservoir. If A and C are open and B is closed, it takes 24 hours to fill the reservoir. How many hours does it take to fill the reservoir if only A is open?

Exercises 43–45 ***Linear Programming***

43. A producer of lawn fertilizer makes two different kinds. Type A contains 20 percent nitrogen and 10 percent potash, while type B contains 10 percent nitrogen and 4 percent potash. The firm has a sufficient supply of each and wishes to put together a mixture that contains a total of at least 240 kg. The mixture should also contain at least 15 percent nitrogen and not more than 8 percent potash. The costs per kilogram of A and B are 20 cents and 15 cents, respectively.

(a) Draw a graph showing the amounts of each that will give the desired mixture.

(b) For each corner point of the graph, find the corresponding cost of the mixture.

44. A computer manufacturer has orders from two retail stores, one in Harmony and one in Gladstone. The Harmony store has ordered 50 computers and the Gladstone store needs 60. The manufacturer has supplies of computers in two warehouses, 80 computers in Salem and 40 in Trent. Shipping costs (in dollars per computer) are shown in the table.

	Harmony	*Gladstone*
Salem	\$20	\$12
Trent	\$16	\$10

Let x be the number of computers shipped from the Salem warehouse to the Harmony store and let y be the number shipped from Salem to Gladstone. How many must be shipped from Trent to each retail store? If the manufacturer wants to minimize shipping costs, how many computers should be sent from each warehouse to each retailer?

45. Solve the problem in Exercise 44 if the shipping costs are

	Harmony	*Gladstone*
Salem	\$10	\$16
Trent	\$12	\$ 8

Exercises 46–50 ***Fitting Points*** *Find an equation for the geometric figure that contains the given points.* (*Hint*: *Assume an equation for a parabola of the form* $y = ax^2 + bx + c$, and $x^2 + y^2 + ax + by + c = 0$ for a circle.)

46. Parabola; $P(1, 3)$, $Q(-2, 21)$, $R(0.5, 3.5)$

47. Parabola; $P(1, 11)$, $Q(-1, 3)$, $R(0.5, 7.5)$

48. Circle; $P(3, -2)$, $Q(0, -1)$, $R(0, -5)$

49. Circle; $P(-1, 5)$, $Q(-1, -1)$, $R(2, 2)$

50. Circle; $P(4, 2)$, $Q(1, 3)$, $R(6, -2)$

10 ANALYTIC GEOMETRY

ANALYTIC GEOMETRY IS THE NAME given to the marriage of algebra and geometry. The early Greeks developed a rich geometry, but their algebra was limited. The algebra-free geometry that came through Euclid is called **synthetic geometry.** Algebra developed independently, with little connection to geometry. Not until the early 1600s were the two melded, primarily by René Descartes (after whom Cartesian coordinates are named) and Pierre de Fermat.

The idea that a geometric picture can illuminate an equation is not new. Graphs are an integral part of our thinking, learning, and understanding. Most of our work has started with an equation or functional relation; in this chapter we more often begin with a geometric property and use algebra to interpret the geometry.

In the first section we consider new ways to make geometric proofs and extend our tools for computing distance, with special applications to lines and circles. Then we look more closely at a topic we have already used extensively, parametric representations of geometric figures and curves. The next sections deal with conic sections, a classical part of analytic geometry, and we end with a different way to describe sets of points in the plane using polar coordinates.

10.1 ALGEBRAIC METHODS FOR GEOMETRY

> The physical theory of general relativity could not have evolved were it not for the work of many generations of mathematicians . . . who were able to free geometry from its earlier imprisonment in Euclidean rigidity.
>
> *Roger Penrose*

Early Greek mathematicians explored geometric relationships with great ingenuity, but with no algebraic tools comparable to ours. They used congruence, similarity, and ratios. Where we apply the Pythagorean theorem to a triangle in terms of

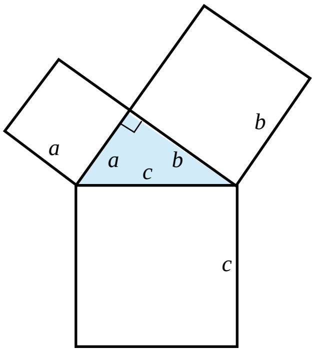


FIGURE 1

. . . [B]y the time I was in the sixth grade I understood algebra and geometry fairly well. I knew the rudiments of calculus and a smattering of number theory, which I liked very much. I felt rather isolated. A lot of teachers are very threatened when they find a child is studying advanced things. And I was reluctant at that time to talk to other children because I felt they found my interest in math somewhat strange.

Paul Cohen

an equation, say $a^2 + b^2 = c^2$, the Pythagoreans saw a triangle with actual squares on the sides. See Figure 1. They found ways to cut up the region and show that the sum of the areas of the two smaller squares is actually equal to the area of the large square.

Theorems from Greek geometry continue as a living part of our mathematical heritage. When we can draw a picture to model a problem, we often use geometry to relate variables. A good part of geometry has become the bedrock on which we build today. But powerful and enduring as the methods of synthetic geometry have proved over the centuries, there are times when we must be able to go beyond. Analytic geometry is one of the tools we need.

Deriving Equations for Geometric Figures

We already have some experience with analytic geometry. A circle is defined in terms of points and a distance, and we may represent it by a rough sketch or by a more careful drawing with a compass. For most purposes in algebra, though, we want an equation. The most fundamental step is putting a coordinate system on the plane. In the coordinate plane, we can use the synthetic definition of a circle as the set of all points equidistant from a fixed point. From the distance formula, we can write an equation that is satisfied by just those points that lie on the circle. As a familiar example, the set of points satisfying the equation $x^2 + y^2 = 1$ is the circle with center at the origin and radius 1.

Since many geometric definitions involve distance, the distance formula is used over and over again. The distance formula involves a square root, so we frequently need to square both sides of an equation. We must be alert to the possibility of squaring introducing extraneous points. We have the following useful criterion.

Squaring property

If U and V are expressions in x and y and both are nonnegative for the x, y values being considered, then these are equivalent equations:

$$U = V \quad \text{and} \quad U^2 = V^2$$

►EXAMPLE 1 ***Theorem from geometry*** Given points $A(1, 2)$ and $B(3, -2)$, show that the set of points that are equidistant from A and B is the perpendicular bisector of the segment $\overline{AB}$.

Strategy: Draw a diagram to visualize the problem, including a typical point $P(x, y)$ that is equidistant from A and B. Get an equation by setting the distances from P to A and P to B equal and simplifying.

Solution

Follow the strategy. First draw the diagram shown in Figure 2, where d_A and d_B denote the distances from P to A and P to B, respectively. Let S be the set of all points for which $d_A = d_B$. The distance formula for d_A and d_B gives:

$$\sqrt{(x-1)^2 + (y-2)^2} = \sqrt{(x-3)^2 + (y+2)^2} \tag{1}$$

To simplify Equation (1), since both sides are nonnegative, we can use the squaring property.

$$(x-1)^2 + (y-2)^2 = (x-3)^2 + (y+2)^2 \tag{2}$$

Expanding and simplifying, Equation (2) reduces to $x - 2y - 2 = 0$, an equation of a line L. Thus all points in S are on the line L. Reversing the above steps would also show that every point on the line L satisfies Equation (1) and hence is equidistant from A and B.

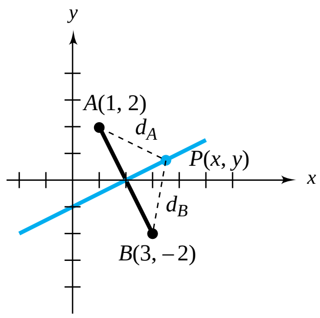


FIGURE 2

To show that L is the perpendicular bisector of $\overline{AB}$, show that **(i)** L contains the midpoint M of $\overline{AB}$ and **(ii)** L is perpendicular to the line through A and B.

(i) The coordinates of the midpoint M are

$$x_m = \frac{1+3}{2} = 2 \quad \text{and} \quad y_m = \frac{2+(-2)}{2} = 0,$$

so M is point $(2, 0)$. Substituting into the equation for L shows that M is a point on L.

(ii) The slope m_1 of the line through A and B is given by

$$m_1 = \frac{-2-2}{3-1} = -2.$$

The slope m_2 of line L can be found by putting the equation for L into point-slope form (that is, by solving for y): $y = \frac{1}{2}x - 1$, so $m_2 = \frac{1}{2}$. Since $m_1 m_2 = -1$, line L is perpendicular to the line through A and B. ◀

▶EXAMPLE 2 ***Find an equation*** For the given points $A(-4, -1)$ and $B(2, 5)$, let K be the set of points P such that the distance $|\overline{AP}|$ is twice the distance $|\overline{BP}|$. Find an equation for K and sketch the graph.

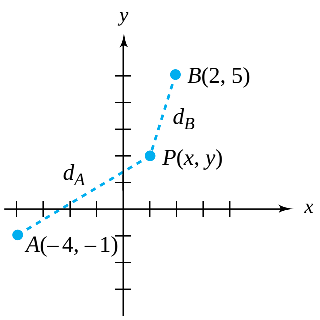


FIGURE 3

Solution

We begin with a figure showing A and B and a typical point $P(x, y)$ belonging to K (see Figure 3). The condition that must be satisfied for P to belong to K is that $d_A = 2d_B$. Expressing d_A and d_B in terms of coordinates, we get the following equation:

$$\sqrt{(x+4)^2 + (y+1)^2} = 2\sqrt{(x-2)^2 + (y-5)^2}.$$

Applying the squaring property, expanding, and simplifying, we have

$$(x^2 + 8x + 16) + (y^2 + 2y + 1) = 4[(x^2 - 4x + 4) + (y^2 - 10y + 25)]$$
$$3x^2 - 24x + 3y^2 - 42y + 99 = 0.$$

Now we divide through by 3 and complete squares.

$$(x^2 - 8x + 16) + (y^2 - 14y + 49) = -33 + 16 + 49$$
$$(x-4)^2 + (y-7)^2 = 32 \qquad \textbf{(3)}$$

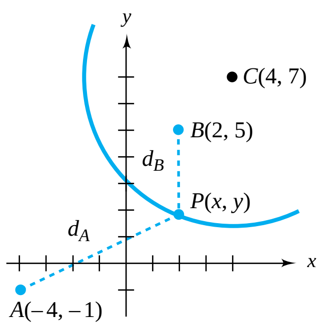
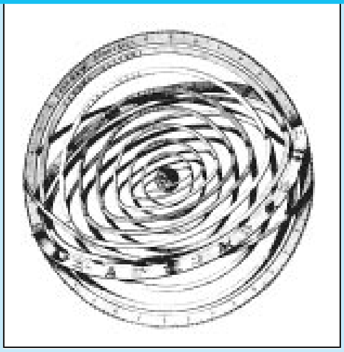


FIGURE 4

We recognize Equation (3) as an equation for the circle C with center at $C(4, 7)$ and radius $4\sqrt{2}$ (see Figure 4). Thus every point in K is on the circle. Conversely, we leave it to the reader to show that by reversing the above steps, every point P on the circle C has the property that $|\overline{AP}| = 2|\overline{BP}|$ and hence belongs to the set K. Therefore K consists precisely of those points on the circle. ◀

Examples 1 and 2 were stated in terms of specific points, which assumes a given coordinate system. The next example shows how a judicious choice of a coordinate system can simplify the proof of a geometric theorem.

HISTORICAL NOTE — A NEW VIEW OF THE WORLD



Astronomers used principles of analytic geometry to disprove the Ptolemaic view of the universe, in which the planets orbited in circles around a stationary earth.

Analytic geometry and the curves it describes have profoundly affected the way we think about our universe. To the ancient Greeks with their love of beauty and ideal form, it was unthinkable that the motion of the sun and planets could involve anything except circles. Careful observations, however, revealed that the planets do not move around the earth in smooth circular paths. At least from the earth, some planets even occasionally move backward! To harmonize observations with the perfection of circles, elaborate schemes were developed. Ptolemy (150 A.D.) described circles rolling around on circles, all rotating about an ideal point somewhere off in space.

By the middle ages, dogma was more important than observation and dictated circular orbits centered about the earth. Copernicus proposed (1543) that the earth and planets orbited a stationary sun, but the idea was heretical. When Galileo reported (1610) that through his newly invented telescope he had *seen* the moons of Jupiter orbiting a heavenly body other than the earth, he was forced to recant, but not before his widely read "Dialogue" spread Copernicanism.

Johannes Kepler, a "closet Copernican," was invited to assist the Danish astronomer Tyche Brahe, undoubtedly the most patient and accurate observer of his (or most any) age. At Brahe's death (1601), Kepler inherited the mountains of data Brahe had collected in over 20 years of watching the night sky. More than ten years of prodigious calculations with Brahe's data forced Kepler to the conclusion that the orbit of Mars is not a circle but an ellipse having the sun at one focus. Ten years of further computation with Brahe's observations ultimately yielded Kepler's laws about times of revolution and distances from the sun.

By Newton's time, mathematics had progressed to the point that when Halley (of Halley's comet) asked Newton about the curve that would describe the motion of planets, assuming Newton's formulation of gravitational force. Newton immediately replied, "An ellipse." And how did he know it? "Why I have calculated it."

Strategy: Choose a convenient location for an arbitrary parallelogram, say with a vertex at the origin and one side along the positive x-axis. Give coordinates to the other vertices and finally try to show that the diagonals have the same midpoints.

►*EXAMPLE 3* ***Theorem from geometry*** Show that the diagonals of a parallelogram bisect each other.

Solution

Follow the strategy and draw a typical parallelogram like the one in Figure 5. Assuming coordinates $A(0, 0)$ and $D(t, 0)$ makes it easy to get coordinates for B and C. With no special assumption about the x-coordinate of B, $\overline{BC}$ is parallel to $\overline{AD}$, so B and C must have the same y-coordinate, and the lengths of $\overline{AD}$ and $\overline{BC}$ are the same, so the difference between x-coordinates of B and C must be the same as the difference between A and D, namely t. Thus, if B has coordinates (r, s), then C must have coordinates $(r + t, s)$, as shown in Figure 5.

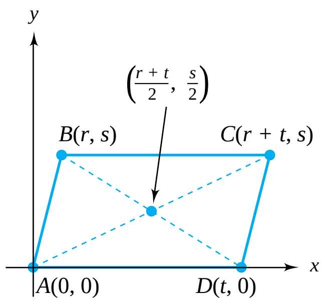


FIGURE 5

In terms of the coordinates shown in Figure 5, find the midpoints of the diagonals $\overline{AC}$ and $\overline{BD}$. Denote the midpoints of $\overline{AC}$ and $\overline{BD}$ by M_{AC} and M_{BD}, respectively.

$$M_{AC} = \left(\frac{0 + r + t}{2}, \frac{0 + s}{2}\right) = \left(\frac{r + t}{2}, \frac{s}{2}\right) \quad \text{and} \quad M_{BD} = \left(\frac{r + t}{2}, \frac{s}{2}\right)$$

Since the midpoints are the same point, the diagonals bisect each other. ◀

Forms of Equations; Distance from a Point to a Line

We have often observed the utility of changing the form of an equation. Different forms display different information. Taking a line for instance, we have a number of options.

$y - y_1 = m(x - x_1)$	*Point-slope form,* given point (x_1, y_1), slope m.
$Ax + By = C$	*Standard form,* can be multiplied by any nonzero constant.
$y = mx + b$	*Slope-intercept form;* solving for y displays slope, y-intercept.
$\frac{x}{a} + \frac{y}{b} = 1$	*Intercept-form*; dividing by C displays x-intercept a and y-intercept b.

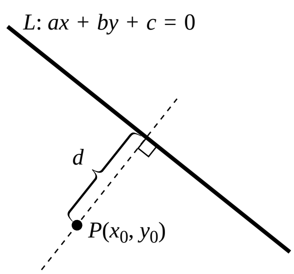


FIGURE 6

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

We haven't previously encountered the intercept form, but it is a simple matter to verify that when we take a standard form and divide through to get 1 on the right side, the x-intercept point is $(a, 0)$ and the y-intercept point is $(0, b)$.

There are several ways to derive the formula for the distance between a given point P and a given line L, whose equation we write in the form

$$ax + by + c = 0.$$

The distance d, shown in Figure 6, is measured along the perpendicular from P to L. There is an obvious way to think about finding the distance: write an equation for the line perpendicular to L containing P, find the coordinates of the point of intersection, and then use the familiar distance formula to find d. This process, while it may be obvious, is messy, and we do not include any details. We invite any interested reader to take up the challenge of filling in the missing steps. There are less messy derivations that involve triangles or vectors, but we simply state the result:

$$d^2 = \frac{(ax_0 + by_0 + c)^2}{a^2 + b^2}.$$

Taking the positive square root of both sides we have the following formula.

Distance from a point to a line

Given a line L with equation $ax + by + c = 0$ and a point $P(x_0, y_0)$, to find the distance d from P to L, substitute the coordinates of P into the left side of the equation for L, take the absolute value, and divide by $\sqrt{a^2 + b^2}$:

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}. \qquad \textbf{(4)}$$

Strategy: Draw a diagram. To show the lines are parallel, compare slopes. The distance between parallel lines is the distance from any point on one line to the other line.

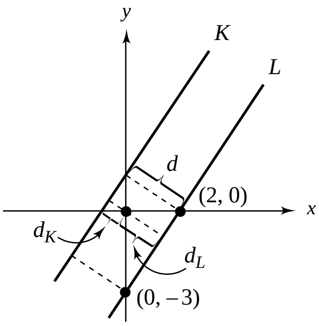


FIGURE 7

▶EXAMPLE 4 *Distances between lines* Consider lines L: $3x - 2y = 6$ and K: $6x - 4y + 5 = 0$.

(a) Show that L and K are parallel.
(b) Find the distance d between L and K.
(c) Find the distance from the origin O to each line.

Solution

(a) Follow the strategy. See Figure 7. Write each equation in point-slope form:

$$L\colon y = \frac{3}{2}x - 3 \qquad K\colon y = \frac{3}{2}x + \frac{5}{4}$$

Since each line has slope $\frac{3}{2}$, the lines are parallel.

(b) The equation for K is already in the proper form. Two convenient points on L are the intercept points. If we use $A(2, 0)$, then substituting its coordinates into Equation (4) yields

$$d = \frac{|6 \cdot 2 - 4 \cdot 0 + 5|}{\sqrt{6^2 + 4^2}} = \frac{17}{2\sqrt{13}}.$$

If we use the other intercept point $B(0, -3)$, Equation (4) becomes

$$d = \frac{|6 \cdot 0 - 4 \cdot (-3) + 5|}{\sqrt{6^2 + 4^2}} = \frac{17}{2\sqrt{13}},$$

as we would expect.

(c) To get the distances from O to each line we substitute 0 for both coordinates and get two distances.

$$d_L = \frac{|3 \cdot 0 - 2 \cdot 0 - 6|}{\sqrt{3^2 + 2^2}} = \frac{6}{\sqrt{13}}, \quad d_K = \frac{|6 \cdot 0 - 4 \cdot 0 + 5|}{\sqrt{6^2 + 4^2}} = \frac{5}{2\sqrt{13}}.$$

From the diagram in Figure 7, it is apparent that the distance between L and K should equal the sum of the distances from O to each line, and from our calculations we can see that $d = d_L + d_K = \frac{6}{\sqrt{13}} + \frac{5}{2\sqrt{13}} = \frac{17}{2\sqrt{13}}$. ◀

▶EXAMPLE 5 *Equation of a circle* Find an equation for the circle of radius 3 with center on the line $y = 2x$ that is tangent to the y-axis.

Solution

It is essential to begin with a picture, as the diagram in Figure 8, where we start with the line $y = 2x$. From the picture, we can see two circles, each of which has an equation of the form

$$(x - h)^2 + (y - k)^2 = 3^2.$$

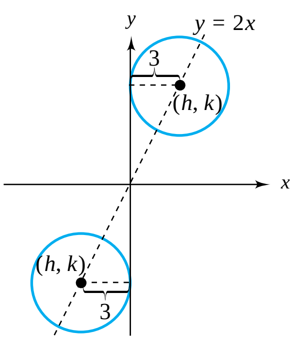


FIGURE 8

We want to find the coordinates (h, k) of the centers. The center of each circle must lie on the line $y = 2x$ and hence its coordinates satisfy the equation of the line. Therefore, $k = 2h$, so that the center has coordinates $(h, 2h)$. Furthermore, for a circle that is tangent to the y-axis, the distance from the center to the y-axis must equal 3. In this instance, we don't need the distance formula; the distance from (h, k) to the y-axis is $|h|$. Setting $|h| = 3$, we have $h = 3$ or $h = -3$.

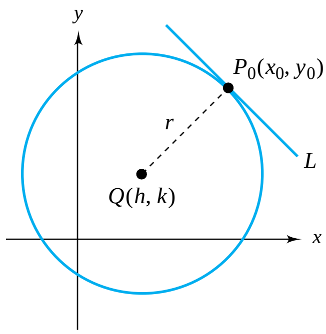


FIGURE 9

Therefore we have two circles satisfying the given conditions, one with center (3, 6) and the other with center $(-3, -6)$. Equations are

$$(x-3)^2 + (y-6)^2 = 3^2 \quad \text{and} \quad (x+3)^2 + (y+6)^2 = 3^2. \blacktriangleleft$$

Lines Tangent to a Circle

Geometrically, it is clear that a line and a circle can intersect in exactly two points, exactly one point, or not at all. When a line and a circle have only one point P_0 in common, the line is **tangent** to the circle at P_0, and point P_0 is the **point of tangency.** Figure 9 shows line L tangent at point $P_0(x_0, y_0)$ to the circle with center at $Q(h, k)$ and radius r. Recall from geometry that the line through P_0 and Q is perpendicular to L and the length of the line segment $\overline{P_0Q}$ is equal to r. We have the following theorem.

Tangent line theorem

A line L is tangent to the circle with center $Q(h, k)$ and radius r if and only if the distance from the center $Q(h, k)$ to line L is equal to r.

Strategy: A general line through $(-2, 8)$ can be written $y - 8 = m(x + 2)$. Rewrite this in standard form and use Equation (4) to find the slope m for which the distance to the center of C equals 2.

▶*EXAMPLE 6* ***Lines tangent to a circle*** Line L: $3x + 4y = 26$ contains point $(-2, 8)$.

(a) Show that L is tangent to circle C: $(x-8)^2 + (y-3)^2 = 4$.
(b) Find the other line that contains $(-2, 8)$ and is tangent to C.

Solution

(a) We could find the intersection of the line and circle by solving the equations simultaneously. If there is only one solution, then L is tangent to C. It is easier, however, to calculate the distance from the center of the circle, $C(8, 3)$, to L. By the Tangent Line theorem, L is tangent if the distance equals the radius, 2.

Use Equation (4) for the distance d from $C(8, 3)$ to L, first writing the equation for L in standard form.

$$d = \frac{|3 \cdot 8 + 4 \cdot 3 - 26|}{\sqrt{9 + 16}} = \frac{|24 + 12 - 26|}{5} = 2$$

Since $d = 2$ and $r = 2$, L is tangent to C.

(b) The other line L' contains $(-2, 8)$ and is tangent to C, as shown in Figure 10. In terms of the slope m, write an equation for L' as $y - 8 = m(x + 2)$, or in standard form,

$$L': mx - y + (2m + 8) = 0$$

Again by the Tangent Line theorem, for L' to be tangent to C the distance from the center $C(8, 3)$ to the line must equal 2. Express d in terms of the slope m, and then find the values of m for which the distance from C to the line equals 2. Substitute the coordinates (8, 3) into Equation (4) and divide by $\sqrt{A^2 + B^2} = (\sqrt{m^2 + 1})$,

$$d = \frac{|m \cdot 8 - 1 \cdot 3 + (2m + 8)|}{\sqrt{m^2 + 1}} = 2.$$

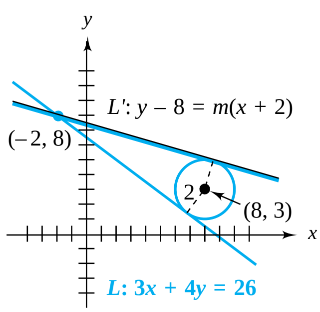


FIGURE 10

Square, clear fractions, and simplify:

$$\begin{aligned}(10m + 5)^2 &= 4(m^2 + 1)\\ 96m^2 + 100m + 21 &= 0\\ (4m + 3)(24m + 7) &= 0.\end{aligned}$$

Thus $m = \frac{-3}{4}$ or $m = \frac{-7}{24}$. Line L in part **(a)** has slope $\frac{-3}{4}$, so the slope of the other line is $\frac{-7}{24}$. An equation for L' is

$$y - 8 = \left(-\frac{7}{24}\right)(x + 2) \quad \text{or} \quad 7x + 24y - 178 = 0. \quad ◀$$

EXERCISES 10.1

Check Your Understanding

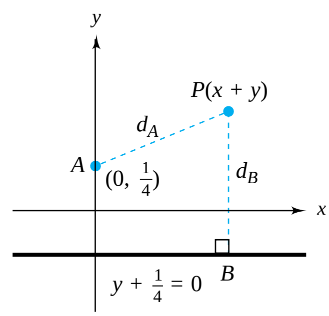
Exercises 1–10 True or False. Give reasons.

1. The distance from point $(3, -4)$ to line $y - 2 = 0$ is equal to 6.
2. Line $3x + 4y + 25 = 0$ is tangent to circle $x^2 + y^2 = 25$ at point $(3, 4)$.
3. Line $y = 2x$ is not tangent to circle $(x - 10)^2 + y^2 = 80$.
4. The distance from point $(-2, 4)$ to line $y = 2x$ is greater than 3.7.
5. Line $y = x$ is a perpendicular bisector of the line segment with endpoints at $(4, 0)$ and $(0, 4)$.
6. If line $3x - 2y = 4$ is tangent to a circle with center at $(0, 6)$ then the radius of the circle must equal 4.
7. The distance between the parallel lines $y = x$ and $y = x + 2$ equals 2.
8. The distance between the x-intercept points of the circle $(x - 2)^2 + y^2 = 4$ is equal to 4.
9. If A is the point $(-3, 0)$ and B is the point $(7, 0)$, then the line segment $\overline{AB}$ is a diameter of the circle $(x - 2)^2 + y^2 = 25$.
10. There are two lines, both of which contain the point $(4, 4)$ and are tangent to circle $x^2 + y^2 = 16$.

Develop Mastery

Exercises 1–8 ***Verbal to Equation*** *Find an equation for the set of points $P(x, y)$ that satisfy the condition.*

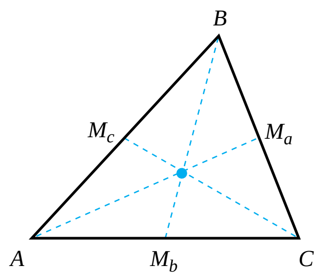
1. Equidistant from $A(-3, 0)$ and $B(0, 3)$.
2. Equidistant from $A(-3, 0)$ and $B(3, 0)$.
3. Equidistant from $A(3, -1)$ and $B(1, 5)$.
4. $|\overline{PA}| = 2|\overline{PB}|$, for $A(6, 0)$ and $B(0, 0)$.
5. $|\overline{PA}| = 2|\overline{PB}|$, for $A(-1, -4)$ and $B(5, 8)$.
6. $|\overline{PA}| = 3|\overline{PB}|$, for $A(-8, 5)$ and $B(8, -3)$.
7. $|\overline{PA}| = 3|\overline{PB}|$, for $A(6, 4)$ and $B(2, 0)$.
8. The distance $|\overline{PA}|$ for $A(0, \frac{1}{4})$ equals the distance from P to line $y + \frac{1}{4} = 0$. (*Hint:* What are the coordinates of point B in the diagram?)



Exercises 9–16 ***Theorems from Geometry*** *For the geometric theorem, first draw a figure and then prove the theorem analytically. See Example 3.*

9. The medians to the equal sides of an isosceles triangle are equal in length. (*Hint:* Locate the base on the x-axis with the opposite vertex on the positive y-axis.)
10. The midpoint of the hypotenuse of a right triangle is equidistant from all vertices of the triangle.
11. The line segment that joins the midpoints of two sides of a triangle is parallel to the third side and half the length of the third side.
12. The diagonals of a square are perpendicular to each other.
13. If the diagonals of a rectangle are perpendicular to each other, then the rectangle is a square.
14. The line segments that join midpoints of opposite sides of a quadrilateral bisect each other.
15. Given a quadrilateral $ABCD$, let R, S, T, and U be the midpoints of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Segments $\overline{RS}$ and $\overline{TU}$ are parallel.

16. The medians of any triangle are concurrent. That is, given $\triangle ABC$ with midpoints of opposite sides M_a, M_b, M_c as in the diagram, the segments $\overline{AM_a}$, $\overline{BM_b}$, $\overline{CM_c}$ all have a common point.



Exercises 17–25 ***Equations of Circles*** *Find an equation for the circle that satisfies the conditions. (Hint: Draw a figure.)*

17. Center (3, 2), contains the origin.

18. Center (3, 2), tangent to the x-axis.

19. Center (3, 2), tangent to the y-axis.

20. Center (3, 2), tangent to the line $x + y = 0$.

21. Contains (2, 9), tangent to both axes.

22. A diameter is the line segment that joins $A(1, 1)$ and $B(7, -5)$.

23. Center on the x-axis, contains $A(3, 5)$ and $B(-1, 7)$.

24. Center on line $x - y = 0$, contains $A(3, 5)$ and $B(-1, 7)$.

25. Center on line $3x - 2y + 3 = 0$, tangent to lines $x = 1$ and $x = 5$.

Exercises 26–29 ***Circle Circumscribing a Triangle*** *Find an equation for the circle that circumscribes the triangle whose vertices are the points of intersection of the given three lines. (Hint: First check to see if the triangle is a right triangle. The hypotenuse of a right triangle inscribed in a circle is a diameter of the circle.)*

26. The coordinate axes and line $3x + 4y = 12$

27. $5x + y = 22$, $x - 5y + 6 = 0$, and $2x + 3y = 1$

28. $2x - y = 0$, $x + 2y = 0$, and $3x - 4y + 10 = 0$

29. $3x - 4y = 6$, $7x - y = 39$, and $x + 7y + 23 = 0$

Exercises 30–35 ***Tangent Lines*** *Find an equation for the line or lines that are tangent to the circle as specified.*

30. $x^2 + y^2 = 17$, at $A(-4, 1)$.

31. $x^2 + y^2 - 6x - 2y + 8 = 0$, where the circle meets the x-axis.

32. $x^2 + y^2 = 8$, perpendicular to $y = x + 2$.

33. $x^2 + y^2 = 1$, contains point $A(4, 1)$.

34. $x^2 + y^2 = 10$, where the circle meets $x^2 + y^2 - 12x - 4y + 30 = 0$.

35. $x^2 + y^2 = 2$, contains the intersection of lines $x - 2y = 1$ and $x + y = 4$.

Exercises 36–39 ***Distance Between Lines*** ***(a)*** *Determine if lines L and K are parallel.* ***(b)*** *If so, find the distance between L and K.* ***(c)*** *Find the distance from the origin to each line.*

36. L: $2x - 3y = 6$, K: $6y - 4x + 3 = 0$

37. L: $2x + 4y = 5$, K: $x + 2y = 4$

38. L: $21y + 20x = 20$, K: $-400x = 420y + 725$

39. L: $x = 17$, K: $x + 24 = 0$

Exercises 40–41 ***Distance from a Point to a Line*** *Find an equation for* ***(a)*** *the line L containing A and B, and* ***(b)*** *the line K perpendicular to L and containing P.* ***(c)*** *Find the intersection Q of lines L and K.* ***(d)*** *Find the distance from P to Q and compare the result with the distance as given by Equation (4).*

40. $A(-1, 2)$, $B(1, 3)$; $P(4, 2)$

41. $A(3, 2)$, $B(6, 1)$; $P(1, 6)$

Exercises 42–43 ***Explore and Your Choice***

42. Sketch the circles C_1: $x^2 + y^2 - 14x - 2y + 25 = 0$ and C_2: $x^2 + y^2 - 25 = 0$ on the same axes and find the points of intersection.

- **(a)** Find an equation for line L_c through the centers of the two circles.
- **(b)** Find an equation for line L_i through the intersections of the two circles. How are lines L_c and L_i related?
- **(c)** If we subtract the equation of C_1 from the equation of C_2, we obtain equation E: $14x + 2y - 50 = 0$. Add the graph of equation E to your sketch.

43. Repeat Exercise 42 with circles C_1: $(x - 1)^2 + y^2 = 1$ and C_2: $x^2 + (y + 1)^2 = 1$, or with another pair of intersecting circles of your choice.

Exercises 44–47 ***Distance from Point to Line*** *Find the distance from the point to the line.*

44. $P(1, 3)$; $x - 4y + 5 = 0$

45. $P(2, -1)$: $3x + y - 2 = 0$

46. Origin; line through $A(3, 2)$ and $B(6, -4)$.

47. Origin; line through $A(-1, 3)$ and $B(-2, -4)$.

Exercises 48–51 ***Finding Altitude and Area*** ***(a)*** *For $\triangle ABC$, find the length of the altitude from the vertex A to side $\overline{BC}$, and* ***(b)*** *find the area of the triangle.*

48. $A(0, 0)$, $B(1, 8)$, $C(6, -2)$

49. $A(1, -3)$, $B(2, -3)$, $C(6, 5)$

50. $A(-2, -4)$, $B(1, -2)$, $C(3, 6)$

51. $A(-1, 2)$, $B(2, -1)$, $C(6, 3)$

Exercises 52–55 ***Minimum Distance*** *Let $Q(x, f(x))$ be any point on the line $y = f(x)$.* ***(a)*** *Find a formula for the distance $D(x)$ from P to Q. Graph $y = D(x)$ and find the minimum value of $D(x)$ (1 decimal place).* ***(b)*** *Compare your result with the distance as given by Equation (4).*

52. $f(x) = 2x - 3, \quad P(2, -3)$

53. $f(x) = x + 4, \quad P(1, -3)$

54. $f(x) = x - 2, \quad P(-1, 4)$

55. $f(x) = -2x + 3, \quad P(-1, -3)$

56. ***A Set in the Complex Plane*** The four roots of 1 are $1, -1, i, -i$. These four points in the complex plane are the corners of a square S.

(a) Sketch the square S and the set F consisting of all points P outside of S such that P is exactly 1 unit from the nearest point of S.

(b) Find the total area enclosed by the set F.

10.2 PARAMETRIC EQUATIONS

Our experience hitherto justifies us in believing that nature is the realization of the simplest conceivable mathematical ideas.

Albert Einstein

So there was one year spent largely on ordinary differential equations. I had a taste of real life and found that mathematics could actually be used for something.

Irving Kaplansky

We have used parametric equations as a convenient way to represent curves, particularly graphs of inverse functions (Section 2.7) and circles (Section 6.1). Such curves illustrate only a fraction of the kinds of behavior parametric graphing can illuminate, where functions of the form $y = f(x)$ are just too limited. In this section we explore additional aspects of parametric graphing, but we just scratch the surface of this vital topic.

The path shown in Figure 11 fails both the horizontal line test and the vertical line test, so we cannot hope to describe the curve in the form $y = f(x)$ or $x = g(y)$ where f and g are functions. The graph can, however, show the path of a point moving in the plane in the direction indicated by the arrows. As the point moves, its location and coordinates are functions of t. These equations describe the motion for $t \geq -2$:

$$x = t^2 - t - 2 \quad \text{and} \quad y = t^3 - 3t. \tag{1}$$

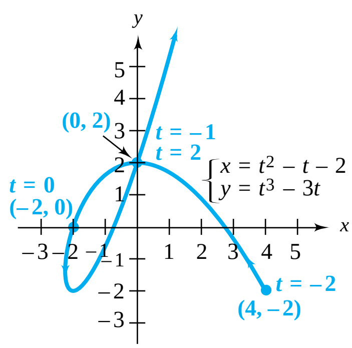


FIGURE 11
Graph of $x = t^2 - t - 2$, for $-2 \leq t \leq 3$, $y = t^3 - 3t$

Equations (1) are **parametric equations** for the curve in Figure 11 and the variable t is called a **parameter.** In parametric mode, with a t-range starting at $t = -2$, you should be able to watch your calculator generate the graph in Figure 11. Trace, and see that the graph goes through the point $(0, 2)$ when $t = -1$ and again when $t = 2$.

Definition: parametric equations for a curve

If f and g are functions defined on an interval $[a, b]$, then

$$x = f(t) \quad \text{and} \quad y = g(t) \qquad t \in [a, b] \tag{2}$$

are parametric equations for the curve C which consists of all points $P(t)$ where

$$P(t) = (f(t), g(t)) \text{ for } t \in [a, b].$$

In the above definition the interval for t may be the set of all real numbers, but if the interval is finite, then $P(a)$ is the **initial point** and $P(b)$ is the **terminal point** of the curve.

Sometimes it is helpful to relate x and y directly by eliminating the parameter between the equations $x = f(t)$, $y = g(t)$. It may be possible to eliminate the parameter by solving one equation for the parameter and substituting into the other equation, or, perhaps, using trigonometric identities. One advantage of parametric equations, however, is that they can define a specified portion of a given curve. Different parametric representations may describe different portions of the same curve, or portions traversed in different directions, as illustrated in the following example.

Strategy: To verify that the given parametric equations satisfy the equation of the circle, calculate the quantity $x^2 + y^2$ to verify that $x^2 + y^2 = 4$. For the direction and portion of the circle, identify the starting and ending points (for **(a)** and **(b)**) and plot some points.

►EXAMPLE 1 ***Parametric representations of a circle*** Show that each of the following parametric equations represents a portion of the circle $x^2 + y^2 = 4$. In each case, draw a graph and identify the portion of the circle and the direction in which the point $P(x, y)$ traverses the curve as the parameter increases.

(a) $x = 2\cos t,\ y = 2\sin t, \qquad 0 \le t \le \dfrac{\pi}{2}$

(b) $x = 2\cos \pi t,\ y = -2\sin \pi t, \qquad 0 \le t \le 1$

(c) $x = \dfrac{2}{\sqrt{m^2 + 1}},\ y = \dfrac{2m}{\sqrt{m^2 + 1}} \qquad -\infty < m < \infty$

Solution

Follow the strategy.

(a)
$$\begin{aligned} x^2 + y^2 &= (2\cos t)^2 + (2\sin t)^2 \\ &= 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4 \cdot 1 = 4. \end{aligned}$$

Thus each point $P(x, y)$ whose coordinates are given by

$$x = 2\cos t \qquad y = 2\sin t$$

lies on the circle $x^2 + y^2 = 4$. Parts **(b)** and **(c)** can be treated in a similar manner. See Exercises 39 and 40.

To identify the portion of the curve in each case, plot some points and examine what happens as the parameter changes. In **(a)** the parameter is restricted to the interval $[0, \frac{\pi}{2}]$, so plot $P(t)$ for values such as $t = 0$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$, as shown in Figure 12a. It should be clear that as t increases from 0 to $\frac{\pi}{2}$, $P(t)$ moves counterclockwise around a quarter-circle.

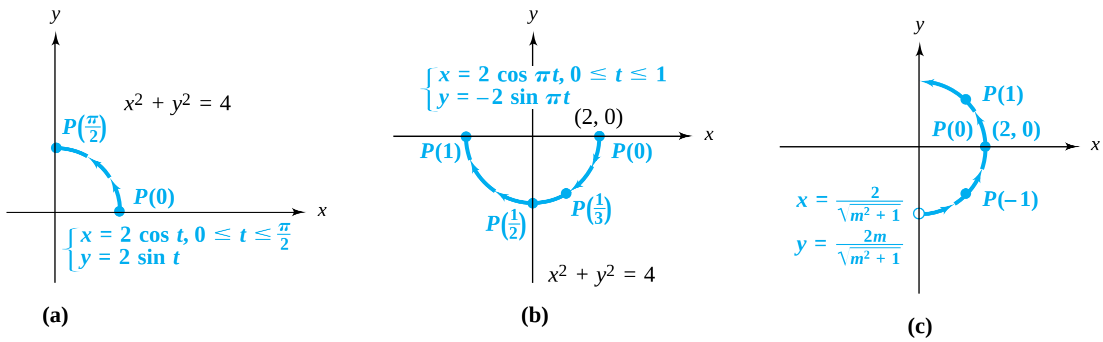


FIGURE 12

(b) Again, plot $P(t)$ for several values of t in the interval $[0, 1]$:

$$P(0) = (2\cos 0, -2\sin 0) = (2, 0)$$

$$P\left(\frac{1}{3}\right) = \left(2\cos\frac{\pi}{3}, -2\sin\frac{\pi}{3}\right) = (1, -\sqrt{3})$$

$$P\left(\frac{1}{2}\right) = \left(2\cos\frac{\pi}{2}, -2\sin\frac{\pi}{2}\right) = (0, -2)$$

$$P(1) = (2\cos\pi, -2\sin\pi) = (-2, 0)$$

As t varies from 0 to 1, point $P(t)$ moves clockwise around the lower half-circle shown in Figure 12b.

(c) The parameter m is not restricted, but the x-coordinate, $\frac{2}{\sqrt{m^2+1}}$, cannot be negative, so the graph must lie in the portion of the plane where $x > 0$. Since the m-range is infinite, we cannot hope to see everything on a calculator graph. We know that the graph lies on the circle $x^2 + y^2 = 4$, so we can experiment with a reasonably large t-range and see what happens. Most graphing calculators require that we use t as the variable for parametric graphing, so we graph X = 2/√(T² + 1), Y = 2T√(T² + 1) in a decimal window with a t-range of perhaps $[-10, 10]$. As expected, we see the right half of the circle $x^2 + y^2 = 4$, traced out counterclockwise. To see just how much of the circle is actually included with this parameterization, we must consider what happens as m (or t) gets very large. As m^2 gets large, x approaches 0, so as m varies over all real numbers $P(m)$ traces the entire right half of the circle, not including the points $(0, \pm 1)$. See Figure 12c. ◀

The sets of parametric equations in Example 1 give no indication of where they came from or how to interpret parameters. It is easy to think of the pairs of equations in **(a)** and **(b)** as functions of a time variable t. In **(a)** t could as easily represent the radian measure of the central angle. The equations in **(c)** are expressed in terms of the variable m, which we have often associated with the slope of a line. As a matter of fact, these equations can be derived in terms of slope. The point of intersection of line $y = mx$ with the right half of circle $x^2 + y^2 = 4$ (see Exercise 40) has coordinates given by

$$x = \frac{2}{\sqrt{m^2+1}} \qquad y = \frac{2m}{\sqrt{m^2+1}}.$$

Strategy: **(a)** Solve one equation for t and substitute into the other equation, checking for limitations in the parametric form that are not apparent in rectangular form.
(b) With sines and cosines, the Pythagorean identity is often helpful for eliminating the parameter. Note that sines and cosines have limited ranges, so x and y will also be limited.

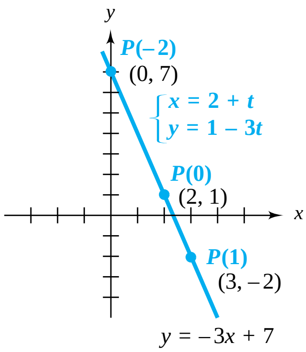


FIGURE 13

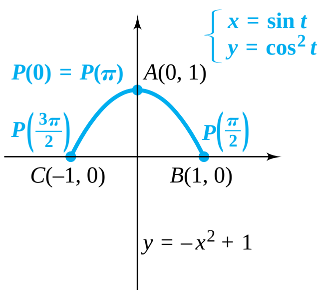


FIGURE 14

▶EXAMPLE 2 ***Eliminating the parameter*** For each of the curves defined parametrically, eliminate the parameter to find an equation in rectangular coordinates that represents the curve, then sketch the graph.

(a) $x = 2 + t,\ y = 1 - 3t$ **(b)** $x = \sin t,\ y = \cos^2 t$

Solution

(a) Follow the strategy. From $x = 2 + t$, $t = x - 2$. By substitution, $y = 1 - 3(x - 2) = 1 - 3x + 6$, or $y = -3x + 7$. Therefore curve C is all or part of line $3x + y = 7$. Since there is no restriction on t, both x and y take on all real values as t varies over the set of all real numbers; consequently the given parametric equations give the entire line (see Figure 13).

(b) Follow the strategy. Since $x = \sin t$ and $y = \cos^2 t$,

$$x^2 + y = \sin^2 t + \cos^2 t = 1.$$

Therefore, the curve contains points $P(t)$ on the parabola $y = -x^2 + 1$. However, whatever the value of t,

$$-1 \le \sin t \le 1 \quad \text{and} \quad 0 \le \cos^2 t \le 1.$$

Thus $P(t)$ is restricted to the portion of the parabola where

$$-1 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1.$$

The curve for the parametric equations is the arc of the parabola shown in Figure 14. Think of $P(t)$ moving in time when $t \ge 0$; visualize the point starting at $A(0, 1)$ $(t = 0)$, and moving along the parabola to $B(1, 0)$ at time $t = \frac{\pi}{2}$, then back to A at $t = \pi$, and on to $C(-1, 0)$ when $t = \frac{3\pi}{2}$, reversing direction again, and continuing indefinitely. ◀

Strategy: Start with an equation for L in rectangular coordinates. Among many different choices, you could let either x or y equal t and then solve for the other variable.

▶EXAMPLE 3 ***Representing a line*** Find two pairs of parametric equations for the line L that contains point $(3, 1)$ and has slope 2.

Solution

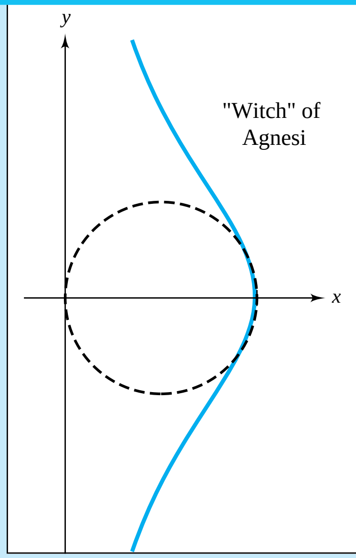
Using the point-slope formula for a line, an equation for L is $y - 1 = 2(x - 3)$. One set of parametric equations can be found by setting $x - 3$ equal to t, from which $x = 3 + t$. By substitution, $y - 1 = 2t$, or $y = 1 + 2t$. Thus one set of parametric equations for L is

$$x = 3 + t$$
$$y = 1 + 2t.$$

As in the strategy, let $y = t$, then substituting t for y and solving for x gives $x = \frac{t+5}{2}$. Another set of equations for L is thus

$$x = \frac{t + 5}{2}$$
$$y = t. \quad ◀$$

HISTORICAL NOTE THE WITCH OF AGNESI



Women today play an important part in the growth and development of mathematics. Throughout much of history, however, circumstances and the attitudes of society severely limited the role of women. One notable exception to the repression of women occurred in renaissance Italy. An atmosphere of encouragement resulted in many women making contributions to all areas of learning, including mathematics.

One of the most remarkable of these women was Maria Gaetana Agnesi, whose name, through an unfortunate twist, is forever linked in English with the word *witch.*

Agnesi began work at the age of 20 on a comprehensive treatment of calculus. Her two-volume work, *Instituzioni Analitiche (Analytic Institutions),* appeared ten years later, in 1748. It was the first major text to pull together the calculus of both Newton and Leibnitz. The book was translated into French and English and it influenced European mathematicians for much of the century.

One curve that Agnesi treated in the analytic geometry portion of her text is shown in the diagram. (See also Exercise 46 in this section.) The curve was called the *versieri,* from the Latin word that means turning. The word *versieri* is similar to the Italian word *avversieri* (which means wife of the devil). Whether a pun or simply a mistranslation, an 1801 translation of Agnesi's book into English rendered the name of the curve as *witch,* and the curve has become widely known as the Witch of Agnesi.

It should be clear from Example 3 that from an equation in rectangular coordinates there are many different substitutions we could make to express one variable in terms of a parameter. Solving for the other variable then yields a pair of parametric equations. Parametric representations are never unique.

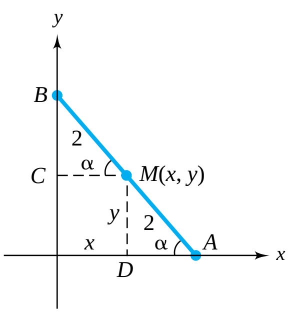


FIGURE 15

Strategy: To express x and y as functions of α, note that x is a side of $\triangle MCB$ and y is a side of $\triangle ADM$, and both right triangles have a hypotenuse of 2.

►EXAMPLE 4 ***Point on moving segment*** Suppose a line segment $\overline{AB}$ of length 4 is moving in such a way that point B is always on the positive y-axis and A is always on the positive x-axis. Think of a ladder propped against a wall with the lower end being pulled away from the wall. Use angle α in Figure 15 to find parametric equations for the midpoint $M(x, y)$ of line segment $\overline{AB}$.

Solution

Follow the strategy. In $\triangle MCB$, $\cos\alpha = \frac{x}{2}$, and in $\triangle ADM$ $\sin\alpha = \frac{y}{2}$. Therefore, the equations

$$x = 2\cos\alpha \quad \text{and} \quad y = 2\sin\alpha$$

express the coordinates of M in terms of the parameter α. In the first quadrant, α varies from $\frac{\pi}{2}$ (when A is at the origin) to 0 (when B is at the origin). Parametric equations are

$$x = 2\cos\alpha,\ y = 2\sin\alpha, \qquad \alpha \text{ decreases from } \tfrac{\pi}{2} \text{ to } 0.$$ ◄

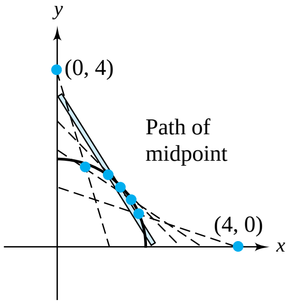


FIGURE 16
Midpoint spot on a sliding ladder.

If we think of Example 4 in terms of a sliding ladder, the conclusion of the example indicates that the midpoint of the ladder moves in the curve given by

$$x = 2\cos\alpha,\ y = 2\sin\alpha, \qquad 0 \le \alpha \le \tfrac{\pi}{2}.$$

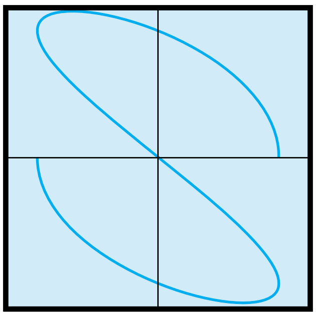
In Example 1 the same parametric equations defined a quarter-circle. Thus a spot by the middle rung of the ladder would move in a quarter-circle path from the wall to the ground. See Figure 16. Exercises 41–43 ask you to consider other questions related to the same setting.

Varieties of Parametric Graphs

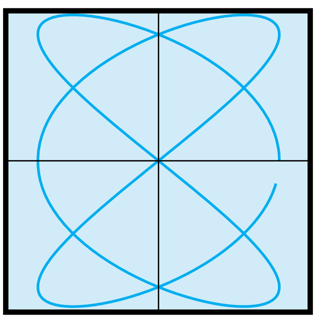
There is literally no limit to the variety of curves that can be represented parametrically. Any function of the form $y = f(x)$ can be parameterized in at least one way by setting $x = t$, $y = f(t)$, so we can obviously write parametric equations for any nonvertical line. The vertical line $x = 3$ could be described by $x = 3$, $y = t$, for example. The Pythagorean identity is another handy tool for parameterizing many curves. From $(\sin t)^2 + (\cos t)^2 = 1$, we can parameterize the circle $(x - h)^2 + (y - k)^2 = r^2$ by setting $x - h = r\sin t$ and $y - k = r\cos t$, or something similar. Figure 11 shows one example of the kinds of curves obtainable when we use polynomial functions for x and y.

The next example shows one of a family of closed curves called *Bowditch,* or *Lissajous, curves.*

▶EXAMPLE 5 ***A Bowditch (Lissajous) curve*** **(a)** Sketch the curve $x = 4\cos 3t$, $y = 3\sin 2t$. **(b)** Find the smallest value p for which the entire curve is traced out when $0 \le t \le p$ and **(c)** determine what kinds of symmetry the curve has.



[−5, 5] by [−3.1, 3.1]
(a)



[−5, 5] by [−3.1, 3.1]
(b)

FIGURE 17
A Bowditch (Lissajous) curve
$x = 4\cos 3t$, $y = 3\sin 2t$.
(a) $0 \le t \le \pi$
(b) $0 \le t \le 2\pi$

Solution

(a) Graphing the curve parametrically in a decimal window with the t-interval $[0, \pi]$ and t-step of 0.1 shows something like Figure 17a, certainly not a closed curve. If we increase the interval to $[0, 2\pi]$, we cannot tell from the calculator whether the graph closes or not. There appears to be a gap near the x-axis. See Figure 17b. Increasing the t-range to 6.5 closes the curve.

(b) To determine whether or not the entire graph is traced out on the interval $[0, 2\pi]$, we calculate the point $P(2\pi) = (4\cos 6\pi, 3\sin 4\pi) = (4, 0)$, and $P(0) = (4, 0)$, so $P(2\pi) = P(0)$ and the curve really is closed. See the Technology Tip that follows this example.

(c) The graph appears to be symmetric about both coordinate axes and the origin. Following the suggestion given in the Strategy, suppose $P(t) = (a, b)$; that is, $4\cos t = a$, $3\sin t = b$. Then

$$\begin{aligned} P(-t) &= (4\cos 3(-t), 3\sin 2(-t)) \\ &= (4\cos 3t, -3\sin 2t) = (a, -b). \end{aligned}$$

Thus the curve is symmetric about the x-axis. Using reduction formulas,

$$\begin{aligned} P(t + \pi) &= (4\cos(3t + 3\pi), 3\sin(2t + 2\pi)) \\ &= (-4\cos 3t, 3\sin 2t), \end{aligned}$$

so if $P(t) = (a, b)$, then $P(t + \pi) = (-a, b)$, and we have symmetry about the y-axis. Together, symmetry about both coordinate axes implies symmetry about the origin as well. ◀

Strategy: **(b)** Try various t-intervals $[0, p]$ until you get a closed curve. **(c)** To check symmetry compare, say, $P(-t)$ or $P(t + \pi)$ with $P(t)$.

TECHNOLOGY TIP ◆ *Importance of t-step*

The fact that the calculator graph in Figure 17b is not closed is an artifact of the t-step, which determines how many points are graphed in parametric mode. A large t-step means that the t-interval is divided into only a few pieces and the graph is likely to consist of relatively few points connected by segments. A very small t-step may require plotting so many points that the graph is very slow. We are used to having our calculators plot about a hundred points, so taking about one hundredth of the t-range will give a plotting speed comparable to what we normally expect.

Some calculators set a t-step automatically when you set a t-range. If your calculator allows you to set your own t-step, try graphing the Bowditch curve of Example 5 with a t-range of $[0, 2\pi]$ and a t-step of 1. Then try a t-step of $\frac{\pi}{50}$, and then of 0.11.

Problem Solving with Parametric Graphs

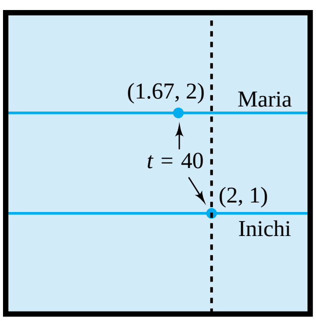
Some problems that involve time are particularly well suited for analysis with parametric representation, as suggested by the following example. In Example 2 of Section 1.6 we used analytic tools to find out if Maria would catch her roommate Inichi on the way to school. By plotting the progress of the girls on parallel lines, we can look at a parametric representation of their travel and use visual methods to get the same results we found in Chapter 1.

►*EXAMPLE 6* ***Travel in parametric form*** Inichi and Maria share an apartment 2 miles from campus, where they have the same 8:45 AM class. Inichi leaves home at 8:00, walking at her usual 3 mph pace, while Maria is still in the shower. Maria, who has missed class three days in a row, knows that she can jog all the way at a 5 mph pace. If she gets out the door by 8:20, will that pace allow Maria to **(a)** catch up with Inichi on the way or **(b)** get to class on time?

Solution

(a) and **(b)** Let t denote the number of minutes after 8:00. We plot Inichi's progress on one horizontal line and Maria's on another. Minutes are essential here, so we calculate both speeds in miles per minute:

$$\text{Inichi: } 3\,\frac{\text{mi}}{\text{hr}} = 3\,\frac{\text{mi}}{60\text{ min}} = \frac{1}{20}\,\frac{\text{mi}}{\text{min}} \qquad \text{Maria: } 5\,\frac{\text{mi}}{\text{hr}} = 5\,\frac{\text{mi}}{60\text{ min}} = \frac{1}{12}\,\frac{\text{mi}}{\text{min}}.$$



[0, 3] by [0, 3]

FIGURE 18
Inichi reaches school $(x = 2)$ when Maria is still $\frac{1}{3}$ of a mile away.

To plot Inichi's travel on the line $y = 1$, we can use $x = \frac{1}{20}t$, $y = 1$. Then we must take into account that Maria doesn't start until 20 minutes after Inichi, so her time must be $t - 20$. Accordingly we plot another parametric graph on the line $y = 2$ with $x = \frac{1}{12}(t - 20)$, $y = 2$.

In a $[0, 3] \times [0, 3]$ window (or any x-range that includes $[0, 2]$) and a t-range, say, of $[0, 60]$, when we graph we see Inichi's progress while Maria is still at home. Then Maria leaves (on the upper horizontal line) and begins to catch up. This is a graph in which the dynamic view is essential. After the graph is complete, all we see are two horizontal lines. See Figure 18. However, by tracing and jumping from one line to the other, we can read the time and see how far behind Maria is. Since the distance to school is 2 miles, where $x = 2$, we can read that Inichi reaches school when $t = 40$, when Maria is still a third of a mile away, but Maria gets to class (when her x-coordinate equals 2) 44 minutes after Inichi left home, at 1 minute before 8:45.

We can see some of this more vividly by setting a t-range of $[0, 40]$, effectively stopping time when Inichi gets to class, or by extending the t-range to 60 and tracing to see that if they had further to go, Maria would catch Inichi in 30 minutes of jogging time, 2.5 miles from their apartment. ◀

Distance from a Point to a Graph

In Section 10.1 we gave a formula for the distance from a point to a line. With parametric equations and the aid of technology, we can do lots more. In particular, we can approximate the distance from a point to any graph that we can represent parametrically, and we may even be able to see the point on the curve nearest the given point.

Distance from a point to a graph

Given a point $P_0(x_0, y_0)$ and the graph of a curve given parametrically by $x = f(t)$, $y = g(t)$. Then each point on the graph has coordinates $(f(t), g(t))$, and we can calculate the distance to P_0 as a function of t,

$$d(t) = \sqrt{(x_0 - f(t))^2 + (y_0 - g(t))^2}.$$

See Figure 19. By graphing the distance function, we can find its minimum value, which is what we mean by the distance from the point to the curve.

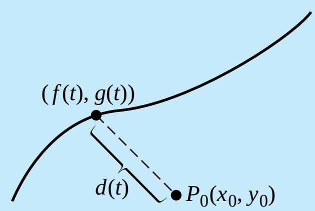


FIGURE 19

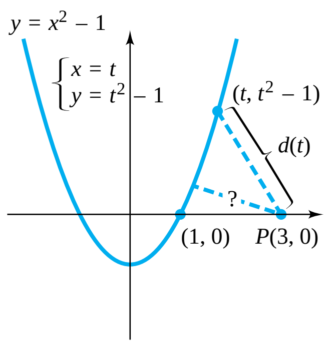
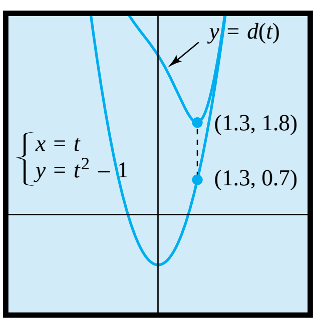


FIGURE 20



[−5, 5] by [−2, 4]

FIGURE 21

►EXAMPLE 7 ***Distance to a parabola*** Find the distance from the point $P(3, 0)$ to the parabola $y = x^2 - 1$.

Solution

The diagram in Figure 20 suggests that the closest point on the parabola is probably somewhere to the right of the x-intercept point $(1, 0)$. Following the general strategy outlined above, we use the parameterization $x = t$, $y = t^2 - 1$. The distance from a general point $(t, t^2 - 1)$ to P is given by

$$d(t) = \sqrt{(3 - t)^2 + (0 - (t^2 - 1))^2} = \sqrt{(3 - t)^2 + (t^2 - 1)^2}.$$

We graph the distance function and trace to find the minimum value. Graphing either in function mode $(y = \sqrt{(3 - x)^2 + (x^2 - 1)^2})$ or parametric mode $(x = t,\ y = \sqrt{(3 - t)^2 + (t^2 - 1)^2})$, we can find the minimum distance. If we graph the parabola on the same screen, by jumping from one curve to the other, we can locate the point on the curve nearest the point P (see Figure 21). The point on the parabola nearest P is about $(1.3, 0.7)$ and distance is just a little more than 1.83 units. ◀

EXERCISES 10.2

Check Your Understanding

Exercises 1–4 True or False. Give reasons.

1. The graph of $x = 1 - t, \quad y = 3 + t$ is a line.
2. The graph of $x = 1 - \cos t, \quad y = \cos t$ is a line segment with endpoints $(0, 0)$ and $(2, -1)$.
3. The graph of $x = \sin t, \quad y = \cos^2 t$ is a parabola.
4. Point $(2, 0)$ is on the graph of $x = 1 + t, \quad y = 1 - t^2$.

Exercises 5–10 Fill in the blank by identifying the graph so that the resulting statement is true.

5. The graph of $x = \sqrt{t}, y = \sqrt{1 - t}$ is ______.
6. The graph of $x = \sin t, y = -\cos t$ is ______.
7. The graph of $x = 2^t, y = 4^t$ is ______.
8. The graph of $x = 1 + t^2, y = 1 - t^2$ is ______.
9. The graph of $x = \cos t, y = -\cos t$ is ______.
10. The graph of $x = e^t, y = -e^t$ is ______.

Develop Mastery

Exercises 1–18 ***Parametric to Rectangular*** *Sketch the graph of the curve defined by the parametric equations. Find an equation in rectangular coordinates for each curve and give any restrictions on x and y.*

1. $x = t, \quad y = 4 - t^2$
2. $x = t, \quad y = 2 - \sqrt{t}$
3. $x = t, \quad y = \sqrt{4 - t^2}$
4. $x = \sqrt{t}, \quad y = \sqrt{t} - 2$
5. $x = \sqrt{t}, \quad y = \sqrt{4 - t}$
6. $x = t, \quad y = \sqrt{t^2 - 4}$
7. $x = 5 - t, \quad y = 2 + t$
8. $x = 3t, \quad y = -\frac{1}{2}t - 1$
9. $x = 1 - 3t, \quad y = 5t + 2$
10. $x = 3 \cos t, \quad y = -3 \sin t$
11. $x = 1 + \sin t, \quad y = 1 - \cos t$
12. $x = 2 \cos t, \quad y = 3 \sin t$
13. $x = \sin t, \quad y = -\sin t$
14. $x = 1 + \cos t, \quad y = 1 - \cos t$
15. $x = 1 + e^t, \quad y = 1 - e^t$
16. $x = \cos t, \quad y = \sin^2 t$
17. $x = -\cos t, \quad y = -\sin^2 t$
18. $x = 2^t, \quad y = -2^t$

Exercises 19–26 ***Portions of a Curve*** *The parametric equations in **(a)** and **(b)** define portions of the same curve. Sketch the graph and indicate the portion of the curve defined and the direction in which the point P(t) moves as t increases.*

19. **(a)** $x = \sin t, \quad y = -\sin t$
    (b) $x = \cos^2 t, \quad y = -\cos^2 t$
20. **(a)** $x = e^t, \quad y = -e^t$
    (b) $x = -e^t, \quad y = e^t$
21. **(a)** $x = e^t, \quad y = e^{-t}$
    (b) $x = -e^t, \quad y = -e^{-t}$
22. **(a)** $x = 2^t, \quad y = 4^t$
    (b) $x = -2^t, \quad y = 4^t$
23. **(a)** $x = -\cos t, \quad y = \sin t, \quad 0 \le t \le \pi$
    (b) $x = \sin \frac{\pi}{2}t, \quad y = \cos \frac{\pi}{2}t, \quad 0 \le t \le 2$
24. **(a)** $x = \sqrt{1 - t^2}, \quad y = -t, \quad -1 \le t \le 1$
    (b) $x = \sqrt{t}, \quad y = \sqrt{1 - t}, \quad 0 \le t \le 1$
25. **(a)** $x = 1 - t, \quad y = 3t - 2, \quad 0 \le t \le 2$
    (b) $x = t - 2, \quad y = 7 - 3t, \quad 1 \le t \le 3$
26. **(a)** $x = t, \quad y = \ln t, \quad t > 1$
    (b) $x = e^t, \quad y = t, \quad t > 0$

Exercises 27–34 ***Eliminate Parameter*** *Eliminate the parameter and give an equation in rectangular coordinates to describe the curve. Indicate any restrictions if the parametric equations define only a portion of the curve.*

27. $x = 1 + \cos t, \quad y = 2 - \sin t$
28. $x = -\cos t, \quad y = 3 \sin t$
29. $x = 1 + \cos t, \quad y = 2 - \cos t$
30. $x = 2 \cos^2 t, \quad y = 1 + 3 \sin t$
31. $x = 4 \sec t, \quad y = 5 \tan t$
32. $x = 2 \tan t, \quad y = \sec^2 t$
33. $x = 1 + 4 \sec t, \quad y = 5 \tan t$
34. $x = \cos t, \quad y = \cos 2t$

Exercises 35–38 ***Parameterizing a Curve*** *Find two sets of parametric equations for the curve. See Example 3. Answers are not unique.*

35. The line that contains $A(-2, 4)$ and is parallel to $3x - y = 4$.
36. The line that contains $(0, 0)$ and is perpendicular to $3x - y = 4$.
37. The line that is tangent to $x^2 + y^2 = 25$ at the point $(-3, 4)$.
38. The line is tangent to $x^2 + y^2 = 25$ at the point $(-4, 3)$.

Exercises 39–40 ***Parameterizing a Circle***

39. Verify (by substitution) that each pair of parametric equations in Example 1 satisfies the equation $x^2 + y^2 = 4$.

40. Find parametric equations for the right half of the circle $x^2 + y^2 = 4$ by finding the coordinates of the intersection of line $y = mx$ with the right half of the circle. Your parametric equations should agree with those in Example 1.

Exercises 41–43 ***Point on a Moving Segment***

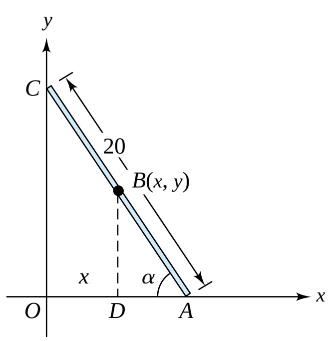
41. Given the line segment $\overline{AB}$ of Example 4, let $P(x, y)$ be the point that is always 1 unit from A. Find parametric equations (using angle α in Figure 15) for the curve traced out by point P as the segment moves from vertical to horizontal. Graph the curve traced out by P.

42. Let $Q(x, y)$ be the point that is always 1 unit from B on the line segment $\overline{AB}$ of Example 4. Find parametric equations for the curve traced out by Q and graph. See Exercise 41.

43. Let $T(x, y)$ be the point at some fixed distance a from A on line segment $\overline{AB}$ of Example 4. Find parametric equations for the curve traced out by T. See Exercises 41 and 42. For what values of a is the path of T part of a circle?

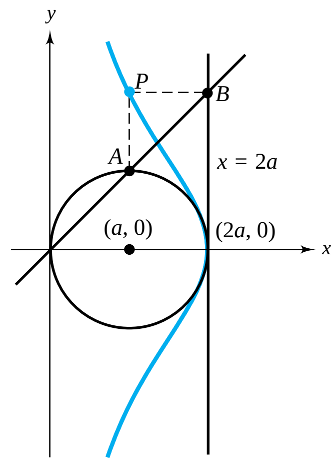
Exercises 44–45 ***Bug on a Ladder***

44. A 20-foot ladder is resting against a vertical wall with a ladybug at the lower end. Suppose the lower end of the ladder is being pulled horizontally away from the wall at the rate of 1 foot per minute, while at the same time the ladybug is crawling upward along the ladder at the rate of 1 foot per minute until it reaches the other end. If the bug is located at point $B(x, y)$ at time t minutes (see the diagram), find formulas that give x and y as functions of t. Find the position of the bug at time $t = 2$, 5, 10, and 20. For what values of t are your formulas valid?



45. Suppose the ladder in Exercise 44 is being pulled from the wall at the rate of 2 feet per minute while the bug still crawls 1 foot per minute. Answer the same questions as Exercise 44.

46. ***The Versieri (Witch of Agnesi)*** The curve called the *versieri* in Maria Agnesi's 1748 calculus treatise (see the Historical Note, "The Witch of Agnesi") can be most easily described parametrically. Take a circle of radius a as shown in the diagram, tangent to the y-axis and to line $x = 2a$. A line through the origin intersects the circle at point A and the vertical line at point B. Point P has the same x-coordinate as A and the same y-coordinate as B. The *versieri* is the set of all such points P.



(a) Using the line through the origin, $y = mx$, find the coordinates of point A in terms of parameter m.

(b) Find the coordinates of point B in terms of m.

(c) Give parametric equations for the *versieri* (that is, express coordinates of point P in terms of m).

(d) The usual equation for the *versieri* is $xy^2 = d^2(d - x)$, where d is the diameter of the circle. Show that the values of x and y given by your parametric equations satisfy this equation.

(e) If x and y are interchanged and the radius of the circle is $\frac{1}{2}$, show that the *versieri* is given by the equation $y = \frac{1}{1 + x^2}$.

Exercises 47–48 ***Exploring Bowditch Curves*** *Use a window at least as large as* $[-5, 5] \times [-3, 3]$ *and a t-range of* $[-4\pi, 4\pi]$.

47. (a) Graph $x = 5\sin(t/4 + k)$, $y = 3\cos t$ for $k = 1, 0.5, 0$.

(b) Explain why the curve for $k = 0$ does not appear to be closed.

48. (a) Repeat Exercise 47 using $x = 5\sin(t/3 + k)$, $y = 3\cos t$.

(b) Approximate a value of k for which the graph does not appear to be closed.

Exercises 49–52 ***Bowditch Curves*** ***(a)*** *Find the size of the smallest t-interval for which the Bowditch curve is closed.* ***(b)*** *Describe any symmetries of the curve.*

49. $x = 5\sin(3t + 1), \quad y = 3\cos 2t$

50. $x = 5\sin 4t, \quad y = 3\cos 2t$

51. $x = 5\sin(2t + 1), \quad y = 3\cos t$

52. $x = 5\sin(2t + 1), \quad y = 3\cos(2t + 1)$

Exercises 53–57 ***Distance from a Point to a Curve*** *Find the distance (in exact form where possible, to 1 decimal place otherwise) from the point P to the curve.*

53. $P(4, 1)$; $y = -x^2 + 2x$ **54.** $P(3, -1)$; $y = x^2 - 1$

55. $P(-5, 1)$; $x = t^2 - t - 2, \quad y = t^3 - 3t$ (See Figure 11.)

56. $P(0, 0)$; $x = 3 + \sin t, \quad y = 1 - \cos t$

57. $P(1, -1)$; $x = 1 + 4t, \quad y = 1 - 2t$

Exercises 58–59 ***Exploring Distance***

58. **(a)** Identify the curve in Exercise 56.
(b) Explain how to use geometry to get the same result.

59. **(a)** Identify the curve in Exercise 57.
(b) Explain how to get the same result by another method.

Exercises 60–64 ***Intercept Points*** *For the graph of the given parametric equations, find* ***(a)*** *the x-intercept point(s) and* ***(b)*** *the y-intercept point(s).*

60. $x = 4 - 2t, \quad y = t^2 - 2t - 3$

61. $x = 1 - \cos^2 t, \quad y = 1 + \sin t$

62. $x = 1 - 2|\cos t|, \quad y = 1 - |\sin t|$

63. $x = t^2 - t + 4, \quad y = t^3 - 4t$

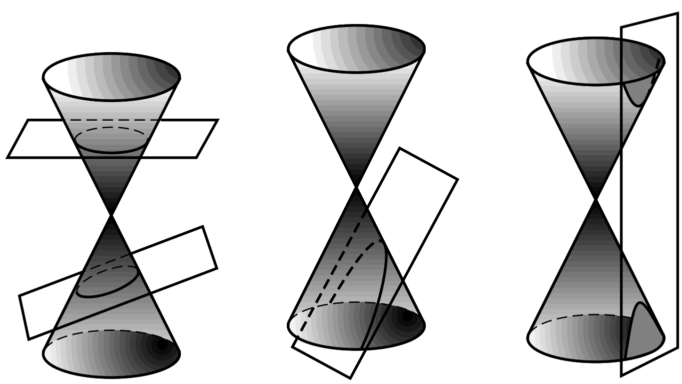
64. $x = 2^t - 8, \quad y = t^2 - 6t + 8$

10.3 CONIC SECTIONS

> Let's take . . . a step up the ladder from the circle. I mean the conic sections, especially the ellipse. These curves were studied by Apollonius of Perga (262–200 B.C.) as the "sections" of a right circular cone. This is "pure mathematics" in the sense that it has no contact with science or technology. The interesting thing is that nearly 2000 years later, Kepler announced that the planetary orbits are ellipses.
>
> *Reuben Hersh*

Analytic geometry was great. It began with a description of Descartes' great victory, the insight that made algebra out of geometry and vice versa. It was all about graphs, and mainly about conics. I thought it was all great stuff and in my letters home I wrote enthusiastically about my mathematics course; it was a beauty, I said.
Paul Halmos

The conic sections mentioned in the epigraph above have influenced human ideas about the universe (see the Historical Note, "Conic Sections"). The name comes from the fact that each of the conic sections is the intersection of a plane with a cone, as suggested in Figure 22. Thus the conic sections are the circle, ellipse, parabola, and hyperbola, with some additional special cases.



(a) (b) (c)

FIGURE 22

We have had extensive experience with the circle, and we have studied parabolas as graphs of quadratic functions. The circle is defined as the set of points equidistant from a single fixed point. In this section we give definitions for the other conics as sets of points satisfying distance relations. We derive standard form equations for each.

Parabolas

Definition: parabola

A **parabola** is the set of all points equidistant from a given point F and a line D that does not contain F. Point F is a **focus** and line D is the **directrix** of the parabola.

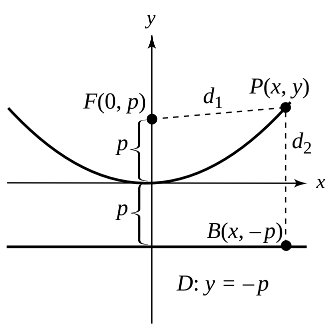


FIGURE 23

To get a simple equation for the parabola, we choose to put the origin midway between F and D, with D parallel to one of the axes. If $2p$ denotes the distance from F to D and we put F on the positive y-axis, then we get the diagram shown in Figure 23, where F has coordinates $(0, p)$ and D has the equation $y = -p$. If a point $P(x, y)$ falls on the parabola, the distances d_1 and d_2 in the diagram must be equal:

$$d_1 = \sqrt{(x - 0)^2 + (y - p)^2} \quad \text{and} \quad d_2 = \sqrt{(x - x)^2 + (y + p)^2}.$$

Setting d_1 equal to d_2, squaring both sides (according to the Squaring Property in Section 10.1), and simplifying, gives

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 \quad \text{or}$$
$$x^2 = 4py. \tag{1}$$

Figure 23 clearly shows that the parabola is symmetric about a line through the focus and perpendicular to the directrix (in this case, the y-axis). This line of symmetry is the **axis** of the parabola. The **vertex** is the point midway between the focus and the directrix, where the parabola meets its axis.

Standard Form Equations for Parabolas

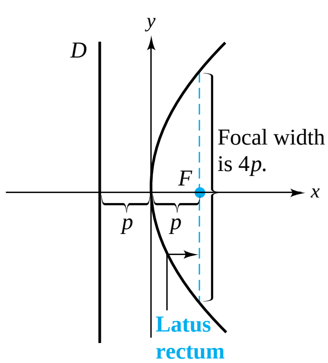


FIGURE 24

With the vertex at $(0, 0)$, Equation (1) is one of four standard forms for an equation of a parabola. Using $2p$ to denote the distance between the focus and the directrix (so that p is positive), we keep the vertex at the origin and can locate the focus F at either $(\pm p, 0)$ or $(0, \pm p)$. In each case we have one of the coordinate axes as the axis of the parabola and an equation of one of the following standard forms: $x^2 = \pm 4py$ or $y^2 = \pm 4px$.

The number $4p$ that appears in the standard form for parabolas has geometric significance. For a given curve we define a **chord** as any line segment that has both endpoints on the curve. A **focal chord** of a parabola is a chord that contains the focus. The focal chord parallel to the directrix is the **latus rectum.** The length of the latus rectum is $4p$. (See Figure 24 and Exercise 8.) The length of the latus rectum is also called the **focal width** of the parabola because it measures the width of the parabola opening.

Standard form equations for parabolas

Let p be a positive number. The graph of each of the following is a parabola with vertex at the origin and *focal width* (length of *latus rectum*) equal to $4p$.

$x^2 = \pm 4py$: vertical axis;
$+$sign: opens *up,* focus $F(0, p)$, directrix D: $y = -p$
$-$sign: opens *down,* focus $F(0, -p)$, directrix D: $y = p$ **(2)**
$y^2 = \pm 4px$: horizontal axis;
$+$sign: opens *right,* focus $F(p, 0)$, directrix D: $x = -p$
$-$sign: opens *left,* focus $F(-p, 0)$, directrix D: $x = p$ **(3)**

Strategy: All specified items are defined in terms of the number p in Equations (2) and (3). Begin with each equation in standard form by solving for the squared term and reading off p, $x^2 = -\frac{1}{2}y$; $p = \frac{1}{8}$, and $y^2 = -2x$; $p = \frac{1}{2}$.

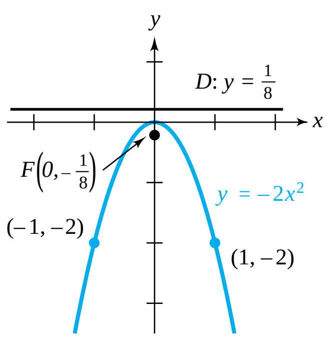


FIGURE 25

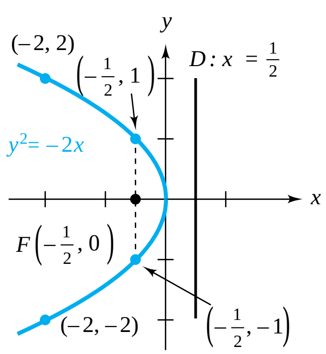


FIGURE 26

▶EXAMPLE 1 *Identifying features of a parabola* For each parabola, find the focus, directrix, and focal width, and sketch the graph.

(a) $y = -2x^2$ **(b)** $y^2 = -2x$

Solution

(a) The equation $y = -2x^2$ is a quadratic function of the type we graphed in Section 2.5. The vertex is at $(0, 0)$ and the parabola opens downward. We locate a point on the parabola, say $(1, -2)$, and by symmetry $(-1, -2)$ is also on the graph (see Figure 25). Identify p from a standard form. First express the equation in standard form (Equation (2) with a minus sign) by dividing both sides by -2,

$$x^2 = -\frac{1}{2}y.$$

Thus $-4p = -\frac{1}{2}$. Hence $p = \frac{1}{8}$ and the length of the latus rectum, the focal width, is $4p$, or $\frac{1}{2}$. Since the parabola opens downward, the focus is $\frac{1}{8}$ unit *below* the vertex, at point $F(0, -\frac{1}{8})$, and an equation for the directrix is D: $y = \frac{1}{8}$.

(b) Comparing standard forms, $y^2 = -2x$ is already in the form of Equation (3) with $4p = 2$. Thus $p = \frac{1}{2}$ and the focal width is 2. The parabola opens to the left and has its vertex at the origin. The focus is $F(-\frac{1}{2}, 0)$, and the directrix is D: $x = \frac{1}{2}$. See Figure 26. ◀

In Figures 25 and 26 the latus rectum visually indicates the focal width. For the parabola $y = -2x^2$, the focal width is $\frac{1}{2}$ and the parabola is quite narrow. In contrast, the wider parabola $y^2 = -2x$ has a focal width of 2.

Applications of Parabolas

The name *focus* comes from one of the properties that makes parabolas important in physical applications. A basic law of physics states that the angle of reflection of light or sound is the same as the angle of incidence. It is proved in calculus that all light rays parallel to the axis of a parabola will be reflected through the focus of the parabola. See Figure 27. This is the principle on which telescopes work. A parabolic mirror gathers light waves from a distance at the focus where the eye-piece is located. The light source of an automobile headlight is located near the focus of the parabolic reflector to send the light in essentially parallel rays. Using

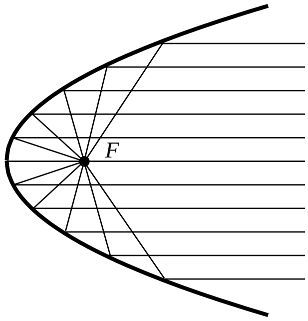


FIGURE 27

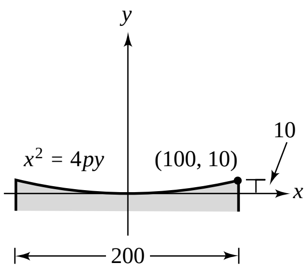


FIGURE 28

a similar principle, a parabolic antenna picks up sounds from a distance (as from a football huddle) or signals from an orbiting satellite.

The path of an object moving near the surface of the earth under the influence of gravity, such as a kicked ball or a thrown baseball, is very nearly parabolic.

►EXAMPLE 2 ***Telescope design*** The diameter of a parabolic mirror of a telescope is 200 centimeters, and the mirror is 10 centimeters deep at its center. How far is the focus of the vertex, that is, how far above the vertex should the eyepiece be located?

Solution

The cross section of the mirror is part of a parabola, as shown schematically in Figure 28. On the coordinate system, (100, 10) is a point on the parabola whose equation in standard form is $x^2 = 4py$. Substituting the coordinates (100, 10) into the equation,

$$100^2 = 4p \cdot 10 \quad \text{or} \quad p = \frac{10{,}000}{4 \cdot 10} = 250.$$

The focus is at (0, 250), or 250 centimeters above the center of the mirror. ◄

Ellipse and Hyperbola

Both the ellipse and the hyperbola involve two focus points, called **foci.** In Figure 29, F_1 and F_2 are the fixed focus points and P is an arbitrary point. In terms of the distances shown, we can define the ellipse and hyperbola. By treating them together, we emphasize both similarities and differences.

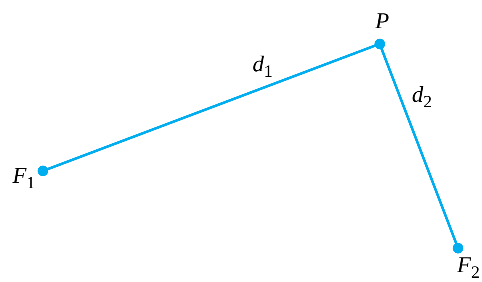


FIGURE 29

Definition: ellipse and hyperbola

Given two points F_1 and F_2 and a fixed positive number k.

An **ellipse** is the set of all points P such that the *sum* of the distances from P to F_1 and from P to F_2 is equal to k, that is, if $d_1 = |\overline{PF_1}|$ and $d_2 = |\overline{PF_2}|$, then

$$d_1 + d_2 = k.$$

A **hyperbola** is the set of all points P such that the *difference* of the distances from P to F_1 and from P to F_2 is equal to k, that is, if $d_1 = |\overline{PF_1}|$ and $d_2 = |\overline{PF_2}|$, then either

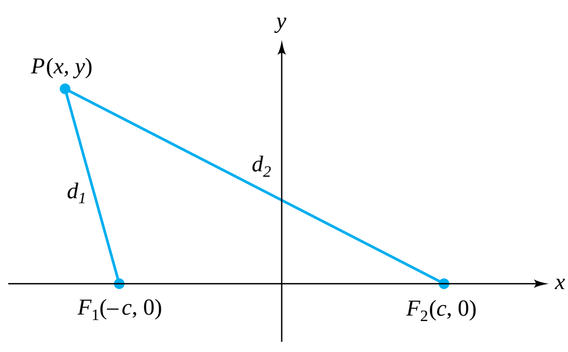
$$d_1 - d_2 = k \qquad \text{or} \qquad d_2 - d_1 = k.$$

Together, $|d_1 - d_2| = k$.

To get equations for the ellipse and hyperbola, we need a coordinate system. The standard equations assume both foci are on one of the coordinate axes. Take the focus points on the x-axis, symmetric to the origin, say at $F_1(-c, 0)$ and $F_2(c, 0)$. For convenience, take the constant k as $2a$. (See Figure 30.) Then we want equations that must be satisfied for a point $P(x, y)$ to lie on an ellipse or a hyperbola. We organize the derivations in parallel columns to make them easier to compare.

Standard Form Equations for Ellipse and Hyperbola

Ellipse $(a > c)$	***Hyperbola*** $(a < c)$
$d_1 + d_2 = 2a$	$d_1 - d_2 = \pm 2a$
$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$	$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a$
First separate the radicals	First separate the radicals
$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$	$\sqrt{(x+c)^2 + y^2} = \pm 2a + \sqrt{(x-c)^2 + y^2}$
Square both sides and simplify; isolate the radical	Square both sides and simplify; isolate the radical
$a\sqrt{(x-c)^2 + y^2} = a^2 - cx$	$\pm a\sqrt{(x-c)^2 + y^2} = cx - a^2$
Square both sides, simplify, and rearrange	Square both sides, simplify, and rearrange
$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$	$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$
Substitute b^2 for $a^2 - c^2$	Substitute b^2 for $c^2 - a^2$
$b^2x^2 + a^2y^2 = a^2b^2$	$b^2x^2 - a^2y^2 = a^2b^2$
Divide through by a^2b^2	Divide through by a^2b^2
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad$ **(4)**	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \qquad$ **(5)**



ellipse: $d_1 + d_2 + 2a$
hyperbola: $d_1 - d_2 = \pm 2a$

FIGURE 30

Note that we introduced no extraneous points squaring both sides (see the Squaring Property discussed in Section 10.1). Having derived an equation for the ellipse and the hyperbola, we look at each separately.

The Ellipse

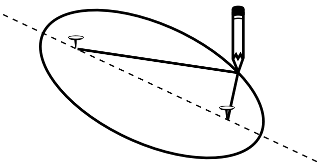


FIGURE 31

The definition suggests a simple way to draw a very good ellipse. Stick two tacks into a drawing board and tie an end of a piece of string to each tack. Moving a pencil around so as to keep the string taut defines a set of points such that the sum of the distances to the two foci (the tacks) equals the constant string-length. See Figure 31. If the foci are located at $(\pm c, 0)$, the graph of Equation (4) is clearly symmetric with respect to both axes and the origin.

Choosing our coordinate system so that the foci are at $F_1(-c, 0)$ and $F_2(c, 0)$ reveals the significance of the constants in the equation. When $y = 0$, $\frac{x^2}{a^2} = 1$ or $x = \pm a$ and the x-intercept points are $(-a, 0)$ and $(a, 0)$. Setting $x = 0$, we find that the y-intercept points are $(0, -b)$ and $(0, b)$. Since b was defined by $b^2 = a^2 - c^2$, we always have $b < a$.

The chord through the foci is called the **major axis** (length $2a$) and its midpoint is the **center** of the ellipse. The endpoints of the major axis are called the **vertices** of the ellipse. The chord that runs perpendicular to the major axis through the center is called the **minor axis** (length $2b$). Because $b < a$, the minor axis is always shorter than the major axis. Each focal chord (that is, each chord that passes through a focus) perpendicular to the major axis is called a **latus rectum.** These relations are shown in Figure 32.

Had we chosen the coordinate system with the foci on the y-axis at $F_1(0, c)$ and $F_2(0, -c)$, then the major axis would fall on the y-axis. The same derivation would lead to an equation of the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \tag{6}$$

and the terminology shown in Figure 33. Ellipses whose equations can be written in the form of either Equation (4) or Equation (6) are in standard position, and those equations are in standard form for ellipses.

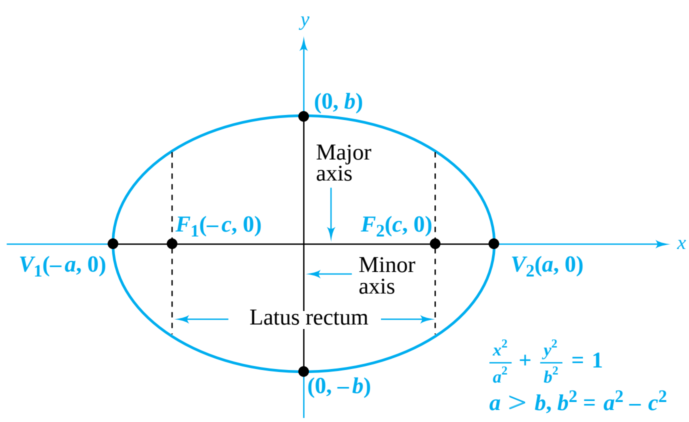


FIGURE 32

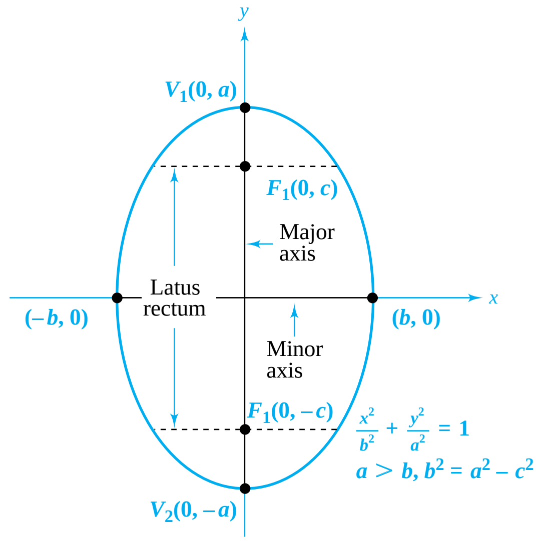


FIGURE 33

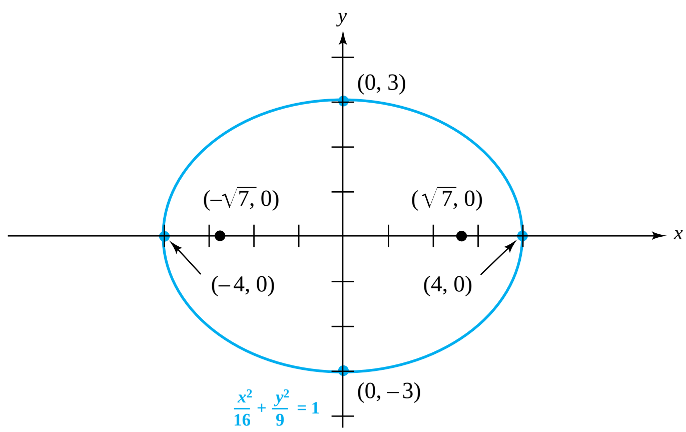


FIGURE 34

▶EXAMPLE 3 ***Identifying features of an ellipse*** Identify the vertices and foci, find the lengths of the major and minor axes, and sketch the graph.

(a) $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ **(b)** $\dfrac{x^2}{6} + \dfrac{y^2}{16} = 1$ **(c)** $x^2 + 9y^2 = 144.$

Solution

(a) Since $16 > 9$, the given equation is in the form of Equation (4); the foci and major axis are on the x-axis. Comparing with Equation (4) $a^2 = 16$ and $b^2 = 9$, and since $b^2 = a^2 - c^2$, $c^2 = a^2 - b^2 = 7$. Therefore $a = 4$, $b = 3$, and $c = \sqrt{7}$. The vertices are at $(\pm 4, 0)$ and the foci are at $(\pm\sqrt{7}, 0)$. The y-intercepts are $(0, \pm 3)$ as shown in Figure 34. The major and minor axes have lengths 8 and 6, respectively.

Strategy: **(c)** First divide through by 144 to get an equation in the form of Equation (4), from which we get a and b. From $c^2 = a^2 - b^2$, get c and the foci.

(b) Since $6 < 16$, the equation has the form of Equation (6). The major axis is on the y-axis and $a^2 = 16$. Thus $a = 4$, and the vertices are $(0, \pm 4)$. Since $b^2 = 6$, $b = \sqrt{6}$, and $c^2 = a^2 - b^2 = 16 - 6 = 10$. The foci are also on the

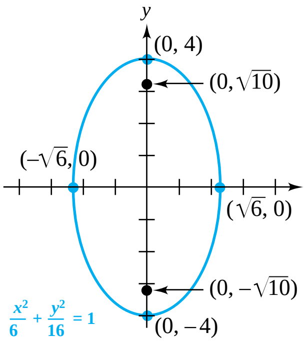


FIGURE 35

y-axis, at $(0, \pm\sqrt{10})$, and the x-intercepts are $(\pm\sqrt{6}, 0)$. See Figure 35. Lengths of major and minor axes are, respectively, 8 and $2\sqrt{6}$.

(c) The equation $x^2 + 9y^2 = 144$ does not have the form of either Equation (4) or (6). Following the strategy,

$$\frac{x^2}{144} + \frac{y^2}{16} = 1.$$

This equation has the form of Equation (4) with $a^2 = 144$, $b^2 = 16$, and $c^2 = 128$. Thus $a = 12$, $b = 4$, and $c = 8\sqrt{2}$. With x-intercept points $(\pm 12, 0)$ and y-intercept points $(0, \pm 4)$, the ellipse is long and thin (see Figure 36). The foci are $(\pm 8\sqrt{2}, 0)$, near the ends of the major axis. The major axis is 24 units long, while the minor axis is 8. ◀

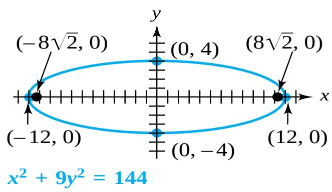


FIGURE 36

The ellipse is something of a "squashed circle," the shape we see when looking at a circular disk from an angle. The three ellipses in Example 3 demonstrate the considerable variation possible in the amount of distortion from a circle. One measure of the distortion is the ratio $\frac{b}{a}$. When a is much larger than b, the ellipse is long and narrow; when the ratio $\frac{b}{a}$ is near 1, the lengths of the major and minor axes are more nearly equal and the ellipse more closely resembles a circle. Historically, however, rather than the ratio $\frac{b}{a}$, the measure used to indicate the distortion of an ellipse is called the **eccentricity,** defined by $\epsilon = \frac{c}{a}$.

▶***EXAMPLE 4*** ***Find an equation for an ellipse*** Find an equation for the ellipse in standard position that has a major axis of length 8 and that passes through point (2, 3).

Strategy: Drawing diagrams of ellipses through (2, 3) suggests two solutions, one with a horizontal major axis and one with a vertical major axis. Using the form of Equation (4) with $a = 4$ and substituting the coordinates (2, 3) into the equation should set up the problem to solve for b for the horizontal case. Use Equation (6) for the vertical case.

Solution

Following the strategy, we expect two solutions. See Figure 37. For each ellipse, $2a = 8$, or $a = 4$. If the major axis is on the x-axis, then use Equation (4) with $a = 4$:

$$\frac{x^2}{16} + \frac{y^2}{b^2} = 1.$$

Substitute the coordinates of the given point into the equation,

$$\frac{2^2}{16} + \frac{3^2}{b^2} = 1$$

and solve for b^2. We get $b^2 = 12, b = \sqrt{12} = 2\sqrt{3}$. Therefore an equation for the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{12} = 1 \quad \text{or} \quad 3x^2 + 4y^2 = 48.$$

Similarly, if the major axis is on the y-axis, use Equation (6):

$$\frac{x^2}{b^2} + \frac{y^2}{16} = 1, \qquad \frac{2^2}{b^2} + \frac{3^2}{16} = 1, \qquad b^2 = \frac{64}{7}.$$

Thus, the desired equation becomes

$$\frac{x^2}{64/7} + \frac{y^2}{16} = 1 \quad \text{or} \quad 7x^2 + 4y^2 = 64. \quad ◀$$

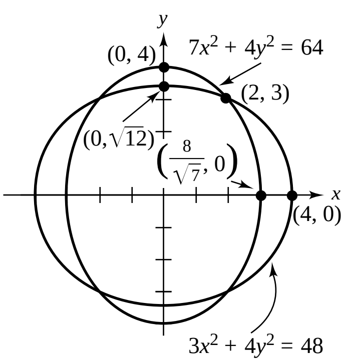


FIGURE 37

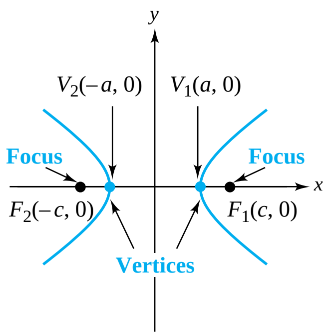


FIGURE 38
Hyperbola with vertices and foci

Hyperbolas

The derivation of the first standard equation for the hyperbola places the foci at $F_1(-c, 0)$ and $F_2(c, 0)$. Setting $y = 0$ in Equation (5),

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

we find the x-intercept points at $(-a, 0)$ and $(a, 0)$. Since $a < c$ for the hyperbola, the intercepts, called the **vertices** of the hyperbola, are between the foci. See Figure 38. From the form of Equation (5), the graph has symmetry similar to that of the ellipse. When we know the graph in one quadrant, the rest of the graph comes from reflections through the coordinate axes and the origin. The **center** of the hyperbola is the midpoint of the segment that joins the vertices.

If we solve Equation (5) for y, we obtain

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2} \tag{7}$$

Equation (7) demonstrates that there is no y-value when x is between $-a$ and a. Furthermore, we may rewrite Equation (7) in the form

$$y = \pm \frac{bx}{a}\sqrt{1 - \frac{a^2}{x^2}}$$

Clearly, as $|x|$ becomes larger, the quantity $\frac{a^2}{x^2}$ approaches 0, and y approaches $\frac{\pm bx}{a}$. This shows that lines $y = \frac{bx}{a}$ and $y = \frac{-bx}{a}$ are oblique **asymptotes** for the hyperbola. The vertices and the asymptotes make graphing the hyperbola simple. Point $P(a, b)$ in the first quadrant is one corner of what is called the **auxiliary rectangle.** The other corners are symmetric to $P(a, b)$, as shown in Figure 34. The auxiliary rectangle is not part of the graph of the hyperbola, but it aids graphing. The lines that contain the diagonals of the rectangle are the asymptotes, and the vertices of the hyperbola are the midpoints of opposite sides of the rectangle. The hyperbola is shown in Figure 39.

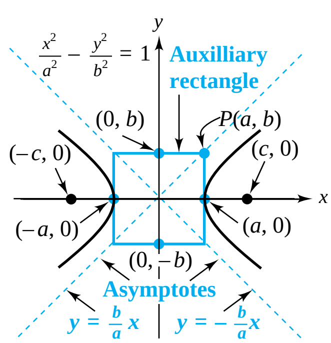


FIGURE 39

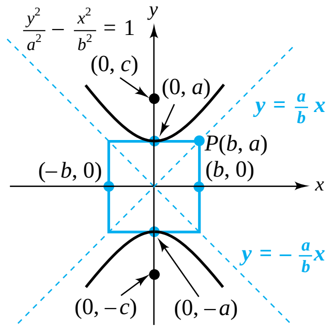


FIGURE 40

If the foci are on the y-axis, at $F_1(0, -c)$ and $F_2(0, c)$, then, by interchanging the roles of x and y, we obtain the other standard form for a hyperbola,

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1. \tag{8}$$

Again $b^2 = c^2 - a^2$. The vertices are at $(0, a)$ and $(0, -a)$, and the asymptotes are $y = \pm\frac{ax}{b}$. The same kind of auxiliary rectangle facilitates drawing this graph, shown in Figure 40.

▶EXAMPLE 5 ***Identifying features of a hyperbola*** Find the coordinates of the foci and the vertices, give equations for the asymptotes, and sketch the graph.

(a) $\frac{y^2}{25} - \frac{x^2}{9} = 1$ **(b)** $\frac{x^2}{9} - \frac{y^2}{25} = 1$

Solution

(a) The first equation has the form of Equation (8), where $a = 5$, $b = 3$, and the foci are on the y-axis. Since $b^2 = c^2 - a^2$, $c^2 = a^2 + b^2 = 25 + 9 = 34$, or $c = \sqrt{34}$. Thus the foci are at $(0, \pm\sqrt{34})$ and the vertices are at $(0, \pm 5)$. The asymptotes are $y = \pm\frac{ax}{b} = \pm\frac{5x}{3}$, and the graph is shown in Figure 41.

(b) The given equation has the form of Equation (5), so the foci and vertices are on the x-axis, even though the denominator of the y^2-term is larger. Since $a = 3$, and $b = 5$, $c = \sqrt{a^2 + b^2} = \sqrt{34}$. The asymptotes are $y = \pm\frac{bx}{a} = \pm\frac{5x}{3}$. The vertices, foci, and graph are shown in Figure 42. ◀

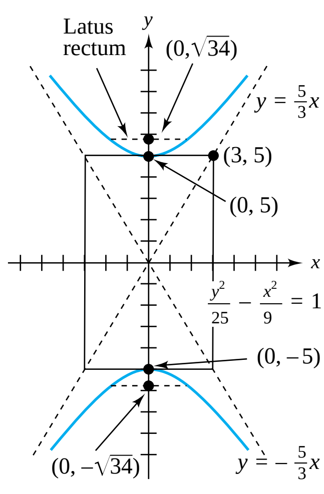


FIGURE 41

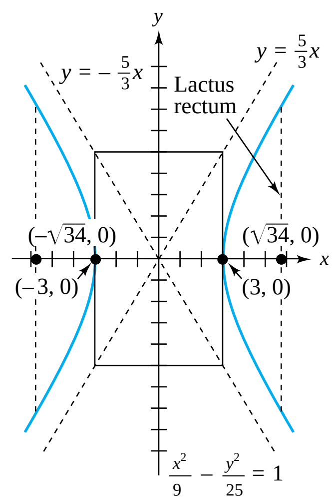


FIGURE 42

Note that the hyperbola in Figure 41 has a much narrower opening than the one in Figure 42. This is conveniently measured by a latus rectum, as for parabolas and ellipses. A latus rectum is a chord through a focus perpendicular to the line that contains the foci and vertices. As with parabolas and ellipses, a latus rectum is easily drawn (see Figures 41 and 42) and indicates the width of the hyperbola at the focus. We leave it to the reader to show that the length of the latus rectum in Figure 41 is $\frac{18}{5}$, while that in Figure 42 is $\frac{50}{3}$.

▶EXAMPLE 6 ***Finding equations for conics*** Find **(a)** the foci and endpoints of the right latus rectum of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, and **(b)** an equation for the parabola in standard position (with its vertex at the origin) that passes through the endpoints of the right latus rectum of the hyperbola.

Strategy: **(b)** Standard form for the equation of the parabola is $y^2 = 4px$, so substitute coordinates of the endpoints of the latus rectum into the equation.

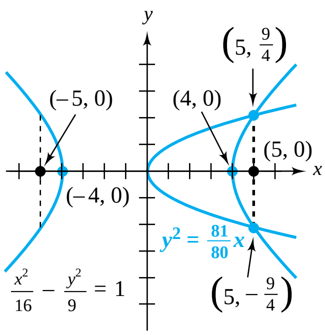


FIGURE 43

Solution

(a) For the given hyperbola, $a = 4$, $b = 3$, from which $c^2 = 16 + 9 = 25$. Thus $c = 5$ and the foci of the hyperbola are at $(\pm 5, 0)$. The right latus rectum is the vertical chord through the right focus, $(5, 0)$. When $x = 5$,

$$\frac{25}{16} - \frac{y^2}{9} = 1 \quad \text{or} \quad y = \pm\frac{9}{4}.$$

The ends of the right latus rectum are thus $(5, \pm\frac{9}{4})$, as in Figure 43.

(b) Follow the strategy. Substituting the coordinates $(5, \frac{9}{4})$ into the equation $y^2 = 4px$, we find, $4p = \frac{81}{80}$. Therefore an equation for the parabola shown in Figure 43 is $y^2 = \frac{81}{80}x$. ◀

Applications of Ellipses and Hyperbolas

The fact that the angle of incidence equals the angle of reflection makes parabolic reflectors useful in many settings. Elliptical reflectors also have applications. An ellipse reflects sound or light from one focus back to the other focus. Buildings in the shape of elliptical domes are often called *whispering galleries*. The Mormon Tabernacle in Salt Lake City and the Statuary Hall in the capitol building in Washington, D.C. are both whispering galleries. A whisper or a dropped pin near one focus can be heard clearly at the other focus. Historical rumor suggests that John C. Calhoun was aware of this property in Statuary Hall, where the House of Representatives met in his time, and he used the knowledge to eavesdrop on his adversaries.

Elliptical paths determine the most efficient changes in the orbits of spacecraft about the earth. Knowledge of conic sections has been profoundly significant in understanding orbiting bodies.

Hyperbolas provide the basis for location and navigation instruments. If three receivers in different places all record the times when a sound is heard from a common source, then the time differences determine hyperbolas on which the source must be located. Plotting the intersection of the hyperbolas allows observers to pinpoint the location. The same principle, in reverse, allows a submarine, say, to locate itself relative to three known sound beacons, which underlies LORAN navigation.

A summary of the significant relations and properties for ellipses and hyperbolas in standard position may be useful.

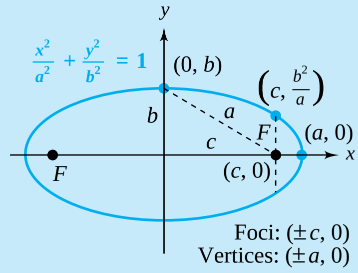
Ellipses and hyperbolas in standard position

Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ or $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1,$

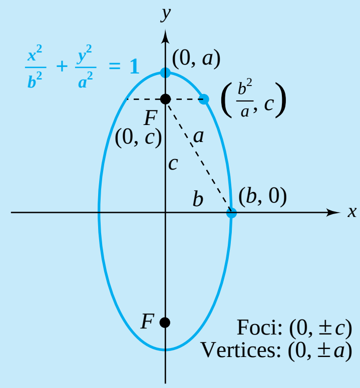
$a > c,\ b^2 = a^2 - c^2$

Latus rectum length $\dfrac{2b^2}{a}$

Major axis length $2a$
Minor axis length $2b$



(a)

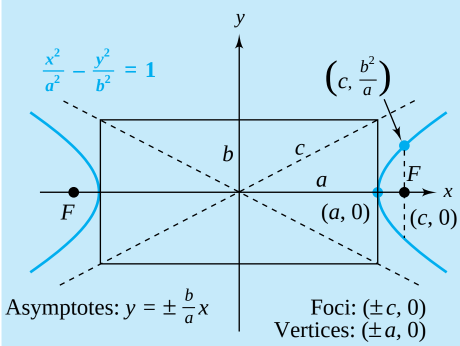


(b)

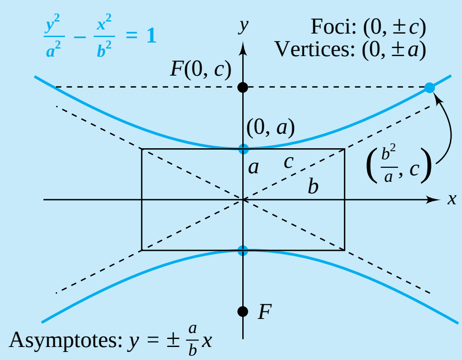
Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1,$ or $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1,$

$c > a,\ b^2 = c^2 - a^2$

Latus rectum length $\dfrac{2b^2}{a}$



(c)



(d)

Graphing Conic Sections with Technology

Since standard equations for all of the conic sections involve at most a y^2-term, we can always solve for y and, if necessary, graph two functions. In Section 6.1, we saw that parametric equations often give a more satisfactory graph of a circle. That is, it may be easier to get a good picture of the circle $(x - 1)^2 + (y + 2)^2 = 4$ by graphing $x = 1 + 2 \cos t$, $y = -2 + 2 \sin t$, $0 \le t \le 2\pi$ in parametric mode than by solving for y and graphing

$$y = -2 + \sqrt{4 - (x - 1)^2} \quad \text{and} \quad y = -2 - \sqrt{4 - (x - 1)^2}.$$

We urge the reader to try both approaches.

Because the circle, ellipse, and hyperbola all involve sums or differences of squares equal to 1, the basic trigonometric identities (I-4) and (I-5),

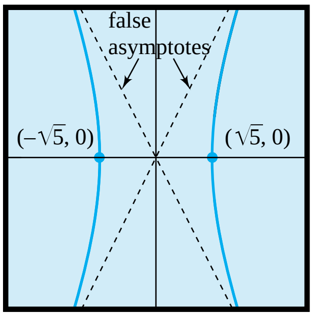
$$\cos^2 t + \sin^2 t = 1, \text{ and } \sec^2 t - \tan^2 t = 1,$$

lend themselves most conveniently to parametric graphing.

For example, to graph an ellipse such as

$$\frac{x^2}{4} + \frac{y^2}{9} = 1, \text{ set } \left(\frac{x}{2}\right)^2 = \cos^2 t, \left(\frac{y}{3}\right)^2 = \sin^2 t.$$

That is, use (I–4) and graph $x = 2 \cos t$, $y = 3 \sin t$ in parametric mode. In a similar fashion, for the hyperbola



$[-6, 6]$ by $[-3, 3]$

FIGURE 44
Calculator graph of the hyperbola $x = \sqrt{5}/\cos t$, $y = 2\sqrt{2} \tan t$, showing false asymptotes.

$$\frac{x^2}{5} - \frac{y^2}{8} = 1, \text{ set } \left(\frac{x}{\sqrt{5}}\right)^2 = \sec^2 t, \left(\frac{y}{2\sqrt{2}}\right)^2 = \tan^2 t,$$

and use (I–5), graphing $x = \sqrt{5} \sec t = \dfrac{\sqrt{5}}{\cos t}$, $y = 2\sqrt{2} \tan t$. For both the ellipse and the hyperbola, we use for the t-range the interval $[0, 2\pi]$ even though the secant and tangent functions are not defined for the whole interval.

An interesting thing often happens when graphing a hyperbola in parametric mode. The graphing calculator is programmed to connect successive pixels, so we may see false asymptotes, (see Figure 44) which are really excellent approximations to the asymptotes of the hyperbola, the lines $y = \pm \frac{b}{a}x$. If false asymptotes do not appear on your graph and you wish to see the asymptotes, it is an easy matter to add parametric equations $x = t$, $y = (\frac{b}{a})t$ and $x = t$, $y = -(\frac{b}{a})t$.

For parabolas that open up or down, there is no need for parametric graphing, although $x = t$, $y = (\frac{1}{4p})t^2$ works. For parabolas opening left or right, we interchange the roles of the variables in parametric form, as we did to graph inverses: $x = (\frac{1}{4p})t^2$, $y = t$, and choose a t-range to match the y-range.

Parametric equations for conic sections

Circle: $(x - h)^2 + (y - k)^2 = r^2$;
$x = h + r \cos t, \quad y = k + r \sin t, \quad 0 \le t \le 2\pi$

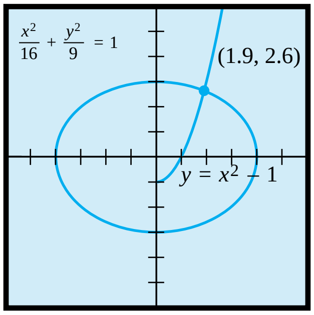
Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1; \quad x = a \cos t, \quad y = b \sin t, \quad 0 \le t \le 2\pi$

Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1; \quad x = \dfrac{a}{\cos t}, \quad y = b \tan t, \quad 0 \le t \le 2\pi$

Parabola: $y^2 = \pm 4px; \quad x = \left(\pm\dfrac{1}{4p}\right)t^2, \quad y = t.$

►EXAMPLE 7 ***Intersections from graphs*** Find (1 decimal place) all intersections of the ellipse and parabola.

$$\frac{x^2}{16} + \frac{y^2}{9} = 1, \qquad y = x^2 - 1$$



[−6, 6] by [−6, 6]

FIGURE 45

Solution

If we were to try to solve this problem algebraically, substitution of $x^2 - 1$ for y would lead to a fourth degree polynomial, for which, in general, we would need technology to find the real zeros. See Exercises 60 and 61. Using parametric graphing, we use the equations from the box above for the ellipse, $x = 4 \cos t$, $y = 3 \sin t$, $0 \le t \le 2\pi$. We could also use the parametric equations for the parabola given in the box, $x = t$, $y = t^2 - 1$, and obtain something like Figure 45, which shows only half of the parabola. The reason for the half parabola is that we have a t-range with only non-negative values. It is also difficult to trace along both curves to check the intersection because the t-values at the intersection are very different for the two curves.

There are several ways around this difficulty. We can extend the t-range or, better, we can change the parameterization. To have the intersection at the same t-value on both curves in this example, we use the same x-equation, $x = 4 \cos t$, and since $y = x^2 - 1$, for the parabola we set $y = (4 \cos t)^2 - 1$. We will then see all of the parabola that fits on the screen. We can also use the symmetry of both the ellipse and parabola, knowing that if we locate the intersection in Figure 45, there is another intersection with the negative of our x-value. Tracing along either curve, we find that the intersection is very near $(1.9, 2.6)$. By symmetry, the two intersections are approximately $(1.9, 2.6)$ and $(-1.9, 2.6)$. ◀

TECHNOLOGY TIP ◆ ***Finding intersections of parametric graphs***

We are used to zooming in to locate intersections graphically. Calculators zoom differently in parametric mode than we may expect, changing only the x- and y-ranges, for example, and not altering the t-range or t-step. Experiment with your own calculator. You may want to zoom in on a point of interest and then cut down the t-range and refine the t-step to show only the point of interest. This requires having the same t-value for both curves. You may also be able to simply read the pixel coordinates for the intersection.

EXERCISES 10.3

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. Point $(2, 1)$ is on the parabola with a focus at $(1, 0)$ and directrix given by $x = -1$.
2. If the directrix of a parabola is a horizontal line and the focus is below the directrix, then the parabola opens downward.
3. The graph of $x^2 + 2x + 1 = y^2 - 8y$ is a circle.
4. The graph of $x^2 = y^2 + 4$ is an ellipse.
5. The graph of $x^2 = 5 - y^2$ is a hyperbola.
6. The graph of $y^2 = x^2 + 1$ is a hyperbola with foci on the y-axis.

Exercises 7–10 Fill in the blanks so that the resulting statement is true.

7. The foci for the graph $x^2 = 2y^2 + 6$ are the points ______.
8. The vertices for the graph of $2x^2 + y^2 = 4$ are the points ______.
9. The circle $x^2 + y^2 = 4$ meets the ellipse $4x^2 + 9y^2 = 36$ in exactly ______ points.
10. The vertices for the graph of $25x^2 - 9y^2 = 225$ are ______.

Develop Mastery

Exercises 1–2 ***Identify Parabola Features*** *An equation for a parabola is given. Give the coordinates of the vertex and focus, and an equation for the directrix. Sketch the graph.*

1. $y^2 - 8x = 0$

2. $x^2 + 8y = 0$

Exercises 3–7 ***Equation for Parabola*** *The given conditions determine one or more parabolas, each with axis parallel to one of the coordinate axes. Find an equation in the form of Equation (2) or (3) for each parabola and sketch a graph.*

3. $F(2, 0)$; D: $x = -2$

4. $F(0, -3)$; D: $y = 3$

5. $F(0, -\frac{1}{2})$; D: $y = \frac{1}{2}$

6. Vertex $(0, 0)$; contains $(3, -2)$

7. Contains points $(1, 2)$, $(\frac{1}{2}, -\sqrt{2})$, and $(2, 2\sqrt{2})$

Exercises 8–10 ***Latus Rectum***

8. Show that the latus rectum of the parabola $x^2 = 4py$ has length $4p$. (*Hint:* Draw the parabola with its latus rectum. Each endpoint of the latus rectum is equidistant from F and D.)

9. Given parabola $4py = x^2$, let T_p be the triangle with vertices at the origin and at the ends of the latus rectum.
(a) Find the area of triangle T_p for parabola $y = x^2$.
(b) For what focal width does the area of T_p equal 2? 1?

10. Find an equation for the circle that contains the ends of the latus rectum of the parabola $2x = -y^2$ and that is tangent to the directrix of the parabola.

Exercises 11–12 ***Applications of Parabolas***

11. The parabolic mirror for the Mount Palomar telescope is 200 inches in diameter; the mirror is 3.75 inches deep at the center. How far from the center of the mirror is the focal point?

12. A parabolic headlight is 4 inches deep and has a maximum diameter of 4 inches. How far from the vertex should the light source be placed to produce a beam of light parallel to the axis of the parabola?

Exercises 13–14 ***Reflection Properties of Parabola***

13. (a) Write an equation in point-slope form for line L_m with slope m and containing point $(1, 1)$.
(b) For what positive slope m is the line in part ***(a)*** tangent to the parabola $y = x^2$? (For what value of m does L_m intersect the parabola in a single point?) Note that a line parallel to the axis of the parabola is not a tangent line.

14. (a) Find the coordinates of the right endpoint R of the latus rectum of the parabola $y = x^2$.
(b) Find the slope of the line tangent to the parabola $y = x^2$ at point R. See Exercise 13.

Exercises 15–16 ***Distances to Foci***

15. Given points $F_1(-3, 0)$ and $F_2(3, 0)$ and $P(x, y)$, suppose that the sum of the distances from P to F_1 and from P to F_2 is 10. From the equation $|\overline{PF_1}| + |\overline{PF_2}| = 10$, show that coordinates of $P(x, y)$ must satisfy the equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

16. Given points $F_1(0, -3)$ and $F_2(0, 3)$ and $P(x, y)$, suppose that the differences of the distances $|\overline{PF_1}| - |\overline{PF_2}|$ is ± 4. From the equation $|\overline{PF_1}| - |\overline{PF_2}| = \pm 4$, show that the coordinates of $P(x, y)$ must satisfy the equation $\frac{y^2}{4} - \frac{x^2}{5} = 1$.

Exercises 17–26 ***Features of Ellipse, Hyperbola*** *An equation is given for either an ellipse or a hyperbola in standard position.* ***(a)*** *Identify the curve, and find the coordinates of the vertices and the foci.* ***(b)*** *For an ellipse give the lengths of the major and minor axes; for a hyperbola, find equations for the asymptotes.* ***(c)*** *Sketch the graph.*

17. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

18. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

19. $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

20. $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

21. $4x^2 + 5y^2 = 20$

22. $4x^2 - 5y^2 = 20$

23. $5x^2 - 4y^2 = 20$

24. $5x^2 + 4y^2 = 20$

25. $x^2 = 9y^2 - 144$

26. $x^2 = 144 - 9y^2$

Exercises 27–33 ***Ellipse Equations*** *Find an equation in standard form for the ellipse satisfying the given conditions.*

27. Foci $(\pm 3, 0)$, vertices $(\pm 5, 0)$

28. Foci $(0, \pm 2)$, vertices $(0, \pm 4)$

29. Foci $(\pm 2, 0)$, major axis 6

30. Vertices $(\pm 5, 0)$, minor axis 4

31. Vertices $(\pm 5, 0)$, contains $(4, -1)$

32. Contains $(2, 1)$ and $(1, -\frac{\sqrt{7}}{2})$

33. Minor axis 12, contains $(5, 4)$ (two solutions)

Exercises 34–40 ***Hyperbola Equations*** *Find an equation for the hyperbola in standard position satisfying the given conditions.*

34. Foci $(\pm 3, 0)$, vertices $(\pm 2, 0)$

35. Foci $(0, \pm 4)$, vertices $(0, \pm 2)$

36. Foci $(\pm 3, 0)$, asymptote $y = x$

37. Vertices $(\pm 1, 0)$, asymptote $y = 2x$

38. Vertices $(0, \pm 1)$, asymptote $y = \dfrac{3}{2}x$

39. Contains $(2, -1)$ and $(\sqrt{10}, 5)$

40. Vertices $(0, \pm 4)$, asymptotes are perpendicular to each other

Exercises 41–44 ***Intersections of Conics*** *Find the coordinates of all intersection points of each pair of curves, and show the solutions graphically.*

41. $4x^2 + 3y^2 = 12$, with the line containing $(\frac{3}{2}, -1)$ and the upper focus point of the ellipse.

42. $x^2 - y^2 = 8$, with the line containing $(-3, 1)$ and the right focus point of the hyperbola.

43. $x^2 + 3y^2 = 1$; $3x^2 + y^2 = 1$

44. $4y^2 - x^2 = 36$; $32y = x^2 + 96$

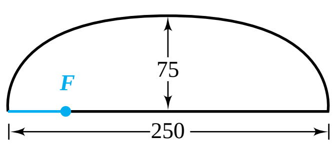
Exercises 45–48 ***Applications of Ellipse, Hyperbola***

45. Let C be the circle having as diameter the segment with ends at the foci of the hyperbola $5x^2 - 4y^2 = 9$. Show that C also contains the foci of the conjugate hyperbola $4y^2 - 5x^2 = 9$.

46. An elliptical garden is to be laid out in a rectangular area 16 feet by 20 feet by driving stakes at the foci and tying an end of a rope to each stake. Where should the stakes be placed, and how long a rope is needed to make the largest possible ellipse in the available area?

47. “The Ellipse” in Washington, D.C., is an elliptical grassy area between the White House and the Washington Monument. The major axis is approximately 500 yards and the minor axis is approximately 425 yards. How far are the foci from the vertices of the ellipse?

48. Suppose an auditorium is to be built with cross sections in the shape of a half-ellipse, as in the diagram. The building is to be 250 feet long and 75 feet high. If a speaking platform is located at one focus of the ellipse, how far from the nearest end of the building should it be?



Exercises 49–53 ***Explore: Graphing Pairs of Conics***

49. Graph the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ on the same set of axes.

50. Find another pair (an ellipse and a hyperbola with foci on the x-axis) whose graphs are related to each other as the pair in Exercise 49.

51. Find another pair with foci on the y-axis whose graphs are related to each other in the same way as the pair in Exercise 49.

52. Graph the conjugate hyperbolas $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and $\frac{x^2}{9} - \frac{y^2}{16} = 1$ on the same set of axes.

53. Repeat Exercise 52 for the conjugate hyperbolas $\frac{x^2}{144} - \frac{y^2}{9} = 1$ and $\frac{x^2}{9} - \frac{y^2}{144} = 1$. (*Hint:* What kind of window will you need?)

Exercises 54–57 ***Intersections of Graphs*** ***(a)*** *Graph parametrically and approximate all intersections (1 decimal place).* ***(b)*** *Eliminate a variable and find the intersections in exact form.*

54. $\dfrac{x^2}{9} + \dfrac{y^2}{9} = 1, \qquad \dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

55. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1, \qquad \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

56. $y^2 = \dfrac{x}{4}, \qquad \dfrac{x^2}{4} - \dfrac{y^2}{2} = 1$

57. $x = t^2 + t,\ y = t^3 - t$
$x = t^2 + t,\ y = -t^4 - 2t^3 + t^2 + 2t$
(*Hint*: Equate y-values and solve for t.)

58. **(a)** ***Explore*** Show that line $\frac{-4x}{18} + \frac{1y}{9} = 1$ is tangent to the ellipse $\frac{x^2}{18} + \frac{y^2}{9} = 1$ at the point $(-4, 1)$.
(b) Show that line $\frac{2x}{8} + \frac{5y}{50} = 1$ is tangent to the ellipse $\frac{x^2}{8} + \frac{y^2}{50} = 1$ at point $(2, 5)$.
(c) Make a guess for an equation of the line tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point (m, n). Test your guess for point $(3, \frac{-8}{5})$ on the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$.
(d) Does your guess also work for hyperbolas?

59. ***Explore: Modify a Definition*** Given points $F_1(1, 0)$, $F_2(-1, 0)$, the set of points such that the *sum of the distances to* F_1 *and* F_2 is 4 is an ellipse. Find an equation for the set of points such that the sum of the *squares* of the distances to F_1 and F_2 equals 4 (that is, $|\overline{PF_1}|^2 + |\overline{PF_2}|^2 = 4$).

Exercises 60–61 ***Intersection Points*** *Find the intersection points of the graphs of the two equations by* ***(a)*** *eliminating y and solving a fourth degree equation in x (2 decimal places), and* ***(b)*** *eliminating x and solving a quadratic in y (4 decimal places).*

60. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1,\ y = x^2 - 1$ (See Example 7)

61. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1,\ y = x^2 - 2$

10.4 TRANSLATIONS AND COORDINATE TRANSFORMATIONS

. . . [T]his is . . . a mathematical form that has survived several scientific revolutions! Cartesian coordinates imply continuity, as well as the notion of space as a backdrop against which objects move.

F. David Peat

Some years ago, after I had given a talk, somebody said, "You seem to make mathematics sound like so much fun." I was inspired to reply, "If it isn't fun, why do it?"

Ralph P. Boas

We have given considerable attention to basic transformation of graphs of functions of the form $y = f(x)$. Of the variety of function transformations, vertical and horizontal shifts, reflections, dilations, and composition with absolute value functions, only vertical and horizontal shifts play a significant role in the graphing of conic sections in general.

Because of symmetry, reflections are less important for circles, ellipses and hyperbolas centered at the origin. Dilations and absolute value compositions change distances and hence change the very nature of conic section definitions. A compressed circle becomes an ellipse, for example. A dilated parabola remains a parabola, but the shape changes; it may well have a different focus, directrix, and focal width.

The kinds of transformations we consider in this section also apply to more general kinds of equations than those of the form $y = f(x)$. To allow us to include equations of the standard defining forms for conic sections (and many others as well), we allow any equation that can be written in the form $F(x, y) = c$, or since we can subtract a constant from both sides, of the form $F(x, y) = 0$.

Changing Coordinates and Translating Conics

The model that makes the general situation easy to remember is the circle. The standard form equation for the circle centered at the origin is

$$x^2 + y^2 = r^2. \tag{1}$$

The same circle, shifted to the center point $C(h, k)$ is described by the equation

$$(x - h)^2 + (y - k)^2 = r^2. \tag{2}$$

We can think about this in terms of shifts right or left and up or down if we wish, but that isn't usually the way we think about circles. We see Equation (2) and recognize that the center of the circle is at $C(h, k)$. That is, replacing x by $x - h$ and y by $y - k$ shifts what was the origin (the center of the circle in Equation (1)) to the new center. The effect is as if we had moved the whole coordinate system to a new system where the origin is now at the point $C(h, k)$.

The same reasoning applies to any equation relating x and y that can be written in the form $F(x, y) = 0$. In particular, any equation for one of the conic sections can be written in such a form.

When we can rewrite an equation completely in terms of $x - h$ and $y - k$, we sometimes replace the expressions $(x - h)$ and $(y - k)$ by new names, X and Y respectively. Thus, for example, by completing squares, the equation

$$x^2 + y^2 - 2x + 6y + 6 = 0$$

can be rewritten in the form

$$(x - 1)^2 + (y + 3)^2 = 4.$$

Replacing $(x - 1)$ by X and $(y + 3)$ by Y gives

$$X^2 + Y^2 = 4.$$

The advantage of this procedure is that the XY-equation is a standard form for the circle centered at the origin of the XY-system, that is, at the point $(1, -3)$.

Changing coordinates

Given an equation in variables x and y of the form $F(x, y) = 0$, we can rewrite the equation completely in terms of $x - h$ and $y - k$, then make the replacements $X = x - h$ and $Y = y - k$.

The graph of the new equation **relative to the XY-coordinate system** is centered at the point $C(h, k)$.

Identifying the graph of a translated conic may be done by completing any squared terms and matching one of the standard forms from Section 10.3.

►EXAMPLE 1 ***Identify a conic section*** Complete the squares in x and y and identify the graph of the equation $4x^2 + 9y^2 + 16x - 54y + 61 = 0$ as a conic section by translating axes. Sketch the graph.

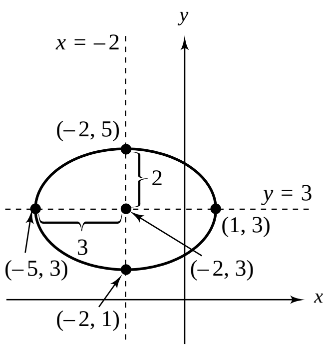


FIGURE 46
Ellipse
$$\frac{(x + 2)^2}{9} + \frac{(y - 3)^2}{4} = 1.$$

Solution

First collect the x-terms and y-terms and factor out the coefficients of x^2 and y^2.

$$4(x^2 + 4x) + 9(y^2 - 6y) = -61$$

Now complete the squares by adding the same quantities to both sides of the equation. Note that this implies adding $4 \cdot 4$ and $9 \cdot 9$, not just 4 and 9.

$$4(x^2 + 4x + 4) + 9(y^2 - 6y + 9) = -61 + 4 \cdot 4 + 9 \cdot 9$$
$$4(x + 2)^2 + 9(y - 3)^2 = 36.$$

Translate axes to a new system with center at $(-2, 3)$ and rename the variables.

$$4X^2 + 9Y^2 = 36.$$

Dividing through by 36 gives an equation for an ellipse in standard form.

$$\frac{X^2}{9} + \frac{Y^2}{4} = 1$$

The origin of the new coordinate system (the XY-system), the center of the ellipse, is at the point $(-2, 3)$. The vertices are 3 units right and left of the center, at $(-5, 3)$ and $(1, 3)$, and the ends of the minor axis are 2 units above and below the center, at $(-2, 5)$ and $(-2, 1)$. See Figure 46. ◄

Strategy: First complete the square on the y-terms and write the equation in the form of Equation (3) from Sec. 10.3, from which find p and the vertex. The focus is on the axis, p units from the vertex.

►EXAMPLE 2 ***A translated parabola*** Find the focus, the directrix, and the ends of the latus rectum for the parabola with equation $y^2 + 2y - 4x + 9 = 0$ and sketch the graph.

Solution

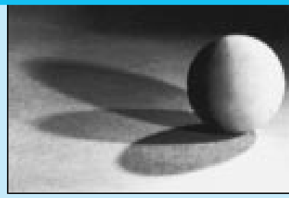
Follow the strategy.

$$y^2 + 2y = 4x - 9$$
$$y^2 + 2y + 1 = 4x - 8$$
$$(y + 1)^2 = 4(x - 2), \text{ or } Y^2 = 4X$$

HISTORICAL NOTE CONIC SECTIONS

The name *conic section* comes from the idea of sections or slices of a right, circular cone. If you could take slices through an ice cream cone at various angles, each slice would result in a conic section, as for instance, the circular section at the top of the cone where the ice cream rests.

The shadows in the photo are examples of conic sections. You can also observe models of conic sections created by the cone of light coming from the top of a lamp. The shadow of the lampshade on the wall forms conic sections. When the top is pointed directly at the wall the shadow is a circle. As the lamp is tipped, the shadow becomes a more and more elongated ellipse, until finally the shadow is no longer closed and becomes a parabola. Tipped further, the shadow becomes a branch of a hyperbola.



Note how the direction of the light source creates increasingly elongated shadows.

At least as far back as Euclid, the Greeks recognized and studied the conics. Apollonius wrote a treatise on conics (200 B.C.) that included our modern definitions in terms of distances from foci. His work remained essentially the last word on conics through the Middle Ages. At age 16 (in 1640), Pascal announced a remarkable theorem regarding what he called "mystic hexagons": Any six points on a conic section determine a hexagon with three pairs of opposite sides. If opposite sides are extended so that they intersect, then the three points of intersection all lie on one line.

Nature seems to like conic sections; we observe them all about us. Without conics, Kepler's discovery that planetary orbits are elliptical would have been unlikely. It is impossible to guess what effect that might have had on Sir Isacc Newton's physics and mathematics. The *Scientific American* reports that the Levy-Shoemaker comet that crashed into Jupiter in 1994 had been highly unstable, with some nearly circular orbits, and some narrow elliptical paths, one of which took it so close to Jupiter that it broke apart. Space exploration would be impossible without an understanding of conic sections.

This equation has the form of Equation (3) with a plus sign, so the vertex is at $(2, -1)$, the parabola opens to the right, and $4p = 4$, $p = 1$. The focus is 1 unit to the right of the vertex, at $F(3, -1)$, and the directrix is the vertical line D: $x = 1$. The focal width is 4, so the ends of the latus rectum are at $(3, 1)$ and $(3, -3)$. The graph is shown in Figure 47. ◀

Strategy: The vertex lies on the axis, so the axis is the vertical line $x = 2$. The focus must be the point where the line $x + y = 1$ crosses the axis. The distance from the vertex to the focus is p, from which we can write an equation in the form of Equation (2) from Section 10.3.

▶EXAMPLE 3 ***Verbal to Equation*** Find an equation for the parabola with a vertical axis, vertex at $(2, -2)$, and focus on the line $x + y = 1$.

Solution

Begin with a diagram that shows the vertex and the line $x + y = 1$. See Figure 48a. The vertical line $x = 2$ intersects line $x + y = 1$ at point $(2, -1)$, so that is the focus of the parabola. Since the focus is 1 unit from the vertex, $p = 1$, the vertex is at $(2, -2)$, and the parabola opens upward. As suggested in the Strategy, an equation for the parabola is $(x - 2)^2 = 4(y + 2)$. The parabola is shown in Figure 48b. ◀

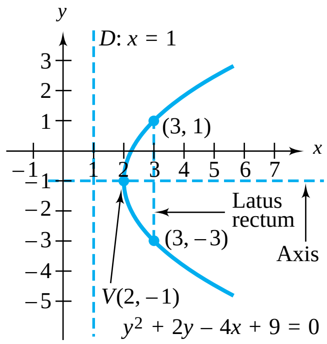


FIGURE 47

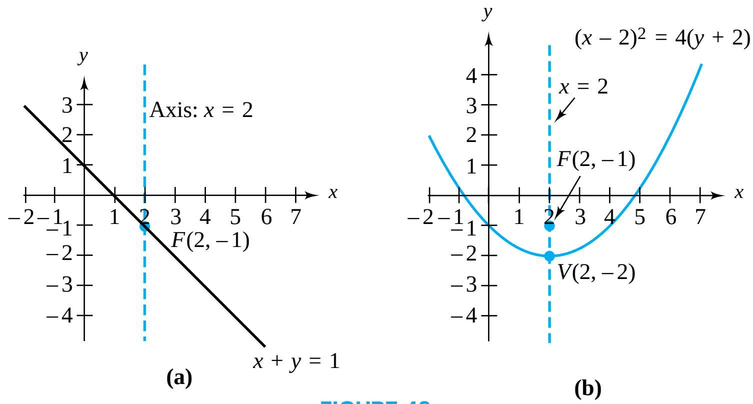


FIGURE 48

Strategy: From the figure, the hyperbola is in standard position relative to the coordinate system centered at (2, 1). Knowing $a = 2$ and $c = 3$, find b.

►EXAMPLE 4 ***A translated hyperbola*** Find an equation for the hyperbola shown in Figure 49 with center at (2, 1), vertices at (0, 1), (4, 1) and foci at (−1, 1), (5, 1).

Solution

Follow the strategy. With respect to the XY-system centered at (2, 1), the equation has the form $\dfrac{X^2}{a^2} - \dfrac{Y^2}{b^2} = 1$ with vertices at (0, 1) and (4, 1) and foci at (−1, 1) and (5, 1). With $c = 3$, $a = 2$, $b^2 = c^2 - a^2 = 3^2 - 2^2 = 5$. The XY-equation is

$$\frac{X^2}{4} - \frac{Y^2}{5} = 1.$$

Replacing X by $x - 2$ and Y by $y - 1$, the desired equation is

$$\frac{(x-2)^2}{4} - \frac{(y-1)^2}{5} = 1. \blacktriangleleft$$

Examples 2, 3, and 4 show a consistent pattern in equations for the entire family of conic sections.

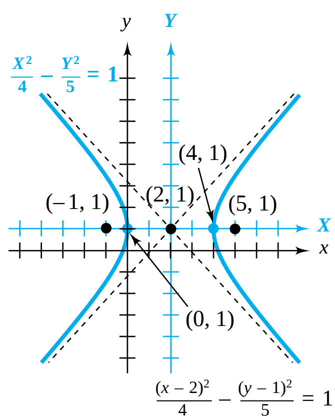


FIGURE 49

Conic section standard form equations

Suppose a conic section is in standard position relative to a translated coordinate system centered at (h, k). The conic section has an equation in one of these standard forms:

Circle: $(x - h)^2 + (y - k)^2 = r^2$

Parabola: $(x - h)^2 = \pm 4p(y - k)$

or $(y - k)^2 = \pm 4p(x - h)$

Ellipse: $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

or $$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

Hyperbola: $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

or $$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Graphing Translated Conic Sections Parametrically

The same ideas that allow us to easily graph standard conic sections in parametric form work just as well for translated conics. Except for the parabola, all of the standard form equations in the box above can be graphed by making use of identities (I–4) and (I–5) in almost the same way as we did in the previous section, following the same pattern we use for circles. The following box displays only the forms for ellipse and hyperbolas with horizontal axis through the vertices, but the adjustment for a vertical major axis should be obvious. For the ellipse, interchange a and b; for the hyperbola, interchange x and y in the standard equation, and hence the roles of sec t, tan t in the parametric equations.

Parametric equations for translated conic sections

Circle: $(x - h)^2 + (y - k)^2 = r^2$;

$x = h + r\cos t, \quad y = k + r\sin t, \quad 0 \le t \le 2\pi$

Ellipse: $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1;\ a > b$$

$x = h + a\cos t, \quad y = k + b\sin t, \quad 0 \le t \le 2\pi$

Hyperbola: $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1;$$

$$x = h + \frac{a}{\cos t}, \quad y = k + b\tan t, \quad 0 \le t \le 2\pi$$

Parabola: $$(y - k)^2 = 4p(x - h); \quad x = h + \frac{1}{4p}t^2, \quad y = k + t.$$

▶EXAMPLE 5 *Identify and graph a conic section*

(a) Write the equation in standard form and identify the graph.
(b) Draw a calculator graph.

$$3x^2 + y^2 - 12x + 2y = 0$$

Solution

(a) Complete squares on both x and y and divide to get a 1 on the right.

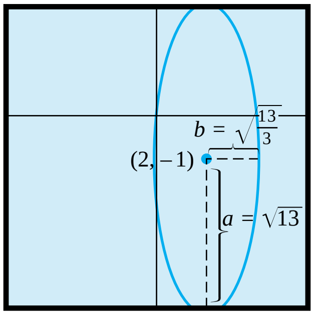
$$3(x^2 - 4x + 4) + (y^2 + 2y + 1) = 3 \cdot 4 + 1$$
$$3(x - 2)^2 + (y + 1)^2 = 13$$
$$\frac{(x-2)^2}{\left(\sqrt{\frac{13}{3}}\right)^2} + \frac{(y+1)^2}{(\sqrt{13})^2} = 1$$

From the last equation we recognize an ellipse centered at $(2, -1)$ with $a = \sqrt{13}$, $b = \sqrt{13/3}$.

(b) A calculator graph, using the parametric equations

$$x = 2 + \sqrt{13/3}\cos t,\ y = -1 + \sqrt{13}\sin t,\ 0 \le t \le 2\pi,$$

is shown in Figure 50. ◀



[−6, 6] by [−4.5, 2.5]

FIGURE 50
$3x^2 + y^2 - 12x + 2y = 0.$

▶EXAMPLE 6 *Identify and graph a conic section*

Repeat Example 5 for the equation

$$3x^2 - y^2 - 12x + 2y = 0.$$

Solution

(a) The equation is the same as the one in Example 5 except for the negative sign on the y^2. We proceed as we did in Example 5.

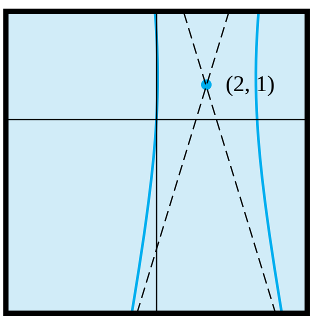
$$3(x^2 - 4x + 4) - (y^2 - 2y + 1) = 3 \cdot 4 - 1$$
$$3(x - 2)^2 - (y - 1)^2 = 11$$
$$\frac{(x-2)^2}{\left(\sqrt{\frac{11}{3}}\right)^2} - \frac{(y-1)^2}{(\sqrt{11})^2} = 1$$

From the last equation we recognize a hyperbola opening right and left, centered at $(2, 1)$ with $a = \sqrt{11/3}$, $b = \sqrt{11}$.

(b) A calculator graph, using

$$x = 2 + \frac{\sqrt{11/3}}{\cos t},\ y = 1 + \sqrt{11}\tan t,\ 0 \le t \le 2\pi,$$

is shown in Figure 51. Note the false asymptotes. Moving the cursor to their intersection (You cannot trace along the asymptotes; *why not?*) indicates that they intersect very near $(2, 1)$, the center of the hyperbola. ◀



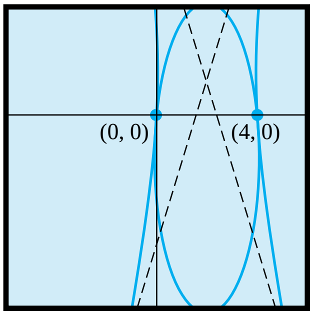
[−6, 6] by [−4.5, 2.5]

FIGURE 51
$3x^2 - y^2 - 12x + 2y = 0.$

►EXAMPLE 7 Intersections of conic sections

(a) Graph the equations of both Examples 5 and 6 on the same screen and find the approximate coordinates of all intersection points.
(b) Confirm algebraically.

Solution



[−6, 6] by [−4.5, 2.5]

FIGURE 52

$3x^2 + y^2 - 12x + 2y = 0$
$3x^2 - y^2 - 12x + 2y = 0$

(a) A calculator graph of both equations is shown in Figure 52. The ellipse and the hyperbola (remember that the asymptotes are not part of the graph of the hyperbola) appear to intersect near the points (0, 0) and (4, 0), but in parametric mode tracing is limited to t-values and and it is difficult to tell more than that both graphs do cross the x-axis very near those two points.

(b) In this case, where it appears that the intersections may have integer coordinates, we can substitute those values and verify that both (0, 0) and (4, 0) lie on both graphs. A more dependable approach in general requires us to try to solve a system of nonlinear equations.

$$3x^2 + y^2 - 12x + 2y = 0$$
$$3x^2 - y^2 - 12x + 2y = 0$$

We can eliminate x by subtracting the second from the first, getting

$$2y^2 = 0, \quad \text{or} \quad y = 0.$$

Substituting $y = 0$ into either of the original two equations and solving, we get $x = 0, 4$, confirming that the intersections are (0, 0) and (4, 0). ◄

General Second Degree Equation in Two Variables

On the basis of our experience with completing squares, we have come to expect that the graph of any equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \qquad \textbf{(3)}$$

is a conic section. If we allow for what are called degenerate cases such as a line (when A and C are both zero) and point- or imaginary circles, it is true that any such equation does represent a conic section. The kind of conic is determined by A and C. For example, we know that we have a circle if $A = C$ and a parabola if there is only one squared term. Rather than classifying all possibilities of A and C, however, we suggest that when you have such an equation that you complete the squares and write the equation in one of the standard forms, from which you can read all the pertinent information about the graph.

You may observe that there is a missing term in Equation (3). The more general possibility for a second degree equation in two variables includes a term of the form Bxy. It remains true that the graph of any equation of the form of Equation (3), even including a nonzero xy-term, is some kind of conic section. When there is a nonzero xy-term, the axes of the conic are rotated from the original coordinate axes.

Rotation Transformations and Graphing

In the graph of general second degree equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

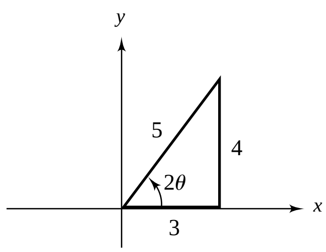
with $B \neq 0$, the axes of the conic section are not horizontal and vertical. By rotating the axes through an angle θ, we get a new set of XY-axes. The new XY-variables are related to x and y by transformation equations:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{4}$$

If we choose the angle θ so that $\tan 2\theta = \frac{B}{A - C}$, then the resulting equation in the new variables will contain no XY-term. The transformation equations require $\sin\theta$ and $\cos\theta$, which we can get from identities. Draw a reference triangle showing 2θ, find $\cos 2\theta$, and use the identities

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}}, \qquad \cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}}.$$

Once we identify the conic in the new coordinate-system, we can use the transformation equations for parametric representation of the rotated conic.



(a)

$+2n\ 2\theta = \frac{4}{3}$

▶EXAMPLE 8 ***A rotated conic*** Use transformation equations (4) to rotate the axes to eliminate the xy-term and graph

$$52x^2 - 28xy + 73y^2 = 720.$$

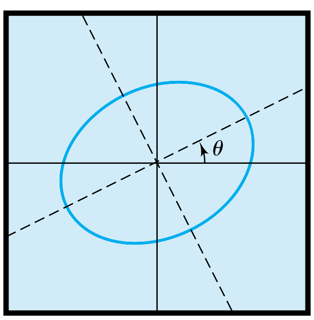
Solution

Setting $\tan 2\theta = \frac{B}{A - C} = \frac{-28}{52 - 73} = \frac{4}{3}$, we draw the diagram in Figure 53a, from which $\cos 2\theta = \frac{3}{5}$. Then the above identities yield

$$\sin\theta = \sqrt{\frac{1 - \frac{3}{5}}{2}} = \frac{1}{\sqrt{5}} \quad \text{and} \quad \cos\theta = \sqrt{\frac{1 + \frac{3}{5}}{2}} = \frac{2}{\sqrt{5}}.$$

Thus we rotate through an angle given by $\theta = \mathrm{Cos}^{-1} \frac{2}{\sqrt{5}} \approx 26.6°$. Substituting these values into the transformation equations, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2X - Y \\ X + 2Y \end{bmatrix}$$



[− 6, 6] by [− 6, 6]

(b)

FIGURE 53
$52x^2 - 28xy + 73y^2 = 720$

so $x = \frac{2X - Y}{\sqrt{5}}$ and $y = \frac{X + 2Y}{\sqrt{5}}$. Now we have some messy algebra, which we leave for you to check. Substituting for x and y, expanding, multiplying through by 5, and collecting terms, we get

$$225X^2 + 0 \cdot XY + 400Y^2 = 3600.$$

Finally, we can write this last equation in the form $\frac{X^2}{16} + \frac{Y^2}{9} = 1$, an equation for an ellipse with major axis of 8 and minor axis of 6.

To see the graph of the original equation, we use the idea that *in the new coordinate system,* we would use the parametric equations

$$X = 4\cos t,\ Y = 3\sin t.$$

Since $x = (2X - Y)/\sqrt{5}$ and $y = (X + 2Y)/\sqrt{5}$, we have

$$x = \frac{(8 \cos t - 3 \sin t)}{\sqrt{5}}, \; y = \frac{(4 \cos t + 6 \sin t)}{\sqrt{5}}.$$

Graphing these parametric equations with a t-range of $[0, 2\pi]$ shows something like Figure 53b, where we have drawn in dashed lines to show the rotated X- and Y-axes. ◀

EXERCISES 10.4

Check Your Understanding

Exercises 1–5 True or False. Give reasons

1. If $(1, 3)$ is the focus of a parabola and $(1, -1)$ is its vertex, then the directrix is the horizontal line $y = -3$.
2. The vertex of the parabola $(y - 2)^2 = 4(x + 1)$ is a point in the fourth quadrant.
3. The graph of $y^2 = 4(x - 2)$ contains points in the first and fourth quadrants.
4. The graph of $y^2 = 2(1 - x)$ contains points in all four quadrants.
5. The ellipse $3(x - 2)^2 + 4(y - 1)^2 = 12$ contains no points in the third or fourth quadrants.

Exercises 6–10 Fill in the blank with the quadrant(s) so that the resulting statement is true.

6. The vertex of the parabola $x^2 + 2x + 4y - 7 = 0$ is in ______.
7. The hyperbola $(x - 2)^2 - (y + 1)^2 = 1$ has its vertices in ______.
8. The graph of the parabola $y = x^2 + 2x + 1$ contains points in ______.
9. The graph of the parabola $(y - 1)^2 - 4(x - 2) = 0$ contains points in ______.
10. The intersections of $(x + 3)^2 + 4(y - 1)^2 = 4$ and $(x + 3)^2 - 4(y - 1)^2 = 4$ lie in ______.

Develop Mastery

Exercises 1–6 ***Features of a Parabola*** *Find coordinates of vertex and focus, and an equation for the directrix.*

1. $4x + y^2 - 2y + 9 = 0$
2. $x^2 + 2x - 8y + 9 = 0$
3. $x^2 - 2x + 2y - 3 = 0$
4. $9y^2 + 24y - 12x + 28 = 0$
5. $x^2 + 2x - 6y - 17 = 0$
6. $x^2 - 6x - 2y + 1 = 0$

Exercises 7–16 ***Equation of Parabola*** *Find an equation for the parabola or parabolas satisfying the conditions.*

7. Vertex $(2, -1)$; D: $x = 1$
8. Vertex $(2, -1)$; D: $y = 0$
9. Vertex $(-1, -2)$; contains $(1, 3)$
10. Vertex $(2, 1)$; focus on the line $x + y = 1$
11. Vertex $(2, 1)$; focus on the line $y = 2x + 1$
12. D: $x = 2$; bottom endpoint of latus rectum at $(6, 2)$
13. D: $x = 2$; upper endpoint of the latus rectum at $(6, 2)$
14. $F(1, 2)$; one endpoint of latus rectum at $(-1, 2)$
15. The latus rectum is the common chord of the two circles $x^2 + y^2 - 4x + 2 = 0$ and $x^2 + y^2 - 4x - 6y + 8 = 0$.
16. The latus rectum is the vertical diameter of the circle $x^2 + y^2 - 2x + 2y = 2$.

Exercises 17–21 ***Equation of Ellipse*** *Find an equation in x and y for the ellipse specified by the given conditions.*

17. Foci $(3, 2)$, $(3, -2)$; major axis 6
18. Vertices $(1, 5)$, $(1, 1)$; minor axis 2
19. Vertices $(3, -1)$, $(-1, -1)$, contains $(1, 0)$
20. Center $(3, -1)$, vertex $(5, -1)$, focus $(\frac{9}{2}, -1)$
21. Center $(-3, 0)$, vertex $(-3, 3)$, minor axis 4

Exercises 22–26 ***Equation of Hyperbola*** *Find an equation in x and y for the hyperbola specified by the given conditions.*

22. Vertices $(3, 2)$, $(3, -2)$, contains $(4, 4)$
23. Center $(-2, 1)$, vertex $(-2, 3)$, focus $(-2, 4)$
24. Center $(0, -1)$, vertex $(2, -1)$, focus $(3, -1)$
25. Foci $(4, 2)$, $(-4, 2)$, vertex $(2, 2)$
26. Vertex $(1, 1)$, asymptotes $y = x - 1$, $y = -x + 3$

Exercises 27–40 ***Identify Translated Conics*** *Identify and sketch the graph of the conic section that corresponds to the equation. Give coordinates of center and vertices.*

27. $\dfrac{(x - 1)^2}{4} - \dfrac{(y + 1)^2}{9} = 1$
28. $\dfrac{(x - 1)^2}{4} + \dfrac{(y + 1)^2}{16} = 1$

29. $\dfrac{x^2}{8} + \dfrac{(y-1)^2}{9} = 1$

30. $\dfrac{(x+2)^2}{4} - \dfrac{(y-2)^2}{16} = 1$

31. $\left(x + \dfrac{1}{2}\right)^2 + 16\left(y - \dfrac{3}{2}\right)^2 = 16$

32. $4(x + 1.25)^2 - (y - 2.5)^2 = 4$

33. $x^2 + 6x - y + 4 = 0$

34. $y^2 + 4y - 3x + 1 = 0$

35. $9x^2 + 4y^2 - 18x - 27 = 0$

36. $4x^2 + 9y^2 - 8x + 54y + 49 = 0$

37. $x^2 + y^2 + 6x - 2y + 6 = 0$

38. $x^2 + 4y^2 - 4x - 8y - 8 = 0$

39. $x^2 - 4y^2 + 2x + 16y - 19 = 0$

40. $8x^2 - 4y^2 + 8x + 12y - 23 = 0$

Exercises 41–48 ***Features of Translated Conics*** *Complete the squares and then write the equation in standard form for a conic section, and graph.*

41. $x^2 - 6x - y + 4 = 0$

42. $x^2 + 4x - 3y + 1 = 0$

43. $4x^2 + y^2 - 8x + 4y - 8 = 0$

44. $4x^2 + 9y^2 - 18y - 27 = 0$

45. $x^2 + y^2 - 6x + 2y + 6 = 0$

46. $x^2 - 4y^2 - 2x - 16y - 19 = 0$

47. $x^2 - y^2 - 4x - 6y - 9 = 0$

48. $y^2 - x^2 + 6x + 4y - 9 = 0$

Exercises 49–51 ***Parabolas***

49. Suppose a golf ball driven off the tee travels 200 meters down the fairway, and during its flight it reaches a maximum height of 50 meters. Taking the tee as the origin of a coordinate system with positive x-axis along the ground in the direction of the drive, find an equation that describes the ball's parabolic path.

50. Find an equation for the parabola with vertex at the left focus of the ellipse $9x^2 + 25y^2 = 900$ and that contains the endpoints of the right latus rectum of the same ellipse.

51. Show that the equation $y = ax^2 + bx + c$, where $a \neq 0$, can be written in the form $(y - k) = a(x - h)^2$, where $h = \frac{-b}{2a}$ and $k = c - \frac{b^2}{4a}$.

Exercises 52–53 ***Latus Rectum***

52. Show that the length of each latus rectum of an ellipse is $\frac{2b^2}{a}$. (*Hint:* Find the coordinates of the endpoints of the vertical chord through the focus.)

53. Show that the length of each latus rectum of a hyperbola is $\frac{2b^2}{a}$. (*Hint:* Find the coordinates of the end points of the vertical chord through the focus.)

Exercises 54–61 ***Rotation of Axes*** *Transformation Equation (4) shows how to change variables when rotating axes through an angle θ.* ***(a)*** *Draw a diagram with an angle θ between 0° and 90° such that $\tan 2\theta = \frac{B}{A-C}$ and then find $\cos 2\theta$.* ***(b)*** *Use the identities $\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}$, $\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$ to write out transformation equations.* ***(c)*** *Find a standard form equation for the curve relative to the XY-coordinate system, identify the conic section, and draw a graph, using the appropriate parametric equations. See Example 8.*

54. $x^2 + 4xy - 2y^2 - 6 = 0$

55. $x^2 - 4xy - 2y^2 - 6 = 0$

56. $x^2 - \sqrt{3}xy + 2y^2 - 10 = 0$

57. $4x^2 - 3xy - 18 = 0$

58. $x^2 + 8xy + 7y^2 - 1 = 0$

59. $x^2 + \sqrt{3}xy + 2y^2 - 5 = 0$

60. $9x^2 - 6xy + 17y^2 - 72 = 0$

61. $x^2 + 3xy + y^2 - 10 = 0$

Exercises 62–65 ***Explore***

62. Find an equation for the set of points P that are equidistant from $F_1(1, 1)$ and the line D: $x + y + 2 = 0$. (*Hint:* Use Equation (4) from Section 10.1 for the distance from a point to a line.)

63. Given $F_1(1, 1)$ and $F_2(-1, -1)$, find an equation for the set of points P such that the sum of the distances, $|\overline{PF_1}| + |\overline{PF_2}|$, equals 4.

64. **(a)** From the definitions of conic sections, describe the kinds of curves you should have for the graphs of your equations in Exercises 62 and 63.
    (b) Use the transformation Equations (4) for rotation of axes and identify the graph of the rotated conic in Exercise 62.
    (c) Repeat part **(b)** for the conic in Exercise 63.

65. ***Distance to an Ellipse***
    (a) Use the techniques of Section 10.2 to find the distance (1 decimal place) from the point $P(4, 2)$ to the ellipse $x^2 + 25y^2 = 25$ and the point on the ellipse nearest P.
    (b) Explain why the point on the ellipse nearest P is not on the line from P to the center O of the ellipse.

10.5 POLAR COORDINATES

> In such examples as Lobachevsky's non-Euclidean geometry, or Cayley's matrix theory, or Galois' and Jordan's group theory, or the algebraic topology of the mid-twentieth century, pure mathematics seemed to have left far behind any physical interpretation or utility. And yet, in the cases mentioned here, and many others, physicists later found in these "useless" mathematical abstractions just the tools they needed.
>
> *Reuben Hersh*

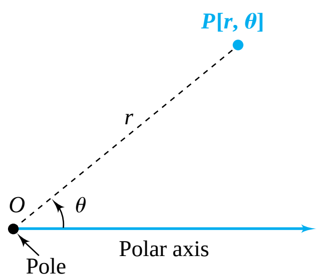
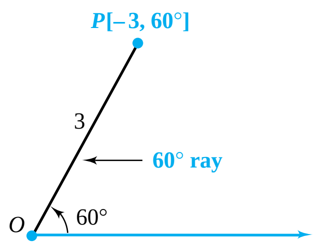
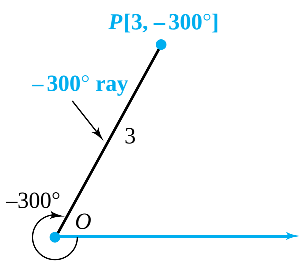


FIGURE 54



(a)



(b)

FIGURE 55

In Section 10.4 we considered alternative ways to get equations for given plane curves by changing coordinate systems from one rectangular system to another. Now we consider an entirely different way to name points using distances and angles: **polar coordinates.**

For polar coordinates, we begin with a single point O, which we call the **pole.** From the pole we take an initial ray called the **polar axis,** as shown in Figure 54. To each point P in the plane we assign an ordered pair $[r, \theta]$, where r is the **directed distance** from O to P, and θ is a **directed angle** from the polar axis to $\overline{OP}$. We use the convention for positive and negative angles from Chapter 5 (counterclockwise is positive, clockwise is negative). We may use either radian- or degree-measure for θ. The ray OP from the pole through P is the **θ-ray.**

Rectangular coordinates allow precisely one ordered pair (x, y) for every point in the plane. Polar coordinates for a given point are never unique. For instance, if P is the point 3 units from O along the 60° ray, then both $[3, 60°]$ (or in radian measure $[3, \frac{\pi}{3}]$) and $[3, -300°]$ are names for P. See Figure 55. In fact, because infinitely many different angles are coterminal with $\overline{OP}$, there are infinitely many polar coordinate names for P.

Furthermore, in addition to multiple angle names, another option arises. We said that r is a *directed* distance, implying that r can assume positive or negative values. For a negative number r, to reach the point $[r, \theta]$, we go r units in the *opposite* direction. We can reach the point $P[3, 60°]$ by going -3 units along the 240° ray, so that $[-3, 240°]$ is yet another name for P. See Figure 56. For the pole, $[0, \theta]$ names the pole for any angle θ.

►EXAMPLE 1 ***Polar coordinate names for a point*** Point P is 4 units from the pole O along the $-30°$ ray. Describe all possible polar coordinate names (using degree measure) for P.

Solution

First draw a diagram to show P. See Figure 57. Clearly, one name for P is $[4, -30°]$, but an angle name can be any angle coterminal with $-30°$, so P can be named by any of the pairs $[4, -30° + k \cdot 360°]$ where k is any integer. We can also reach P by going 4 units in the opposite direction on the 150° ray. Hence P can be named as well by $[-4, 150° + k \cdot 360°]$ for any integer k. ◄

Relating Polar Coordinates and Rectangular Coordinates

If we want the option of choosing either polar or rectangular coordinate equations for a given curve, we must be able to relate the two systems. Take the pole O at the origin of the xy system and the positive x axis as the polar axis. With the diagram

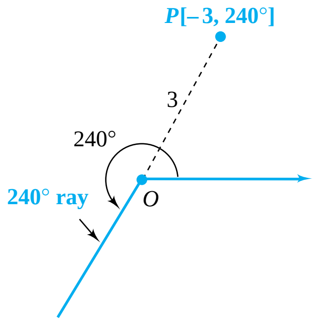


FIGURE 56

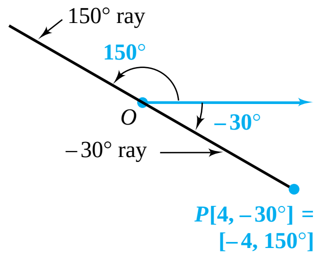


FIGURE 57

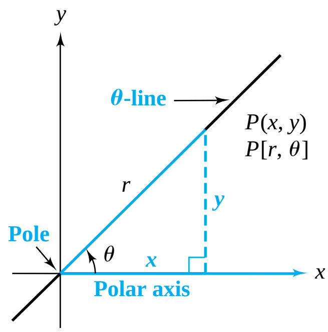


FIGURE 58

in Figure 58, we can read off the relations between polar and rectangular coordinates for a given point P.

Polar–rectangular coordinate transformation equations

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{1}$$

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \tag{2}$$

Strategy: Begin with a diagram that shows P and Q, and one for A and B, and use Equations (1) and (2).

▶EXAMPLE 2 ***Polar⟷rectangular*** Use the polar–rectangular coordinate transformation equations to

(a) express $P[4, \frac{\pi}{6}]$ and $Q[-2, 90°]$ in rectangular coordinates, and

(b) express $A(-2, 2)$ and $B(-1, 0)$ in polar coordinates.

Solution

(a) Follow the strategy. See Figure 59a. From transformation Equations (1), for P

$$x = 4\cos\left(\frac{\pi}{6}\right) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3} \qquad y = 4\sin\left(\frac{\pi}{6}\right) = 4 \cdot \frac{1}{2} = 2.$$

At that time [in high school] I became interested in chemistry. When I discovered that I could integrate things such as $(1 + ax + bx^2)^{-1/2}$, a specialist . . . recommended that I read some calculus books. However, books of all kinds were unavailable locally.

Lucien Le Cam

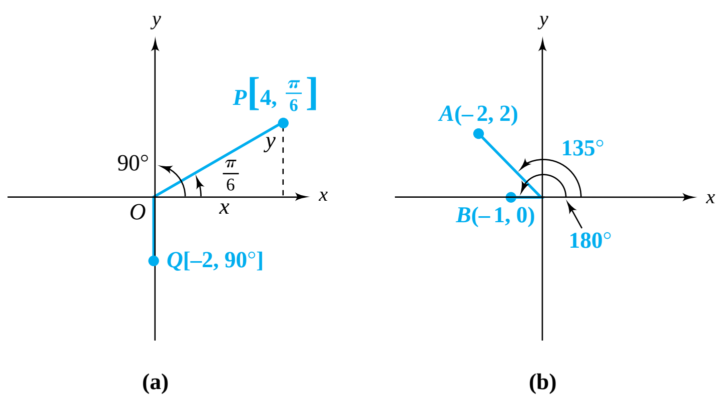


FIGURE 59

Similarly, for Q,

$$x = -2 \cos 90° = -2(0) = 0 \qquad y = -2 \sin 90° = -2(1) = -2.$$

Thus both $[4, \frac{\pi}{6}]$ and $(2\sqrt{3}, 2)$ are names for point P, while $[-2, 90°]$ and $(0, -2)$ both represent point Q.

(b) From transformation Equations (2), for A choose r and θ to satisfy

$$r^2 = (-2)^2 + 2^2 = 8 \quad \text{and} \quad \tan \theta = (-2)/2 = -1.$$

One such pair is $r = 2\sqrt{2}$ and $\theta = 135°$, so point A has names $(-2, 2)$ and $[2\sqrt{2}, 135°]$. For B,

$$r^2 = (-1)^2 + 0^2 = 1 \quad \text{and} \quad \tan \theta = 0/(-1) = 0.$$

We can take $r = 1$ and $\theta = 180°$, or take r to be -1 and any value of θ that is coterminal with 0°, so $(-1, 0)$, $[1, 180°]$, and $[-1, 0°]$ are all names for B. See Figure 58b. ◀

▶EXAMPLE 3 ***Rectangular to polar equation*** Express the equation $x^2 + y^2 - 2y = 0$ in polar coordinates.

Strategy: Use $x = r \cos \theta$ and $y = r \sin \theta$. Simplify.

Solution

Follow the strategy. Using the identity $\cos^2 \theta + \sin^2 \theta = 1$,

$$(r \cos \theta)^2 + (r \sin \theta)^2 - 2(r \sin \theta) = 0,$$
$$r^2(\cos^2 \theta + \sin^2 \theta) - 2r \sin \theta = 0, \qquad r^2 - 2r \sin \theta = 0.$$

Simplifying and dividing by r,

$$r = 2 \sin \theta.$$

Since we cannot divide by 0, we must check to see that we do not lose the point where $r = 0$ (the pole) when we divide by r. The pole $(r = 0)$ satisfies $r^2 - 2r \sin \theta = 0$, so we want to see if there is still a value of θ for which the pole is on the graph. When $\theta = 0$, then $r = 2 \sin 0 = 0$, so $[0, 0]$ is still on the graph. ◀

▶EXAMPLE 4 ***Polar to rectangular equation*** Express the polar equation $r = \frac{4}{2 \sin \theta - \cos \theta}$ in terms of rectangular coordinates.

Solution

Write the given equation as $r(2 \sin \theta - \cos \theta) = 4$, or as $2(r \sin \theta) - (r \cos \theta) = 4$. Using transformation Equations (1), replace $r \sin \theta$ by y and $r \cos \theta$ by x to get

$$2y - x = 4.$$

Since the graph of $2y - x = 4$ is a line, the graph of $r = \frac{4}{2 \sin \theta - \cos \theta}$ is the same line. ◀

Polar Functions, Graphs, and Technology

So far in this course we have devoted considerable attention to drawing graphs of functions in rectangular coordinates. The analogous situation in polar coordinates is graphing functions expressible as $r = f(\theta)$. In many cases we can express the polar equation in rectangular form and draw the graph using familiar techniques. For many polar functions, though, the equation in x and y is difficult to handle and it is easier to draw the graph directly from the polar equation. We will look at examples of both kinds of equations, those that are fairly easy to translate into recognizable rectangular equivalents, and those for which it is easier to draw the graph directly in polar form.

Your graphing calculator graphs beautifully and efficiently in polar coordinates, as it does in rectangular coordinates. For most calculators there is a POLAR mode, chosen in much the same way as you choose the FUNCTION or PARAMETRIC mode.

Once you have set POLAR mode, you simply enter the function in the form $r = f(\theta)$, choose an appropriate θ-range, and graph. The exceptions are the TI-81 and the HP-48, as described in the Technology Tip that follows.

TECHNOLOGY TIP ◆ ***Polar graphing on the HP-48 and TI-81***

HP-48 There is a θ-character available, but most of us find it easier to use x as the independent variable. Choose POLAR as the plot type and enter the function to be graphed in tick marks, as usual.

TI-81 There is no separate polar graphing mode, so polar equations are entered in parametric mode with the variable t, using the transformation equations $x = r \cos t$, $y = r \sin t$. For a function $r = f(\theta)$, enter $x = f(t) \cos t$, $y = f(t) \sin t$. Thus, to graph $r = \sin 2\theta$, use

$$x = (\sin 2t) \cos t, \; y = (\sin 2t) \sin t.$$

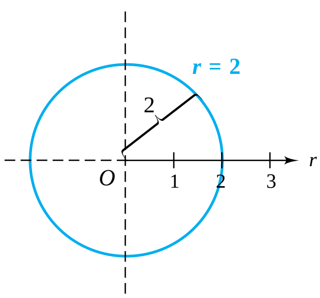


FIGURE 60
Graph of $r = 2$.

►*EXAMPLE 5* *Polar graphing*

Draw a graph of the equation $r = 2$.

Solution

Since the r-coordinate of a point measures the distance from the pole, the equation $r = 2$ is satisfied by all points that are 2 units from the pole, precisely the condition for the circle centered at the origin with radius 2. See Figure 60. In this case, we can also translate to rectangular coordinates. Squaring both sides of the equation and using transformation Equations (2),

$$r^2 = 4 \quad \text{or} \quad x^2 + y^2 = 4,$$

the familiar form for the equation of the circle. We must be careful about introducing extraneous points when squaring both sides, but in this instance we already saw that all points of the circle satisfy the original equation.

With a graphing calculator we can graph the equation $r = 2$ directly. Make sure that the θ-range includes a full revolution (for example, $[0, 2\pi]$ or $[-\pi, \pi]$). You should check to see that graphing with a smaller θ-range produces only a part of the circle. ◄

Strategy: Try some equivalent forms, looking for $r\cos\theta$, $r\sin\theta$, or r^2.

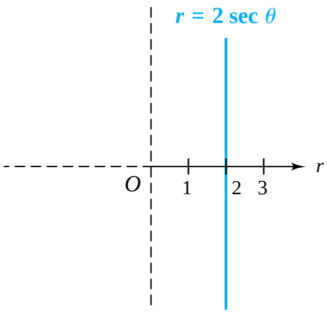


FIGURE 61
Graph of $r = 2\sec\theta$. $r = 2\sin\theta$, $r = 2$, $0 \le \theta \le \pi$.

▶EXAMPLE 6 ***Polar graphing*** Draw a graph of the function $r = 2\sec\theta$.

Solution

The transformation equations involve certain expressions listed in the Strategy. Since $\sec\theta = \frac{1}{\cos\theta}$, multiply through by $\cos\theta$:

$$r(\cos\theta) = (2\sec\theta)(\cos\theta) \quad \text{or} \quad r\cos\theta = 2.$$

For this equation, the transformation Equations (1) yield

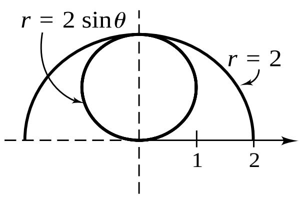
$$x = 2,$$

an equation that describes the vertical line in Figure 61.

Again, the calculator draws the graph without our identification of the graph in rectangular coordinates. If you use the θ-range of $[0, 2\pi]$, observe where the graph begins and ends. ◀

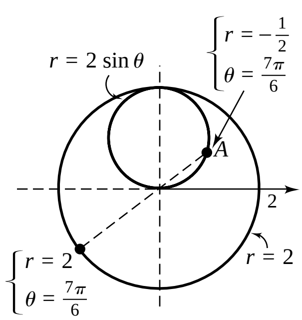
Dynamic graphing. One of the great benefits of graphing technology is being able to *watch* the graph being drawn, rather than simply looking at a set of points after the graph is finished. This is particularly important in the case of polar coordinate graphing because we want to see how r varies with θ. We are used to thinking about the independent variable, the x-coordinate, increasing from left to right, as we see a graph drawn in rectangular coordinates. In contrast, when graphing a function of the form $r = f(\theta)$, the independent variable is θ, so the graph is traced out in a path moving counterclockwise about the pole.

In many examples of polar coordinate graphing, it is helpful to have a second graph to show the way the θ-variable changes, as illustrated in the next couple of examples.



(a)

$r = 2\sin\theta$, $r = 2$, $0 \le \theta \le \pi$.



(b)

$r = 2\sin\theta$, $r = 2$, $0 \le \theta \le 2\pi$.

FIGURE 62

▶EXAMPLE 7 ***Another circle*** For the function $r = 2\sin\theta$,

(a) change to rectangular coordinates and identify the graph, and
(b) draw a calculator graph of $r = 2\sin\theta$ and of $r = 2$ simultaneously on the same screen, first using a θ-range of $[0, \pi]$ and then with a θ-range of $[0, 2\pi]$. Explain the differences you observe.

Solution

(a) We saw in Example 3 that the equation $r = 2\sin\theta$ is equivalent to $x^2 + y^2 - 2y = 0$. Without having just seen that example, how would we go about changing to rectangular coordinates? As we observed in Example 6, the most convenient expressions involve $r\cos\theta$, $r\sin\theta$, or r^2. In this case, multiplying both sides by r gives two such expressions:

$$r^2 = 2r\sin\theta$$

from which

$$x^2 + y^2 = 2y.$$

Completing the square yields $x^2 + (y - 1)^2 = 1$, whose graph is a circle of radius 1.

(b) Entering $r = 2\sin\theta$ and $r = 2$ and graphing with a θ-range of $[0, \pi]$, we see a small circle tangent to a semicircle, as in Figure 62a. Using $[0, 2\pi]$ for the θ-range, we get the two circles in Figure 62b.

The inner, smaller circle is completed as θ increases from 0 to π, but the larger circle, $r = 2$, requires a full revolution, $[0, 2\pi]$. What happens with the smaller circle as θ varies from π to 2π? We can see the answer as we trace around the smaller circle. As we noted in Figure 62a, we complete the circle as θ goes from 0 to π. Then, as θ increases around to 2π, *the circle is traced out again.* It is instructive to jump back and forth between the two circles, noticing that we always remain on the same line through the origin, on the same or opposite sides of the same θ-ray.

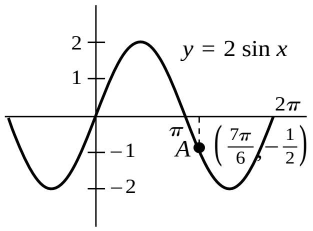


FIGURE 63

Compare the sine curve $y = 2 \sin x$ in Figure 63. The polar graph plots the y-value as the r-distance out along the θ-ray. In the first two quadrants, where y, and hence r, is positive, we get the entire circle $r = 2 \sin \theta$ in Figure 62a. Then, in the third and fourth quadrants, r is negative, so the smaller circle is tranced out again with negative values of r. As a specific example, we show a point A on the graph of $r = 2 \sin \theta$ and the corresponding points on the graph $r = 2$ (Figure 62b) and on the graph of $y = 2 \sin x$. ◀

The dynamic view is even more important in the next example. There is no equivalent rectangular equation that is any easier to graph than the relatively simple polar equation $r = \sin 2\theta$. You may check this by using the identity $\sin 2\theta = 2 \sin \theta \cos \theta$ and multiplying through by r^2. This yields

$$r^3 = 2(r \cos \theta)(r \sin \theta),$$

which is equivalent to

$$(x^2 + y^2)^{3/2} = 2xy,$$

certainly not much of an improvement in terms of graphing.

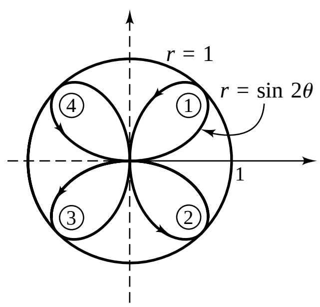


FIGURE 64
Graph of $r = \sin 2\theta$.

▶EXAMPLE 8 A "rose" Graph the function $r = \sin 2\theta$, indicating the portions of the graph where r is positive and where r is negative. After graphing $r = \sin 2\theta$, add the graph $r = 1$ for reference.

Solution

The calculator shows a graph like Figure 64. This graph is sometimes called a "4-leafed Rose". We have labeled the leaves of the rose in the order in which they are traced out. When we add the graph $r = 1$, we can see that a leaf (or petal) is traced out as the variable θ goes through each quadrant in turn, giving an additional meaning to the label numbers in Figure 64.

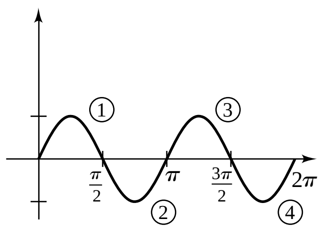


FIGURE 65
Rectangular coordinate graph of $y = \sin 2x$.

By comparing the polar graph to the sine curve $y = \sin 2x$ in Figure 65, we can see that as x increases from 0 to $\frac{\pi}{2}$, y increases from 0 to 1 and then back to 0. The corresponding portion of the polar graph is the first leaf, where r goes from 0 (at the pole), out to 1, and back to the pole. Then in the next quadrant, y (and hence r) is negative; r decreases from 0 to -1 (1 unit away from the pole in the negative direction), and then back to the pole at π. Tracing along the rose or the enclosing circle $r = 1$ and jumping back and forth makes it clear that r is positive in the first and third quadrants, negative in the second and fourth. ◀

Other Polar Graphs

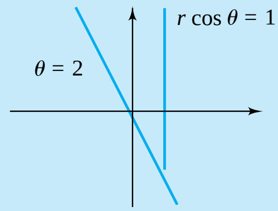
No brief introduction can do more than touch on the tremendous variety of useful polar graphs. The kinds of polar curves you are most likely to encounter in calculus are illustrated in the Brief Catalog. All of the graphs in the catalog can be sketched as illustrated in this section.

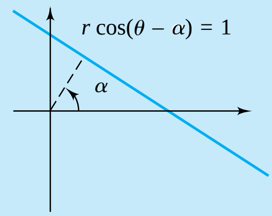
Brief catalog of curves and graphs in polar coordinates

Assume that a, b, d, and α are given constants.

Line: $r\cos(\theta - \alpha) = d$

$\theta = a$

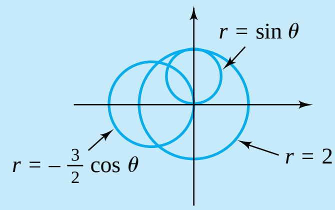




Circle: $r = \pm a$

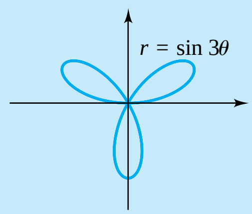
$r = \pm a \sin\theta$

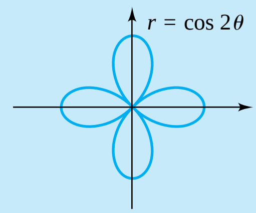
$r = \pm a \cos\theta$



Rose: $r = a \sin n\theta$

$r = a \cos n\theta$

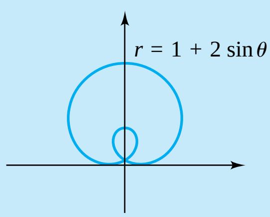


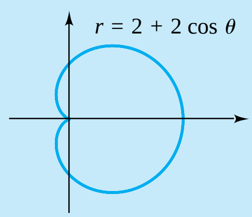


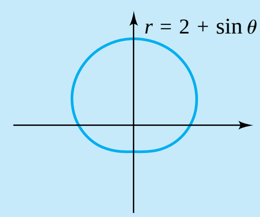
For $0 \le \theta \le 2\pi$, there are n petals if n is odd (each traced twice); $2n$ petals if n is even (each traced once).

Limaçon and cardioid: $r = a \pm b \sin\theta$

$r = a \pm b \cos\theta$



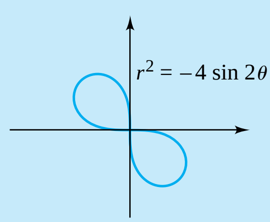


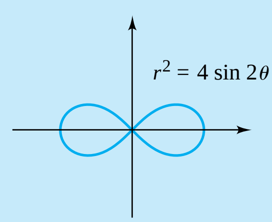


There is an inside loop if $a < b$, just an indentation if $a > b$, and if $a = b$, the curve is a cardioid (heart-shaped).

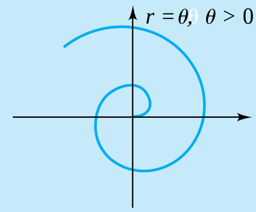
Lemniscate: $r^2 = \pm a^2 \sin 2\theta$

$r^2 = \pm a^2 \cos 2\theta$





Spiral: $r = \pm a\theta$



Looking Ahead to Calculus: Intersections of Polar Curves

A frequent source of difficulty in calculus is the problem of finding intersections of polar curves and of describing regions for computing area. The proper use of technology can make both ideas much more tractable. We illustrate in the next example.

▶EXAMPLE 9 ***Intersections and inequalities*** Consider the circle and the rose with respective equations

$$r = 1 \quad \text{and} \quad r = 2\cos 2\theta.$$

(a) Find coordinates of all points of intersection of the two curves.
(b) Write a system of inequalities to describe the set of points that are outside the circle but inside the right-most leaf of the rose.

Solution

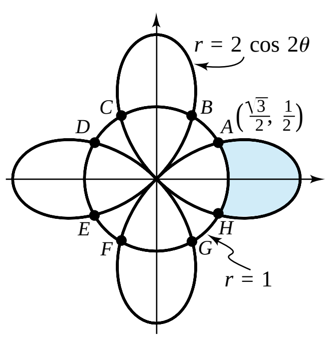


FIGURE 66

$B\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ has polar coordinates $\left[1, \frac{\pi}{3}\right]$ on the circle and $\left[-1, \frac{4\pi}{3}\right]$ on the rose.

(a) We begin by drawing the two functions on the same screen. See Figure 66. There are clearly eight intersection points. To solve the two equations, we can eliminate the variable r by setting the two equal, obtaining $2\cos 2\theta = 1$. We use techniques from Chapter 6 to find that the solutions in the interval $[0, 2\pi]$ are given by $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$. These four solutions lie on the two horizontal leaves of the rose. Where are the intersection points on the vertical leaves?

If we trace along the curves, jumping back and forth from one to the other, the answer becomes apparent. As we approach the point marked A, the rectangular coordinates on both curves approach $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, which is equivalent to $\left[1, \frac{\pi}{6}\right]$. Continuing *along the circle* toward point B, the coordinates near $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, but when we jump to the rose, the cursor nears the point $F\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. If we stay on the rose, we don't reach B until θ nears 4.19, about $\frac{4\pi}{3}$. The rectangular coordinates are again near $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, but the point B has different polar coordinates on the rose than it does on the circle, a common occurrence with polar graphs. Polar coordinates for B on the circle are $\left[1, \frac{\pi}{3}\right]$, and the same point on the rose has polar coordinates $\left[-1, \frac{4\pi}{3}\right]$.

Putting all of this together (and using the obvious symmetries), the eight intersection points in rectangular coordinates are $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(\pm\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $\left(\pm\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

(b) With the intersection points identified, it is fairly straightforward to write inequalities for the desired region, which we have shaded in Figure 66. We want r-values that are greater than the r-values of the circle and less than the

r-values of the rose, between the θ-values of $-\frac{\pi}{6}$ and $\frac{\pi}{6}$, suggesting the inequalities

$$1 < r < 2\cos 2\theta, \qquad -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}.$$

Thus the region is the set

$$\left\{[r, \theta] \,|\, 1 < r < 2\cos 2\theta,\ -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}\right\}.$$ ◄

EXERCISES 10.5

Check Your Understanding

Exercises 1–5 True or False. Give reasons. Throughout, the notation is the same as in the text: (x, y) is the name of a point relative to the rectangular coordinate system and $[r, \theta]$ names a point in polar coordinates.

1. Another name for $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ is $[1, \frac{\pi}{3}]$.
2. Another name for $[-2, \frac{5\pi}{4}]$ is $(-\sqrt{2}, \sqrt{2})$.
3. Point $[1, -\frac{\pi}{3}]$ belongs to the graphs of both $r = 1$ and $r^2 - 4r\cos\theta + 1 = 0$.
4. The graph of $r^2 - 4r\cos\theta = 0$ is a circle.
5. The graph of $r = 2\sin\theta$ contains points in all four quadrants of the rectangular coordinate system.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. Another name for point $[2, \pi]$ is ______.
7. Another name for point $(2, -2)$ is ______.
8. The center of the circle $r = -4\cos\theta$ is ______.
9. The graph of $r(\cos\theta - \sin\theta) = 4$ is a line that contains no points in Quadrant ______.
10. The graph of $\theta = \dfrac{\pi}{4}$ is a line that passes through Quadrants ______.

Develop Mastery

Exercises 1–4 ***Rectangular/Polar Coordinates*** *Draw a diagram that shows the given points. Give both rectangular coordinates and two different sets of polar coordinates for each point.*

1. $A\left[2, \dfrac{\pi}{3}\right]$; $B\left[-2, \dfrac{\pi}{3}\right]$
2. $A(0, -2)$; $B(-2, 0)$
3. $A(\sqrt{3}, -1)$; $B\left[2, -\dfrac{\pi}{4}\right]$
4. $A(1, 1)$; $B[1, 1]$

Exercises 5–8 ***Verbal to Coordinates*** *Draw a diagram that shows the points described. Give both rectangular coordinates and two different sets of polar coordinates. O denotes the pole.*

5. P is 2 units from O on the $\frac{5\pi}{6}$ line; Q is the reflection of P through O.
6. P is -4 units from O on the π-line; Q is the reflection of P through the $\frac{\pi}{4}$ line.
7. A, B, and C are the vertices of the equilateral triangle with sides of length 2, A on the positive y-axis, and the side opposite A is on the x-axis.
8. P and Q are the points of intersection of the circles $x^2 + y^2 = 1$ and $(x - 2)^2 + y^2 = 3$.

Exercises 9–15 ***Rectangular to Polar Equations*** *Express the equation in polar coordinates. If the pole is on the graph, find the smallest nonnegative value of θ for which $[0, \theta]$ satisfies the equation, then sketch the graph.*

9. $x^2 + y^2 = 4$
10. $x^2 + y^2 - 4x = 0$
11. $x = 3$
12. $\sqrt{x^2 + y^2} - 2 = 0$
13. $y = 3x$
14. $x^2 + y^2 - \sqrt{x^2 + y^2} = y$
15. $x^2 + y^2 - 2x + 2y = 0$

Exercises 16–25 ***Polar to Rectangular Equations*** *Express the equation in rectangular coordinates and sketch the graph.*

16. $r = 3$
17. $r = -2$
18. $r = 3\cos\theta$ (*Hint*: Multiply by r.)
19. $r\sec\theta = -4$
20. $\theta = \dfrac{-\pi}{4}$
21. $\theta = \dfrac{3\pi}{4}$
22. $r(\cos\theta - \sin\theta) = 1$

23. $r = \sin^2\theta + \cos^2\theta$

24. $r = 2\sqrt{2}\sin\left(\theta + \dfrac{\pi}{4}\right)$

25. $r^2 - 2\sqrt{3}\, r\cos\theta - 2r\sin\theta = 12$

Exercises 26–32 ***Graph Polar Equations*** *Graph the equation. Use your calculator and the Catalog of Polar Curves. Identify any portion of the curve where r is negative.*

26. $r = 3\cos\theta$ (circle)

27. $r = \cos 3\theta$ (rose)

28. $r = 2 - \cos\theta$ (limaçon)

29. $r = 2 + 2\cos\theta$ (cardioid)

30. $r^2 = \sin 2\theta$ (lemniscate)

31. $r = \cos^2\theta - \sin^2\theta$ (rose)

32. $r + \theta = 0$ (spiral)

Exercises 33–39 ***Compare Polar Graphs*** *Compare the graphs of the pair of equations and write a brief paragraph describing your observations, including the dynamic aspects of the graph.*

33. $r = 2$, $r = -2$

34. $r = 3\cos\theta$, $r = \cos 3\theta$

35. $r = \sin\theta$, $r = \sin\theta + 1$

36. $r = 1 + \cos\theta$, $r = 1 - \cos\theta$

37. $r = 2 + 2\cos\theta$, $r = 3 + 2\cos\theta$

38. $r = 1 + 2\sin\theta$, $r = 2 + 1\sin\theta$

39. $r = \theta$, $r = -\theta$

Exercises 40–45 ***Intersections of Polar Curves*** *First sketch the graphs of the pair of equations and observe that the graphs intersect. Find points of intersection.*

40. $r = \cos\theta$, $r = -\sin\theta$

41. $r = 1 + \cos\theta$, $r = 1 - \cos\theta$

42. $r = 2\sin 2\theta$, $r = 1$

43. $r = 2 + 4\cos\theta$, $r = 3$

44. $r = 2 + 4\cos\theta$, $r = 2\cos\theta$

45. $r^2 = -4\sin 2\theta$, $r = 1$

Exercises 46–50 ***Intersecting Polar Graphs*** *Give the quadrants in which the graphs of the two equations intersect.*

46. $r = 3,\ r = \dfrac{2}{\cos\theta - \sin\theta}$

47. $r = 1 + \cos\theta,\ r = \dfrac{3}{3\cos\theta - 2\sin\theta}$

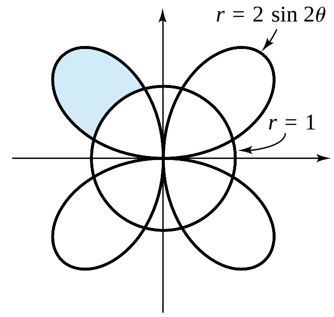
48. $r = -\sec\theta,\ r = 2 + \sin\theta$

49. $r = 2\cos\theta,\ r = \dfrac{2}{1 + \cos\theta}$

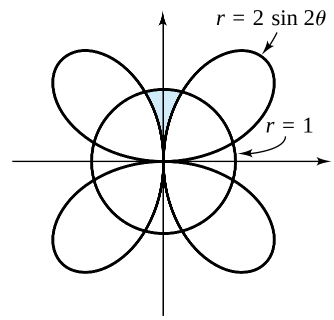
50. $r = \dfrac{4}{1 - \cos\theta},\ r = \dfrac{3}{1 + 0.5\cos\theta}$

Exercises 51–56 ***Inequalities for Regions*** *Write a set of inequalities to describe the shaded region. It may be necessary to use two separate inequalities for different θ-ranges.*

51.



52.



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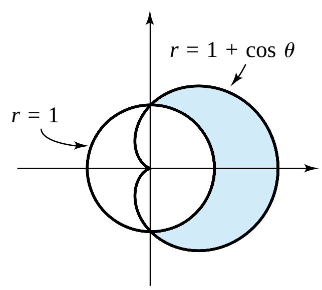

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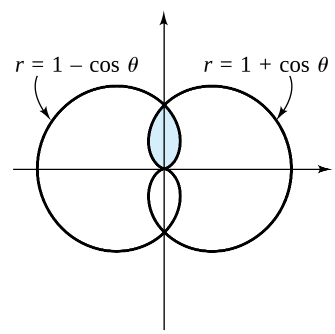

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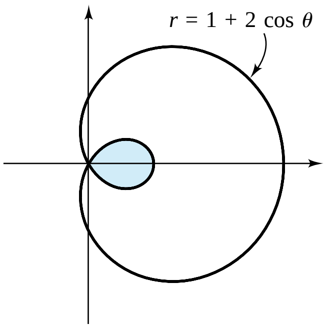

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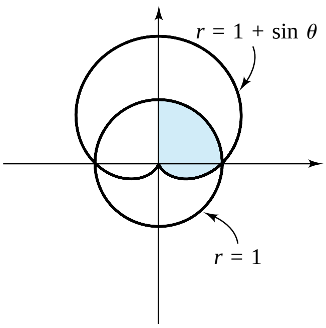


Exercises 57–58 ***Explore: Conic Sections in Polar Coordinates.***

57. (a) Convert each equation to rectangular coordinates and identify the conic section:

$$r = \frac{2}{1 - \cos\theta} \qquad r = \frac{2}{1 + \cos\theta}$$

(b) Do the same for

$$r = \frac{2}{1 + \left(\frac{1}{2}\right)\cos\theta} \qquad r = \frac{2}{1 + 2\cos\theta}$$

58. What conic section is represented by the equation

$$r = \frac{2}{1 + b\cos\theta}$$

for various values of b? Write a brief paragraph to give your guess and describe your reasons. What is the effect of using a different numerator, say 1 or -2? What would happen if $\sin\theta$ replaced $\cos\theta$?

Exercises 59–61 ***Explore New Forms***

59. Use an identity to convert the polar equation $r\cos(\theta + \frac{\pi}{6}) = 2$ to rectangular coordinates in the form $ax + by = c$. Sketch the graph.

60. Use an identity to convert the polar equation $r\cos(\theta - \alpha) = d$ into rectangular coordinates. The result is an equation for a line called *normal form*.

61. By converting to rectangular coordinates, show that $r = a\cos\theta + b\sin\theta$ is an equation for a circle. Find the center and the radius.

CHAPTER 10 REVIEW

Test Your Understanding

Exercises 1–30 True or False. Give reasons.

1. The graph of $x^2 + y^2 - 4 = 0$ is a circle.

2. The graph of $y^2 = x^2 - 1$ is a hyperbola.

3. The graph of $y^2 = 9 + 4x^2$ is an ellipse.

4. The graph of $x^2 + y + 4 = 0$ is a parabola.

5. The graph of $x^2 + y^2 = -1$ is a circle.

6. The graph of $x^2 - y^2 = 0$ is a hyperbola.

7. The graph of $9x^2 - y^2 = 0$ consists of two intersecting lines.

8. The graph of $y = \sqrt{4 - x^2}$ is a semicircle.

9. The graph of $y = \sqrt{1 + 4x^2}$ is half of an ellipse.

10. The graph of $y = -\sqrt{1 - 4x^2}$ is half of an ellipse.

11. The graph of $x^2 + y^2 - 2x + 4y + 6 = 0$ is a circle.

12. The graph of $0.5x^2 + 0.5y^2 = 2$ is a circle of radius 2.

13. The graph of $xy = 4$ is a hyperbola.

14. The graph of $x^2 + 2y^2 - 2x + 4y - 8 = 0$ is an ellipse.

15. Every hyperbola has two asymptotes that intersect at the point midway between the foci of the hyperbola.

16. The line $y = x$ is an asymptote for the hyperbola $(x - 1)^2 - (y - 1)^2 = 1$.

17. Every point inside the circle $x^2 + y^2 = 4$ is also inside the ellipse $x^2 + 2y^2 = 2$.

18. Every point inside the ellipse $x^2 + 4y^2 = 4$ is also inside the circle $x^2 + y^2 = 4$.

19. The graphs of $x^2 + 4y^2 = 4$ and $4x^2 + y^2 = 4$ intersect in four points.

20. Point $(1, 2)$ is inside the graph of $3x^2 + 4y^2 = 12$.

21. The graph of $r = \cos\theta$ is a circle of radius 0.5.

22. The graph of $r = |\cos\theta|$ is a semicircle.

23. The graph of $r = \dfrac{1}{\sin\theta + \cos\theta}$ is a line.

24. The polar coordinates $[-1, \frac{\pi}{4}]$ and $[1, -\frac{3\pi}{4}]$ represent the same point.

25. Point $[-1, \pi]$ is inside the circle $r = 2\cos\theta$.

26. The graph of $\begin{cases} x = \sin t \\ y = \cos^2 t \end{cases}$ is part of a parabola.

27. The graph of $\begin{cases} x = \sin t \\ y = |\cos t| \end{cases}$ is a circle with its center at the origin.

28. The graph of $\begin{cases} x = |\sin t| \\ y = |\cos t| \end{cases}$ is a quarter-circle.

29. The graph of $\begin{cases} x = 2t \\ y = t^2 \end{cases}$ is a parabola with its focus at $(-1, 0)$.

30. The graph of $\begin{cases} x = 2 + \sin t \\ y = 1 + 2\cos t \end{cases}$ is an ellipse with its center at $(2, 1)$.

Review for Mastery

Exercises 1–6 ***Foci and Vertices*** *Find the vertices and foci for the conic section.*

1. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

2. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

3. $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$

4. $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$

5. $\dfrac{(x-1)^2}{9} + \dfrac{(y+2)^2}{25} = 1$

6. $\dfrac{x^2}{9} - \dfrac{(y+1)^2}{16} = 1$

Exercises 7–17 ***Verbal to Equation*** *For the conic section specified by the given information, write an equation in standard form. Some may have more than one solution.*

7. Circle: center $(-2, 1)$, radius 3

8. Circle: center $(0, -3)$, radius $\sqrt{5}$

9. Parabola: focus $(3, 0)$, vertex $(0, 0)$

10. Parabola: directrix $x = 5$, axis $y = 1$, contains $(0, 4)$

11. Ellipse: center $(1, 4)$, focus $(1, 2)$, vertex $(1, 0)$

12. Ellipse: foci $(4, -1)$ and $(0, -1)$ vertex $(5, -1)$

13. Hyperbola: center $(1, -1)$, focus $(4, -1)$, vertex $(3, -1)$

14. Hyperbola: vertices $(1, 3)$ and $(1, -1)$, focus $(1, -2)$

15. Parabola: focus $(3, 1)$, directrix $x = 2$

16. Parabola: focus $(3, 1)$, directrix $y = 2$

17. Parabola: vertex $(3, -1)$, contains the ends of a diameter of the circle $x^2 + y^2 - 6x - 4y + 9 = 0$

Exercises 18–23 ***Identify Conic Sections*** *Identify the type of conic section defined by the equation and sketch the graph. For a circle, give the center and radius; for a parabola, give the focus and vertex; for an ellipse, give the center and the lengths of the major and minor axes; for a hyperbola, give the center, vertices, and asymptotes.*

18. $x^2 + y^2 + 2x - 4y + 1 = 0$

19. $x^2 - 2x + 2y = 5$

20. $9x^2 + 4y^2 - 8y - 32 = 0$

21. $x^2 + y^2 = 2y + 2$

22. $x^2 - 9y^2 - 4x - 5 = 0$

23. $x^2 = 2x + y$

24. ***Explore: Derive Equations*** Find an equation for the set of points $P(x, y)$ such that the distance $|\overline{PF}| = kd$, where F is point $(1, 2)$ and d is the distance from P to the x-axis, for the values of k: **(a)** $k = 1$, **(b)** $k = 2$, **(c)** $k = \frac{2}{3}$. Make a guess about the kind of curve defined.

Exercises 25–30 ***Graphing Parametrically*** *Sketch the graph of the curve determined by the parametric equations. Identify the kind of curve defined.*

25. $x = 2t,\ y = \sqrt{4 - 4t^2}$

26. $x = 2t - 1,\ y = 3 - 6t$

27. $x = \sqrt{t-1} - 1,\ y = 2\sqrt{t-1} + 3$

28. $x = \cos t,\ y = -\cos t$

29. $x = 4^t,\ y = 2^t$

30. $x = 1 + \sin t,\ y = 1 - \sin t$

Exercises 31–40 ***Your Choice*** *Find a standard form equation for a conic section satisfying the conditions.*

31. Ellipse; foci are in the first and fourth quadrants.

32. Ellipse; foci are in the third and fourth quadrants.

33. Ellipse; foci are both in the second quadrant.

34. Parabola; vertex in the fourth quadrant, graph contains points in all quadrants.

35. Parabola; vertex in the third quadrant, graph contains no points in the first quadrant.

36. Parabola; vertex in the third quadrant, graph contains no points in the second quadrant.

37. Hyperbola; vertices are in the first and second quadrants, graph is symmetric about the line $x = 1$.

38. Hyperbola; vertices are in the third quadrant, graph does not meet line $y = -2$.

39. Circle; center in the second quadrant, tangent to both axes.

40. Circle; center in fourth quadrant, interior contains points in just three quadrants.

Exercises 41–44 ***Equations for Lines*** *Find an equation for the set of points equidistant from the given lines. (Hint: Draw a picture, and use the formula for the distance from an arbitrary point $P(x, y)$ to each line.)*

41. $x + 2y = 4$
$2x + 4y = 15$

42. $y = 2x$
$y = \frac{x}{2}$

43. $y = x$
$y = 0$

44. $y = 2x$
$4x - 2y = 6$

Exercises 45–48 ***Distance from Point to a Conic*** *Find the distance from the point to the given conic, and find the point on the conic nearest the given point.*

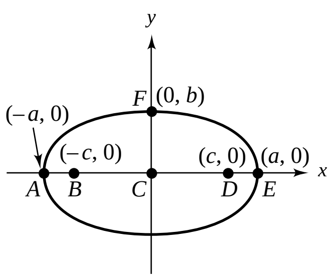
45. $P(4, 0)$; circle $(x - 1)^2 + (y - 1)^2 = 1$

46. $P(-2, 5)$; parabola $x = t^2 - 1, y = t$

47. $P(-4, -1)$; parabola $y + 4 = (x - 2)^2$

48. $P(1, -\frac{7}{2})$; rotated hyperbola $y = \frac{1}{x}$

49. ***Property of an Ellipse*** Given the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ labeled as shown in the diagram, show that $\frac{|\overline{AB}|}{|\overline{CF}|} = \frac{|\overline{CF}|}{|\overline{BE}|}$.



Exercises 50–54 ***Polar Coordinate Graphs*** *Sketch the graph of the curve defined by the polar coordinate equation and indicate the portion where r is positive.*

50. $r^2 = 4 \cos 2\theta$

51. $r = 2 - \sqrt{3} \cos \theta$

52. $r = \sqrt{3} - 2 \cos \theta$

53. $r = \sqrt{3} + \sqrt{3} \sin \theta$

54. $r = \sqrt{3} - \sqrt{3} \sin \theta$

Exercises 55–56 ***Intersections of Polar Curves*** *Sketch the two curves and find their points of intersection.*

55. $r = 2 \cos \theta$; $r = \sin \theta + \cos \theta$

56. $r^2 = 2 \sin 2\theta$; $r = 1$

Exercises 57–60 ***Intersection Points*** *Find the intersection point(s) of the curves C_1 and C_2 given by parametric equations.*

57. $C_1 : x = 1 + 2t, y = 3 - 4t, C_2 : x = 2 - 3t, y = 1 + t$

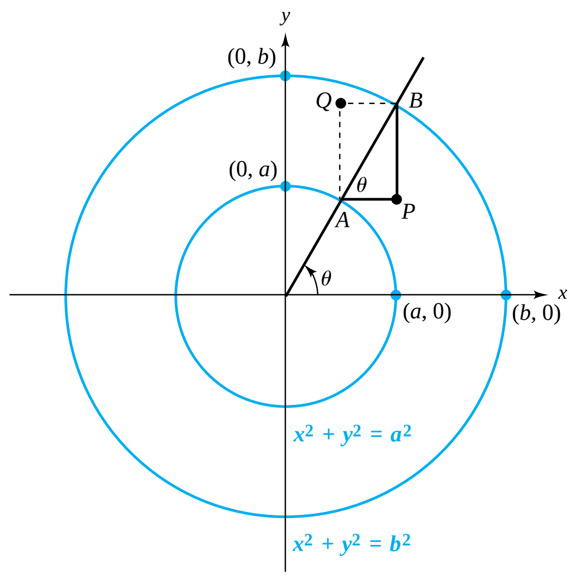
58. $C_1 : x = 2 + 5 \cos t, y = 1 - 5 \sin t, C_2 : x = 5t, y = -3t$

59. $C_1 : x = \sin t, y = \cos^2 t, C_2 : x = t, y = 2t$

60. $C_1 : x = 2 + \sin t, y = -4 + \cos^2 t, C_2 : x = t, y = -4t$

Exercises 61–62 ***Deriving Parametric Equations***

61. Given the two concentric circles in the diagram, a line at angle θ from the positive x-axis intersects the circles in points A and B. Point $P(x, y)$ has the same y-coordinate as A and the same x-coordinate as B. Using the right triangles, find parametric equations in terms of θ for the curve traced out by P.



62. Let Q be the point shown in the diagram for Exercise 46. Find parametric equations in terms of θ for the curve traced out by Q.

APPENDIX

How They Came to Mathematics

The quotations included in the margins throughout the book are all from prominent contemporary mathematicians, most of them still active professionally. Too often we get the impression that mathematics somewhow sprang full-blown from the brow of Zeus, without human intervention or participation. Quite to the contrary—mathematics is an intensely human creation, requiring the passionate involvement of people who find satisfaction (and fun) in thinking about its puzzles. The field is growing today faster than at any time in the past. The people who contribute to the creation of mathematics vary as much as those engaged in any other human endeavor. No one has succeeded very well in defining mathematical talent, but it is clear that talent is not limited to individuals of any particular gender or race, and mathematical discoveries have been made by high school students as well as by mature professionals.

The statements we quote are all taken from larger contexts, from interviews or from writings. They are intended to give just a flavor of the variety of backgrounds and interests of those who decided to spend much of their lives with mathematics. Each person came to mathematics differently; some got caught almost accidentally, others found the subject fascinating from the beginning. The capsule biographies that follow cannot do justice to the rich and complex lives of the individuals quoted, but they may give an idea of the stature of people who began significant careers in ordinary ways. Many of our quotations were taken from a series of interviews conducted by Donald J. Albers and Gerald L. Alexanderson, now available in two books: *Mathematical People* (published by Birkaüser Boston) and *More Mathematical People* (Harcourt, Brace, Jovanovich, with Constance Reid as coeditor).

Lipman Bers was born in what is now Latvia. He fled fascism several times in his life, first to study mathematics in Germany and then in Czechoslovakia, and ultimately to the United States. He taught for years at Columbia University and wrote technical books (*Partial Differential Equations, Mathematical Aspects of Subsonic and Transonic Gas Dynamics*) as well as *Calculus*. He was president of the American Mathematical Society and was a member of the American Academy of Arts and Sciences.

Garrett Birkhoff is the son of one of the first American mathematicians to achieve international recognition. The younger Dr. Birkhoff became a world-renowned mathematician in his own right and taught at Harvard for 45 years. His book, *A*

Survey of Modern Algebra, co-authored with Saunders MacLane, placed an indelible stamp on mathematics curricula and affected the training offered by colleges and universities throughout the nation.

David Blackwell planned originally to be an elementary school teacher but earned instead a Ph.D. in mathematics. After a dozen years at Howard University, he was invited to teach mathematical statistics at the University of California at Berkeley. He is deeply interested in understanding ideas thoroughly enough to communicate them to his students, who have responded by making Dr. Blackwell one of the most honored teachers at Berkeley.

Ralph P. Boas, Jr. was born in Walla Walla, Washington. He "drifted" into mathematics because, he claimed, he "was too clumsy to be a chemist." He did his doctoral work at Harvard, with postdoctoral study at Princeton. In addition to chairing the Mathematics Department at Northwestern University in Chicago for many years, he edited both *Math Reviews* and the *American Mathematical Monthly* and served as president of the Mathematical Association of America.

Paul Cohen grew up in Brooklyn. He was one of four national Westinghouse Science Talent Search winners from his high school graduating class. Now at Stanford, Dr. Cohen has won two of the most prestigious awards available to mathematicians: the Bôcher Prize and the Fields Medal (the mathematical equivalent of the Nobel Prize). Much of his international reputation comes from his proof of the independence of the continuum hypothesis, one of the most fundamental problems in set theory, where he showed that we can neither prove nor disprove Cantor's conjecture.

John Horton Conway is at home on two continents (at Cambridge and Princeton universities). His delight in mathematics as a glorious game infects almost everyone who works with him. An accomplished Rubik's cubologist, he analyzes the mathematical games of others as well as inventing his own. He loves to sit in the Commons Room and lock horns with all comers. Much of his serious mathematics grows out of his interest in recreational mathematics, where his contributions are, according to Martin Gardner, "unique in their combination of depth, elegance, and humor."

George B. Dantzig is best known as the inventor of the simplex method, which makes linear programming such an incredibly powerful tool for management and production. While the simplex method and linear programming are among the most important of all applied mathematics, Dr. Dantzig doesn't believe there is any real difference between applied and pure mathematics. He was just 33 when he developed the simplex algorithm, but his fame as the "Father of Linear Programming" led people to expect a much older man. He was given the Presidential Medal of Science by former President Gerald Ford.

Freeman Dyson has one of the widest ranging imaginations among current physicists and mathematicians. Growing up in Britain, he worked for its War Office during World War II doing statistical analysis of the effectiveness of bombing. (He learned that aerial bombing is *very* inefficient.) One of his earliest mathematical memories is of adding up the infinite geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$ (and getting a sum of 2). He writes widely on public policy and disarmament issues (*Disturbing the Universe*), has spent considerable time at the Institute for Advanced Study (Princeton), and has made fundamental contributions to both physics and mathematics.

Richard Feynman has been called "perhaps the most original genius produced in theoretical physics [in] his generation." He worked on the Manhattan Project in the development of the atomic bomb and won the Nobel Prize for physics in 1965 for his work in the fundamental nature of matter. He was deeply concerned with how we learn physics and developed a radically new approach to teach introductory physics during his years at Cal Tech. A delightful and very personal book is *"Surely You're Joking, Mr. Feynmann!"*

Andrew Gleason finished his master's degree at Yale and immediately went into the Navy to work on cryptanalysis. After the war ended, he was appointed as a Junior Fellow at Harvard. He remained at Harvard for the next forty years, without a Ph.D., while directing a number of doctoral students of his own. Some of his early reputation was established by his part in the solution of Hilbert's Fifth Problem, one of a famous list of twenty challenges posed by David Hilbert at the beginning of the twentieth century.

Bill Gosper was one of the original MIT "hackers", who were fascinated by computers and convinced that they should have access to any and all information about the way the world works. His approach to computing millions of decimals of the number π used new techniques of continued fractions. He continues to be fascinated by all kinds of "toys," from the Aerobie to supercomputers, which he says will never be "big enough or fast enough."

Paul Halmos grew up in Hungary but received most of his schooling in America. A gifted research mathematician and expositor, he taught for many years in the midwest (at the universities of Chicago, Indiana, Illinois, Michigan, and Syracuse) as well as in California and Hawaii. As editor of the *American Mathematical Monthly* and in his own writing (he calls his life story, *I Want to Be A Mathematician,* an "Automathography"), he has worked tirelessly to improve the quality of writing both of and about mathematics.

Mark Kac went to high school in Poland but became "profoundly American." Less than a year after he came to America, "the world exploded and much of my part of it was consumed by flames. Millions, including my parents and my brother, were murdered by the Germans and many disappeared without a trace. . . ." Dr. Kac made basic and profound contributions to probability theory and inspired hundreds of students during his years of teaching at Cornell and Rockefeller University. Many thousands more have seen his Mathematical Association of America film, "Can One Hear the Shape of a Drum?"

Irving Kaplansky grew up and went to college in Toronto, Canada, before going to Harvard for his Ph.D. in algebra. Complementing his years of research at the University of Chicago are many books, including the very readable *Matters Mathematical*. More recently he moved to Berkeley to direct the Mathematical Sciences Research Institute. He is a member of the National Academy of Sciences and served as president of the American Mathematical Society.

Peter Lax was born in Hungary. He emigrated to New York when he was fifteen. While he was still a student in high school, Paul Erdös introduced him to Albert Einstein. Dr. Lax has also been president of the American Mathematical Society and was the director of the Courant Institute at New York University, where he still teaches. His wife Anneli is also a professor of mathematics (they met in a graduate course in complex variables), and they work together to improve mathematics education.

Lucien Le Cam grew up in rural France before World War II. He interrupted his schooling to join the French underground in resisting the Nazis. After the war he worked with the organization that set up the modern electric system of France. He came to Berkeley to work with the statistician Jerzy Neyman, completed his doctorate in less than two years, and has remained at Berkeley ever since. Called "brilliant" by colleagues, he sometimes refuses to share his discoveries. When he claimed in a colloquium that he had proved the speaker's result years earlier, the speaker said, "Ah, but you didn't publish it!"

Saunders MacLane has been a "towering figure in American mathematics for over half a century." The University of Gottingen in Germany attracted students from all over the world, from the latter part of the nineteenth century until the rise of the Nazis before World War II. Saunders MacLane was one of the last Americans to study at Gottingen before the war. Some of his best-known work has been done in collaboration with others, including Garrett Birkhoff. A long-time professor at the University of Chicago, MacLane is one of only five people who have served as president of both the American Mathematical Society and the Mathematical Association of America.

Cathleen S. Morawetz is of Irish parentage and is a grand-niece of the Irish playwright J. M. Synge. She grew up in Toronto before leaving for her graduate study at the Massachusetts Institute of Technology. She got her Ph.D. in partial differential equations at New York University, where she has taught for many years and where she also directed the Courant Institute of Mathematical Sciences. She is the only woman to have been invited to give the Gibbs Lecture to the American Mathematical Society and is a member of the National Academy of Sciences.

Frederick Mosteller was trained as a mathematician and statistician with a Ph.D. from Princeton, but he considers himself a "scientific generalist." Reflecting his concern for public education, he designed and taught a statistics course for the Continential Classroom, a long-running and popular program on public television. He also co-authored the popular text that has also been used in hundreds of college classes, *Probability, A First Course.* He has been president of both the American Statistical Association and the American Association for the Advancement of Science.

Ivan Niven grew up in the northwest, going to high school and college in British Columbia. He went on to his Ph.D. in number theory at the University of Chicago, but after a few years in the midwest he returned to Oregon for the rest of his professional career. He received considerable attention early for his one-page *Simple Proof that π Is Irrational*—simple in that it only uses calculus. He wrote *Mathematics of Choice* for the Mathematical Association of America's New Mathematical Library series. Dr. Niven has been president of the Mathematical Association of America and was recognized by the association for Distinguished Service to Mathematics.

Roger Penrose claims to be unable to decide whether he is a mathematician or a physicist, but he makes important contributions to both fields. He and Stephen Hawking together showed that black holes are an inevitable part of our universe. His fascination with patterns led to his discovery of two shapes of tiles that together can cover the plane without any repetition. Long an Oxford don, he has given much thought to computers and the human mind. In his book *The Emperor's New Mind,* he asserts that the mind is forever beyond the capabilities of any computer.

I. I. Rabi grew up on the Lower East Side of New York and went to Cornell to study chemistry and physics. He earned a doctorate in physics from Columbia but had to go to Europe to learn about quantum mechanics. He returned to teach at Columbia, where he earned the prestigious rank of University Professor. He won the Nobel Prize in 1944 and during World War II was a key player at the Massachusetts Institute of Technology Radiation Laboratory in the development of radar. A recent biography is titled *Rabi, Scientist and Citizen.*

Julia Robinson's life was a series of "only" or "first" accomplishments. She was the only girl taking mathematics, the only one taking physics in her high school (San Diego), the only woman taking a number theory course at Berkeley (from Rafael Robinson, who later became her husband). She was one of the three people who collectively solved Hilbert's Tenth Problem. She was elected to the National Academy of Sciences and served as president of the American Mathematical Society (the first woman mathematician in both instances). She died of leukemia in 1985.

Mary Ellen Rudin is another mathematician who married a mathematician. She was recruited in college by the famous "Texas topologist," R. L. Moore. Like many Moore students, she went on to a remarkably productive career. She lives in a house designed by Frank Lloyd Wright. While her husband Walter does some of his mathematics in his well-appointed study, she works on hers in the living room where, over the years, she has simultaneously been able to watch over their children. After years of doing mathematics without formal academic affiliation, she was recognized with an endowed chair at the University of Wisconsin.

Claude Shannon is known as the father of information theory, the mathematical theory of communication. Widely respected as a creative engineer of daring imagination for his work at Bell Laboratories, Dr. Shannon laid the groundwork for the improvement of signal transmission, especially telephone and television communication. He pioneered the programming of machines for complicated tasks (including chess playing and juggling). His love of toys (e.g., a two-seated unicycle) and games of imagination is reflected in his work, keeping him young at heart while he continues to do serious mathematics and engineering.

William Thurston is a Princeton topologist who deals with high-level geometry. He is known around Princeton for his "uniform" of jeans and plaid shirts, and his friends wondered if he would conform to formal dress standards when he received an international award in Finland. Bill did appear formal (for him); he had pressed his plaid shirt. Many people are surprised by how much use he makes of the computer in very abstract mathematics. The quality of his research is reflected in the number of awards he has won, including the Fields Medal and the Veblen Prize.

Stanislaw (Stan) Ulam is internationally known for his work with the Manhattan Project in World War II and for his contributions to the design of the "super," as the fusion H-bomb was known. He grew up in Poland but spent his professional life in America, teaching for many years at Colorado, Florida, and California. Before he received his doctorate, he studied number theory, topology, and set theory. He later broadened the scope of his studies to contribute to the fields of probability, computers, biology, and coding theory. According to Gian-Carlos Rota, “to generations of mathematicians, Ulam’s problems were the door that led them into the new, to the first sweet taste of discovery.”

Robin Wilson is the son of a two-time Prime Minister of England. He spends much of his time as a mathematician at the Open University. This national university serves thousands of adults all over Britain who have never done university work or who have been away from it for many years. Much of his teaching is done on camera or over the air, and he writes supporting materials for his students. He also devotes time to his love of music, both as a performer and as an author of books as diverse as *Graph Theory* and *Gilbert and Sullivan.*

ACKNOWLEDGMENTS

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ANSWERS TO SELECTED EXERCISES

CHAPTER 1

EXERCISES 1.1 *(page 7)*

Check Your Understanding **1.** F **3.** F **5.** T **7.** **(a)** 3, **(b)** 2, **(c)** 2 **9.** $\frac{355}{113}$

Develop Mastery **17.** **(a)** 8.07×10^2, **(b)** 8.07×10^3, **(c)** 8.070×10^2, **(d)** 8.070×10^{-3} **19.** **(a)** 6400, **(b)** 0.00706, **(c)** 0.03470, **(d)** 56000 **21.** **(a)** 81, **(b)** 0.36, **(c)** 0.036, **(d)** 250000 **23.** **(a)** 11.6 da, **(b)** 31.7 yr, **(c)** 31,700 yr **25.** **(a)** 1.73, **(b)** 0.628, **(c)** 8.86, **(d)** 4.88 **27.** **(a)** 95.7, **(b)** 44.3, **(c)** 4.64 **29.** **(a)** 22.9 ft., **(b)** 41.6 sq. ft. **31.** $1.30 \times 10^5 \text{in}^3$ **33.** **(a)** 9.33 ft./sec., **(b)** 560 ft./min., **(c)** 6.36 mi./hr. **35.** 1,040 mi./hr. **37.** 15,600 mi./hr. **39.** 4.7 miles **41.** 89 miles **43.** 8 miles compared to 9.8 miles **45.** **(a)** 8.19 min, 8.46 min, **(b)** 16 sec **47.** 41 oz; 48 oz; yes **49.** **(a)** 2.2 qt, **(b)** 2.7 qt; yes **51.** 39 percent

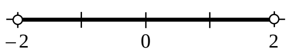
EXERCISES 1.2 *(page 15)*

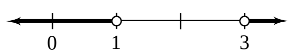
Check Your Understanding **1.** F **3.** T **5.** F **7.** 1 **9.** 0.564

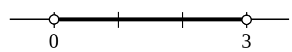
Develop Mastery **1.** **(a)** F, **(b)** F **3.** **(a)** T, **(b)** F **5.** **(a)** T, **(b)** T **7.** **(a)** F, **(b)** F **9.** **(a)** Q, **(b)** N, I, E, Q, **(c)** H, **(d)** N, I, E, Q **11.** **(a)** 0.625, **(b)** $0.41\overline{6}$ **13.** **(a)** 0.82, **(b)** $0.\overline{769230}$ **15.** **(a)** $\frac{63}{100}$, **(b)** $\frac{7}{11}$ **17.** **(a)** $\frac{5}{6}$, **(b)** $\frac{83}{99}$ **19.** **(a)** 0.344, **(b)** 0.344 **21.** **(a)** 3.118, **(b)** 3.118 **23.** 6.928203 **25.** 1.366025 **27.** 3.162278 **29.** 3.968119 **31.** 169 **33.** **(a)** F, **(b)** T, **(c)** T **35.** **(a)** $\sqrt{2} + \sqrt{3}$, **(b)** $(3 + \sqrt{2}) + (1 - \sqrt{2})$, **(c)** $(2 + \sqrt{2})(2 - \sqrt{2})$, **(d)** $\dfrac{\sqrt{3}}{\sqrt{3}}$ **37.** $x = \sqrt{6}$; irrational **39.** **(a)** 3.142857143, **(b)** 3.141509434, **(c)** 3.141592920 **41.** 80 characters per line of length 4 inches; 1,783 miles

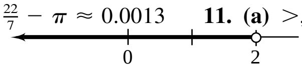
EXERCISES 1.3 *(page 24)*

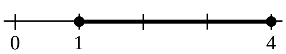
Check Your Understanding **1.** T **3.** F **5.** F **7.** -1 **9.** 19

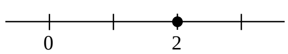
Develop Mastery **1.** 

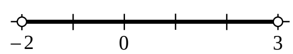
3. 

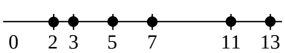
5. 

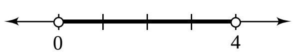
7. **(a)** $\frac{5}{4} = 1.25$, **(b)** $\frac{3}{2} = 1.5$ **9.** **(a)** $\pi - 3 \approx 0.1416$, **(b)** $\frac{22}{7} - \pi \approx 0.0013$ **11.** **(a)** $>$, **(b)** $<$ **13.** **(a)** $=$, **(b)** $>$ **15.** $y < z < x$ **17.** $y < x < z$ **19.** **(a)** T, **(b)** F **21.** 

23. 

25. 

27. 

29. 

31. 

33. 5 **35.** 16 **37.** $8 - 4i$ **39.** $3 + 3i$ **41.** $7 + i$ **43.** 10 **45.** $3 - i$ **47.** $-2 + 2\sqrt{3}i$ **49.** **(a)** -1, **(b)** -1, **(c)** 1, **(d)** -1 **51.** $-5 + 2i$ **53.** $-1 - 3i$ **55.** $-1.2 - 0.6i$ **57.** **(a)** $(-3, 2)$, **(b)** $(-2, -1)$, **(c)** $(-3, -2)$, **(d)** $(-5, 1)$, **(e)** $(8, -1)$ **59.** **(a)** $(5, -1)$, **(b)** $(-1, 1)$, **(c)** $(5, 1)$, **(d)** $(4, 0)$, **(e)** $(-4, 6)$ **61.** All points are on the circle of radius 1 and center at (0, 0). **63.** All points are on the circle with center at the origin and radius 1.

EXERCISES 1.4 *(page 35)*

Check Your Understanding **1.** T **3.** T **5.** F **7.** T **9.** QIV

Develop Mastery **1.** **(a)** $\sqrt{10}$, **(b)** $(-\frac{1}{2}, \frac{7}{2})$ **3.** **(a)** $\frac{\sqrt{277}}{6}$, **(b)** $(\frac{1}{4}, \frac{5}{6})$ **5.** **(a)** 8, **(b)** $(0, -\sqrt{2})$ **7.** Right triangle
9. Isosceles triangle **11.** Right triangle **13.** $x^2 + y^2 - 2x - 2y - 1 = 0$ **15.** $4x^2 + 4y^2 + 8x - 40y + 103 = 0$
17. $x^2 + y^2 + 4x + 2y + 1 = 0$ **19.** $x^2 + y^2 - 8x - 2y + 7 = 0$ **21.** $x^2 + y^2 - 8x - 5y + 16 = 0$
23. **(a)** (3, 4), (3, −2), **(b)** (6, 1), (0, 1) **25.** **(a)** $(3 \pm 2\sqrt{2}, 0)$, (0, 1), **(b)** $4\sqrt{2}$ **(c)** 0
27. **(a)** (−4, 0), (0, 3), **(b)** 5, **(c)** $\frac{3}{4}$ **29.** **(a)** (2, 0), (0, −1.5), **(b)** 2.5, **(c)** $\frac{3}{4}$ **31.** $x = 3 \pm \sqrt{39}$ **33.** $y = -1 \pm \sqrt{5}$
35. **(a)** Line, **(b)** (4, 0); (0, 4) **37.** **(a)** Line, **(b)** $(\frac{2}{3}, 0)$, (0, −2) **39.** **(a)** Circle, **(b)** (0, 0), $\sqrt{7}$
41. **(a)** Circle, **(b)** (−1, 0), 1 **43.** **(a)** Circle, **(b)** (2, −1), 2 **57.** (3.2, 1.8) **59.** (1.6, −2.2) **61.** *iii* **63.** *iii*
65. *iii* **67.** QIII **69.** QII **73.** (1,4), (2,2) **75.** (3, 2) **79.** (6, 6.5) **81.** (4, −7), (−5, −2); $\sqrt{106}$
83. **(a)** (0, −4), **(b)** (3, 0), **(c)** (−4, 3), **(d)** (4, −3) **85.** $\frac{1}{2}$

EXERCISES 1.5 *(page 47)*

Check Your Understanding **1.** F **3.** T **5.** F **7.** 23 **9.** 3

Develop Mastery **1.** $-\frac{1}{2}$ **3.** $\frac{3}{2}$ **5.** $-1, \frac{1}{3}$ **7.** $-\frac{2}{3}, \frac{4}{3}$ **9.** $-\frac{5}{2}, \frac{3}{2}$ **11.** $x > 2$ **13.** $-0.55 \le x \le -0.45$
15. $-1 \le x \le 2$ **17.** $x < -3$ or $-2 \le x \le 2$ **19.** $1 \le x \le 3$ **21.** $x < 0$ **23.** $x > 4$ or $x < -1$
25. $x \le 2$ **27.** $(-\infty, -2)$ **29.** $(-3, -2) \cup (3, \infty)$ **31.** $(-1, 3)$ **33.** $(1 - \sqrt{2}, \sqrt{2} - 1)$ **35.** Two real roots
37. $\pm\sqrt{2}$ **39.** −3, 13 **41.** ±3 **43.** No solution **45.** ±3 **47.** {0, 1, 2} **49.** {0, ±1, ±2}
51. $-3 \le x \le -1$ **53.** $x < -3$ or $x > -2$ **55.** {−2, 3} **57.** $\{x \mid -5 < x < 1\}$ **59.** $x^2 + 6x + 8 = 0$
61. $x^2 - 2x - 1 = 0$ **63.** **(a)** 8, **(b)** 12 **65.** {−3, 2, 3} **67.** 6; 13 **69.** −6 **71.** 4 **73.** 17 **75.** $5 + \sqrt{69}$
77. $41° < F < 68°$ **79.** 16.2 minutes **81.** $A = 100x - x^2$; $0 < x < 100$ **83.** **(b)** $0 < x < 4$, **(c)** 1.6; 67.6

EXERCISES 1.6 *(page 54)*

Check Your Understanding **1.** T **3.** T **5.** T **7.** greater than **9.** equal to

Develop Mastery **1.** $\frac{1}{12}$ **3.** 64 **5.** 2.4 hours **7.** **(a)** Runs 6 miles, walks 14 miles, **(b)** Runs 1 hour, walks 3.5 hours

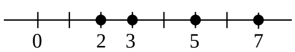
9. 1.5 sq. mi. **11.** $\dfrac{-1 \pm \sqrt{5}}{2}$ **13.** Yes **15.** $4\sqrt{3}$ **19.** 2 **21.** 37.5 mph **23.** 200 mph **25.** $\frac{4\pi}{3}$

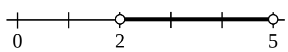
27. **(b)** $y = 2 \pm \sqrt{25 - (x + 3)^2}$ **29.** **(b)** $y = 4 \pm \sqrt{64 - (x - 2)^2}$ **31.** **(b)** $y = 2 \pm \sqrt{25 - (x + 3)^2}$, $y = \pm\sqrt{25 - x^2}$
33. 32 **35.** Any point $(u, 1)$ where u is a positive integer **37.** 6 ft. **39.** $A = 36\pi$ for any value of r.
41. **(a)** $2\sqrt{2}, 2\sqrt{3}, 2\sqrt{4}, 2\sqrt{5}$, **(b)** h **43.** Two

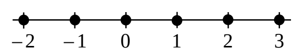
CHAPTER 1 REVIEW *(page 57)*

Test Your Understanding **1.** F **3.** T **5.** T **7.** F **9.** F **11.** F **13.** F **15.** F **17.** F **19.** T
21. F **23.** F **25.** T **27.** T **29.** F **31.** T **33.** F **35.** T **37.** F **39.** T **41.** T
43. **(a)** F, **(b)** F, **(c)** F **45.** QIV **47.** QIV **49.** Two

Review for Mastery **1.** No **3.** No **5.** **(a)** $\frac{1}{7}$, **(b)** $3 - 2\sqrt{2}$, **(c)** $\frac{1}{275}$ **7.** **(a)** >, **(b)** >, **(c)** =

9. **(a)** 

(b) 

(c) 

11. $x \le 3$ **13.** $\{\frac{8}{3}\}$ **15.** {0, −2} **17.** $\left\{\dfrac{2 \pm \sqrt{14}}{2}\right\}$ **19.** {−3, 1} **21.** {−1, 4}. **23.** {−3, 3} **25.** $5i$

27. $-2\sqrt{6}$ **29.** $\{x \mid x < 3\}$ **31.** $\{x \mid x < -2$ or $x > \frac{1}{2}\}$ **33.** $\{x \mid -1 < x < 0$ or $x > \frac{5}{2}\}$
35. $\{x \mid -3 \le x \le 1\}$ **37.** **(a)** {−1}, **(b)** $\{x \mid x > -1\}$ **39.** **(a)** $\{x \mid x \ge 0\}$, **(b)** $\{x \mid x < 0\}$ **41.** $-5 \le x \le 1$
43. $x^2 + y^2 + 6x - 4y + 12 = 0$ **45.** **(a)** Line, **(b)** (2, 0), **(c)** (0, 3) **47.** **(a)** Circle, $C(3, -1)$, $r = 2$,
(b) $(3 \pm \sqrt{3}, 0)$, **(c)** None **49.** **(a)** Line, **(b)** $(\sqrt{3}, 0)$, **(c)** (0, 3) **51.** **(b)** $y_1 = \sqrt{9 - x^2}$, $y_2 = 2x$, **(c)** QI
53. **(b)** $y_1 = -3 - \sqrt{16 - (x - 2)^2}$, $y_2 = 2x - 12$, **(c)** QIV **55.** **(a)** Circle, $C(3, 2)$, $r = 2$,
(b) A is outside, B is inside, C is on the circle **57.** **(a)** 40 percent, **(b)** 2.5 quarts **59.** 3 seconds **61.** $64\pi - 48\sqrt{3}$
63. $4\sqrt{7}$ **65.** **(a)** $8\sqrt{3}$ ft., **(b)** 12 sq. ft.

CHAPTER 2

EXERCISES 2.1 *(page 66)*

Check Your Understanding **1.** F **3.** F **5.** T **7.** Two **9.** 6

Develop Mastery **1.** {−4, 0, 4} **3.** $\{\frac{2}{3}, \frac{9}{11}\}$ **5.** **(a)** 1, **(b)** R, **(c)** $\frac{-4}{3}$ **7.** **(a)** $\frac{5}{17}$, **(b)** R, **(c)** −1

9. **(a)** -2, **(b)** $\{x \mid x \le 1 \text{ and } x \ne -2\}$, **(c)** 1 **11.** **(a)** $\sqrt{6}$, **(b)** $\{x \mid x \le -4 \text{ or } x \ge 1\}$, **(c)** -4, 1

13. $\dfrac{x+2}{x^2}$ **15.** $\dfrac{-x}{\sqrt{x^2+4}}$ **17.** -18.27; 0 **19.** 3; 4.83 **21.** 3 **23.** $2x + h - 2$ **25.** $\dfrac{-1}{x(x+h)}$

27. Triple x and add 4 to the result. **29.** Take square root of x and subtract from 4. **31.** 9; 4; $\sqrt{5} - 2$ **33.** 1; -1; -1
35. $D = [-2.3, 4.3]$ **37.** $D = [\frac{-4}{3}, 4]$ **39.** **(a)** 5, **(b)** -1, **(c)** $x^4 - 4$, **(d)** $x^2 - 2x - 3$

41. **(a)** 2, **(b)** 5, **(c)** 5, **(d)** 43 **43.** **(a)** $\frac{2}{3}$, **(b)** $\frac{5}{8}$, **(c)** 12, **(d)** $\dfrac{1 \pm \sqrt{41}}{4}$ **45.** **(a)** $f(u) = 2u^2 - u - 3$, **(b)** $-1, \frac{3}{2}$

47. **(a)** 7, **(b)** 3, **(c)** 3, **(d)** $2\sqrt{2} + 3$; $g(x) = \begin{cases} 1 - 2x & \text{if } x \le -\frac{1}{2} \\ 2x + 3 & \text{if } x > -\frac{1}{2} \end{cases}$ **49.** $P = 4s$ **51.** $A = \dfrac{P^2}{16}$

53. **(a)** $\sqrt{99}$; $25\sqrt{3}$; $9\sqrt{19}$, **(b)** $A = x\sqrt{100 - x^2}$ **55.** **(a)** $d = |x|$, **(b)** 1; 3, **(c)** R

57. **(a)** \$576; \$936, **(b)** $W = \begin{cases} 18x \text{ if } 0 \le x \le 40 \\ 27x - 360 \text{ if } 40 < x \le 168 \end{cases}$ **59.** **(a)** $L = \frac{3x}{5}$, **(b)** 10 ft.

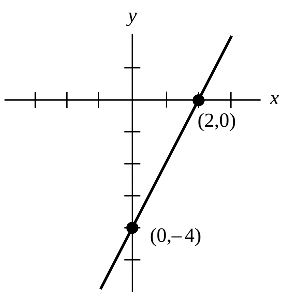
61. $A = 8x - \dfrac{8x^2}{15}$, $0 < x < 15$

EXERCISES 2.2 *(page 75)*

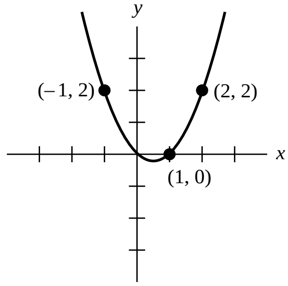
Check Your Understanding **1.** T **3.** **(a)** F, **(b)** T **5.** T **7.** One **9.** Two

Develop Mastery **1.** Graph consists of three points: $(-1, -3)$, $(2, 3)$, $(3, 5)$. $R = \{-3, 3, 5\}$
3. Graph consists of five points: $(-2, -6)$, $(-1, 0)$, $(0, 0)$, $(1, 0)$, $(2, 6)$. $R = \{-6, 0, 6\}$

5.



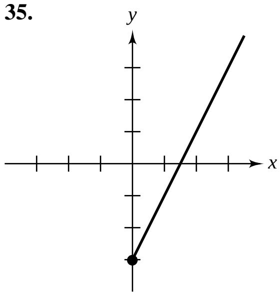
7.



9. -7 **11.** 2 **13.** f is even; g is neither

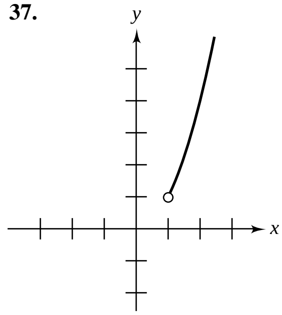
15. f is neither; g is odd **17.** $(-2, 0)$; $(0, 4)$ **19.** $(-2, 0)$; $(0, \sqrt{2})$ **21.** $(0, 0)$; $(2, 0)$ **23.** $(-1 \pm \sqrt{2}, 0)$; $(0, -1)$
25. $(-2, 0)$; $(0, -4)$ **27.** $(\pm 2, 0)$; $(0, 2)$ **29.** *(iii)*

35.



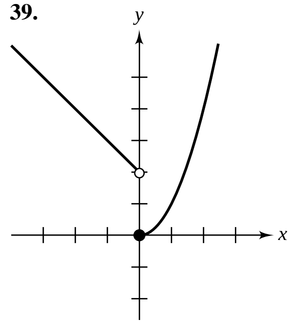
$f(x) = 2x - 3,\ x \ge 0$
(b) $R = [-3, \infty)$

37.



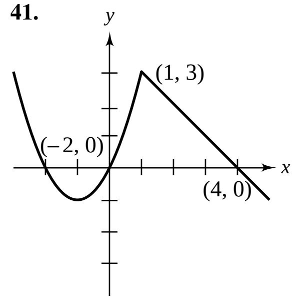
$f(x) = x^2,\ x > 1$
(b) $R = (1, \infty)$

39.



$f(x) = \begin{cases} x^2, & x \ge 0 \\ 2 - x, & x < 0 \end{cases}$
(b) $R = [0, \infty)$

41.



$f(x) = \begin{cases} x^2 + 2x, & x < 1 \\ 4 - x, & x \ge 1 \end{cases}$
(b) $R = \mathbb{R}$

43. $f(3) < f(4.5) < f(-1) < f(0.5)$ **45.** $(1, 3)$; $(3, -2)$ **47.** **(a)** $-2 \le x < -1$ or $2 < x < 5$,
(b) $-1 < x < 2$ or $5 < x \le 6$ **49.** Only **(a)** **51.** **(a)** $[-200, \infty)$, **(c)** $f(x) = 4$ and $g(x) = 4$ for $-200 \le x \le 200$, **(d)** Yes, **(e)** ϕ; $[-200, 200]$ **53.** **(b)** $[-8, 0] \cup [6, \infty)$

55. **(a)** 17; 11; 7; 1.5, **(b)** $f(x) = \begin{cases} -2x + 7 & \text{if } x \le 3 \\ x - 2 & \text{if } x > 3 \end{cases}$ **57.** **(a)** $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$, **(b)** $[6, 7)$
59. **(a)** $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$, **(b)** $[4.5, 5.5)$ **61.** $[3, 4)$ **63.** $\{0, \pm 2, \pm 4, \ldots\}$ **65.** $[0, 1)$ **67.** $(-\infty, 4)$
69. $x = \dfrac{\sqrt{5} - d}{\sqrt{5}}, y = \dfrac{2d}{\sqrt{5}}$ **71.** **(b)** \$7.20, **(c)** $2 \le x < 2.25$ **73.** Up to 9 minutes

EXERCISES 2.3 *(page 86)*

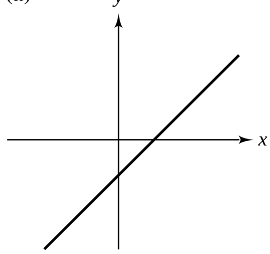
Check Your Understanding **1.** T **3.** T **5.** T **7.** QIV **9.** QI

Develop Mastery **1.** **(a)** $(-1, 4)$ and $(5, -5)$, **(b)** $(-4, 4)$ and $(2, -5)$ **3.** **(a)** $(-1, 4)$ and $(2, -5)$, **(b)** $(-4, 4)$ and $(8, -5)$
5. **(a)** $(0, 7)$ and $(6, -2)$, **(b)** $(-2, 0)$ and $(4, 9)$ **7.** **(a)** $g(x) = x^2 + 3x$, **(c)** For g shift the graph of f 2 units left.
9. **(a)** $g(x) = x^2 - 7x + 10$, **(c)** For g shift the graph of f 3 units right. **11.** **(a)** $g(x) = x^2 + x$, **(c)** For g reflect the graph of f about the y axis, then shift up 2 units.
13. **(a)** Line through $(1, -3)$ and $(3, 1)$, **(b)** Line though $(3, -3)$ and $(5, 1)$, **(c)** Line through $(-1, -3)$ and $(-3, 1)$
15. **(a)** $D = [2, 4], R = [-2, 2]$, **(b)** $D = [5, 7], R = [-2, 2]$, **(c)** $D = [2, 4], R = [-2, 2]$
17. **(a)** $D = [-5, 1], R = [-1, 2]$, **(b)** $D = [-3, 3], R = [-1, 2]$, **(c)** $D = [-3, 3], R = [-2, 1]$
19. Translate 2 units left and 2 units down **21.** Reflect about the x-axis, stretch away from the x-axis by a factor of 2, and then shift up 1 unit. **23.** $g(x) = -x^2 - 2x$ **25.** $g(x) = 2x^2 - 1$ **27.** **(a)** For g: shift right 2 units. For h: Reflect about the x-axis and then shift up 3 units. **(b)** $g(x) = (x - 2)^2 + 1$; $h(x) = -x^2 + 2$. **29.** **(a)** For g: Reflect about the x-axis, shift left 2 units, then down 1 unit. For h: shift right 2 units and down 1 unit. **(b)** $g(x) = -|x + 2| - 1$, $h(x) = |x - 2| - 1$
31. **(a)** $D = [-1, 7], R = [4, 8]$, **(b)** $D = [-4, 4], R = [4, 8]$
33. **(a)** $D = (-\infty, 4], R = [-6, 4]$, **(b)** $D = [-4, \infty), R = [-4, 6]$ **35.** $D = [0, 6], R = [-3, 5]$
37. $D = [6, 12], R = [-1, 4]$ **39.** **(a)** $(-1, 0)$; $(4, 0)$, **(b)** $(2, 0)$; $(7, 0)$, **(c)** $(0, 14)$
41. **(a)** $(\pm 2, 0)$, **(b)** $(\pm 1, 0)$, **(c)** $(0, -4)$ **45.** $(-1, -4), (1, -2), (2, 0), (3, -3)$
47. **(a)** $(-5, 3), (1.4, 5.6)$, **(b)** $(-5, 6), (1.4, 8.6)$

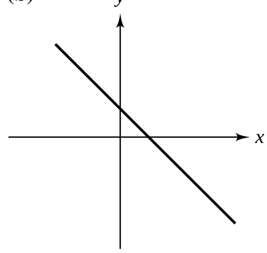
EXERCISES 2.4 *(page 95)*

Check Your Understanding **1.** F **3.** T **5.** QII, QIV **7.** QIV **9.** QI

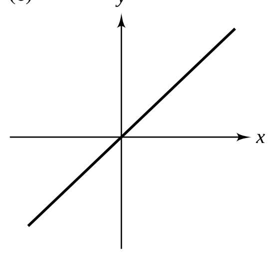
Develop Mastery **1.** $-\frac{3}{2}$ **3.** $\frac{5}{9}$ **5.** 0 **7.** $y = -2x - 2$ **9.** $y = -\frac{2x}{3}$ **11.** $y = -\frac{3}{2}$ **13.** $-\frac{3}{2}$; $(2, 0), (0, 3)$
15. -2; $(2, 0), (0, 4)$ **17.** -3; $(2, 0), (0, 6)$ **19.** **(b)** $y = \dfrac{x}{2} + \dfrac{7}{2}$, **(c)** $y = -2x + 1$
21. **(b)** $y = -\frac{2x}{3} - 2$, **(c)** $y = \frac{3x}{2} - 2$ **23.** **(b)** $y = 2$, **(c)** $x = -1$ **25.** Not collinear **27.** Collinear
29. I, III, IV **31.** II, IV **33.** **(a)**



(b)



(c)



35. **(a)** $(0, 4)$, **(b)** $y = -2x + 4$ **37.** **(a)** $(0, 2)$, **(b)** $y = \frac{3x}{2} + 2$ **39.** $y = -\frac{x}{4} + \frac{3}{2}$ **41.** $y = \frac{x}{3} + \frac{10}{3}$ **43.** Yes
45. No **47.** 2 **49.** **(a)** $m_{AC} = -1, m_{AB} = 1$, **(b)** $x^2 + y^2 - 8x - 20y + 96 = 0$ **51.** **(a)** $C = 60 + 0.20x$, **(b)** Not more than 200 mi. **53.** **(a)** $C = 1{,}200 + 10x$, **(b)** $R = 16x$, **(c)** $P = 6x - 1{,}200, x > 200$ **55.** 4
57. **(a)** \$64,000, **(b)** \$4,000 **59.** **(a)** $L = 0.002T + 124.91$, **(b)** 124.95 cm, **(c)** 130°C **61.** $\pm\sqrt{3}$ **63.** $\frac{2}{3}$ **65.** π
67. $4 - 2\sqrt{2}$

EXERCISES 2.5 *(page 104)*

Check Your Understanding **1.** T **3.** F **5.** F **7.** zero **9.** one

Develop Mastery **1.** $(0, -3), (\pm\sqrt{3}, 0)$; $V(0, -3)$ **3.** $(0, 2), (1, 0), V(1,0)$ **5.** $(0, -2), (-1 \pm \sqrt{3}, 0)$; $V(-1, -3)$
7. $(0, 2), (-1 \pm \sqrt{3}, 0)$; $V(-1, 3)$ **9.** $(0, 2),(1, 0)$; $V(1, 0)$ **11.** $(0, 0), (-4, 0)$; $V(-2, -2)$ **13.** Graphs all pass through $(0, -1)$; two x intercepts, one positive and one negative; vertex in IV. **15.** Graphs all pass through $(0, 4)$. If $b < 4$, no zeros; if $b = 4$, two zeros; if $b > 4$, four zeros. **17.** $y = -2x + 1$ **19.** $y = -4x + 3$ **21.** I, II, and IV **23.** I and II **25.** $2\sqrt{7}$

27. $\sqrt{2}$ **29.** $\{x \mid x < 1 \quad \text{or} \quad x > 3\}$ **31.** $\{x \mid -1 < x < \frac{3}{2}\}$ **33.** $V(1,-4)$ **35. (a)** $[-6.25, \infty)$ **37.** $(-\infty, 9]$ **43.** $f(x) = -x^2 + 4x + 5$ **45.** $f(x) = x^2 + 6x + 7$ **47.** 15 **49.** 24 **51. (a)** Shift right 2 units and up 1 unit. **(b)** $y = x^2 - 4x + 5$ **53. (a)** Reflect graph of $y = x^2$ about the x-axis, then shift down 2 units. **(b)** $y = -x^2 - 2$ **55.** $\{y \mid 4 \le y \le 8\}$ **57.** $\{y \mid -1 < y \le 8\}$ **59.** Min is $-\frac{25}{4}$, no max **61.** Min is $-\frac{1}{4}$, max is 12 **63.** Max is 9, no min **65.** Shift up 3 units and left 2 units. **67.** Max is 2, min is 0 **69.** Min is $\sqrt{2}$, no max **71. (c)** Graph of G is an absolute value graph, $y = |x + 1|$ **73.** 1.60 **75. (a)** $A = 2u - \dfrac{u^2}{2}$, **(b)** $(2, 2)$, **(c)** 2 **77. (a)** $A = x(18 - x)$, **(b)** $0 < x < 18$ **79.** $A = -x^2 + 410x$; $D = \{x \mid 0 < x < 390\}$ when $x = 205$, A is max.

81. $T = \begin{cases} 1{,}400x & \text{if } 0 \le x \le 120 \\ 2{,}600x - 10x^2 & \text{if } 120 < x \le 150; \end{cases}$ T is max when $x = 130$.

83. (a) $A = \dfrac{\sqrt{3}}{36}x^2 + \left(25 - \dfrac{x}{4}\right)^2$, **(b)** $A_{\min} \approx 272$ **85. (a)** $A = 10x - 2x^2$, $D = \{x \mid 0 < x < 5\}$, **(b)** $\frac{5}{2}$

87. (a) 1.7 miles from P, **(b)** 2.8 hours

EXERCISES 2.6 *(page 114)*

Check Your Understanding **1.** F **3.** F **5.** T **7.** 9 **9.** 0, 2.5

Develop Mastery **1. (a)** -5, **(b)** 0 **3. (a)** 3, **(b)** 2.25 **5. (a)** $(f + g)(x) = 2x - \dfrac{1}{x}$, $x \ne 0$, **(b)** $\left(\dfrac{f}{g}\right)(x) = 1 - \dfrac{1}{x^2}$, $x \ne 0$ **7. (a)** $(f + g)(x) = -1$, $x \ge 0$, **(b)** $\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{x} - 2}{1 - \sqrt{x}}$, $x \ge 0$ and $x \ne 1$ **9. (a)** 5, **(b)** 15, **(c)** 8

11. (a)

x	3	-1	0	1	3
$(g \circ f)(x)$	-2	-1	u	3	4

(b) $\{-3, -1, 1, 3\}$, **(c)** $\{0, 3\}$ **13.** $(g \circ f)(x) = x - 4$, $D = \{x \mid x \ge 0\}$ **15.** 0, -1 **17.** $-1, \frac{9}{2}$ **19.** $\dfrac{2 \pm \sqrt{31}}{3}$ **21.** $\{x \mid -\sqrt{2} \le x \le \sqrt{2}\}$ **23.** $\{x \mid x \ge \frac{11}{7}\}$ **25.** $\mathbb{R}$ **27. (b)** $(0, 0)$; $(1, 0)$, $(2, 0)$

29. $(0, -7)$; $(\pm 1, 0)$, $(\pm\sqrt{7}, 0)$ **31. (a)** T, **(b)** T **33. (a)** Reflect graph of f about y-axis, **(b)** Reflect graph of g about x-axis. **35. (a)** Shift graph of f right 2 units. **(b)** Shift graph of g down 2 units. **37. (a)** $(f \circ g)(x) = x$, $(g \circ f)(x) = x$, **(b)** Yes **39. (a)** $(f \circ g)(x) = x$ for $x \ge 1$; $(g \circ f)(x) = |x|$ for x in $\mathbb{R}$, **(b)** No **41.** $g(x) = \dfrac{x + 5}{2}$ **43.** $g(x) = \dfrac{2x}{x - 2}$

45. 3.68 **47.** 1.81 **49.** $f(x) = \dfrac{1}{x}$, $g(x) = x^2 + 5$ **51.** $f(x) = |x|$, $g(x) = 5x + 3$ **57. (a)** Empty set, **(b)** $[-2.5, -2)$ **59. (a)** Shift graph of g right 2 units. **(b)** Compress graph of g towards y-axis. **(c)** Stretch graph of g horizontally. **61. (a)** Shift graph of g right two units. **(b)** Shift graph of g left 2 units. **(c)** Reflect graph of g about y-axis. **63. (a)** Shift graph of f up 2 units. **(b)** Keep portion of the graph of g that is above or on x-axis and reflect the portion below about x-axis. **65. (a)** $(1, 4)$, **(b)** $(-1, -2)$ **67.** $F(x) = (f \circ k)(x)$ **69.** $H(x) = (g \circ h)(x)$ **71.** 5 **73.** 3 **75. (a)** 3, -1, 3, -1, **(b)** -1, 3 **77. (a)** $C = 80 + 192t - 16t^2$, **(b)** \$592, **(c)** 6 hrs.

79. (a) $V = (\frac{\pi}{48})t^3$, **(b)** 6.74 sec. **81. (a)** $A = \dfrac{\pi}{(t + 1)^2}$, **(b)** $\frac{\pi}{4}$ sq. ft.; $\frac{\pi}{9}$ sq. ft., **(c)** 4 min.

83. (a) 3 ft., **(b)** $V = (\frac{4\pi}{3})(0.25t + 3)^3$, **(c)** 697 cu. ft., **(d)** 6.3 sec.

EXERCISES 2.7 *(page 128)*

Check Your Understanding **1.** T **3.** T **5.** T **7.** QI, QIV **9.** $(-5, 2)$

Develop Mastery **1. (a)** $\{(-1, 0), (3, 1), (5, 2)\}$, **(b)** Yes **3. (a)** $\{(4, -3), (2, -1), (1, 1), (2, 3)\}$, **(b)** No **5.** Yes **7.** No **9. (b)** $(1.4, 1.4)$ **11. (b)** $(1.6, 1.6)$ **13. (a)** No, **(b)** Yes **15.** $f^{-1}(x) = \dfrac{x - 5}{2}$

17. $f^{-1}(x) = \dfrac{1}{x - 1}$ **19.** $f^{-1}(x) = \dfrac{2}{x + 1}$ **21.** $f^{-1}(x) = x$ **23.** $f^{-1}(x) = 1 + \sqrt{x}$ **25. (b)** f is one–one, **(c)** Yes **27. (b)** f is not one–one, **(c)** No **29. (b)** f is one–one, **(c)** Yes **31. (a)** $f(x) = 0.5x + 4$, **(b)** $f^{-1}(x) = 2x - 8$ **33. (a)** $f(x) = -x + 1$; $[-3, -1]$, **(b)** $f^{-1}(x) = -x + 1$; $[2, 4]$ **35. (a)** $f^{-1}(x) = -0.6x + 2.8$, **(b)** $f(x) = \frac{-5}{3}x + \frac{14}{3}$ **37. (a)** $g(x) = -2x + 7$; $(0, 7)$, $(3.5, 0)$, **(b)** $h(x) = -2x + 11$; $(0, 11)$, $(5.5, 0)$ **39. (a)** $f^{-1}(x) = x - 3$

41. **(a)** $f^{-1}(x) = \sqrt{x+4}$ **43.** **(a)** $f(x) = -3 + \sqrt{1+4x}$, $x \geq 0$ **45.** 2 **47.** -2 **49.** **(a)** Decreasing, **(b)** Yes **51.** **(a)** Decreasing, **(b)** Yes **53.** **(a)** Increasing, **(b)** Yes **57.** $y = -x$ **59.** $[-19, 36]$ **61.** **(b)** 2 **65.** Graph is the line segment with endpoints **(a)** $(-2, -1)$, and $(7, 2)$, **(b)** $(0, -3)$ and $(3, 6)$, **(c)** $(-3, 1)$ and $(6, 4)$, **(d)** $(-1, -1)$ and $(2, 8)$ **67.** **(a)** $D = [-4, 4]$, $R = [-5, 2]$, **(b)** $D = [-5, 2]$, $R = [-4, 4]$, **(c)** -4; -2; 4 **69.** 23 **71.** 4 **73.** **(a)** $r = \sqrt[3]{\frac{3V}{4\pi}}$, **(b)** 0.94, 1.05, 1.11 **75.** **(a)** $K = \sqrt{2}(2x - x^2)$, **(b)** $D = \{x \mid 0 < x < 2\}$, $R = \{y \mid 0 < y \leq \sqrt{2}\}$, **(c)** f is not one–one, **(d)** 0.196, 1.804 **77.** $r = \sqrt{\frac{V}{3\pi}}$

EXERCISES 2.8 *(page 136)*

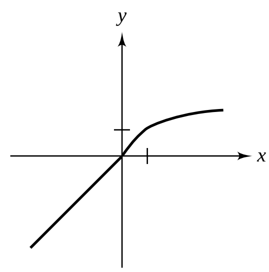
Check Your Understanding **1.** T **3.** T **5.** T **7.** T **9.** T

Develop Mastery **1.** $\sqrt{10} \approx 3.16$ sec. **3.** **(a)** 6.31 sec., **(b)** 177 ft./sec. **5.** **(a)** 42 ft., **(b)** 3.12 sec. **7.** **(a)** $s(t) = 128t - 16t^2$, **(b)** 192 ft., **(c)** 256 ft. **9.** **(c)** 287 ft. **11.** **(a)** 9.6 sec., **(b)** 307 ft./sec. **13.** $8\sqrt{2.4} \approx 12.4$ ft./sec. **15.** **(a)** 6.62 sec., **(b)** 212 ft./sec. $\approx$ 144 mi/hr, **(c)** Model neglects air resistance. **17.** **(a)** 80 ft./sec., **(b)** 80 ft./sec. **19.** 42.1 sec. **21.** 178 ft. **23.** 320 ft. **25.** **(a)** $R = 36x - 0.2x^2$, **(b)** 90 calculators **27.** **(a)** \$6,000; \$6,300; \$6,400, **(b)** \$40 rental rate gives \$6,400 **29.** **(a)** $V = -312.5t + 3{,}000$, **(b)** 9.6 yrs. **31.** 40 by 60 ft. **33.** **(a)** 1,000 cu. in., **(b)** 38.2 sec., **(c)** $\{t \mid 0 \leq t \leq 38.2\}$, **(d)** 17.7 sec.; 20.5 sec. **35.** **(a)** $r = \frac{d}{3}$; $V = (\frac{\pi}{27})d^3$, **(b)** $V = (\frac{\pi}{27})(30 - 5\sqrt{t})^3$, **(c)** $\frac{1{,}000\pi}{27} \approx 116$ cu. ft., **(d)** 36 min. **37.** **(a)** $C = 4{,}000(40 - 2x + 3\sqrt{x^2 + 36})$, **(b)** 216,498; 213,722; 213,823; 216,000; 219,943; 233,866, **(c)** When x is about 5.37 miles, C is about 214 thousand dollars. **39.** **(b)** 2.67 by 2.31; $A \approx 6.16$ **41.** 5.5 by 4.9 feet **43.** $r \approx 4.9$, $h \approx 6.9$, $V \approx 522.4$ **45.** **(b)** $k = b(8 + \sqrt{64 - b^2})$, **(c)** $b \approx 13.9$, $h \approx 12.0$, $k \approx 83.1$

CHAPTER 2 REVIEW *(page 141)*

Test Your Understanding **1.** F **3.** F **5.** T **7.** F **9.** T **11.** T **13.** T **15.** T **17.** T **19.** F **21.** T **23.** F **25.** T **27.** F **29.** T **31.** F **33.** T **35.** F **37.** F **39.** F **41.** T **43.** T **45.** T **47.** T **49.** F

Review for Mastery **1.** Yes **3.** $\mathbb{R}$ **5.** $\{x \mid x \leq 2\}$ **7.** $\{x \mid x \neq -2, x \neq 2\}$ **9.** $2x - 3y + 11 = 0$ **11.** $3x + 2y = 1$ **13.** Graph is a line through $(2, 0)$ and $(0, -4)$. **15.** Graph is a parabola opening up from lowest point $(2, -1)$ and passing through $(1, 0)$, $(3, 0)$, $(0, 3)$. **17.** Graph consists of two half-lines: $y = x$ for $x \geq 1$ and $y = -x + 2$ for $x < 1$. **19.** $\{x \mid x < 3\}$ **21.** $\{x \mid x < 0 \text{ or } x > 2\}$ **23.** $\{x \mid -1 \leq x \leq 4\}$ **25.** $\{-2\}$ **27.** 3 **29.** -7 **31.** 1 **33.** 0, -2 **35.** 0, 1 **37.** **(a)** Neither, **(b)** Not one–one **39.** **(a)** Increasing, **(b)** Is one–one **41.** $f^{-1}(x) = \frac{x+4}{2}$; $D = \mathbb{R}$, $R = \mathbb{R}$ **43.** $f^{-1}(x) = x^2 + 1$; $D = \{x \mid x \geq 0\}$, $R = f\{y \mid y \geq 1\}$ **45.** **(a)**



47. **(a)** Graph is a parabola opening up with lowest point at $(1, 1)$. **(b)** Min is 1, no max **49.** **(a)** Translate graph of $y = \sqrt{x}$ up 1 unit. **(b)** Min is 1, no max **53.** **(a)** Graph is line segment joining $(-4, 3)$ and $(0, -3)$. **(b)** Graph is line segment joining $(-2, -3)$ and $(2, 3)$. **(c)** $g(x) = -1.5x - 3$ for $-4 \leq x \leq 0$, $h(x) = 1.5x$ for $-2 \leq x \leq 2$ **55.** **(b)** $f^{-1}(x) = 0.5(x + 1)$, $D = [-3, 3]$ **57.** 4 **59.** **(a)** $[-1, \infty)$, **(b)** $f^{-1}(x) = 2 - \sqrt{x+1}$, $D = [-1, \infty)$, $R = (-\infty, 2]$ **61.** 14 feet **63.** **(a)** $s = 160 + 48t - 16t^2$, **(b)** 5 sec., **(c)** 196 ft. **65.** 32 min. and 44 sec. after 12 o'clock **67.** -1 **69.** 2 **71.** -4 **73.** **(a)** 1,200 cu. in., **(b)** 1,043.2; 820; 480 cu. in., **(c)** 36.75 sec. **75.** 3.35 sec. **77.** 30 min. **79.** **(a)** $T = \begin{cases} 2400x, & 0 \leq x \leq 100 \\ 3900x - 15x^2, & 100 < x \leq 180, \end{cases}$ **(b)** $\{0, 1, 2, 3, \ldots, 180\}$, **(c)** 130

CHAPTER 3

EXERCISES 3.1 *(page 155)*

Check Your Understanding **1.** T **3.** F **5.** F **7.** T **9.** F

Develop Mastery **1.** Yes, 2 **3.** Yes, 3 **5. (a)** $(f+g)(x) = 2x + 7$, **(b)** deg. 1, l.c. 2, c.t. 7 **7. (a)** $(fg)(x) = -3x^2 + 13x + 10$, **(b)** deg. 2, l.c. -3, c.t. 10 **9. (a)** $(f \circ h)(x) = 6x^2 - 3x + 2$, **(b)** deg. 2, l.c. 6, c.t. 2 **11. (c)** **13. (c)** **15. (a)** $p(x) = x^3 + x^2 - 2x$, **(b)** 0, 1, -2, **(c)** $(-\infty, -2) \cup (0, 1)$ **17. (a)** $p(x) = 2x^3 - x^2 - 2x + 1$, **(b)** -1, 0.5, 1, **(c)** $(-\infty, -1) \cup (0.5, 1)$ **19. (a)** 0, $-\frac{1}{3}$, **(b)** ↖, ↗ **21. (a)** 1, $\dfrac{-1 \pm \sqrt{3}i}{2}$, **(b)** ↙, ↗ **23. (b)** One zero, two turning points, **(c)** QIII, QIV **25. (b)** Five zeros, four turning points, **(c)** QI, QII, QIII, QIV **27.** 3.3 **29.** 2.4 **31. (a)** $(-0.8, 5.2)$, **(b)** ↙, ↗ **33. (a)** $(0.1, 5.0)$, **(b)** ↖, ↗ **35.** max$(-0.8, 5.2)$ **37.** min$(-2.1, -7.1)$ **39.** f **41.** c **43.** g **45.** $g(x) = f(x + 1)$ **47.** $g(x) = f(x - 2)$ **49.** -2 through 10 **51.** 2, 3, 4 **55.** 4.6 by 10.7 **57.** 2.2 by 7.6 by 10.6 **61. (c)** 0.7 **63.** $(-0.8, -1.4)$ **65.** $(2.8, 4.9)$ **67.** ↙, ↘ **69.** ↖, ↘ **71.** ↖, ↗ **73. (a)** -2, 1, **(b)** max$(-2, 23)$, min$(1, -4)$ **75. (a)** $1 \pm \sqrt{3}\,i$, **(b)** no local extrema

EXERCISES 3.2 *(page 167)*

Check Your Understanding **1.** T **3.** F **5.** F **7.** 2 **9.** 2

Develop Mastery **1.** $[-2, -1.5]$ **3.** $[-0.5, 0]$ **5.** $p(x) = (x - 1)(2x^2 + 5x + 4) + 2$; $p(1) = 2$ **7.** $p(x) = (x + 1)(3x^3 - 2x^2 + 1) - 2$; $p(-1) = -2$ **9.** 11 **11.** $-\frac{3}{2}$ **13.** $-\frac{4}{3}$ **15.** $-\frac{3}{2}$ **17.** $-\frac{15}{2}$ **19.** $\pm[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}]$ **21.** $\pm[1, 2, \frac{1}{2}]$ **23.** $\pm[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}]$ **25. (a)** 4, $\pm\sqrt{2}i$, **(b)** $(-\infty, 4)$ **27. (a)** $-\frac{3}{2}$, $2 \pm \sqrt{5}$, **(b)** $(-\infty, -\frac{3}{2}) \cup (2 - \sqrt{5}, 2 + \sqrt{5})$ **29. (a)** $\frac{3}{2}, \frac{5}{3}, -\dfrac{1 \pm \sqrt{3}i}{2}$, **(b)** $(\frac{3}{2}, \frac{5}{3})$ **31. (a)** $\{0, 4\}$, **(b)** $\emptyset$ **33. (a)** $\pm\sqrt{2 - \sqrt{3}}$, $\pm\sqrt{2 + \sqrt{3}}$, **(b)** ± 0.52, ± 1.93 **35. (a)** $\{-1, \frac{1}{2}, 2\}$, **(b)** $\{0, \frac{3}{2}, 3\}$, **(c)** $\{x \mid x \le -1 \text{ or } \frac{1}{2} \le x \le 2\}$ **37. (a)** $\{-2, 1\}$, **(b)** $\{-1, 2\}$, **(c)** $\{x \mid x \le -2 \text{ or } x = 1\}$ **39. (b)** Root of $x^2 - 2 = 0$ **41. (b)** Root of $x^3 + 3x^2 + 3x - 1 = 0$ **43. (a)** $f(x) = x^3 - 2x^2 - 5x + 6$, **(b)** Two **45. (a)** $f(x) = x^4 - 3x^3 + x^2 + 3x - 2$, **(b)** three **47. (a)** Four, **(b)** ± 2, 3, 5, **(c)** Three; QI, QIII, QIV **49.** Yes, $f(x) = -x^3 - x^2 + 4x + 4$ **51.** ↖, ↘, **(b)** $(-0.8, -8.2)$ **53. (a)** ↙, ↗, **(b)** $(0.1, -10.1)$ **55.** $(2.1, 4.1)$ **57.** $(-3.4, 33.8)$ **63.** $[-2, 4]$ **65.** $[-2, 3]$ **67.** $[-4, 3]$ **69.** For $c = 3$, zeros are 1, 1, -2. **71. (a)** $V = \frac{\pi}{3}(16h^2 - h^3)$, **(b)** $h \approx 10.7$, $r \approx 7.5$, $V \approx 635.5$

EXERCISES 3.3 *(page 177)*

Check Your Understanding **1.** T **3.** F **5.** T **7.** T **9.** Four

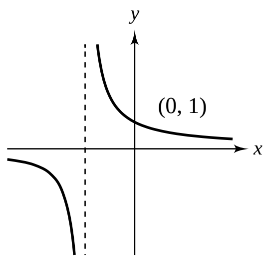
Develop Mastery **1. (a)** $[-1, 0]$, $[0, 1]$, $[3, 4]$, **(b)** 3.1 **3. (a)** $[-2, -1]$, **(b)** -1.1 **5.** $(-2.1, -3)$ **7.** $(1.3, 2.3)$ **9.** $f(x) = x^4 - 4x^3 + 5x^2 - 2x - 2$ **11.** $f(x) = x^3 - 6x + 4$ **13.** 2 and $1 + \sqrt{2}$ **15.** $\frac{1}{2}$ and $2 + \sqrt{5}$ **17.** 4, $\pm\sqrt{2}i$ **19.** $-\frac{3}{2}$, $2 \pm \sqrt{5}$ **21.** $\frac{3}{2}, \frac{5}{3}, \dfrac{-1 \pm \sqrt{3}i}{2}$ **23.** 1, $\frac{3}{2}$, -1, $-\frac{1}{2}$ **25.** $\frac{1}{3}$, $\pm\sqrt{1.5}$ **27.** -1, -1, 1, 2 **29.** 1, 1, -2 **31.** -3, -1, $-\frac{2}{3}$ **33. (a)** $[-3, 1 - \sqrt{6}] \cup [1 + \sqrt{6}, \infty)$, **(b)** $[-2, 2 - \sqrt{6}] \cup [2 + \sqrt{6}, \infty)$ **35. (b)** 1.2 **37. (b)** -0.5 **39.** Four **41. (a)** $c < 0$, **(b)** $c \le -5$ **47. (a)** $k \le -4$, **(b)** $k \ge -2$, **(c)** No value **49. (a)** $k \le -8$, **(b)** $k \ge -7$, **(c)** $k \ge -7$ **51. (a)** When f has no positive zeros. **(b)** When f has exactly one positive zero. **(c)** When f has two distinct positive zeros. **(d)** When f has three distinct positive zeros. **53. (b)** If $f(x) \ge 0$ for every x. **55. (a)** $2 < c < 6$, **(b)** $c = 2$ or $c = 6$, **(c)** $c > 6$ **57.** 1.675 130 871 **59.** 1.926 743 498 **61. (a)** $V = 4x^3 - 42x^2 + 108x$, $D = (0, 4.5)$, **(b)** 1.5, 1.902 **63.** 2.8 **65.** 8.75 **67.** $u \approx 19$ ft, $v \approx 32.6$ ft, $d \approx 23.2$ ft. **69.** $34.1 \le h \le 41.9$ when $6.7 \le x \le 11.9$

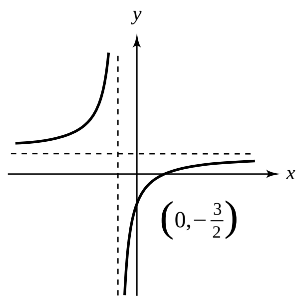
EXERCISES 3.4 *(page 187)*

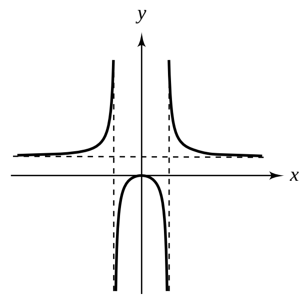
Check Your Understanding **1.** F **3.** F **5.** T **7.** T **9.** F

Develop Mastery **1. (a)** $f(0.92) \approx 122$, $f(0.97) \approx 1{,}014$, **(b)** $f(1.11) \approx 113$, $f(1.03) \approx 1{,}214$ **3. (a)** From above, **(d)** Yes, at $(1, 1)$. **5.** $y = -\dfrac{1}{x - 1}$. Reflect graph of $y = \frac{1}{x}$ about the x-axis, then translate to the right 1 unit. **7.** $y = -\frac{2}{x - 3}$. Reflect graph of $y = \frac{1}{x}$ about the x-axis, translate to the right 3 units, and then stretch vertically away from the x-axis by a factor of 2. **9.** $y = 1 - \frac{1}{x}$. Reflect graph of $y = \frac{1}{x}$ about the x-axis, then translate up 1 unit. **11. (a)** $g(x) = f(x) - 2$, **(b)** None **13. (a)** $g(x) = f(x - 2)$, **(b)** $(1, 1)$

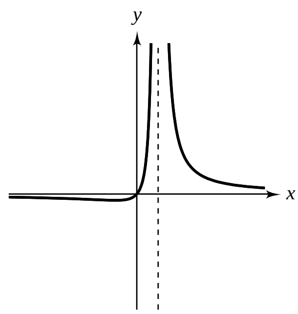
15. *V.A.* $x = -2$; *H.A.* $y = 0$ **17.** *V.A.* $x = -2$; *H.A.* $y = 2$ **19.** *V.A.* $x = \pm 2$; *H.A.* $y = 1$







21. *V.A.* $x = 1$; *H.A.* $y = 0$ **23.** **(a)** None, **(b)** $\mathbb{R}$, **(c)** $(0, 2]$ **25.** **(a)** None, **(b)** $\mathbb{R}$, **(c)** $(1, 2]$



27. **(a)** $(2, 0)$, **(b)** $\{x \mid x \neq -1, x \neq 0, x \neq 4\}$, **(c)** $\mathbb{R}$ **29.** **(a)** $(-\infty, -2)$ and $(-2, \infty)$, **(b)** None
31. **(a)** None, **(b)** $(-\infty, -1), (-1, 2), (2, \infty)$ **33.** $\{\frac{2}{3}, 1\}$ **35.** $\{-1, -\frac{1}{2}, 3\}$ **37.** $\{x \mid -1 < x < -\frac{1}{2}\}$
39. $\{x \mid -1.5 \leq x \leq 1\}$ **41.** Graph $y = \dfrac{1}{x + 2}$ with point $(0, \frac{1}{2})$ missing **43.** Graph $y = \dfrac{x - 1}{x + 3}$ wtih point $(1, 0)$ missing
45. $y = 1$; $(3, 1)$ **47.** $y = 1$; $(-1, 1)$ **49.** $y = x + 1$; graph does not cross asymptote **51.** $y = 2x + 3$, graph does not cross asymptote **53.** Min is 3 when $x = \pm 1$ **55.** $f(x) = \dfrac{2(x - 2)}{x + 1}$ **57.** $(1, -5)$ **59.** $(-1, 3)$ **61.** d **63.** f
65. b **67.** c **69.** $\{x \mid -1 < x < 3\}$ **71.** $(-\infty, -2) \cup (-1, 1)$ **73.** **(b)** above, **(c)** Yes, $(2, 7)$, **(d)** $x \geq 98$
77. **(a)** No, **(b)** Two **79.** 12.65 by 12.65 **81.** $r \approx 2.4$, $h \approx 4.8$ **83.** 0.79 **85.** **(a)** 8.2 by 24.6 by 12.2, **(b)** 10.3 by 30.9 by 7.7 **87.** **(a)** $C(x) = 600 + 3x + \dfrac{x^2}{240000}$, **(b)** 12,000 **89**. $y = -0.8x + 8$

CHAPTER 3 REVIEW *(page 190)*

Test Your Understanding **1.** F **3.** T **5.** T **7.** T **9.** F **11.** F **13.** T **15.** F **17.** F **19.** T
21. T **23.** T **25.** T **27.** F **29.** T **31.** T **33.** T **35.** T **37.** T **39.** F **41.** T **43.** T
45. T **47.** T **49.** F **51.** T **53.** F **55.** T **57.** T **59.** T **61.** F

Review for Mastery **1.** $q(x) = 3x^2 - 7x + 6$; $r = -5$ **3.** 12
5. $(-2, 0), (1, 0), (1, 0), (0, 2)$; $x \to \infty, f(x) \to \infty$; $x \to -\infty, f(x) \to -\infty$ **7.** **(a)** $\pm[1, 2, 3, 5, 6, 10, 15, 30, 0.5, 1.5, 2.5, 7.5]$,
(b) Eliminate all but $-2.5, -1.5$. **(c)** $\dfrac{-5}{2}, \dfrac{3 \pm \sqrt{33}}{2}$ **9.** $-3, -2, \frac{1}{2}$
11. **(a)** $[-3, -2], [0, 1], [1, 2]$, **(b)** $[-2.5, -2.4], [0.6, 0.7], [1.8, 1.9]$, **(c)** 1.83
13. **(a)** $0, 0, \pm\sqrt{3}$, **(b)** Graph is tangent to the x-axis at $(0, 0)$ and crosses at $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$.
15. $f(x) = x^4 - 8x^2 + 16$ **17.** $y = (x + 1)(x - 1)(x - 3)$; $(-1, 0), (1, 0), (3, 0), (0, 3)$ **19.** **(a)** Function is odd; $(-2, 0)$, $(0, 0)$, $(2, 0)$, **(b)** Translate graph in **(a)** down 1 unit. **(c)** Translate graph in **(a)** to the right 1 unit.
21. **(a)** $(-3, 0), (0, 1.5)$, **(b)** *V.A.* $x = 2$; *H.A.* $y = -1$ **23.** **(a)** $(1, 0), (1, 0)$, no y-intercept, **(b)** *V.A.* $x = 0, x = 4$; *H.A.* $y = 1$ **25.** $\{x \mid x \geq 3\}$ **27.** $\{x \mid x \leq -1 \text{ or } x = 2 \text{ or } x > 3\}$ **29.** **(a)** 2, **(b)** -2 **31.** **(a)** $\{-1, 0.5, 3\}$,
(b) $\{0, 1.5, 4\}$, **(c)** $\{-3, -1.5, 1\}$ **33.** **(a)** $\{x \mid -3 \leq x \leq -0.5 \text{ or } x \geq 1\}$, **(b)** $\{x \mid -2 \leq x \leq 0.5 \text{ or } x \geq 2\}$
35. $(-2, -1)$ **37.** $x > 1.5$ **39.** $x > -2$ **41.** 4 **43.** $(2.54, 0.88)$ **45.** $(-2, -2)$ **47.** **(a)** Yes, **(b)** $(-2, 6.33)$
49. **(a)** $y = 2x - 1$, **(c)** Yes, $(2, 3)$ **51.** $u = 1.6$, $v = 5.3$ **53.** 5.25 in. by 7.62 in. **55.** 11.31 in. by 11.31 in.

CHAPTER 4

EXERCISES 4.1 *(page 207)*

Check Your Understanding **1.** F **3.** T **5.** F **7.** one **9.** QIII

Develop Mastery **1. (a)** $\frac{7}{2}$, **(b)** $\frac{512}{27}$ **3. (a)** -140, **(b)** $\frac{1}{\sqrt{3}}$ **5. (a)** 0.5664, **(b)** 0.5234 **7. (a)** $2\sqrt{5}-1$, **(b)** $\sqrt{x}+2$ **9.** $|x|$ **11.** $x, x>0$ **13.** 3^{9x-5} **15.** $\{\frac{1}{2}, 1\}$ **17.** $\{\frac{2}{5}\}$ **19.** $\{x \mid x \neq 1, x \neq -1\}$
21. $\{\frac{3}{2}\}$ **23.** Yes **25.** No **27. (a)** -0.37, **(b)** 5.82 **29.** $3^{\sqrt{3}}$ **31. (a)** $\{y \mid y<0\}$, **(b)** Decreasing
33. (a) $\{y \mid y>0\}$, **(b)** Increasing **35. (a)** $\{y \mid y<1\}$, **(b)** Decreasing **37. (a)** Reflect graph of f about x-axis
39. (a) Shift graph of f up 2 units. **41.** ± 2 **43.** $\mathbb{R}$ **45.** $\{x \mid x < -\sqrt{5}$ or $x > \sqrt{5}\}$
47. $f(x)=e^{x+2}$, translate graph of $y=e^x$ to the left 2 units. **49.** $f(x) \to e^2$ as $x \to \infty$ **51.** $f(x) \to e^2$ as $x \to \infty$
53. $b=c$ **55.** 34; 309 **57.** 1 **59. (a)** $\mathbb{R}$ **(b)** *(i)* and *(iii)* **61. (a)** $[0, \infty)$, **(b)** $\mathbb{R}$
63. (a) Shift graph of $y=x^{2/3}$ left 2 units, reflect about x-axis, then shift up 1 unit. **(b)** $-3, -1$, **(c)** $(-2, 1)$
65. (b) 4, 5, 6, 7, 8, 9 **67. (b)** 2, 3 **69. (b)** $x<0$, **(c)** between **71. (a)** Yes; about the origin, **(c)** Yes, increasing
73. (1, 3), (−1.87, 0.13) **75.** −0.49, 3.32 **77. (a)** 40 sec., **(b)** 28 sec., **(c)** 17 sec.
79. (a) \$4.87, **(b)** \$1,869.16 **81. (a)** 5.7 trillion dollars, **(b)** between 2001 and 2002 **83.** About 10,000 years
85. $-1.6 \leq x \leq 1.6$

EXERCISES 4.2 *(page 217)*

Check Your Understanding **1.** T **3.** T **5.** T **7.** QII **9.** (0, 1)

Develop Mastery **1. (a)** $\log_5 125 = 3$, **(b)** $\log_4(\frac{1}{16}) = -2$, **(c)** $\log_3 5 = x-1$ **3. (a)** 2, **(b)** −2 **5. (a)** −2, **(b)** 5
7. (a) 0, **(b)** $\frac{1}{3}$ **9. (a)** $\sqrt{e}$, **(b)** 1 **11. (a)** 16, **(b)** 17 **13. (a)** $x-2$ for any x, **(b)** $x-2$ for $x>2$
15. (a) x for any x, **(b)** x for any x **17. (a)** −1, 3, **(b)** $2 \pm \sqrt{13}$ **19.** $\frac{1}{3}$, 27 **23.** $b=(\frac{2}{3})\,c$ **25. (a)** $\sqrt{2}$, **(b)** $\frac{1}{49}$
27. (a) 2 and 3, **(b)** 3 and 4 **29. (a)** $\log_5 36$, **(b)** $\log_2 0.4$ **31. (a)** 4, **(b)** 4 **33. (a)** Four, **(b)** Two
35. (a) Shift graph of $y=\log_2 x$ up 2 units. **(b)** Reflect graph of $y=\log_2 x$ about y-axis. **37. (a)** $D=\mathbb{R}$, **(b)** $y=-x^2$
39. (a) $D=(-\infty, 2)$, **(b)** $y=2-x$ for $x<2$ **41. (a)** $D=(-\infty, 0)$, **(b)** $D=\{x \mid x<0$ or $x>2\}$
43. (a) Yes, **(b)** $f^{-1}(x)=-\ln x$, **(c)** 0.14; −1.39 **45. (a)** Yes, **(b)** $f^{-1}(x)=\ln(6-x)$, **(c)** −1.39; 0.69
47. (a) $f^{-1}(x)=1+e^{-x}$, **(c)** Yes; QI **49. (a)** $f^{-1}(x)=e^{x-1}$, **(c)** Yes; QI and QIII **51. (b)** (4.84, 0.42)
53. (b) (−1.84, 1.61) **55. (a)** Reflect about x-axis and shift up 2 units. **(b)** (e^2, ∞) **57. (b)** 4, 5, 6
61. (a) $f(x)=1-\log_2 x$, **(b)** (1,1) **63. (a)** $D=\mathbb{R}$; $R=(0, 1)$, **(b)** $f^{-1}(x)=\log_3\left(\dfrac{x}{1-x}\right)$, $D=(0, 1)$, $R=\mathbb{R}$
65. [6, 7) **67.** b **69.** d

EXERCISES 4.3 *(page 225)*

Check Your Understanding **1.** F **3.** T **5.** F **7.** 17 **9.** Four

Develop Mastery **1. (a)** $\frac{3}{2}$, **(b)** $\frac{5}{3}$ **3. (a)** 1, **(b)** $\log_7 54$ **5. (a)** $\log_{10} 2$, **(b)** $\log_2 9 - 6$
7. (a) $\log_5 x + 0.5\log_5(x^2+4)$, **(b)** $2+\log_5 x + 0.5\log_5(x^2+1)$ **9. (a)** $\log_3\left(\dfrac{x^2}{x+2}\right)$, **(b)** $\log_5(3\sqrt{x})$ **11. (a)** $1-u$,
(b) $u-1$ **13. (a)** $(\frac{1}{3})(u+2v)$, **(b)** $(\frac{1}{2})(3u+v)$ **15. (a)** $v-u$, **(b)** $(\frac{1}{2})(v-u)$ **17. (a)** $2u$, **(b)** $(\frac{1}{2})(3v-u)$
19. $\frac{1}{2}$ **21.** $\frac{3}{14}$ **23.** 2 **25.** 1 **27. (a)** $\{x \mid x>3\}$, **(b)** $\{x \mid x<0$ or $x>2\}$ **29.** $x>4$ **31.** (2.62, 1.93)
33. (1.65, 2.50) **35.** $1+e^{0.5} \approx 2.65$ **37. (a)** 4.41, **(b)** 1.59, 4.41 **39. (a)** 2.73, **(b)** −0.73, 2.73
41. (a) Different domains, **(b)** $\{x \mid x>0\}$ **45.** $\log_2(ab)=7$, $(\log_2 a)(\log_2 b)=12$ **47.** $\log_2 c^n = 6$, $(\log_2 c)^n = 8$
49. (a) $\emptyset$, **(b)** $\mathbb{R}$ **51.** $\sqrt{3}+\sqrt{2}=\frac{1}{\sqrt{3}-\sqrt{2}}$ **53.** $\sqrt{k+1}+\sqrt{k}=\dfrac{1}{\sqrt{k+1}-\sqrt{k}}$ **55.** $f^{-1}(x)=(\frac{1}{2})(3^{-x}-3^x)$, $D=\mathbb{R}$;
57. (a) 54.7, 140.6, **(b)** 91.1, 39.5 **59. (a)** T, **(b)** F **61. (a)** $c<0$, **(b)** $c \geq 6$
65. Not a function; $-x^2+2x-3<0$ for every x.

EXERCISES 4.4 *(page 234)*

Check Your Understanding **1.** T **3.** F **5.** T **7.** 20 **9.** QI and QII

Develop Mastery **1. (a)** 1.6094, **(b)** 1.1931 **3. (a)** 0.6826, **(b)** −0.5108, **(c)** 1.5440 **5. (a)** 0.3466, **(b)** 0.8326,
(c) 0.6931 **7. (a)** 0.4136, **(b)** Undefined, **(c)** 0.5774 **9. (a)** $\sqrt{5}$; 2.24, **(b)** $\frac{1}{\sqrt{5}}$; 0.45
11. (a) $\sqrt{3}$; 1.73, **(b)** $\frac{1}{6}$; 0.17 **13.** $>$ **15.** $=$ **17.** $>$ **19.** $b<a<c$ **21.** No, different domains
23. No, different domains **25. (a)** No, **(b)** $\mathbb{R}$; $(0, \infty)$ **27. (a)** No, **(b)** $\mathbb{R}$; $(0, \infty)$ **29. (a)** $f^{-1}(x)=-\ln(x-2)$,
(b) (2.1, 2.1) **31. (a)** $f^{-1}(x)=1-\log(0.25x)$, **(b)** (1.4, 1.4) **33. (b)** $f^{-1}(x)=1+e^x$, **(c)** $D=\mathbb{R}$, $R=(1, \infty)$
35. (b) $f^{-1}(x)=\dfrac{e^x}{e^x-1}$, **(c)** $D=(0, \infty)$, $R=(1, \infty)$ **37.** $\sqrt[3]{e}$; 1.396 **39.** $\dfrac{e+10}{15}$; 0.848 **41.** $\dfrac{10+\sqrt{10}}{9}$; 1.462

43. $\frac{\ln 4}{\ln 3}$; 1.262 **45.** $\frac{\ln(\ln 4)}{\ln 3}$; 0.297 **47.** $-\ln(\ln 8 - 1)$; -0.076. **49.** $\frac{-\ln 6}{\ln 35}$; -0.504 **51.** $(0.66, 0)$, $(0, -3)$
53. $(-0.48, 0)(0, 2)$ **55.** **(b)** $\{x \mid x \geq -1\}$ **57.** **(b)** $\{x \mid x \leq -\ln 2\}$ **59.** 1, $\frac{\ln 2}{\ln 5}$ **61.** 1, $e \approx 2.718$
63. No solution **65.** $\{-0.9, 1.2, 16\}$ **67.** 36.3 **69.** 790 billion miles **71.** $\{2\}$ **73.** $\{10, 1000\}$ **75.** **(a)** $\{10\}$, **(b)** $\{e^{-2}, e^2\}$ **77.** $\{x \mid x > 0 \text{ and } x \neq 1\}$ **79.** $\{x \mid x > 0 \text{ and } x \neq 1\}$ **81.** $10^{-3} w/m^2$ **83.** **(a)** 95 dB, **(b)** 27 percent
85. $f^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$

EXERCISES 4.5 *(page 245)*

Check Your Understanding **1.** f **3.** a **5.** c **7.** a **9.** b

Develop Mastery **1.** **(a)** 16,000, **(b)** 25,000, **(c)** 5 hrs. **3.** **(a)** 6.4 billion, **(b)** Year 2005 **5.** 270 million
7. 6 percent **9.** $8\frac{2}{3}$ percent **11.** \$1,934.79 **13.** **(a)** 9.69 gm., **(b)** 6.2 gm. **15.** 7.3 lb. **17.** 3.9 yrs.
19. 7,500 yrs. **21.** 1,340 yrs. **23.** 250 times as great **25.** 7.7 **27.** 19.5 km. **29.** 7.35, slightly basic
31. 26 yrs. **33.** **(a)** 111 lb., **(b)** 110 min. **35.** **(a)** 16,800, **(b)** 17 days **37.** **(a)** 25 percent, **(b)** 1.2 percent

CHAPTER 4 REVIEW *(page 247)*

Test Your Understanding **1.** F **3.** F **5.** F **7.** T **9.** F **11.** T **13.** F **15.** T **17.** F **19.** F
21. T **23.** F **25.** T **27.** F **29.** T **31.** T **33.** T **35.** F **37.** F **39.** F **41.** T **43.** T **45.** F
47. T **49.** T **51.** F **53.** F **55.** T **57.** T **59.** T **61.** T **63.** F **65.** F **67.** F **69.** c **71.** i
73. e

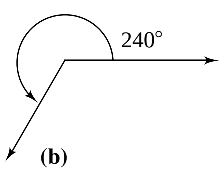
Review for Mastery **1.** $\frac{6}{5}$ **3.** $\frac{3}{2}$ **5.** 0 **7.** $\frac{1}{49}$ **9.** $\frac{1}{2}$ **11.** 3.850 **13.** 0.235 **15.** 23.141 **17.** Three
19. $\sqrt{2}$ **21.** 2.08; -0.54 **23.** -2.35; 23.10 **25.** $\frac{9}{2}$ **27.** 4 **29.** 1 **31.** $\frac{e}{e-1}$ **33.** 0.86 **35.** 1.20
37. 0.29 **39.** 3.72 **41.** 0 **43.** $\{x \mid x < 0 \text{ or } x > 2\}$ **45.** $\{x \mid x > 0\}$ **47.** $\{x \mid x > 1\}$ **49.** Translate graph of $y = \ln x$ up 1 unit. **51.** Reflect graph of $y = e^x$ about the y-axis, then translate up 1 unit.
53. Draw graph of $y = x$ for $x > 0$. **55.** $(\ln 2, 0)$ **57.** $(\frac{5}{2}, 0)$ **59.** No; domains are different **63.** $(2, 3)$
65. **(a)** $f^{-1}(x) = e^{4-x}$, **(b)** $(2.9, 2.9)$ **67.** **(a)** $f^{-1}(x) = -1 + \log_2(x - 1)$, **(b)** None **69.** 0.1, 4.5
71. -1.8, 1.1 **75.** **(a)** \$1318.98 **(b)** 36.65 years **77.** 77 percent; 18 percent **79.** Nine years
81. 20; 70; 100; 120; 140

CHAPTER 5

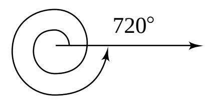
EXERCISES 5.1 *(page 264)*

Check Your Understanding **1.** F **3.** T **5.** F **7.** $\frac{7\pi}{6}$ **9.** $\frac{36}{\pi}$

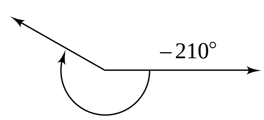
Develop Mastery **1.** **(a)**



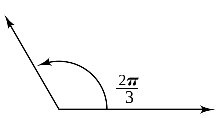
(b)



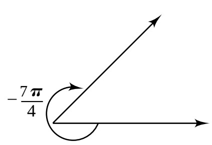
(c)



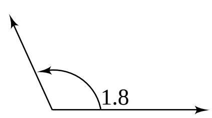
3. **(a)**



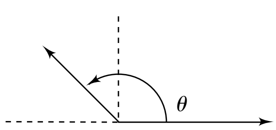
(b)



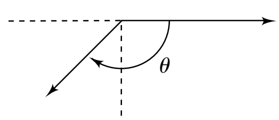
(c)



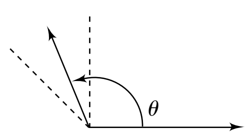
5. **(a)**



(b)



(c)



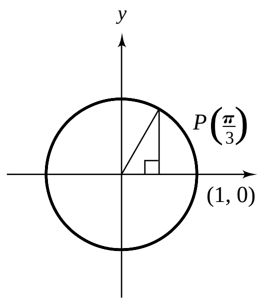
7. **(a)** 23.633°, **(b)** 143.273°, **(c)** −95.517° **9.** **(a)** $\frac{\pi}{3}$, 1.05, **(b)** $\frac{11\pi}{6}$; 5.76, **(c)** $\frac{\pi}{8}$; 0.39, **(d)** $\frac{7\pi}{12}$; 1.83 **11.** **(a)** 120°, **(b)** 75°, **(c)** 720°, **(d)** 206.3° **13.** $\gamma < \alpha < \beta$ **15.** $\gamma < \beta < \alpha$ **17.** 49° **19.** $\frac{\pi}{3}$ **21.** **(a)** 4π, **(b)** 48π **23.** **(a)** 733, **(b)** 60,100 **25.** **(a)** 150, **(b)** 2,800 **27.** **(a)** 2.33, **(b)** 133.5° **29.** **(a)** 1.26, **(b)** 117 million mi. **31.** **(a)** 105°, **(b)** 172.5° **33.** **(a)** 9.42 in., **(b)** 311 in. **35.** **(a)** 0.009 rad./min., **(b)** 0.105 rad./min. **37.** **(a)** 23 mi./hr, **(b)** 80; 92, **(c)** 0.023 rad **39.** 1357 cubic in. **41.** 0.40 radians (23°), 2.32 radians (133°) **43.** **(a)** $144\sqrt{3}$ cm.^{2}, **(b)** 96π cm.^{2}, **(c)** 192π cm.^{2} **45.** 380 m./min. **47.** 66700 mi./hr. **49.** $390\,\dfrac{\text{rev.}}{\text{min.}}$

51. **(a)** $296\,\dfrac{\text{rev.}}{\text{min.}}$, **(b)** $1900\,\dfrac{\text{km.}}{\text{hr.}}$

EXERCISES 5.2 *(page 274)*

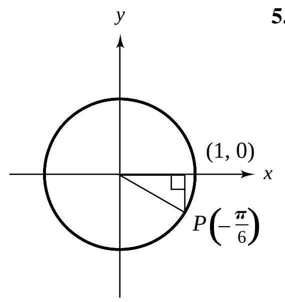
Check Your Understanding **1.** F **3.** T **5.** F **7.** QII or QIII **9.** $<$

Develop Mastery **1.** **(a)**



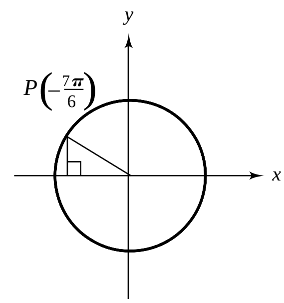
(b) All are positive.

3. **(a)**



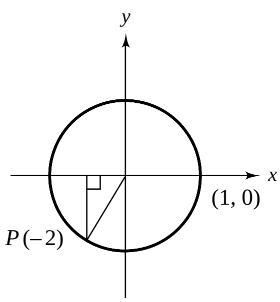
(b) $\cos(-\frac{\pi}{6})$ is positive, the other two are negative.

5. **(a)**



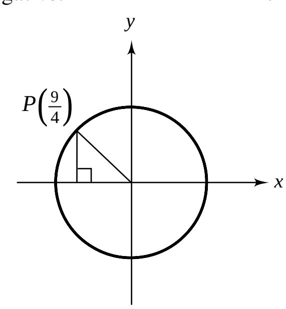
(b) $\sin(-\frac{7\pi}{6})$ is positive; the other two are negative.

7. **(a)**



(b) $\tan(-2)$ is positive; the other two are negative.

9. **(a)**



(b) $\sin(\frac{9}{4})$ is positive; the other two are negative.

11. $P(\frac{5\pi}{2})$ is (0, 1); $\cos(\frac{5\pi}{2}) = 0$; $\sin(\frac{5\pi}{2}) = 1$, $\csc(\frac{5\pi}{2}) = 1$, $\cot(\frac{5\pi}{2}) = 0$
13. $P(-3\pi)$ is (−1, 0); $\cos(-3\pi) = -1$, $\sin(-3\pi) = 0$, $\sec(-3\pi) = -1$, $\tan(-3\pi) = 0$
15. $P(-\frac{15\pi}{2})$ is (0, 1); $\cos(-\frac{15\pi}{2}) = 0$, $\sin(-\frac{15\pi}{2}) = 1$, $\csc(-\frac{15\pi}{2}) = 1$, $\cot(-\frac{15\pi}{2}) = 0$

Exercise	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
17.	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2
19.	$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
21.	$-\frac{11\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
23.	$\frac{13\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$

25. $\{t \mid t = k \cdot 2\pi\}$ **27.** $\left\{t \mid t = \frac{\pi}{3} + k \cdot 2\pi \text{ or } t = \frac{5\pi}{3} + k \cdot 2\pi\right\}$ **29.** $\left\{t \mid t = \frac{\pi}{6} + k \cdot \pi\right\}$
31. $\left\{t \mid t = \frac{3\pi}{4} + k \cdot \pi\right\}$ **33.** $\left\{t \mid t = \frac{5\pi}{6} + k \cdot 2\pi\right\}$ **35.** $\left\{t \mid t = \frac{3\pi}{4} + k \cdot 2\pi\right\}$
37. For $t = \frac{2\pi}{3}$: $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}$. For $t = \frac{3\pi}{4}$: $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1$. For $t = \frac{5\pi}{6}$: $\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$ **39.** IV **41.** II or III
43. **(a)** negative, **(b)** negative **45.** **(a)** negative, **(b)** positive **47.** $P(t) = (-\frac{3}{5}, \frac{4}{5})$; $\sin t = \frac{4}{5}$, $\cos t = -\frac{3}{5}$, $\tan t = -\frac{4}{3}$, $\cot t = -\frac{3}{4}$, $\sec t = -\frac{5}{3}$, $\csc t = \frac{5}{4}$ **49.** $P(t) = (\frac{7}{25}, -\frac{24}{25})$; $\sin t = -\frac{24}{25}$, $\cos t = \frac{7}{25}$ **53.** **(a)** $x = \pm\frac{1}{\sqrt{2}}$, **(b)** $\cos t = \pm\frac{1}{\sqrt{2}}$, $\sin t = \mp\frac{1}{\sqrt{2}}$ **55.** **(a)** $y = \pm\frac{1}{\sqrt{5}}$, **(b)** $\cos t = \pm\frac{2}{\sqrt{5}}$, $\sin t = \pm\frac{1}{\sqrt{5}}$
57. $-\frac{\pi}{6}$ **59.** $\frac{7\pi}{6}$ **61.** $-\frac{3\pi}{2}$ **63.** $\frac{5\pi}{4}$ **65.** $\frac{\pi}{4}$ **67.** $\cos t = -\frac{\sqrt{7}}{4}$, $\sin t = -\frac{3}{4}$, **69.** $\cos t = -\frac{3}{5}$, $\sin t = \frac{4}{5}$,
71. 1 **73.** 1 **75.** **(a)** -1, **(b)** $\sqrt{2}$, **(c)** -1; $\sin 2t = 2 \sin t \cos t$ **77.** **(a)** 0, **(b)** 2, **(c)** 0; $\sin 2t = 2 \sin t \cos t$
79. The weight oscillates between 3 and -3 for y and makes a complete oscillation every $\frac{1}{2}$ second.
81. **(d)** 42, 40, 58; 120, 119, 169

EXERCISES 5.3 *(page 285)*

Check Your Understanding **1.** F **3.** T **5.** F **7.** II **9.** $\frac{3}{\sqrt{58}}$

Develop Mastery **1.** $\sin\theta = \frac{4}{5}$, $\cos\theta = -\frac{3}{5}$, $\tan\theta = -\frac{4}{3}$, **3.** $\sin\theta = \frac{12}{13}$, $\cos\theta = \frac{5}{13}$, $\tan\theta = \frac{12}{5}$
5. $\sin\theta = -\frac{24}{25}$, $\cos\theta = -\frac{7}{25}$, $\tan\theta = \frac{24}{7}$ **7.** $\sin\theta = -\frac{1}{\sqrt{2}}$, $\cos\theta = \frac{1}{\sqrt{2}}$, $\tan\theta = -1$
9. $\sin\theta = -\frac{2}{\sqrt{5}}$, $\cos\theta = -\frac{1}{\sqrt{5}}$, $\tan\theta = 2$ **11.** $\sin\theta = \frac{3}{\sqrt{13}}$, $\cos\theta = \frac{2}{\sqrt{13}}$, $\tan\theta = \frac{3}{2}$
13. $\sin\theta = -\frac{2}{3}$, $\cos\theta = \frac{\sqrt{5}}{3}$, $\tan\theta = -\frac{2}{\sqrt{5}}$ **15.** $\sin\theta = \frac{4}{5}$, $\cos\theta = -\frac{3}{5}$, $\tan\theta = -\frac{4}{3}$
17. $\sin\phi \approx -0.45$, $\cos\phi \approx -0.89$, $\tan\phi = 0.50$ **19.** $\sin\phi \approx 0.40$, $\cos\phi \approx -0.92$, $\tan\phi \approx -0.44$
21. $\sin\phi \approx -0.60$, $\cos\phi = 0.80$, $\tan\phi \approx -0.75$ **23.** $\sin\phi \approx -0.87$, $\cos\phi = 0.50$, $\tan\phi \approx -1.73$
25. $\sin\phi \approx -0.98$, $\cos\phi \approx 0.20$, $\tan\phi = -5.00$ **27.** 0.595 **29.** 0.972 **31.** -0.130 **33.** -3.381
35. 1.110 **37.** -1.323 **39.** **(a)** $(-0.15, 0.99)$, **(b)** $\sin t \approx 0.99$, $\cos t \approx -0.15$, $\tan t \approx -6.80$
41. **(a)** $(0.81, -0.59)$, **(b)** $\sin t \approx -0.59$, $\cos t \approx 0.81$, $\tan t \approx -0.73$
43. **(a)** $(-0.54, -0.84)$, **(b)** $\sin t \approx -0.84$, $\cos t \approx -0.54$, $\tan t \approx 1.56$
45. **(a)** $(-0.91, 0.41)$, **(b)** $\sin t \approx 0.41$, $\cos t \approx -0.91$, $\tan t \approx -0.45$ **47.** **(a)** 0.416, **(b)** 7.750
49. **(a)** -0.909, **(b)** 1.342 **51.** $a \approx 31, b \approx 16$ **53.** $b \approx 34, c \approx 38$ **55.** $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$
57. $1 + (\tan\theta)^2 = (\sec\theta)^2$ **59.** $\cos 2\theta \neq 2\cos\theta$, but $\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2 = 2(\cos\theta)^2 - 1$
61. 8 **63.** **(a)** $8\sqrt{3}$, **(b)** $\sqrt{3} - 1$ **65.** Area $\approx$ 7.53 sq. in. **67.** **(a)** $x = 4\cos\left(\frac{2\pi t}{15}\right)$, $y = 4\sin\left(\frac{2\pi t}{15}\right)$,
(b) $Q(x, y)$; $(-2, -3.46)$, $(-2, 3.46)$, $(-2, -3.46)$, $(-2, -3.46)$ **69.** **(a)** 1.04, **(b)** -5.36, **(c)** 0.703
71. $\cos 0.1 \approx 0.9950$, $C(0.1) \approx 0.9950$; $\cos 0.2 \approx 0.9801$, $C(0.2) \approx 0.9801$; $\cos 6.6 \approx 0.8253$, $C(6.6) \approx 0.8254$

EXERCISES 5.4 *(page 297)*

Check Your Understanding **1.** T **3.** T **5.** T **7.** 4 **9.** QIII

Develop Mastery **1.** **(a)** $\cos t$, **(b)** $-\sec t$ **3.** **(a)** $\tan t$, **(b)** $-\csc t$

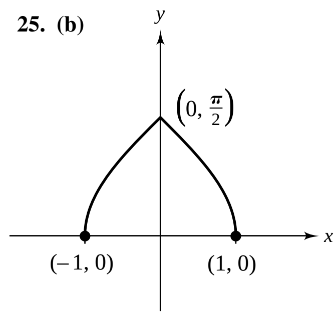
5. For $t = \theta + \frac{\pi}{2}$, $\sin t = -\frac{3}{5}$, $\cos t = -\frac{4}{5}$, $\tan t = \frac{3}{4}$, $\cot = \frac{4}{3}$, $\sec t = -\frac{5}{4}$, $\csc t = -\frac{5}{3}$
7. No **9.** No. **13.** Draw the graph of $y = \sin x$, $-2\pi \leq x \leq 2\pi$. **15.** Draw the graph of $y = \tan x$, $-2\pi \leq x \leq 2\pi$.
17. Starting point (1, 0), counterclockwise.
19. **(a)** Stretch the graph of $y = \sin x$ vertically away from the x-axis by a factor of 2.
(b) Stretch the graph of $y = \cos x$ vertically away from the x-axis by a factor of 2, then reflect about the x-axis.
21. $f(x) = -1$, $D = \{x \mid x \neq (2k - 1)(\frac{\pi}{2})\}$ **23.** $f(x) = 1$, $D = \{x \mid x \neq k(\frac{\pi}{2})\}$
25. **(a)** $-\pi \leq x \leq \pi$, **(b)** f and g are even functions, **(c)** f is periodic, g is not **27.** Sketch graph of $y = 1$ without points $(-\frac{\pi}{2}, 1)$ and $(\frac{\pi}{2}, 1)$. **29.** Sketch graph of $y = -\tan x$ on $[-\pi, \pi]$. **31.** Stretch graph of f vertically away from x-axis by a factor of 2. **33.** Shift graph of f to the right by $\frac{\pi}{2}$ units. **35.** $(-2.6, 0)$, $(-0.5, 0)$, $(3.7, 0)$, $(5.8, 0)$ **37.** 0.8
39. **(a)** Three, **(b)** Three, **(c)** Seven **41.** **(a)** 2.7, **(b)** 2.0 **43.** **(a)** 2.2, **(b)** 1.9
45. **(a)** *(i)* $p = 4\pi$ *(ii)* $D = \mathbb{R}$, $R = [-1, 5]$, **(b)** *(i)* $p = 2\pi$ *(ii)* $D = \mathbb{R}$, $R = [0.37, 2.72]$
47. **(a)** No, **(b)** $f(x) = 1$ for every x, **(c)** $f(x) = 1$ for every x in the domain of f. **49.** For every x where $x \neq (2k - 1)(\frac{\pi}{2})$.
51. **(a)** 0.80, **(b)** 1.04 **53.** 8.42 **55.** $p = \pi$ **57.** **(a)** $(0, 0)$, $(\frac{\pi}{2}, \frac{\pi}{2})$, $(-\frac{3\pi}{2}, -\frac{3\pi}{2})$ **59.** **(a)** $\{0.69\}$,

(b) $\{x \mid 0 < x < 0.69\}$ **61.** **(a)** No, **(b)** $g(x) = -\cos(x + \frac{\pi}{2}) = \cos(x - \frac{\pi}{2})$ **63.** **(a)** $V = 1125\,\pi(\cos\theta)^2 \sin\theta$, **(b)** 0.6 rad, **(c)** 1360 **65.** **(b)** $\theta = \frac{\pi}{4} \approx 0.79$, **(c)** 144 **67.** Two solutions: 0.73 rad, 1.16 rad **69.** $V = 2\pi r^2\sqrt{36 - r^2}$ **71.** x in $[-1.20, 1.20]$

EXERCISES 5.5 *(page 309)*

Check Your Understanding **1.** F **3.** F **5.** T **7.** -2 **9.** π

Develop Mastery **1.** **(a)** $\frac{\pi}{6}$, **(b)** $\frac{5\pi}{6}$ **3.** **(a)** 0, **(b)** $-\frac{\pi}{4}$ **5.** **(a)** 1.160, **(b)** 0.253 **7.** **(a)** 0.779, **(b)** -3 **9.** **(a)** 0.3, **(b)** $-\frac{\pi}{2}$ **11.** **(a)** $\frac{2}{3}$, **(b)** $\frac{1}{3}$ **13.** **(a)** 1.471, **(b)** 0.860 **15.** **(a)** -0.35, **(b)** -0.74 **17.** **(a)** Reflect graph of $y = \text{Cos}^{-1}x$ about the x-axis. **(b)** Shift graph of $y = \text{Cos}^{-1}x$ left one unit. **19.** See graphs in Box on p. 306 **21.** **(a)** 0.80, **(b)** 0.95 **23.** **(b)** $x = \frac{1}{\sqrt{2}}$ **25.** **(b)**



$y = \text{Cos}^{-1}|x|$

27. **(a)** $[-4, 4]$, **(c)** $R = \left\{\frac{\pi}{2}\right\}$

29. **(a)** $D = [-1, 1], R = [0, 1]$, **(c)** Since $0 \le \text{Cos}^{-1}x \le \pi$, $\sin(\text{Cos}^{-1}x) \ge 0$. **31.** **(a)** 2.27, 4.02 **33.** **(a)** 0.321, **(b)** $[0, \pi]$ **35.** Best view: $x \approx 6.8$ feet, $\theta \approx 0.25$ (about 14°). Janet is 1.6 feet from basketball player. **37.** **(a)** $y = \text{Tan}^{-1}\frac{6}{x} - \text{Tan}^{-1}\frac{2}{x}$, **(b)** $x \approx 3.5$, **(c)** $x = \sqrt{12}$ **39.** **(a)** $D = \mathbb{R}, R = [0, \frac{\pi}{2})$ **41.** **(a)** 0.41, 2.73, **(b)** 0.41 **43.** **(a)** $[0, \pi]$, **(b)** $[-\frac{\pi}{2}, \frac{\pi}{2}]$

47. $f(x) = \dfrac{x}{\sqrt{1 + x^2}}$ **53.** 2.04 **55.** **(a)** Yes, **(b)** Yes **57.** **(a)** Even, **(b)** Neither, **(c)** Even, **(d)** Odd

59. $f(x) = \begin{cases} 2x \text{ if } -\frac{\pi}{2} \le x < 0 \\ 0 \text{ if } 0 \le x \le \frac{\pi}{2} \end{cases}$, corners: $(0, 0)$, $(\frac{\pi}{2}, 0)$, $(\pm\pi, -\pi)$, $(-\frac{\pi}{2}, -\pi)$ **61.** $f(x) = \begin{cases} -x - \pi & \text{if } -\pi \le x \le -\frac{\pi}{2} \\ x & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -x + \pi & \text{if } \frac{\pi}{2} \le x \le \pi \end{cases}$

CHAPTER 5 REVIEW *(page 312)*

Test Your Understanding **1.** **(a)** T, **(b)** F, **(c)** F, **(d)** T **3.** F **5.** **(a)** F, **(b)** T, **(c)** T, **(d)** T **7.** F **9.** T **11.** T **13.** F **15.** T **17.** T **19.** **(a)** F, **(b)** F **21.** T **23.** T **25.** T **27.** T **29.** T **31.** T **33.** F **35.** T **37.** T **39.** T **41.** F **43.** F **45.** F

Review for Mastery **1.** $s \approx 13$ cm., $A \approx 150$ cm.^{2} **3.** $r \approx 9.8$, $\theta \approx 1.3$ **5.** **(a)** $P(t) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, **(b)** $\sin t = \frac{\sqrt{2}}{2}$, $\cos t = -\frac{\sqrt{2}}{2}$, $\tan t = -1$ **7.** **(a)** $P(t) = (-\frac{\sqrt{3}}{2}, \frac{1}{2})$, **(b)** $\sin t = \frac{1}{2}$, $\cos t = -\frac{\sqrt{3}}{2}$, $\tan t = -\frac{\sqrt{3}}{3}$ **9.** **(a)** $P(t) = (0.81, -0.59)$, **(b)** $\sin t = -0.59$, $\cos t = 0.81$, $\tan t = -0.73$ **11.** $(2k + 1)\pi$ where k is any integer. **13.** **(a)** Two points, **(b)** $(\frac{1}{4}, \pm\frac{\sqrt{15}}{4})$ **15.** $-\cos t$ **17.** $-\sec t$ **19.** **(a)** $\frac{5\pi}{4}$, **(b)** $\frac{2\pi}{3}$, **(c)** π **21.** **(a)** 0.682, **(b)** -0.532, **(c)** 0.541 **23.** **(a)** $-\frac{3}{5}$, **(b)** $\frac{3}{5}$ **25.** **(b)** $(-\frac{5}{13}, \frac{12}{13})$, **(c)** $\frac{12}{13}$; $-\frac{12}{5}$ **27.** $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$; $\theta \approx 0.93$ **29.** $\sin\theta = -\frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = \frac{3}{4}$; $\theta \approx 3.79$ **31.** **(a)** $\frac{\pi}{4}$, **(b)** $\frac{5\pi}{6}$ **33.** **(a)** $\frac{1}{\sqrt{5}}$, **(b)** $\frac{\pi}{4}$ **35.** **(a)** $-\frac{\pi}{6}$, **(b)** $-\sqrt{5}$ **37.** **(a)** $-\frac{5}{7}$, **(b)** $-\frac{12}{13}$ **39.** **(a)** 0.49, **(b)** 1.82 **41.** $\frac{\sqrt{3}}{2}$ **43.** None **45.** See graph in Figure 45. **47.** Translate graph of $y = \cos x$ up 1 unit. **49.** Draw graph of $y = -\cot x$ by reflecting the graph in Figure 49b about the x-axis. **51.** Graph is the line segment with endpoints $(-1, -1)$ and $(1, 1)$. $D = \{x \mid -1 \le x \le 1\}$, $R = \{y \mid -1 \le y \le 1\}$

53. Translate graph of $y = \text{Cos}^{-1}x$ in Figure 62b down $\frac{\pi}{2}$ units. $D = \{x \mid -1 \le x \le 1\}$, $R = \{y \mid -\frac{\pi}{2} \le y \le \frac{\pi}{2}\}$
55. **(a)** Locate point $(-\frac{1}{2}, \frac{2\pi}{3})$ on the graph. **(b)** $\{x \mid -1 \le x \le -\frac{1}{2}\}$ **57.** $-2\pi \le x \le -\pi$ *or* $0 \le x \le \pi$
59. **(a)** $D = \mathbb{R}, R = [\frac{1}{3}, 3]$, **(b)** Yes **61.** Three; QI and QIII **63.** 0.77 **65.** 671 cubic in.
67. **(a)** $A = 18x - 36\sin\frac{x}{2}\cos\frac{x}{2}$, **(b)** 2.55

CHAPTER 6

EXERCISES 6.1 *(page 326)*

Check Your Understanding 1. T **3.** T **5.** F **7.** $\{x \mid x \ne k\frac{\pi}{2}\}$ **9.** $-\tan x$

Develop Mastery **1.** **(a)** Identity; $\mathbb{R}$, **(b)** Not an identity, **(c)** Identity; $\{x \mid x > 0\}$ **21.** **(a)** $\mathbb{R}$, **(b)** $\{x \mid x \ne (2k-1)\frac{\pi}{2}\}$ **23.** **(a)** $\mathbb{R}$, **(b)** $\{x \mid x \text{ is in QI}\}$ **35.** Not an identity **37.** Identity
39. Not an identity **41.** Identity **43.** Identity **45.** Identity **47.** Identity **49.** $f(x) = \sin x$
51. $f(x) = \sec x$ **53.** $f(x) = \sin x\cos x$ **55.** $f(x) = \sin x$ **57.** $f(x) = \sec x$ **59.** $[0, \pi]$
61. $\{x \mid x$ is in QI or QIV or x is coterminal with 0 or $\pm\frac{\pi}{2}\}$ **63.** $\{x \mid x \text{ is in QI}\}$ **65.** **(a)** $[-4, 4]$, **(b)** $[-100, 100]$
67. **(a)** $D = [-1, 1], R = [1, \pi - 1]$, **(b)** $D = [-1, 1], R = [-1 - \frac{\pi}{2}, 1 + \frac{\pi}{2}]$ **69.** No, try any $x > 10000$.
71. $(f \circ g)(x) = |\csc x|$ **73.** True **75.** **(a)** $(x-1)^2 + (y+2)^2 = 1$ **77.** Graphs coincide for $0 \le x \le 3$. **79.** $S = 0$

EXERCISES 6.2 *(page 339)*

Check Your Understanding **1.** F **3.** T **5.** T **7.** $-\frac{1}{2}$ **9.** $-\sqrt{3}$

Develop Mastery **3.** Functions of the type $f(x) = kx$ are additive. **7.** $\dfrac{\sqrt{6}-\sqrt{2}}{4}$, 0.2588 **9.** $\dfrac{\sqrt{2}-\sqrt{6}}{4}$, -0.2588
11. **(a)** $\dfrac{\sqrt{2}+\sqrt{6}}{4}$, **(b)** $-2-\sqrt{3}$ **13.** $-\frac{33}{65}$ **15.** $\frac{63}{16}$ **17.** $\frac{7}{25}$ **19.** $-\dfrac{3\sqrt{7}}{8}$ **21.** $\dfrac{3\sqrt{7}-4\sqrt{3}}{5}$
23. $\dfrac{3\sqrt{2}-\sqrt{14}}{8}$ **25.** **(a)** $\frac{\sqrt{3}}{2}$, **(b)** 0 **27.** **(a)** $\frac{1}{2}$, **(b)** 2 **37.** **(a)** $D = \mathbb{R}$, **(b)** Identity **39.** **(a)** $D = \mathbb{R}$, **(b)** Not an identity **41.** **(a)** $D = [0, \infty)$, **(b)** Not an identity **43.** $f(x) = -x$ for $-1 \le x \le 1$ **45.** $f(x) = -\sqrt{1-x^2}$ **49.** $\frac{1}{7}$
51. Shift the graph of g to the right $\frac{\pi}{6}$ units to get the graph of f. **53.** $\pm\frac{\sqrt{3}}{2}$ **55.** **(a)** 29.7°, **(b)** 26.6°, 56.3°, **(c)** (2, 3)
57. $\frac{6}{7}$ **59.** 1 **63.** **(c)** $\sin 10° \approx 0.1736481777$, $\sin 50° \approx 0.7660444431$, $\sin 250° \approx -0.9396926208$
69. **(a)** $y = \text{Tan}^{-1}\frac{12.5}{x} - \text{Tan}^{-1}\frac{9}{x}$, **(b)** $x \approx 10.6$ feet **71.** $x = 4.90, y = 0.47$ **73.** **(a)** $D = \{x \mid x \text{ is in QI or QIII}\}$,
(b) $D = \{x \mid x \text{ is in QI}\}$ **75.** **(a)** odd, **(b)** $D = \{x \mid x \ne 0\}, R = [-1.8, 1.8]$

EXERCISES 6.3 *(page 347)*

Check Your Understanding **1.** F **3.** T **5.** T **7.** < **9.** >

Develop Mastery **1.** **(a)** $\dfrac{\sqrt{2-\sqrt{3}}}{2}$, 0.25882, **(b)** $-\dfrac{\sqrt{2-\sqrt{2}}}{2}$, -0.38268 **3.** **(a)** $-\dfrac{\sqrt{2+\sqrt{2}}}{2}$, -0.92388,
(b) $\frac{\sqrt{2-\sqrt{2}}}{2}$, 0.38268 **5.** **(a)** $\frac{5}{\sqrt{26}}$, **(b)** $\frac{1}{\sqrt{26}}$, **(c)** 5 **7.** **(a)** $-\frac{2}{\sqrt{5}}$, **(b)** $-\frac{1}{\sqrt{5}}$, **(c)** 2
9. **(a)** $-, +, \pm$, **(b)** $-, \pm, \pm$ **11.** **(a)** $-, +, +$, **(b)** $-, +, +$ **13.** **(a)** Yes, $\theta = 240°$, **(b)** Yes, $\theta = -120°$
15. **(a)** $-\dfrac{\sqrt{2+\sqrt{3}}}{2} \approx -0.9659$, **(b)** $-\dfrac{\sqrt{2}+\sqrt{6}}{4} \approx -0.9659$ **25.** Not an identity **27.** Not an identity
29. Identity **31.** Identity **33.** Identity **35.** Not an identity; try $x > 10000$ **37.** **(a)** 0.368, **(b)** 0.183, **(c)** 0.449
39. **(a)** $-\frac{3}{4}$, **(b)** $\frac{1}{\sqrt{10}}$, **(c)** $-\frac{3}{\sqrt{10}}$, **41.** **(a)** $\frac{\sqrt{6}}{2}$, **(b)** $\frac{\sqrt{6}}{2}$ **43.** **(a)** Decreasing, **(b)** $D = [-1, 1], R = [0, \infty)$, **(c)** (0, 1)

45. **(a)** Decreasing, **(b)** $D = [-1, 1], R = [0, 1]$ **(c)** $(0, \frac{\sqrt{2}}{2})$ **47.** (1.5, 0.7) **49.** **(a)** $f(x) = \begin{cases} 1 & \text{if } \sin x \ge 0, \sin x < 1, \\ -1 & \text{if } \sin x < 0 \text{ or } \sin x = 1, \end{cases}$
(b) $g(x)$ is defined the same way as $f(x)$, replacing $\sin x$ by $\cos x$. **53.** 1.33 **57.** **(a)** $\frac{1}{2}(\sin 5x + \sin x)$,
(b) $\frac{1}{2}(\cos 4x + \cos 2x)$ **61.** **(a)** $2\cos 2x\sin x$, **(b)** $-2\sin 3x\sin 2x$ **63.** $\tan(\frac{5x}{2})$
65. **(a)** $\cos x = \dfrac{1-u^2}{1+u^2}$, $\tan x = \dfrac{2u}{1-u^2}$, **(b)** $\dfrac{u(1-u^2)}{1+u^2}$

EXERCISES 6.4 *(page 357)*

Check Your Understanding **1.** F **3.** T **5.** T **7.** $\frac{\pi}{2}$ **9.** $\frac{\pi}{4}$

Develop Mastery **1.** $\frac{\pi}{3}, \frac{5\pi}{3}$ **3.** $0, \frac{2\pi}{3}, 2\pi$ **5.** $0, \pi, 2\pi$ **7.** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **9.** $\frac{\pi}{3}, \frac{5\pi}{3}$ **11.** $\frac{\pi}{6}, \frac{5\pi}{6}$ **13.** $\frac{\pi}{2}, \frac{3\pi}{2}$
15. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ **17.** $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ **19.** $\frac{\pi}{3}, \frac{2\pi}{3}$ **21.** $-\pi, -\frac{\pi}{3}, \pi$ **23.** $-\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$
25. $\{x \mid x = \pm\frac{\pi}{4} + k \cdot 2\pi\}$ **27.** $\{x \mid x = \frac{2\pi}{3} + k \cdot 2\pi \text{ or } x = \frac{4\pi}{3} + k \cdot 2\pi\}$

29. $\{x \mid x = \frac{\pi}{2} + k \cdot 2\pi\}$ **31.** $\frac{\pi}{2}, \pi$ **33.** $\frac{\pi}{6}$ **35.** $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ **37.** $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$ **39.** ± 0.8
41. $-0.4, 2.1$ **43.** $-1.9, 1.2$ **45.** ± 0.9 **47.** $0.5, -2.7$ **49.** $0.7, 2.4$ **51.** $0.8, 2.3$ **53.** 0.7
55. **(a)** max $P(1.6, 2.7)$, **(b)** $Q(\frac{\pi}{2}, e)$, P and Q are the same point **57.** **(a)** max $P(2.0, 1.8)$, **(b)** $Q(2.0, 1.8)$, P and Q are the same point **59.** $\frac{3}{4}, \frac{7}{4}$ **61.** $0, \frac{1}{3}, 1, \frac{5}{3}, 2$ **63.** $\frac{1}{4}$ **65.** **(a)** $f(x) = \sqrt{2}\sin(x - \frac{\pi}{4})$, **(b)** $\sqrt{2}$ **(c)** $\frac{3\pi}{4}$
67. **(a)** $f(x) = 2\sin(x - \frac{\pi}{6})$, **(b)** 2, **(c)** $\frac{2\pi}{3}$ **69.** $\frac{33\pi}{2} \approx 51.84$ **71.** $\frac{199\pi}{4}$ **73.** **(a)** 3, **(b)** 0 **75.** **(a)** 13, **(b)** -4
77. **(a)** $V = 243\pi \sin^2 x \cos x$, $V_{max} \approx 294$ in.^{3} **(b)** 0.49(28°) or 1.37(78.5°) **79.** **(a)** $A = (2.75)^2 \csc\theta$,
(b) $D = [\text{Tan}^{-1}\frac{11}{30}, \frac{\pi}{2}]$

EXERCISES 6.5 *(page 371)*

Check Your Understanding **1.** T **3.** F **5.** T **7.** f **9.** e

Develop Mastery **1.** $[0, \frac{2\pi}{3}]$ **3.** $[-\frac{3}{4}, \frac{5}{4}]$ **5.** **(a)** $y = -\cos 2x$, **(b)** $[0, \pi]$, $A = 1$, $p = \pi$ **7.** **(a)** $y = -2\sin x$,
(b) $[0, 2\pi]$, $A = 2$, $p = 2\pi$ **9.** (1): $y = \sin\dfrac{\pi x}{2}$; (2): $FI = [0, 4]$; (3): (0, 0), (2, 0), (4, 0); (4): (1, 1), (3, −1). No phase shift
11. (1): $y = -3\sin(2x - \frac{\pi}{4})$; (2): $FI = [\frac{\pi}{8}, \frac{9\pi}{8}]$; (3): $(\frac{\pi}{8}, 0), (\frac{5\pi}{8}, 0), (\frac{9\pi}{8}, 0)$; (4): $(\frac{3\pi}{8}, -3), (\frac{7\pi}{8}, 3)$. Phase shift $\frac{\pi}{8}$ to the right
13. **(a)** $A = 2$, $y = 2$; phase shift: shift graph of $y = 2\sin \pi x$ left 3 units, not unique.
(b) $g(x) = 2\sin \pi(x + 1)$, not unique. **15.** **(a)** $A = 3$, $p = \pi$; phase shift; shift graph of $y = 3\cos 2x$ to the left 1.5 units.
(b) $g(x) = 3\cos(2x + 3)$, not unique. **17.** **(a)** $A = 2$, $p = 120°$; phase shift: shift graph of $y = 2\sin 3x$ to the right 20°.
(b) $g(x) = 2\sin(3x + 300°)$, not unique. **19.** **(a)** $A = 2$, $p = 120°$; phase shift: shift graph of $y = -2\cos 3x$ to the right 16°.
(b) $g(x) = -2\cos(3x + 312°)$, not unique. **21.** $y = \sqrt{2}\sin(x + \frac{\pi}{4})$, $FI = [-\frac{\pi}{4}, \frac{7\pi}{4}]$
23. $y = 2\sin(x + \frac{\pi}{3})$, $FI = [-\frac{\pi}{3}, \frac{5\pi}{3}]$ **25.** $0.95, -0.62$ **27.** $0.75, -0.25$ **29.** First draw a graph of $y = 2\sin 2x$ with $FI = [0, \pi]$, $A = 2$, then translate upward 1 unit. **31.** Draw graph of $y = \sin \pi x$ with $FI = [0, 2]$, then shift down 2 units.
33. $y = \sin 2x$ **35.** $y = 2 + \sin 2x$. **37.** $y = \cos 4x$.
39. $y = \sin x$, with missing points $(\frac{\pi}{2}, 1), (-\frac{\pi}{2}, -1), (\frac{3\pi}{2}, -1), \ldots$ **41.** $y = \cos x$, where the domain is $\{x \mid \cos x > 0\}$.
43. **(b)** Loc. max$\left(\dfrac{1}{4}, \dfrac{1}{\sqrt[4]{2}}\right), \left(-\dfrac{3}{4}, \dfrac{1}{\sqrt[4]{8}}\right)$ Loc. min.$\left(-\dfrac{1}{4}, -\dfrac{1}{\sqrt[4]{2}}\right), \left(\dfrac{3}{4}, -\dfrac{1}{\sqrt[4]{8}}\right)$, **(c)** The graphs of f and g meet at the local maximum points of f. The graphs of f and h meet at the local minimum points of f.
45. **(a)** Yes, $p = \frac{2\pi}{3}$, **(b)** Yes **47.** **(a)** Yes, $p = 2\pi$, **(b)** No **49.** **(a)** $f(x) = 2\sin(x + \frac{\pi}{6})$, not unique, **(b)** $A = 2$, $p = 2\pi$; shift the graph of $y = 2\sin x$ to the left $\frac{\pi}{6}$ units; not unique. **51.** $f(x) = \sin(2x + 2)$
57. $f(x) = \sin(2x - \frac{2\pi}{3})$ **59.** $f(x) = 1 + \sin(2\pi(x - 0.5))$ **61.** $f(x) = -2\sin(x + \frac{\pi}{4})$
63. $A = 4$, $p = \frac{1}{3}$, $f = 3$, $f(0) = 0$ **65.** $A = \frac{2\sqrt{26}}{5}$, $p = 3$, $f = \frac{1}{3}$, $E(0) = 2$
67. Graph of f meets the envelope curves at $(\frac{\pi}{8}, \sqrt{\frac{\pi}{8}}), (\frac{5\pi}{8}, \sqrt{\frac{5\pi}{8}}), (\frac{3\pi}{8}, -\sqrt{\frac{3\pi}{8}}), (\frac{7\pi}{8}, -\sqrt{\frac{7\pi}{8}})$
69. Graph of f meets the envelope curves at $(0.25, \sqrt{0.5}), (1.25, \sqrt{2.5}), (2.25, \sqrt{4.5}), (0.75, -\sqrt{1.5}), (1.75, -\sqrt{3.5}), (2.75, -\sqrt{5.5})$
71. By (I-22), $f(x) = (2\sin x)\cos 5x$. The graph of f will touch the graph of $y = 2\sin x$ whenever $\cos 5x = 1$, and will touch the graph of $y = -2\sin x$ whenever $\cos 5x = -1$. **73.** **(a)** $I(t) = 4.0 + 0.4\sin\left(\dfrac{2\pi t}{10.8}\right)$, **(b)** $I(0) = 4.0$, $I(4) \approx 4.3$

CHAPTER 6 REVIEW *(page 374)*

Test Your Understanding **1.** T **3.** T **5.** T **7.** T **9.** F **11.** T **13.** F **15.** F **17.** T **19.** F
21. T **23.** T **25.** T **27.** T **29.** T **31.** F **33.** F **35.** F **37.** 0 **39.** 0.8 **41.** $[-1, 1]$
43. QI, QIII, QIV **45.** 7

Review for Mastery **13.** Identity **15.** Not an identity **17.** Identity **19.** $f(x) = \cos x$, $x \neq (2k - 1)(\frac{\pi}{2})$
21. $f(x) = 1$ for every real number x. **23.** $-\frac{4}{3}$ **25.** $-\frac{1}{\sqrt{26}}$ **27.** $\frac{-3}{5}$ **29.** -7 **31.** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **33.** $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$
35. $\frac{3\pi}{4}$ **37.** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **39.** $-1.11, 2.03$ **41.** $-0.97, -2.18$ **43.** 0 **45.** $\{x \mid x$ is in QI or QII or x is coterminal with $0, \frac{\pi}{2}$ or $\pi\}$. **47.** $\{x \mid x$ is in QI or QIV or x is coterminal with 0 or $\pm\frac{\pi}{2}\}$ **49.** $\{x \mid x$ is in QI$\}$ **51.** Graph is a sine curve with $FI = [\frac{\pi}{4}, \frac{9\pi}{4}]$, amplitude 2, period 2π, and phase shift $\frac{\pi}{4}$ to the right **53.** Draw a graph of $y = 2\sin(x - \frac{\pi}{4})$.
55. Draw a graph of $y = 2\cos 2x$, reflect about the x-axis and then shift left $\frac{\pi}{6}$ units. **57.** Shift the graph of g right $\frac{\pi}{6}$ units.
59. Shift the graph of g right $\frac{\pi}{4}$ units. **61.** 1.89 **63.** **(a)** Yes, $p = 2$, **(b)** Yes **65.** **(a)** Yes, $p = \pi$, **(b)** No
67. ± 1.45 **69.** $0.85, 1.32$ **71.** 4.36 **73.** **(a)** 72.0°, **(b)** $(-5, -1)$ **75.** $f^{-1}(x) = -\cos x$, $D = [0, \pi]$, $R = [-1, 1]$
77. $f(x) = \sqrt{2}\sin(x + \frac{\pi}{4})$

CHAPTER 7

EXERCISES 7.1 *(page 385)*

Check Your Understanding **1.** T **3.** F **5.** F **7.** $a(1 + \cot\alpha + \csc\alpha)$ **9.** $30°$

Develop Mastery **1.** $\beta = 54°$, $b = 5.1$, $c = 6.3$ **3.** $\beta = 63°$, $a = 18$, $c = 39$ **5.** $\alpha = 24°40'$, $a = 9.89$, $b = 21.5$ **7.** $c = 92$, $\alpha = 53°$, $\beta = 37°$ **9.** $b = 29.9$, $\alpha = 35.6°$, $\beta = 54.4°$ **11.** 0.078 **13.** 0.588 **15.** 1.45 **17.** $63°$ **19.** 5.3 **21.** 16.66 cm. and 8.397 cm. **23.** 3.55 cm., $42.0°$ **25.** 15 ft. **27.** 17 **29.** **(a)** 8.3, **(b)** 24 **31.** $P = 47.5$ cm., $A = 109$ cm.^{2} **33.** **(a)** 750 cm.^{2}, **(b)** 1,200 cm.^{2}, **(c)** 1,500 cm.^{2}, **(d)** 1,700 cm.^{2} **35.** 51 ft. **39.** 6,280 cm.^{3} **41.** $47.16°$ **43.** **(c)** $\sin 15° = \dfrac{1}{2\sqrt{2+\sqrt{3}}}$, $\cos 15° = \dfrac{\sqrt{2+\sqrt{3}}}{2}$ **45.** $d = \sqrt{1.5h + \left(\frac{h}{5{,}280}\right)^2}$, h in feet, d in miles. **47.** 1,070 mi. **49.** $K = 64\sqrt{2} - 20\pi \approx 27.68$ **51.** $\frac{2}{3}$ **53.** $16\sqrt{2-\sqrt{3}}$ **55.** $K = \left(\frac{3-\sqrt{3}}{4}\right)s^2$ **57.** **(a)** $0 < h \le \frac{c}{2}$, **(b)** The pair $\{a, b\}$ is uniquely determined.

EXERCISES 7.2 *(page 397)*

Check Your Understanding **1.** T **3.** T **5.** T **7.** $\frac{a \sin\gamma}{\sin\alpha}$ **9.** $6\sqrt{3}$

Develop Mastery **1.** $\gamma = 81.0°$, $b = 35.6$, $c = 36.4$ **3.** $\gamma = 21.0°$, $a = 158$, $b = 208$ **5.** $\alpha = 92°\,15'$, $a = 14.2$, $c = 11.9$ **7.** $\gamma = 90.0°$, $a = 28.0$, $c = 53.0$ **9.** Two solutions: $\beta_1 = 82°$, $\gamma_1 = 42°$, $c_1 = 46$; $\beta_2 = 98°$, $\gamma_2 = 26°$, $c_2 = 30$ **11.** $\alpha = 32°$, $\gamma = 24°$, $a = 2.1$ **13.** 364 **15.** 630 **17.** 298 ft. **19.** 374 yds. **21.** 437 ft. **23.** $|\overline{BC}| = 5.5$, $|\overline{CD}| = 2.8$ **25.** $41.4°$ **27.** **(a)** $c = 4\sqrt{2} \pm \sqrt{a^2 - 32}$, **(b)** No solution when $a < 4\sqrt{2}$; one solution when $a = 4\sqrt{2}$ or $a \ge 8$; two solutions when $4\sqrt{2} < a < 8$. **29.** **(a)** $V = \frac{25\pi}{3}\sin^2 x(5\cos x + \sqrt{64 - 25\sin^2 x})$, **(b)** $V_{\max} \approx 189$ when $x \approx 1.22$ (about $70°$) **31.** **(a)** $K = (8\sin x)(\cos x + \sqrt{4 - \sin^2 x})$, **(b)** $K_{\max} \approx 16$ when $x \approx 1.11$ (about $63.6°$) **33.** **(a)** 240, **(b)** 50 **35.** $5.26°$ **37.** 30.2 **39.** **(a)** 63 million mi., 120 million mi.; **(b)** $23°$, one solution **41.** $\theta = \mathrm{Sin}^{-1}\left(\frac{5}{14}\right) \approx 20.9°$ **43.** Yes

EXERCISES 7.3 *(page 405)*

Check Your Understanding **1.** T **3.** T **5.** F **7.** $5\sqrt{3}$ **9.** $90°$

Develop Mastery **1.** $c = 52$, $\alpha = 30°$, $\beta = 102°$ **3.** $b = 114$, $\alpha = 39.5°$, $\gamma = 25.1°$ **5.** $\alpha = 51.3°$, $\beta = 38.6°$, $\gamma = 90.0°$ **7.** $a = 6.30$, $\beta = 72.4°$, $\gamma = 53.9°$ **9.** $\alpha = 90°$, $\beta = 58°$, $\gamma = 32°$ **11.** $\alpha = 47°$, $\beta = 40°$, $\gamma = 93°$ **13.** $c = 53$, $\alpha = 58°$, $\beta = 32°$ **15.** **(a)** 4.4, **(b)** 16 **17.** **(a)** 6.95, **(b)** 18.9 **19.** **(a)** $36.2°$, **(b)** $26.6°$, **(c)** 825 **21.** **(a)** $34.1°$, **(b)** 2.44, **(c)** 4.16 **23.** $\beta = 86°$ **25.** 18 **27.** **(a)** 28, **(b)** $98°$ **29.** **(a)** $(-1, 5)$, **(b)** $13°$, $30°$ **31.** 2.0 **33.** **(a)** $y = \sqrt{369 - 360\cos x}$, **(b)** 16, **(c)** $51.3°$, **(d)** $K_{\max} = 90$ **35.** $c \approx 22.3$, $\gamma \approx 105.3°$, $a = 15.4$, $\alpha = 41.9°$ **37.** 58.0 (for $\lambda > 90°$) **41.** **(a)** $y \approx \dfrac{7.49}{\sin x}$, $48.5° < x < 141.4°$, **(b)** $90°$, **(c)** 7.9, **(d)** Two solutions: $52°$ and $128°$ **47.** **(a)** 15, **(b)** 90 **49.** **(a)** $d = \begin{cases} 4t & \text{if } 0 \le t \le 1 \\ \sqrt{9t^2 - 30t + 37} & \text{if } t > 1, \end{cases}$ **(b)** 7.8 mi., **(c)** 1:30 P.M. **51.** $30°$ **53.** **(a)** 0.769, **(b)** 221 **55.** $78°$ **57.** $2\sqrt{43}$ **59.** 58 m. **61.** $20 + 16\sqrt{2}$ **63.** **(a)** Yes, **(b)** 180 sq. ft.

EXERCISES 7.4 *(page 417)*

Check Your Understanding **1.** F **3.** T **5.** T **7.** QII **9.** Four

Develop Mastery **1.** **(a)** $(-3, 0)$; $3(\cos 180° + i\sin 180°)$, **(b)** $(0, -1)$; $\cos 270° + i\sin 270°$ **3.** **(a)** $(3, 5)$; $\sqrt{34}(\cos\theta + i\sin\theta)$, $\theta = \mathrm{Tan}^{-1}\left(\frac{5}{3}\right)$, **(b)** $(2, -3)$; $\sqrt{13}(\cos\theta + i\sin\theta)$, $\theta = \mathrm{Tan}^{-1}(-1.5)$ **5.** **(a)** $(\sqrt{3}, 2)$; $\sqrt{7}(\cos\theta + i\sin\theta)$, $\theta = \mathrm{Tan}^{-1}\left(\frac{2}{\sqrt{3}}\right)$, **(b)** $(1, 1)$; $\sqrt{2}(\cos 45° + i\sin 45°)$ **7.** **(a)** $(0, -1)$; $\cos 270° + i\sin 270°$, **(b)** $\left(\frac{1}{2}, -\frac{1}{2}\right)$; $\left(\frac{1}{\sqrt{2}}\right)(\cos 315° + i\sin 315°)$ **9.** $\sqrt{2} + \sqrt{2}i$

11. $4i$ **13.** $-\frac{1}{2} - (\frac{\sqrt{3}}{2})i$ **15.** $2(\cos 315° + i \sin 315°)$ **17.** $4(\cos 270° + i \sin 270°)$
19. $\cos(-30°) + i \sin(-30°)$ **21.** $\cos 60° + i \sin 60° = \frac{1}{2} + (\frac{\sqrt{3}}{2})i$ **23.** $\cos 60° + i \sin 60° = \frac{1}{2} + (\frac{\sqrt{3}}{2})i$
25. $2(\cos 150° + i \sin 150°) = -\sqrt{3} + i$ **27.** $2(\cos 270° + i \sin 270°) = -2i$ **29.** $8(\cos 90° + i \sin 90°) = 8i$
31. $\cos 120° + i \sin 120° = -\frac{1}{2} + (\frac{\sqrt{3}}{2})i$ **33.** $\cos 240° + i \sin 240° = -\frac{1}{2} - (\frac{\sqrt{3}}{2})i$
35. $\cos 180° + i \sin 180° = -1$ **37.** $16(\cos 60° + i \sin 60°) = 8 + 8\sqrt{3}\, i$
39. $16(\cos 0° + i \sin 0°) = 16$ **41.** $(\frac{1}{4})(\cos 90° + i \sin 90°) = (\frac{1}{4})i$ **43.** $\frac{1}{64}$ **45.** $\pm 0.87 - 0.5i$, i
47. $\pm(1.44 + 0.41i)$, $\pm(0.41 - 1.44i)$ **49.** $-0.52 + 1.63i$, $-1.15 - 1.26i$, $1.67 - 0.36i$
51. $\pm(1.48 + 0.24i)$, $\pm(0.24 - 1.48i)$ **53.** ± 1, $\pm(0.50 \pm 0.87i)$ **55.** $\pm 1.73 - i$, $2i$
57. 1, $0.31 \pm 0.95i$, $-0.8i \pm 0.59i$ **59.** $\pm(0.38 + 0.92i)$, $\pm(0.92 - 0.38i)$ **61.** $\pm(1.10 \pm 0.46i)$
63. -1, $\pm(0.50 \pm 0.87i)$ **65.** i, $\pm(0.50 \pm 0.87i)$ **67.** $E_n = 2\cos(n \cdot 30°)$, $E_{45} = 0$, $E_{48} = 2$

EXERCISES 7.5 *(page 426)*

Check Your Understanding **1.** F **3.** F **5.** F **7.** $\langle <2\sqrt{2}, -2\sqrt{2}\rangle$ **9.** Two

Develop Mastery **1.** $\langle 4, 4\rangle$ **3.** $\langle -6, 3\rangle$ **5.** $\langle 1, -6\rangle$ **7.** $\langle -10, 13\rangle$ **9.** $\langle -3, 0\rangle$ **11.** $\langle 1, 5\rangle$ **13.** 113°
15. 138° **17.** $(\frac{1}{\sqrt{5}})\langle 1, 2\rangle$, $(\frac{1}{\sqrt{5}})\langle -1, -2\rangle$ **19.** $\sqrt{2}, 2\sqrt{2}$ **21.** $\sqrt{5}, 3\sqrt{5}$ **23.** $\sqrt{2}, k\sqrt{2}$
25. $\sqrt{5}, \sqrt{29}, \sqrt{5a^2 - 18ab + 29b^2}$ **27.** $\langle \frac{8}{\sqrt{5}}, \frac{4}{\sqrt{5}}\rangle$ or $\langle -\frac{8}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\rangle$ **29.** $\langle 8, 2\rangle$ or $\langle -2, 2\rangle$ **31.** Any x, y for which (x, y) is a point on the circle with center $(-3, 1)$ and radius 2 **33.** $(2, 6)$ or $(2, -2)$ **35.** 87.4 m., 16° north of east
37. 8.8 ft., 38° north of east **39.** Downstream at 6 mph, at an angle of 55° from direction of current.
41. **(a)** $\boldsymbol{m}(t) = \langle 0, 2640 - 96t\rangle$, $\boldsymbol{j}(t) = \langle 1760 - 72t, 0\rangle$
(b) $\boldsymbol{m}(20) = \langle 0, 720\rangle$, $\boldsymbol{j}(20) = \langle 320, 0\rangle$
$\boldsymbol{m}(30) = \langle 0, -240\rangle$, $\boldsymbol{j}(30) = \langle -400, 0\rangle$
(c) 176 ft. when $t = 26.4$ secs.
43. **(a)** $|\boldsymbol{v}(x)| = \sqrt{x^2 + 400\sqrt{2}x + 160{,}000}$
(b) $\theta = \text{Tan}^{-1}\left(\dfrac{x}{x + 400\sqrt{2}}\right)$ **(c)** From north east
(d) $x = 40$, 429 mph, 4° east of north
$x = 80$, 460 mph, 7° east of north
$x = 120$, 492 mph, 10° east of north
$x = -50$, 366 mph, 6° west of north
45. 575 mph, 38° north of east
47. **(a)** $T(x) = \dfrac{2400}{\sqrt{x^2 + 500\sqrt{2}x + 125000}}$, **(b)** *(i)* 4.5 hr *(ii)* 4.2 hr *(iii)* 5.2 hr
49. 53° south of east, one hour and 35 minutes.

CHAPTER 7 REVIEW *(page 428)*

Test Your Understanding **1.** F **3.** T **5.** F **7.** T **9.** F **11.** T **13.** F **15.** T **17.** F **19.** T **21.** T **23.** F **25.** T **27.** T **29.** F **31.** T **33.** T **35.** T **37.** T **39.** T **41.** T **43.** F **45.** T **47.** F **49.** F

Review for Mastery **1.** $\alpha = 64°40'$, $a = 33.8$, $b = 16.0$ **3.** 1.96 cm. **5.** 35 cm.^{2} **7.** 8.89 **9.** 29°
11. 2.98 **13.** 200 in.^{2} **15.** 52.2 cm., 641 cm.^{2} **17.** **(a)** 0.99 cm.^{2}, **(b)** 0.099 cm.^{2} **19.** 116 ft.
21. **(a)** $f(x) = 8\cos x + \sqrt{25 - (8\sin x)^2}$, $g(x) = 8\cos x - \sqrt{25 - (8\sin x)^2}$, **(b)** $f(25°) \approx 10.9$, $g(25°) \approx 3.6$, **(c)** $0 < x < \text{Sin}^{-1}(\frac{5}{8})$ **23.** **(a)** $V = (\frac{125\pi}{3})\sin^2 x(\cos x + \sqrt{4 - \sin^2 x})$, **(b)** $x \approx 74.7°$, $V \approx 245.5$
27. **(a)** 150 ft., **(b)** 1,600 sq. ft. **29.** **(a)** 29.5°, **(b)** 13.4, **(c)** 47.5 **31.** **(a)** 70.6°, **(b)** 109.4°
33. 11 **35.** 44 ft. **37.** One **39.** Infinitely many **41.** One **43.** **(a)** $-4 - 4i$, **(b)** $-2{,}035 + 828i$
45. **(a)** 16, **(b)** $12.39 + 8.66i$ **47.** **(a)** $64i$, **(b)** $-i$ **49.** **(a)** $i \pm 1$, **(b)** $\dfrac{2 \pm \sqrt{3}}{2} - \dfrac{i}{2}$
51. $\pm(0.97 - 0.23i)$, $\pm(0.23 + 0.97i)$ **53.** $\pm(1.96 + 0.39i)$, $\pm(0.39 - 1.96i)$ **55.** **(a)** $\langle -1, -1\rangle$, **(b)** $\langle -2, 10\rangle$
57. 162° **59.** **(a)** 445 mph, 51° north of east, **(b)** 2 hrs. and 15 min.
61. $T(x) = \dfrac{1500}{\sqrt{x^2 - 400\sqrt{2}x + 160{,}000}}$, 3.9 hours, 4.3 hours, 4.6 hours

CHAPTER 8

EXERCISES 8.1 *(page 440)*

Check Your Understanding **1.** F **3.** F **5.** T **7.** $\frac{7}{6}$ **9.** 6

Develop Mastery **1.** 4, 7, 10, 13; 25 **3.** $\frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \frac{1}{625}; \frac{1}{390625}$ **5.** 43, 47, 53, 61; 113 **7.** $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}; \frac{255}{256}$
9. 2, 12, 120, 1680; 518918400 **11.** 2, 4, 14, 32; 184 **13.** 3, -1, -5, -9 **15.** 2, 6, 18, 54
17. 1, 2, 2, 4 **19.** 2, 4, 6, 10 **21.** 1, 2, $\frac{3}{2}, \frac{7}{4}$ **23.** 1, $\frac{1}{2}, \frac{1}{12}, \frac{1}{288}$ **25.** 3, 8, 15, 24 **27.** $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$
29. 1, $\frac{3}{2}, \frac{11}{6}, \frac{25}{12}$ **31.** 65 **33.** $-\frac{5}{6}$ **35.** $\frac{49}{16}$ **37.** $a_n = \dfrac{1}{2^n}$ **39.** $a_n = (-1)^{n+1}\dfrac{n}{(n+1)^2}$
41. $a_n = n! + 1$ **43.** $(1, -1), (2, 1), (3, -1), (4, 1), (5, -1)$ **45.** $(1, -2), (2, \frac{3}{2}), (3, -\frac{4}{3}), (4, \frac{5}{4}), (5, -\frac{6}{5})$
47. $\displaystyle\sum_{k=1}^{6} \frac{1}{k+1}$ **49.** $\displaystyle\sum_{k=1}^{4} \frac{1}{k(k+1)}$ **51.** $\displaystyle\sum_{k=1}^{5} \frac{(-1)^{k+1}}{k(k+1)}$ **53.** 15 **55.** 2,450 **57.** $\frac{9!}{4!\,5!}$
59. **(a)** 1, 5, **(b)** 18, 180 **61.** Yes **63.** $a_n = 2n - 1$ **65.** $a_n = 1 - \dfrac{1}{2^n}$ **67.** 1, 2, 5, 29, 866 **69.** 1, 2, 2, 4, 8
71. **(a)** a_n: 3, 5, 9, 15, 23 b_n: 3, 5, 9, 15, 23, **(b)** Yes, **(c)** 3,543 **73.** **(a)** 68, **(b)** 112 terms
75. **(b)** For a_1 equal to 1 or 3, sequence eventually reaches 1, 2 as a loop. For a_1 equal to 5, 7, or 9, the loop is 5,14, 7, 20, 10.
77. $a_n = b_n$ only for $n = 1, 2, 3, 4, 5$. **79.** 771 million years.

EXERCISES 8.2 *(page 448)*

Check Your Understanding **1.** T **3.** T **5.** F **7.** Three **9.** $\sqrt{5}$

Develop Mastery **1.** Diverges **3.** Converges to 2 **5.** Converges to 3 **7.** Converges to $2 + \frac{1}{e}$ **9.** Diverges
11. **(a)** 1.732, 1.126, 1.369, **(b)** 1.302776 **(c)** $\dfrac{-1+\sqrt{13}}{2}$
13. **(a)** 2.646, 2.087, 2.217, **(b)** 2.192582 **(c)** $\dfrac{-1+\sqrt{29}}{2}$
15. **(a)** 1.817, 1.985, 1.999, **(b)** 2 **(c)** 2 is a root of $x^3 - x - 6 = 0$.
17. **(a)** 3, $\frac{10}{3}, \frac{33}{10}$, **(b)** 3.302776, **(c)** $\frac{3+\sqrt{13}}{2}$ **19.** **(a)** 3, $\frac{8}{3}, \frac{21}{8}$, **(b)** 2.618034, **(c)** $\frac{3+\sqrt{5}}{2}$
21. **(a)** 3, $\frac{28}{9}, \frac{2433}{784}$, **(b)** 3.103803, **(c)** Root of $x^3 - 3x^2 - 1 = 0$ **23.** **(a)** 2, $\frac{1}{2}$, -1, 2, $\frac{1}{2}$, -1; No, **(b)** -1, 30
25. **(a)–(d)** If $a_1 = k$, then all terms of $\{a_n\}$ are k. **27.** **(a)** 3, $\frac{8}{3}, \frac{21}{8}, \frac{55}{21}, \frac{144}{55}, \frac{377}{144}$, **(b)** $a_n = \dfrac{f_{2n+2}}{f_{2n}}$
31. For any value of a_1, limit ≈ 1.272020. **33.** **(a)** No, **(b)** $\{a_{2n-1}\}$ converges to -1, $\{a_{2n}\}$ converges to 1.
35. **(a)** No, **(b)** $\{a_{2n}\}$ converges to 0, $\{a_{4n-3}\}$ converges to 1, $\{a_{4n-1}\}$ converges to -1.
41. **(a)** 6, 3, 6, 3, **(b)** 6, 3, **(c)** 90 **43.** **(a)** 4, $\frac{12}{5}$, 4, $\frac{12}{5}$, **(b)** 4, $\frac{12}{5}$, **(c)** 64 **45.** 0.37255950
47. 0.97069872 **49.** 1.74903139 **51.** 58.77010594
53. **(a)** If $a_1 = \frac{1}{2}$, $a_3 = 0$ and a_4 is not defined. **(b)** $\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \ldots$; sequence is $\left\{\dfrac{f_n}{f_{n+1}}\right\}$ for $n \geq 2$. **(c)** If $a_1 = \frac{3}{5}$, $a_5 = 0$ and a_6 is not defined. **(d)** Round off **55.** 2 **57.** $\sqrt{2}$ **59.** 3
61. **(a)** 4.236068, **(b)** 3.732051, **(c)** $\{c_{2n-1}\}$ converges to 3.732051, $\{c_{2n}\}$ converges to 4.236068.

EXERCISES 8.3 *(page 459)*

Check Your Understanding **1.** T **3.** T **5.** F **7.** -8 **9.** 1

Develop Mastery **1.** **(a)** 3, **(b)** 18, 30, **(c)** 165 **3.** **(a)** -5, **(b)** -21, -41, **(c)** -185
5. **(a)** $-\frac{8}{3}$, **(b)** $\frac{20}{3}$, -4, **(c)** 80 **7.** **(a)** $-2\sqrt{2}$, **(b)** $1 - 9\sqrt{2}$, $1 - 17\sqrt{2}$, **(c)** $10 - 80\sqrt{2}$
9. **(a)** ln 2, **(b)** 6 ln 2, 10 ln 2, **(c)** 55 ln 2 **11.** **(a)** $-\frac{1}{3}$, **(b)** $-\frac{1}{243}, -\frac{1}{2.187}$, **(c)** $\frac{61}{81}$
13. **(a)** $\frac{1}{3}$, **(b)** $\frac{2}{27}, \frac{2}{243}$, **(c)** $\frac{242}{9}$ **15.** **(a)** $\sqrt{2}$, **(b)** $4\sqrt{2}, 8\sqrt{2}$, **(c)** $7 + 3\sqrt{2}$ **17.** **(a)** $\frac{1}{2}$, **(b)** $\frac{3}{32}, \frac{3}{128}$, **(c)** $\frac{93}{16}$
19. **(a)** $1 + \sqrt{2}$, **(b)** $17 + 12\sqrt{2}$, $99 + 70\sqrt{2}$, **(c)** $11 + 9\sqrt{2}$ **21.** Neither **23.** Neither
25. Arithmetic, $d = \ln\sqrt{3}$ **27.** Geometric, $r = 0.01$ **29.** $d = -\frac{5}{3}$, $a_1 = \frac{25}{3}$ **31.** $d = 2$, $S_8 = 64$
33. $a_1 = 1$, $S_4 = 16$ **35.** $a_4 = 0$, $a_{16} = 4\pi$, $S_{16} = 24\pi$ **37.** $r = \frac{3}{2}$, $a_6 = \frac{243}{8}$ **39.** $a_1 = \frac{4}{81}$, $S_5 = \frac{211}{324}$
41. $a_5 = \frac{2}{9}$, $S_5 = \frac{242}{9}$ **43.** $a_1 = -\frac{32}{5}$, $S_8 = -\frac{17}{4}$ **45.** 1 **47.** $\pm\sqrt{2}$ **49.** 3 **51.** Arithmetic **53.** Geometric
55. Geometric **57.** **(a)** $r = -\frac{1}{3}$; $S_n = \frac{3}{4}(1 - (-\frac{1}{3})^n)$, **(b)** $\frac{3}{4}$
59. **(a)** $r = \frac{3}{4}$; $S_n = \frac{3}{4}(1 - (\frac{3}{4})^n)$, **(b)** $\frac{3}{4}$ **61.** $\frac{9}{5}$ **63.** **(a)** $S_n = 2.5(1 - (-0.6)^n)$, **(b)** 2.5, **(c)** 16, 25 **65.** **(a)** $\frac{31}{25}$, **(b)** $\frac{41}{33}$
67. **(a)** $\frac{9}{8}$, **(b)** $\frac{1,124}{999}$ **69.** 264π **71.** 10,000 **75.** 60 **77.** $\sqrt{\frac{1+\sqrt{5}}{2}}$ **79.** **(a)** 548 m., **(b)** 850 m.
81. **(a)** 16 ft., **(b)** $a_n = 32n - 16$, arithmetic, **(c)** 2,304, **(d)** 20 terms, sum = 6,400

EXERCISES 8.4 *(page 468)*

Check Your Understanding **1.** T **3.** F **5.** F **7.** T **9.** F

Develop Mastery **1.** **(b)** $f(n)$ is divisible by 4 for every n, and is divisible by 3 and 12 for every even number n. **3.** The units digit of $f(n)$ is 0 for every even n and it is 9 for every odd n. The units digit is never 1. **5.** **(a)** $f(n)$ is prime for all n in the table. **(b)** The units digit of $f(n)$ is 1, 3, or 7. **(c)** $f(n+1) = f(n) + 2n$, **(d)** $f(n)$ is not prime when n is 41 or 82. **7.** **(b)** $f(n) = n(n+1)$ **9.** **(b)** $f(n) = \dfrac{2^n - 1}{2^n}$

11. **(b)** $f(n) = 3^n$, **(c)** $\sum_{k=1}^{n} 3^{k-1} = \dfrac{3^n - 1}{2}$ **13.** **(b)** $f(n) = 2^n$, **(c)** $g(n) = 2^n - 1$

15. $f(n) = \dfrac{n+2}{n+1}$ **17.** **(b)** $f(n) = n(n+1)$ **19.** $f(n) = \dfrac{n(n-1)}{2}$ **21.** Row n has $n + 1$ entries.

25. $f(n) = 2^{n-1}$ **27.** $f(n) = \dbinom{n+1}{2}$ **29.** $f(n) = \dbinom{n+3}{4}$ **31.** $P(n) = 1 + \dbinom{n+1}{2}$

33. $R(4) = 8, R(5) = 16, R(6) = 31, R(7) = 57$ **35.** **(b)** $b_{n+6} = b_n$, **(c)** $2y - x, x + y$

EXERCISES 8.5 *(page 474)*

Check Your Understanding **1.** F **3.** F **5.** T **7.** T **9.** F

Develop Mastery **1.** All are true. **3.** $P(1)$ and $P(2)$ are true; $P(5)$ is false. **5.** $P(1)$ and $P(2)$ are true; $P(5)$ is false. **7.** All are true. **9.** 6 **11.** 3

13. Hyp: $1^3 + 2^3 + \ldots + k^3 = \dfrac{k^2(k+1)^2}{4}$; Concl: $1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = \dfrac{(k+1)^2(k+2)^2}{4}$

15. Hyp: $1 \cdot 2 + 2 \cdot 3 + \ldots + k(k+1) = \dfrac{k(k+1)(k+2)}{3}$;

Concl: $1 \cdot 2 + 2 \cdot 3 + \ldots + k(k+1) + (k+1)(k+2) = \dfrac{(k+1)(k+2)(k+3)}{3}$

17. Hyp: $4^k - 1$ is a multiple of 3; Concl: $4^{k+1} - 1$ is a multiple of 3. **41.** True for every n **43.** True for every n **45.** $P(4)$ is false. **47.** $P(41)$ is false. **49.** True for every n **51.** **(a)** 1, 4, 9, 16, **(b)** $S_n = n^2$ **53.** **(a)–(b)** All Terms are 180. **57.** **(b)** All odd integers greater than 3 and not a multiple of 3; the sequence includes all prime numbers greater than 3.

EXERCISES 8.6 *(page 483)*

Check Your Understanding **1.** F **3.** T **5.** T **7.** 10 **9.** 28

Develop Mastery **1.** **(a)** 84, **(b)** 84 **3.** **(a)** 56, **(b)** 56 **5.** **(a)** 1,330, **(b)** 1,330 **7.** **(a)** 210, **(b)** 210 **9.** **(a)** 4,200, **(b)** 4,200 **11.** **(a)** 720, **(b)** 120 **13.** **(a)** 124, **(b)** 6,700 **15.** **(a)** 735471, **(b)** 443667

17. **(a)** 564300, **(b)** 38 **19.** n **21.** $\dfrac{n(n+1)}{2}$ **23.** $\dfrac{n-k}{k+1}$ **25.** $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$

27. $\dfrac{1}{x^4} - 8\dfrac{y^2}{x^3} + 24\dfrac{y^4}{x^2} - 32\dfrac{y^6}{x} + 16y^8$ **29.** $243x^5 + 405x^2 + \dfrac{270}{x} + \dfrac{90}{x^4} + \dfrac{15}{x^7} + \dfrac{1}{x^{10}}$

31. **(b)** $32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$ **33.** **(b)** $x^{10} + 10x^7 + 40x^4 + 80x + \dfrac{80}{x^2} + \dfrac{32}{x^5}$

35. **(a)** Nine, **(b)** $5{,}670x^8$ **37.** **(a)** Six, **(b)** $10x + 10x\sqrt{x}$ **39.** $x^{20} + 20x^{18} + 190x^{16}$ **41.** $40x^7$ **43.** $-1{,}920x^7y^3$ **45.** 24 **47.** 42,240 **49.** $-3{,}003$ **51.** 15 **53.** 1 **55.** 2 **57.** 8 **59.** 6 **63.** **(a)** $S_8 = 9!$, **(b)** $S_n = (n+1)!$ **65.** **(a)** 10, **(b)** 15, **(c)** 30

CHAPTER 8 REVIEW *(page 485)*

Test Your Understanding **1.** F **3.** T **5.** T **7.** F **9.** F **11.** F **13.** T **15.** T **17.** F **19.** T **21.** T **23.** T **25.** T **27.** T **29.** T **31.** T **33.** T **35.** F **37.** T **39.** F **41.** T **43.** F **45.** T

Review for Mastery **1.** **(a)** $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$, **(b)** $\frac{49}{16}$ **3.** **(a)** 2, 5, 8, 11, **(b)** 26 **5.** 3, 6, 12, 24, 48 **7.** $a_n = 5n - 2$ **9.** $a_n = 2^n - 1$ **11.** $a_n = (-1)^n$ **13.** **(a)** 118, **(b)** 1,452 **15.** -1 or 3 **17.** **(a)** $\frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \frac{17}{16}$, **(b)** No, **(c)** $\frac{79}{16}$ **19.** **(a)** $0.9\overline{3}$, **(b)** $1.\overline{63}$, **(c)** $0.2\overline{142857}$ **21.** 225 **23.** $\frac{1}{2}$ **25.** 47 **27.** $\frac{10}{11}$ **29.** $-\frac{17}{2}$ **31.** $-\frac{1}{128}$ **33.** $\frac{1}{4}$

37. Yes **39.** **(a)** 15, **(b)** 455, **(c)** 84 **41.** $81 + 216x + 216x^2 + 96x^3 + 16x^4$ **43.** $99 - 70\sqrt{2}$

45. $\dfrac{1}{x^6} + 6\dfrac{y^{1/2}}{x^5} + 15\dfrac{y}{x^4} + 20\dfrac{y^{3/2}}{x^3} + 15\dfrac{y^2}{x^2} + 64\dfrac{y^{5/2}}{x} + y^3$ **47.** $(\frac{20}{729})x^8$ **49.** $6435x^6$

51. **(a)** 0.34868, **(b)** 0.36603, **(c)** 0.36770, **(d)** 0.36786, **(e)** 0.36788 **53.** **(a)** 2, 6, 20, **(b)** $\binom{2n}{n}$ **55.** **(a)** 2.236068, **(b)** $\sqrt{5}$ **57.** $\{a_{2n-1}\}$ converges to 5, $\{a_{2n}\}$ converges to 20 **59.** **(a)** 1140, **(b)** 680

CHAPTER 9

EXERCISES 9.1 *(page 498)*

Check Your Understanding **1.** T **3.** T **5.** F **7.** F **9.** $(-3, -1)$

Develop Mastery **1.** $x = 1, y = 3$ **3.** $x = \frac{1}{9}, y = -\frac{1}{3}$ **5.** Inconsistent **7.** $x = 6, y = 3, z = -3$ **9.** $x = 2, y = -1, z = 0$ **11.** Dependent $x = \dfrac{2k + 3}{3}, y = \dfrac{13k + 3}{3}, z = k$ (any number) **13.** $x = 2, y = -3$ **15.** $x = 8, y = -1$ **17.** $x = -3, y = 1, z = -2$ **19.** Dependent $x = -k, y = 0, z = k$ (any number) **21.** Inconsistent **23.** Dependent $x = \dfrac{12 - k}{7}, y = \dfrac{-4 - 2k}{7}, z = k$ (any number) **25.** $x = -1, y = 1, z = 2$ **27.** $x = 1, y = -1, z = 4$ **29.** $x = 5, y = 2, z = -2$ **31.** $x = 1, y = 2, z = -3$ **33.** $x = -2, y = 4, z = 4$ **35.** $x = 2, y = -2, z = 1$ **37.** $x = 21, y = -8$ **39.** $x = -7, y = 15$ **41.** $x = -12, y = 2$ **43.** $x = 1, y = \frac{1}{3}$ **45.** $x = -\frac{1}{6}, y = -\frac{1}{11}$ **47.** $(-1, -2)$ **49.** No real solutions. **51.** **(a)** $8 + 2\sqrt{2} + 2\sqrt{10}$, **(b)** 8 **53.** **(a)** $5 + 3\sqrt{5} + \sqrt{10}$, **(b)** 7.5 **55.** $x = \frac{24}{5}, y = 8, z = 24$ **57.** $x = e, y = \dfrac{1}{e}, z = \sqrt{e}$ **59.** $x = 5, y = 1$ **61.** **(b)** $(1, 1), (3, 2)\ (5, -1)$, **(c)** 97° **63.** 35 cm.^{2} **65.** \$800 and \$1,700 **67.** 882 **69.** $x = 400$ gm., $y = 1{,}600$ gm. **71.** Plane 420 km./hr., wind 60 km./hr. **73.** Sandwich \$1.60, drink \$0.30, pie \$0.60 **75.** $x = 300$ gm., $y = 1{,}200$ gm., $z = 900$ gm. **77.** 2 hrs. and 40 min.

EXERCISES 9.2 *(page 507)*

Check Your Understanding **1.** T **3.** T **5.** T **7.** $x = 1, y = -2$ **9.** $x = 1, y = 2, z = -2$

Develop Mastery **1.** $\begin{aligned} x + 2y &= -1 \\ x - 3y &= 2 \end{aligned}$ **3.** $\begin{aligned} x - y \phantom{{}-4z} &= 1 \\ 2x + 3y - 4z &= 1 \\ -x - 2y + 3z &= 5 \end{aligned}$ **5.** $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & -1 & 0 & 3 \\ -1 & 2 & -1 & 0 \end{bmatrix}$

9. $x = 1, y = 2, z = -2$ **11.** $x = 0, y = -1, z = -2$ **13.** $x = \frac{1}{2}, y = -\frac{5}{2}$ **15.** $x = 2, y = -4$ **17.** Dependent, $x = \dfrac{8k - 5}{4}, y = k$ (any number) **19.** $x = \frac{3}{11}, y = \frac{7}{11}, z = \frac{16}{11}$ **21.** $x = -\frac{4}{3}, y = \frac{1}{3}, z = 0$ **23.** Dependent, $x = \dfrac{5 - 4k}{3}, y = \dfrac{11 - 4k}{3}, z = k$ (any number) **25.** $x^2 + y^2 - 3x + 5y - 14 = 0$ **27.** $y = 1.6x^2 + 0.4x - 6.2$ **29.** $\dfrac{3}{3x + 2} - \dfrac{1}{x - 2}$ **31.** $\dfrac{1}{x} + \dfrac{2}{x + 2} - \dfrac{3}{x - 2}$ **33.** $\dfrac{2}{x - 1} + \dfrac{3x}{x^2 + 1}$ **35.** $\dfrac{4}{x - 2} + \dfrac{1}{(x - 2)^2} - \dfrac{3}{x}$ **37.** 9, 3, 12 **39.** Oatmeal 0.6 cup, milk 0.5 cup, orange juice 1 cup.

EXERCISES 9.3 *(page 511)*

Check Your Understanding **1.** F **3.** F **5.** T **7.** $(\pm 1, 0)$ **9.** $(\pm 2, -2)$

Develop Mastery **1.** $(-1, 1), (4, 16)$ **3.** $(0, 0), (-0.5, 1.5)$ **5.** $(-3, 1), (1.5, -2)$ **7.** No solution **9.** $(0, -5), (3, 4)$ **11.** $(-2.56, -4.56), (1.56, -0.44)$ **13.** $(4, 2)$ **15.** $(-2, -2), (-2, 2), (2, -2), (2, 2)$ **17.** $x = 1, y = 0$ **19.** $x = 2, y = 1.39$ **21.** $x = 3.56, y = 4.75$ **23.** $x = 3, y = 8$ **25.** $(-1, 2), (1, 2)$ **27.** $(2, 1), (-2, -1), (\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$ **29.** $(1, 2), (-1, -2), (2, 1), (-2, -1)$ **31.** $(\frac{\pi}{4}, \frac{1}{\sqrt{2}}), (\frac{5\pi}{4}, -\frac{1}{\sqrt{2}})$ **33.** $(\frac{\pi}{6}, 0), (\frac{\pi}{6}, 2\pi), (\frac{5\pi}{6}, 0), (\frac{5\pi}{6}, 2\pi)$ **35.** $(2, 4), (2, -4), (-2, 4), (-2, -4)$ **37.** $(0, \ln 5)$ **39.** $(e, \frac{1}{e})$ **41.** $(6, -3)$ **43.** 8 cm. by 12 cm. **45.** 66 cm. **47.** $y = x$ **49.** $6 + 4\sqrt{10} + \sqrt{148} \approx 30.81$ cm or $6 + 4\sqrt{10} + \sqrt{244} \approx 34.27$; base 6, altitude 12

EXERCISES 9.4 *(page 518)*

Check Your Understanding **1.** F **3.** F **5.** F **7.** QIV **9.** QII, QIII

Develop Mastery **1.** **(a)** No, **(b)** Yes **3.** **(a)** No, **(b)** No **5.** **(a)** All points below line $x + 2y = 4$ (broken), **(b)** $(0, 0), (1, 1)$ **7.** **(a)** All points on or below line $2x - 3y = 6$ (solid), **(b)** $(2, -1), (4, 0)$ **9.** **(a)** All points below line $x + y = -4$ (broken), **(b)** $(-3, -2), (-6, 0)$ **11.** **(a)** All points on or above line $y = 2x$ (solid), **(b)** $(0, 0), (0, 1)$ **13.** All points below line $x + y = 4$ and above line $2x - y = -1$ (both broken), corner point $(1, 3)$, open circle **15.** All points on or below line $x - 2y = 4$ (solid) and to the right of $x = 2$ or to the left of $x = -2$, corner points $(2, -1), (-2, -3)$, open circles **17.** All points inside the triangle with vertices $(-1, 2), (3, 4)$, and $(1, -2)$, all open circles.

19. All points above the x-axis and above line $x + y = 1$, corner point (1, 0), open circle **21.** All points to the left of the y-axis and above the line $x + y = 1$, corner point (0, 1), open circle **23.** All points inside the triangle with vertices $(2, -1)$, $(2, 1)$, and $(4, -1)$, all open circles **25.** I and II **27.** II and III **29.** All points on or below line $2x - y = 4$ (solid) **31.** All points above line $2x + y = 2$ (broken) **33.** All points on or between lines $y = x + 1$ and $y = x - 1$ (both solid) **35.** All points to the right of the y-axis and above line $y = x$ (broken) **37.** $x + 2y \geq 4$ **39.** $2x - 3y \leq 6$ **41.** $\begin{aligned} 2x + 3y &\leq 6 \\ x - 3y &\leq 3 \\ x - y &\geq -2 \end{aligned}$ **43.** **(b)** $\begin{aligned} x + 3y &> -2 \\ 3x + 2y &< 8 \\ 2x - y &> -4 \end{aligned}$ **45.** **(b)** $\begin{aligned} y &> 0 \\ 2x + y &< 4 \\ 4x - 3y &> -12 \end{aligned}$ **47.** $\begin{aligned} 2x - y &< 1 \\ x + 2y &< 4 \end{aligned}$

49. Minimum 320 at (3, 1); maximum 496 at (2, 5) **51.** Minimum -548 at $(-4, -8)$; maximum 472 at (6, 2) **53.** Minimum 97.2 at (0, 2.4); maximum 696 at (4, 6) **55.** Let x, y denote number of \$6, and \$3 seats, respectively: $0 \leq x \leq 200$, $0 \leq y \leq 600$, $6x + 3y \geq 2{,}100$. All lattice points (x, y) that satisfy the constraints. **57.** 1,000 Rambis, 2,000 Eustis; profit \$28,000 **59.** x units of A, y units of B: $x \geq 0$, $y \geq 0$, $2x + 3y \geq 8$, $5x + 2y \geq 9$, $2x + y \leq 8$ **61.** x units of C, y units of D: $0 \leq x \leq 3$, $y \geq 2$, $3x + y \geq 5$, $3x + 4y \leq 21$ **63.** **(a)** 64 acres of soybeans and 38 acres of corn for a return of \$8300, **(b)** 64 acres of soybeans and 38 acres of corn for a return of \$8940 **65.** One bag of A, four bags of B, cost \$22.50.

EXERCISES 9.5 *(page 529)*

Check Your Understanding **1.** T **3.** T **5.** F **7.** 2 **9.** 2

Develop Mastery **1.** $c_{12} = 1$, $c_{31} = -1$ **3.** $c_{22} = 2\sqrt{3}$, $c_{33} = 10$ **5.** 25 **7.** 1 **9.** -16 **11.** 0 **13.** -132 **15.** 45 **17.** -0.723 **19.** 0 **21.** **(a)** $16x$, **(b)** $\frac{3}{16}$ **23.** **(a)** $e^x - e^2$, **(b)** 2 **25.** **(a)** $-28x$, **(b)** $-\frac{1}{7}$ **27.** **(a)** $0 \cdot x$, **(b)** Any real number **29.** **(a)** $8 \sin x + 5$, **(b)** $\frac{3\pi}{2} + k \cdot 2\pi$ **31.** $1, \dfrac{3 \pm \sqrt{13}}{2}$ **33.** $0, \pm\sqrt{6}$ **35.** $x = 4$, $y = -3$ **37.** $x = 8.5$, $y = -1.5$ **39.** Dependent, $x = k$, $y = k$, $z = k$. **41.** 14 **43.** 0.5 **45.** 31.5 **47.** 4 **49.** **(a)**, **(b)**, **(c)** Each determinant is equal to 0. **51.** **(a)** $-12a$, **(b)** $7k$, **(c)** $-17c$ **53.** $x - 2y + 5 = 0$ **55.** $9\boldsymbol{i} - 5\boldsymbol{i} + 3\boldsymbol{k}$

EXERCISES 9.6 *(page 539)*

Check Your Understanding **1.** T **3.** T **5.** T **7.** F **9.** T

Develop Mastery **1.** **(a)** 2 by 2, **(b)** $a_{12} = -3$, $a_{21} = -1$ **3.** **(a)** 3 by 3, **(b)** $a_{12} = 1$, $a_{21} = 0$

5. $\begin{bmatrix} 9 \\ 1 \end{bmatrix}$ **7.** $\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$ **9.** $\begin{bmatrix} -1 & -4 \\ -4 & -5 \end{bmatrix}$ **11.** $\begin{bmatrix} -7 & 1 & -2 \\ -11 & 2 & -1 \\ -8 & 0 & 10 \end{bmatrix}$ **13.** $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ **15.** $\begin{bmatrix} 1 & 0 \\ -3 & 0.5 \end{bmatrix}$

17. No inverse **19.** $\begin{bmatrix} -3 & -2 & 6 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$ **21.** $\begin{bmatrix} -3 & -1 & 1 \\ 2 & 1 & 2 \\ 5 & 2 & 0 \end{bmatrix}$ **23.** $\begin{bmatrix} -4 & 2 & -3 \\ 10 & -5 & 8 \\ -1 & 1 & -1 \end{bmatrix}$

25. $\begin{bmatrix} 1.5 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ **27.** $\frac{1}{20}\begin{bmatrix} 2 & 1 & 3 & -1 \\ -40 & 50 & -10 & -30 \\ -34 & 33 & -1 & -13 \\ 6 & -7 & -1 & 7 \end{bmatrix}$ **29.** **(a)** $|A| = 1$, **(b)** $x = -30$, $y = 23$

31. **(a)** $|A| = -1$, **(b)** $x = 10$, $y = 17$ **33.** **(a)** $|A| = -1$, **(b)** $x = 13$, $y = 13$, $z = 17$ **35.** **(a)** $|A| = -1$, **(b)** $x = -25$, $y = 6$, $z = 3$ **37.** **(a)** $|A| = 0$, **(b)** Inconsistent **39.** **(a)** $|A| = 238$, **(b)** $x = 2$, $y = 3$, $z = 1$, $w = -1$ **41.** **(a)** $x^2 + y^2 - 4x - 4y = 17$, **(b)** Center (2, 2) radius 5 **43.** **(a)** $x^2 + y^2 - 6x + 4y = 12$, **(b)** Center $(3, -2)$, radius 5 **45.** **(a)** $y = -x^2 + 2x + 2$, **(b)** $(1 - \sqrt{3}, 0)$, $(1 + \sqrt{3}, 0)$; $V(1, 3)$ **47.** **(a)** $a = -2$, $b = 12$, $c = 230$, **(b)** 14.1 sec.

49. **(a)** $AB = \begin{bmatrix} -5 & -6 \\ 4 & 5 \end{bmatrix}$, **(b)** $(AB)^{-1} = \begin{bmatrix} -5 & -6 \\ 4 & 5 \end{bmatrix}$, **(c)** $A^{-1}B^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, $B^{-1}A^{-1} = \begin{bmatrix} -5 & -6 \\ 4 & 5 \end{bmatrix}$

51. **(a)** $AB = \begin{bmatrix} -15 & -5 & 7 \\ 5 & 2 & 1 \\ 1 & 0 & -4 \end{bmatrix}$, **(b)** $(AB)^{-1} = \begin{bmatrix} -8 & -20 & -19 \\ 21 & 53 & 50 \\ -2 & -5 & -5 \end{bmatrix}$, **(c)** $A^{-1}B^{-1} = \begin{bmatrix} 14 & -6 & 11 \\ 24 & -11 & 19 \\ 47 & -21 & 37 \end{bmatrix}$, $B^{-1}A^{-1} = (AB)^{-1}$

53. **(a)** I, **(b)** A, **(c)** I, **(d)** A **55.** **(a)** I, **(b)** A, **(c)** I, **(d)** A **57.** **(a)** A, **(b)** A, **(c)** A, **(d)** A

59. $A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for $n = 2, 3, 4, \ldots$ **61.** $A^n = A$ for $n = 1, 2, 3, \ldots$

63. **(a)** $\begin{bmatrix} -4 & -8 & 0 \\ 2 & 4 & 0 \\ 2 & -4 & 0 \end{bmatrix}$, $A^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for $n = 3, 4, 5, \ldots$

CHAPTER 9 REVIEW *(page 541)*

Test Your Understanding **1.** T **3.** F **5.** F **7.** T **9.** F **11.** T **13.** T **15.** T **17.** F **19.** F **21.** F **23.** F **25.** T **27.** T **29.** F **31.** F **33.** F **35.** F **37.** F **39.** F

Review for Mastery **1.** $x = 7, y = 8$ **3.** $x = 4, y = -6$ **5.** $x = -2, y = -1, z = 3$
7. Dependent, $x = \dfrac{-6k - 3}{2}, y = \dfrac{16k + 5}{4}, z = k$ (any number) **9.** $x = \frac{15}{37}, y = \frac{15}{16}$
11. A line and a circle intersecting at $(-13, 0)$ and $(5, 12)$ **13.** Graphs intersect at $(2, 2)$ and $(-4, -1)$
15. Two parabolas intersecting at $(0, 4)$ and $(3, 1)$ **17.** All points above line $x + y = 1$ and above line $2x - y = 5$ (both broken), lines intersect at $(2, -1)$ **19.** All points below line $2x + y = 4$ (broken) and on or below line $x - 2y = 1$ (solid), intersection point $(\frac{9}{5}, \frac{2}{5})$ not in the graph **21.** All points on or inside the triangle with vertices at $(0, -8)$, $(4, -4)$, and $(3, -2)$
23. 2 **25.** $-\frac{2}{5}$ **27.** **(a)** $\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$, **(b)** $\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$ **29.** **(a)** $\begin{bmatrix} 10 & -13 \\ -23 & 30 \end{bmatrix}$, **(b)** $\begin{bmatrix} 10 & -13 \\ -23 & 30 \end{bmatrix}$
31. $B^{-1} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$; $x = -5, y = 18$ **33.** $x = -6, y = -1, z = -10$
35. **(a)** $(0, 0), (\frac{5}{4}, 0), (\frac{5}{6}, \frac{5}{6})$, **(b)** Maximum $\frac{15}{2}$ when $x = \frac{5}{6}, y = \frac{5}{6}$
37. **(a)** $(0, 2), (1, 1), (3, 5)$, **(b)** Minimum 7 when $x = 1, y = 1$ **39.** 5
41. For x (number of \$5 tickets) and y (number of \$3 tickets), $0 \leq x \leq 500$, $0 \leq y \leq 1{,}000$, and $5x + 3y \geq 3{,}700$
43. At (120, 120) cost is \$42, at (160, 80) cost is \$44. **45.** Minimum cost is \$1,140 when x is 50 and y is 20.
47. $y = 2x^2 + 4x + 5$ **49.** $x^2 + y^2 + 2x - 4y - 4 = 0$

CHAPTER 10

EXERCISES 10.1 *(page 552)*

Check Your Understanding **1.** T **3.** F **5.** T **7.** F **9.** T

Develop Mastery **1.** $x + y = 0$ **3.** $x - 3y = -4$ **5.** $(x - 7)^2 + (y - 12)^2 = 80$
7. $(x - 1.5)^2 + (y + 0.5)^2 = 4.5$ **17.** $x^2 + y^2 - 6x - 4y = 0$ **19.** $x^2 + y^2 - 6x - 4y + 4 = 0$
21. Two circles: $(x - 5)^2 + (y - 5)^2 = 25$, $(x - 17)^2 + (y - 17)^2 = 289$ **23.** $(x + 2)^2 + y^2 = 50$
25. $(x - 3)^2 + (y - 6)^2 = 4$ **27.** $x^2 + y^2 - 4x + 2y - 8 = 0$ **29.** $x^2 + y^2 - 4x - 21 = 0$
31. Two lines: $y = -x + 2$ and $y = x - 4$ **33.** Two lines: $y = 1$ and $8x - 15y = 17$
35. Two lines: $y = x - 2$ and $x + 7y = 10$ **37.** **(a)** Yes, **(b)** $\frac{3}{2\sqrt{5}}$, **(c)** $d_L = \frac{5}{2\sqrt{5}}$, $d_K = \frac{4}{\sqrt{5}}$
39. **(a)** Yes, **(b)** 41, **(c)** $d_L = 17$, $d_K = 24$ **41.** **(a)** $y = -\frac{x}{3} + 3$, **(b)** $y = 3x + 3$, **(c)** $(0, 3)$, **(d)** $\sqrt{10}$
45. $\frac{3}{\sqrt{10}}$ **47.** $\sqrt{2}$ **49.** **(a)** $\frac{2}{\sqrt{5}}$, **(b)** 4 **51.** **(a)** $3\sqrt{2}$, **(b)** 12 **53.** **(a)** $D(x) = \sqrt{(x - 1)^2 + (x + 7)^2}$; 5.7, **(b)** $4\sqrt{2}$ **55.** **(a)** $D(x) = \sqrt{(x + 1)^2 + (-2x + 6)^2}$; 3.6, **(b)** $\frac{8}{\sqrt{5}}$

EXERCISES 10.2 *(page 562)*

Check Your Understanding **1.** T **3.** F **5.** A quarter circle **7.** Half of a parabola **9.** A line segment

Develop Mastery **1.** $y = 4 - x^2$; graph is a parabola that opens down. **3.** $y = \sqrt{4 - x^2}$; graph is the upper half of a circle with center at (0, 0) and radius 2. **5.** $y = \sqrt{4 - x^2}$ where $x \geq 0$ and $y \geq 0$; graph is a quarter-circle. **7.** $x + y = 7$; graph is a line. **9.** $5x + 3y = 11$; graph is a line. **11.** $(x - 1)^2 + (y - 1)^2 = 1$; graph is a circle. **13.** $x + y = 0$ where $-1 \leq x \leq 1$; graph is the line segment with endpoints $(-1, 1)$ and $(1, -1)$. **15.** $x + y = 2$ where $x > 1$; graph is a half-line from endpoint $(1, 1)$. **17.** $y = x^2 - 1$ where $-1 \leq x \leq 1$; graph is part of a parabola with endpoints $(-1, 0)$ and $(1, 0)$.
19. Both graphs are parts of the line $x + y = 0$. **(a)** Line segment with endpoints $(-1, 1)$ and $(1, -1)$, **(b)** Line segment with endpoints $(0, 0)$ and $(1, -1)$ **21.** Graphs are parts of the graph of $y = \dfrac{1}{x}$. **(a)** Graph is the part in the first quadrant. **(b)** Graph is the part in the third quadrant. **23.** Graphs are parts of the circle $x^2 + y^2 = 1$. **(a)** Graph is the upper half. **(b)** Graph is the right half. **25.** Graphs are parts of the line $3x + y = 1$. For both **(a)** and **(b)** the graph is the line segment with endpoints $(1, -2)$ and $(-1, 4)$. **27.** $(x - 1)^2 + (y - 2)^2 = 1$; graph is a circle. **29.** $x + y = 3$ where $0 \leq x \leq 2$; graph is a line segment. **31.** $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1$ **33.** $\dfrac{(x - 1)^2}{16} - \dfrac{y^2}{25} = 1$ **35.** $x = t - 2, y = 3t + 4$, or $x = t, y = 3t + 10$

37. $x = 4t - 3, y = 3t + 4,$ or $x = 4t + 1, y = 3t + 7$ **41.** $x = 3\cos\alpha, y = \sin\alpha$ where $0 \le \alpha \le \frac{\pi}{2}$; equation is $\frac{x^2}{9} + \frac{y^2}{1} = 1.$ **43.** $x = (4 - a)\cos\alpha, y = a\sin\alpha$; when $a = 2$, then the graph is a circle.
45. $x = 2t - \dfrac{t^2}{10}, y = \dfrac{t\sqrt{100 - t^2}}{10}$ for $0 \le t \le 10$; $x = 20 - t, y = 0$ for $10 \le t \le 20.$ **49.** **(a)** 2π, **(b)** About y-axis
51. **(a)** 2π, **(b)** About x-axis. **53.** 2.2 to (2, 0) **55.** 3.2, to $(-2, 0)$ **57.** 1.8 **59.** **(a)** Line, **(b)** $d = \frac{4}{\sqrt{5}}$
61. **(a)** (1, 0), **(b)** (0, 1) **63.** **(a)** (4, 0), (6, 0), (10, 0), **(b)** None

EXERCISES 10.3 *(page 577)*

Check Your Understanding **1.** F **3.** F **5.** F **7.** $(\pm 3, 0)$ **9.** Two

Develop Mastery **1.** $V(0, 0), F(2, 0), D: x = -2$ **3.** $y^2 = 8x$; opens right, $V(0, 0)$ **5.** $x^2 = -2y$; opens down, $V(0, 0)$
7. $y^2 = 4x$ **9.** **(a)** $\frac{1}{8}$, **(b)** $4, 2\sqrt{2}$ **11.** $\frac{10000}{15} \approx 666.7$ in **13.** **(a)** $y = mx + 1 - m$, **(b)** $m = 2$
17. **(a)** Ellipse, $V(\pm 3, 0), F(\pm\sqrt{5}, 0)$, **(b)** Major 6, minor 4 **19.** **(a)** Hyperbola, $V(\pm 3, 0), F(\pm\sqrt{13}, 0)$, **(b)** $y = \pm\frac{2x}{3}$
21. **(a)** Ellipse, $V(\pm\sqrt{5}, 0), F(\pm 1, 0)$, **(b)** Major $2\sqrt{5}$, minor 4 **23.** **(a)** Hyperbola, $V(\pm 2, 0), F(\pm 3, 0)$, **(b)** $y = \pm\frac{\sqrt{5}x}{2}$
25. **(a)** Hyperbola, $V(0, \pm 4), F(0, \pm 4\sqrt{10})$, **(b)** $y = \pm\frac{x}{3}$ **27.** $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ **29.** $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$ **31.** $x^2 + 9y^2 = 25$
33. $\dfrac{x^2}{45} + \dfrac{y^2}{36} = 1$; $16x^2 + 11y^2 = 576$ **35.** $\dfrac{y^2}{4} - \dfrac{x^2}{12} = 1$ **37.** $\dfrac{x^2}{1} - \dfrac{y^2}{4} = 1$ **39.** $4x^2 - y^2 = 15$
41. $(\frac{3}{2}, -1), (-\frac{9}{14}, \frac{13}{7})$ **43.** $(\pm\frac{1}{2}, \frac{1}{2}), (\pm\frac{1}{2}, -\frac{1}{2})$ **45.** $C: x^2 + y^2 = \frac{81}{20}$ **47.** 118 yds. from nearest vertex
55. **(b)** $x = \pm\sqrt{\frac{5}{2}}, y = \pm\sqrt{\frac{27}{8}}$ **57.** **(b)** (0, 0), (2, 0), (6, −24) **59.** $x^2 + y^2 = 1$
61. **(a)** (±2.13, 2.54), **(b)** (±2.1305, 2.5391)

EXERCISES 10.4 *(page 588)*

Check Your Understanding **1.** F **3.** T **5.** F **7.** QIV **9.** QI, QIV

Develop Mastery **1.** $V(-2, 1), F(-3, 1), D: x = -1$ **3.** $V(1, 2), F(1, \frac{3}{2}), D: y = \frac{5}{2}$
5. $V(-1, -3), F(-1, -\frac{3}{2}), D: y = -\frac{9}{2}$ **7.** $(y + 1)^2 = 4(x - 2)$ **9.** $(x + 1)^2 = \frac{4}{5}(y + 2)$, or $(y + 2)^2 = \frac{25}{2}(x + 1)$
11. $(y - 1)^2 = -8(x - 2)$, or $(x - 2)^2 = 16(y - 1)$ **13.** $(y + 2)^2 = 8(x - 4)$
15. $(x - 2)^2 = 2(y - \frac{1}{2})$, or $(x - 2)^2 = -2(y - \frac{3}{2})$ **17.** $\dfrac{(x - 3)^2}{5} + \dfrac{y^2}{9} = 1$ **19.** $\dfrac{(x - 1)^2}{4} + \dfrac{4(y + 1)^2}{7} = 1$
21. $\dfrac{(x + 3)^2}{4} + \dfrac{y^2}{9} = 1$ **23.** $\dfrac{(y - 1)^2}{4} - \dfrac{(x + 2)^2}{5} = 1$ **25.** $\dfrac{x^2}{4} - \dfrac{(y - 2)^2}{12} = 1$
27. Hyperbola, center $(1, -1)$, V:$(-1, -1), (3, -1)$, $a = 2, b = 3$ **29.** Ellipse, center (0, 1), V:(0, −2), (0, 4), $a = 3, b = 2\sqrt{2}$
31. Ellipse, center $(-\frac{1}{2}, \frac{3}{2})$, V:$(-\frac{9}{2}, \frac{3}{2}), (\frac{7}{2}, \frac{3}{2})$, $a = 4, b = 1$ **33.** Parabola, $V(-3, -5)$, opens up
35. Ellipse, center (1, 0), V:(1, 3), (1, −3), $a = 3, b = 2$ **37.** Circle, $C(-3, 1), r = 2$
39. Hyperbola, center (−1, 2), V:(−3, 2), (1, 2), $a = 2, b = 1$ **41.** $(x - 3)^2 = y + 5$, parabola, $V(3, -5)$, opens up
43. $\dfrac{(x - 1)^2}{4} + \dfrac{(y + 2)^2}{16} = 1$; ellipse, center (1, −2), foci $(1, -2 \pm 2\sqrt{3})$ **45.** $(x - 3)^2 + (y + 1)^2 = 4$; circle, $C(3, -1), r = 2$
47. $\dfrac{(x - 2)^2}{4} - \dfrac{(y + 3)^2}{4} = 1$; hyperbola, center (2, −3), foci $(2 \pm 2\sqrt{2}, -3)$ **49.** $-200(y - 50) = (x - 100)^2$
55. **(a)** $\tan 2\theta = -\frac{4}{3}, \cos 2\theta = -\frac{3}{5}$, **(b)** $x = \dfrac{X - 2Y}{\sqrt{5}}, y = \dfrac{2X + Y}{\sqrt{5}}$, **(c)** $2Y^2 - 3X^2 = 6$; hyperbola
57. **(a)** $\tan 2\theta = -\frac{3}{4}, \cos 2\theta = -\frac{4}{5}$, **(b)** $x = \dfrac{X - 3Y}{\sqrt{10}}, y = \dfrac{3X + Y}{\sqrt{10}}$ **(c)** $9Y^2 - X^2 = 36$; hyperbola
59. **(a)** $\tan 2\theta = -\sqrt{3}, \theta = 60°$, **(b)** $x = \dfrac{X - \sqrt{3}\,Y}{2}, y = \dfrac{\sqrt{3}\,X + Y}{2}$, **(c)** $5X^2 + Y^2 = 10$; ellipse
61. **(a)** $\tan 2\theta$ is undefined, so $2\theta = 90°, \theta = 45°$, **(b)** $x = \dfrac{X - Y}{\sqrt{2}}, y = \dfrac{X + Y}{\sqrt{2}}$, **(c)** $5X^2 - Y^2 = 20$; hyperbola
63. $3x^2 + 3y^2 - 2xy = 8$ **65.** $d \approx 1.4$ to (3.7, 0.7)

EXERCISES 10.5 *(page 598)*

Check Your Understanding **1.** T **3.** T **5.** F **7.** $[2\sqrt{2}, -\frac{\pi}{4}]$ **9.** Q II

Develop Mastery **1.** A is $(1, \sqrt{3})$; B is $(-1, -\sqrt{3})$. **3.** A is $[2, -\frac{\pi}{6}]$ or $[2, \frac{11\pi}{6}]$; B is $(\sqrt{2}, -\sqrt{2})$.
5. P is $[2, \frac{5\pi}{6}]$ or $[-2, -\frac{\pi}{6}]$ or $(-\sqrt{3}, 1)$. Q is $[2, \frac{11\pi}{6}]$ or $[2, -\frac{\pi}{6}]$ or $(\sqrt{3}, -1)$.
7. A is $(0, \sqrt{3})$ or $[\sqrt{3}, \frac{\pi}{2}]$; B is $(-1, 0)$ or $[1, \pi]$; C is $(1, 0)$ or $[1, 0]$. **9.** $r = 2$ or $r = -2$; graph is a circle with center at the pole and radius 2. **11.** $r = 3 \sec \theta$; graph is a vertical line. **13.** $\theta = \text{Tan}^{-1} 3$; graph is a line through the origin. **15.** $r = 2 \cos \theta - 2 \sin \theta$; graph is a circle with center at $(1, -1)$ and radius $\sqrt{2}$. Pole is given by $[0, \frac{\pi}{4}]$.
17. $x^2 + y^2 = 4$; graph is a circle with center at (0, 0) and radius 2. **19.** $x^2 + y^2 + 4x = 0$; graph is a circle with a hole at the origin. **21.** $x + y = 0$; graph is a line. **23.** $x^2 + y^2 = 1$; graph is a circle with center at the origin and radius 1. **25.** $x^2 + y^2 - 2\sqrt{3}\,x - 2y = 12$; graph is a circle with center at $(\sqrt{3}, 1)$ and radius 4. **27.** Three-leafed rose is traced on the interval $[0, \pi]$; $r < 0$ on leaf in QIII **29.** Cardioid; $r \geq 0$ for all θ **31.** Four-leafed rose; $r < 0$ on top and bottom leaves. **33.** Graphs are the same circle, starting at opposite ends of a diameter. **35.** Circle ($r = \sin \theta$) is traced out on $[0, \pi]$, the cardioid on $[0, 2\pi]$. **37.** First is a cardioid enclosed in a limaçon. **39.** Graphs are the same spiral. **41.** Two cardioids intersecting at $[1, \frac{\pi}{2}]$, $[1, \frac{3\pi}{2}]$, and the pole. **43.** Circle meets limaçon at $[3, \pm\text{Cos}^{-1} .25]$. **45.** Lemniscate and circle meet at four points, $[\pm 1, 3.015]$, $[\pm 1, 1.697]$. **47.** QI, QIV **49.** QI, QIV **51.** $1 \leq r \leq -2 \sin 2\theta$ on $[\frac{19\pi}{12}, \frac{23\pi}{12}]$ **53.** $1 \leq r \leq 1 + \cos \theta$ on $[\frac{-\pi}{2}, \frac{\pi}{2}]$ **55.** $0 \leq r \leq -(1 + 2 \cos \theta)$ on $[\frac{2\pi}{3}, \frac{4\pi}{3}]$ **57. (a)** $y^2 = 4(x + 1)$, parabola; $y^2 = -4(x - 1)$, parabola, **(b)** $3x^2 + 8x + 4y^2 = 16$ ellipse; $3x^2 - 8x - y^2 + 4 = 0$; hyperbola **59.** Line; $\sqrt{3}\,x - y = 4$ **61.** Center $(\frac{a}{2}, \frac{b}{2})$, radius $\frac{1}{2}\sqrt{a^2 + b^2}$

CHAPTER 10 REVIEW *(page 600)*

Test Your Understanding **1.** T **3.** F **5.** F **7.** T **9.** F **11.** F **13.** T **15.** T **17.** F **19.** T **21.** T **23.** T **25.** T **27.** F **29.** F

Review for Mastery **1.** Ellipse; $V(0, \pm 5)$, $F(0, \pm 4)$ **3.** Hyperbola; $V(\pm 3, 0)$, $F(\pm\sqrt{34}, 0)$
5. Ellipse; $V_1(1, 3)$, $V_2(1, -7)$, $F_1(1, 2)$, $F_2(1, -6)$ **7.** $(x + 2)^2 + (y - 1)^2 = 9$ **9.** $y^2 = 12x$
11. $\dfrac{(x + 1)^2}{12} + \dfrac{(y - 4)^2}{16} = 1$ **13.** $\dfrac{(x - 1)^2}{4} - \dfrac{(y + 1)^2}{5} = 1$ **15.** $(y - 1)^2 = 2(x - 2.5)$
17. $(x - 3)^2 = \dfrac{4(y + 1)}{3}$ **19.** Parabola, opens down, $V(1, 3)$, $F(1, 2.5)$ **21.** Circle, center (0, 1), radius $\sqrt{3}$
23. Parabola, opens up, $V(1, -1)$, $F(1, -\frac{3}{4})$ **25.** $y = \sqrt{4 - x^2}$, upper half of a circle **27.** $y = 2x + 5$; $x = -1$; graph is a half-line with endpoint $(-1, 3)$. **29.** Graph is the upper half of the parabola $y^2 = x$ without (0, 0).
41. $x + 2y = \frac{23}{4}$ **43.** $x - (1 + \sqrt{2})y = 0$, $x - (1 - \sqrt{2})y = 0$ **45.** $d = \sqrt{10} - 1$, to $(1 + \frac{3}{\sqrt{10}}, 1 - \frac{1}{\sqrt{10}})$
47. $d = \sqrt{17}$, to (0, 0) **51.** Limaçon, $r > 0$ for all θ **53.** Cardioid, $r \geq 0$ for all θ
55. Circles intersect at $[\sqrt{2}, \frac{\pi}{4}]$ and the pole. **57.** (2, 1) **59.** $(\sqrt{2} - 1, 2\sqrt{2} - 2)$
61. $x = b \cos \theta$, $y = a \sin \theta$; curve is an ellipse.

PRECALCULUS

PROBLEM SOLVING WITH TECHNOLOGY

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HISTORICAL NOTES

PREFACE

Technology affects almost every aspect of our lives. There has never been a greater need for a population with good mathematical skills and proper training in applying mathematics to solving problems. Technology demands people who can handle new challenges and apply their training to tackle problems never before encountered.

Fortunately, the same technology provides new tools and new ways to help us learn the mathematical skills and ideas needed. ***Precalculus: Problem Solving with Technology*** is a partial response to the changing demands on our students.

The whole area of introductory mathematics has been in ferment in the last decades of this century. There has been a profound revolution in mathematics education, with major national efforts in calculus reform, the adoption of standards for teaching and assessment of public school mathematics, and attempts at better articulation of college mathematics with the disciplines that depend on calculus. One significant driving force of this effort has been technology. With more emphasis on technology has come a re-examination of fundamental principles of pedagogy. Both pedagogical concerns and technology have significantly influenced the design and writing of this book.

Our assumption is that every student has consistent and convenient access to graphing or computational technology. A graphing calculator will certainly be the primary tool for most users, but the book can be used just as well with more sophisticated computer systems. We expect our students to learn to use graphing tools and to use them to explore their own mathematics. We expect active learners. Mathematics has always required student involvement, and this text provides guidance to make that involvement more productive, whether done individually, in small learning groups, or in full classroom discussions.

Our philosophy on the impact of technology on mathematics is summed up in the following two statements.

Some mathematics becomes more important because technology requires it;
some mathematics becomes less important because technology replaces it;
some mathematics becomes possible because technology allows it.

Technology provides powerful tools, but it has unavoidable limitations.

We strongly believe that technology should affect the precalculus curriculum. Many books that attempt to make use of graphing technology still cover every traditional topic, many in the old traditional ways, making technology just an adjunct that permits the treatment of more difficult problems.

Our attitude is that students should learn some topics in entirely new ways. Other topics should receive less emphasis (or be dropped entirely) because they no longer carry the same importance in a world where students have ready access to computer power. Graphing calculators are really hand-held computers, each with more computing power than entire universities had not many years ago.

We realize that not all students have access to the same technology. It is handy, but frequently impossible, for the teacher to have the same calculator or computer for all students. Even in schools where there are classroom sets of a single calculator, one classroom may have Texas Instrument machines and another may be supplied by Hewlett-Packard. We make a strong effort to be "calculator neutral." Our language, illustrations, and instructions are generic, not mimicking any specific calculator. To provide some help to students, however, we include an on-page pedagogical feature called Technology Tips, suggestions about using specific calculators for specific tasks.

We have no intention of making our students calculator experts. They will end up knowing much more than we could ever hope to teach them, anyway. Our aim is limited: we want each student to understand a few tasks well enough to make the calculator a tool for mathematical exploration, a means of visualizing mathematical concepts. Our Technology Tips address most of the general questions students will have about the kinds of calculator skills required, as well as opening up a number of unexpected vistas, ideas not found anywhere else in our experience.

An essential part of learning about calculators is an understanding of some of the limitations of computing technology, so most sections in which the calculator is discussed also include examples to suggest where the calculator cannot take us.

We endorse the principles enunciated in "For Good Measure," the report of the National Summit on Mathematics Assessment, and our exercises and examples specifically address each of the following from that report:

Encourage students to explore

Many exercises are titled *Explore,* directing students toward specific goals while inviting them to delve into open-ended explorations.

Help students to verbalize their mathematical ideas

Students are invited to verbalize appropriate strategies for a particular exercise or to explain why some result might be counterintuitive.

Show students that many mathematical questions have more than one right answer

Exercises titled *Your Choice* ask students to create their own examples satisfying given conditions, testing an understanding of basic underlying concepts.

Teach students, through experience, the importance of careful reasoning and disciplined understanding

Examples, many with strategies outlining the reasoning to be followed, step students carefully through numerous applied problems, followed by discussions of pitfalls or common errors, sometimes looking at the nature of solutions and how meaningful a particular result may be.

Provide evidence that mathematics is alive and exciting

In addition to casting examples in terms of contemporary and interesting applications, we include frequent Historical Notes humanizing some significant mathematical developments, as well as marginal quotations from contemporary mathematicians that suggest how these individuals came to discover mathematics or how they dealt with challenges on their way to their present stature. Capsule biographies appear in an appendix, "How They Came to Mathematics."

Content Highlights

Functions and Graphs Graphing technology can make functional behavior come to life for students. Terminology such as "increasing" or "decreasing" becomes clear in the context of pictures. Students learn functional notation naturally as they explore translations and dilations of graphs and see for themselves the difference between the graphs of $f(x) + c$ and $f(x + c)$. The language of functions is introduced in Chapter 2 and is a unifying theme throughout the text.

Roots and Zeros of Functions The solutions of both equations and inequalities have vivid meaning in terms of graphs. Students find that much of the mystery that has traditionally attended work with absolute values evaporates when they can draw their own graphs, and the algebraic manipulations required make more sense. (See Sections 1.4 and 1.5.) Students see new connections between algebraic and graphical information and find reinforcement with each new kind of function studied.

Polynomial and Rational Functions Some of the powerful theorems that have been developed for finding zeros of polynomial functions take on new meaning in the light of graphing technology. As an example, the rational zeros theorem is not needed as a means of starting the search for zeros of an integer polynomial function, but it yields information about the nature of zeros that the calculator cannot provide. On the other hand, calculators give us excellent decimal approximations beyond reach without technology. Relations between factored and expanded forms have geometric meaning. In exploring asymptotic behavior of rational functions, students learn how important different windows can be. The calculator can alert us to significant aspects of functional behavior, but if we fail to understand the underlying mathematical meaning, the calculator can just as easily hide important, subtle items. When students are asked to construct their own examples of polynomial functions with certain properties of turning points, they see if they really understand polynomial structure.

Parametric Equations Parametric equations allow us to view many topics in a new light. Parametric graphing is introduced in Chapter 2 and is used consistently throughout the text. Inverse functions are most natural when we interchange variables and see how the graph is reflected. Thus relations between the graphs of exponential and logarithmic functions, or graphs of trigonometric and inverse trigonometric functions are more easily seen. Circles and other conic section graphs are easy to produce. After an introduction to matrices (Section 9.6), we get matrix transformations to rotate axes, and with parametric graphing students can graph rotated conics (Section 10.4).

Exponential and Logarithmic Functions After a foundation of transformations of functions (Chapter 2), the essential unity of all exponential functions becomes transparent; any exponential function can be considered as a transformation of any other, and the natural exponential function is our choice as prototype. The same relations apply just as naturally to the function inverses, the logarithmic functions.

Trigonometric Functions The function concept and the dynamic possibilities of graphing provide more unity than has previously been available for introducing the trigonometric functions and their inverses. Graphing technology allows new approaches to identities, but limitations of the technology illustrate for students why there is something to prove.

Optimization and Problem Solving A consistent theme of the text is the proper formulation and solution of applied problems. We examine examples of problems leading to polynomial, exponential, trigonometric, linear, and nonlinear models. We discuss differences between exact and approximate solutions and consider where each may be appropriate or necessary. Students are constantly encouraged to check their results to see if they are reasonable or appropriate, leading naturally to ideas of estimation.

Exercises and Pedagogy Exercises are numerous and varied. Mathematical skills are developed and reinforced. Because we assume that every student uses a graphing calculator regularly, we do not segregate exercises that call on calculators, but most sets of related exercises are identified with capsule titles. Many exercises call for both algebraic and graphical solutions. Each exercise set begins with *Check Your Understanding* exercises, going beyond typical skills-based problems. They allow individuals to test understanding of key ideas, or they can be used for class discussion purposes. *Test Your Understanding* sets at the end of each chapter serve the same ends, pulling together the concepts of the whole chapter, followed by *Review for Mastery* exercises. A frequent feature is *Looking Ahead to Calculus.* These examples and exercises develop the algebraic and technological skills needed for certain kinds of problems normally encountered in a calculus course. Some exercises are labeled *Explore* or *Your Choice,* which may be used individually as enrichment or as a basis for group or written projects. Many exercises ask students to verbalize their responses, to explain or relate ideas, and can be used for varied purposes by the instructor.

Technology Tips All graphing calculators that are now widely available will do many more things than we ask of our students in this text. Several calculators have built-in routines to approximate solutions to transcendental equations or find complex zeros of polynomials or solve systems of linear equations; some have split-screen graphing or tabular function displays. We deal essentially only with calculator capabilities that are common to the following calculators:

TI-81, TI-82, TI-85,
HP 48G and 48GX, and the HP 38G,
Casio fx-7700 (or 9700 or 9900).

Our Technology Tips help students to use all of these calculators to do specific tasks required in the text. Instructors, of course, have the option of teaching their students to use other calculator or computer routines that may be available, and they can make appropriate adjustments in assigning exercises or designing test questions.

This text is based in part on the second edition of our *Precalculus* but it has been rewritten from the ground up. We have re-examined every topic, every section, and every exercise. Many of the changes are those suggested by our work with pilot sections of classes in which all students have graphing calculators. In each such class we get confirmation of the fact that our students have lots to teach us, their teachers, in addition to what they teach each other if we give them an opportunity.

Supplement Package to Accompany the Text

This text is accompanied by the following supplements:

For the Instructor. The *Instructor's Guide* includes solutions to all of the text exercises. A collection of problems for each chapter can be used for tests.

The *HarperCollins Test Generator for Mathematics* is one of the top testing programs on the market for IBM and Macintosh computers. It enables instructors to select questions for any section in the text or to use a ready-made test for each chapter. Instructors may generate tests in multiple-choice or open-response formats, scramble the order of questions while printing, and produce up to 25 versions of each test. The system features printed graphics and accurate mathematical symbols. The program also allows instructors to choose problems randomly from a section or problem type, or to choose problems manually while viewing them on the screen with the option to regenerate variables. The editing feature allows instructors to customize the chapter disks by adding their own problems. This is especially important in designing questions that are appropriate for students.

The *QuizMaster On-Line Testing System,* available in both IBM and Macintosh formats, coordinates with the HarperCollins Test Generator and allows instructors to create tests for students to take at the computer. The test results are stored on disk so the instructor can view or print test results for a student, a class section, or an entire course.

For the Student. The *Student's Solution Manual* contains detailed solutions to the odd-numbered exercises. It is available for student purchase; ask your college bookstore manager to order ISBN 0-673-99971-8.

Precalculus Investigations Using DERIVE (0-673-99097-4) by David Mathews of Longwood College and *Precalculus Investigations Using MAPLE* (0-673-99410-4) by David Mathews and Keith Schwingendorf, of Purdue University-North Central, will help you to integrate technology into your precalculus course. Twelve lab exercises provide carefully structured, interactive learning environments for students. The manual includes real-world applications, concept overviews, and lab reports.

Acknowledgments

We are very appreciative to the reviewers who read substantial portions of the manuscript. They generously shared their ideas and suggestions, for which we are grateful. They include the following:

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Finally, to our families, we are deeply indebted for their patience, understanding, and support during the countless hours we devoted to the project.

Lawrence O. Cannon
Joseph Elich

INDEX

What Size is Your Calculator Display?

Model	*Addressable Pixels (plus axes)*	*Familiar (Decimal) Window*
TI–82, Casio 7700	94 by 62	$[-4.7, 4.7] \times [-3.1, 3.1]$
Casio 9800	94 by 62	$[-4.7, 4.7] \times [-3.1, 3.1]$
TI–81	95 by 63	$[-4.8, 4.7] \times [-3.2, 3.1]$
TI–85	126 by 62	$[-6.3, 6.3] \times [-3.1, 3.1]$
Casio 9700	126 by 74	$[-6.3, 6.3] \times [-3.7, 3.7]$
HP–38G, 48G	130 by 63	$[-6.5, 6.5] \times [-3.1, 3.2]$

Algebra

$x^2 - y^2 = (x + y)(x - y)$

$x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$

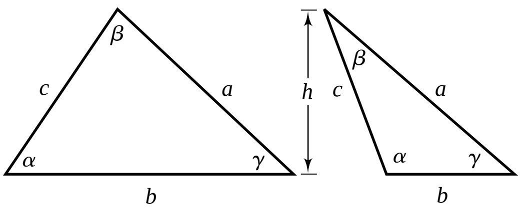
$(x \pm y)^2 = x^2 \pm 2xy + y^2$

$(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$

Quadratic Formula: If $ax^2 + bx + c = 0$, $a \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Geometry

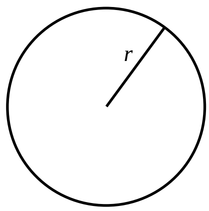
Triangles

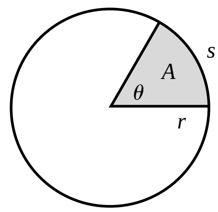


$\alpha + \beta + \gamma = 180°$ Area $= \frac{1}{2}bh$

Perimeter $= a + b + c$

Circle





Circumference $= 2\pi r$

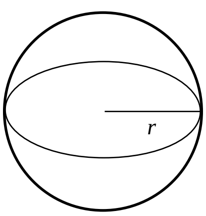
Area $= \pi r^2$

Arc length: $s = r\theta$

Area of sector: $A = \frac{1}{2}r^2\theta = \frac{1}{2}rs$’

where θ must be in radians

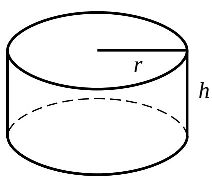
Sphere



Volume $= \frac{4}{3}\pi r^3$

Surface area $= 4\pi r^2$

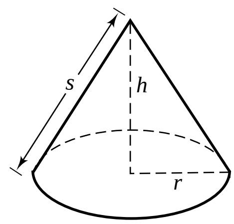
Right Cylinder



Volume $= \pi r^2 h$

Surface are $= 2\pi r(r + h)$

Cone



Volume $= \frac{1}{3}\pi r^2 h$

Slant surface area $= \pi rs$

Absolute Value

$$|u| = \begin{cases} u \text{ if } u \geq 0 \\ -u \text{ if } u < 0 \end{cases} \qquad |u| \geq 0 \qquad \sqrt{u^2} = |u|$$

Pythagorean Theorem

A triangle is a right triangle (with $\gamma = 90°$) if and only if $a^2 + b^2 = c^2$.

Distance Formula

The distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The distance between the point $P(x_1, y_1)$ and line $Ax + By + C = 0$ is given by

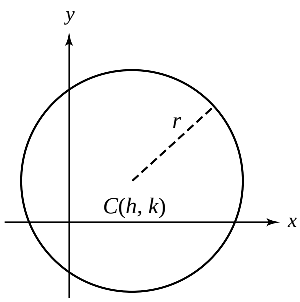
$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Binomial Theorem

$$(a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} ab^{n-1} + b^n$$

Conic Sections with Parametric Equations for Calculator Graphing

Circle

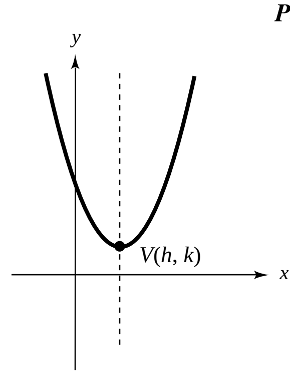


Center: (h, k) Radius: r

$$(x - h)^2 + (y - k)^2 = r^2$$

$$\begin{cases} \mathrm{X = H + R\ COS\ T} \\ \mathrm{Y = K + R\ SIN\ T} \end{cases} \quad 0 \le \mathrm{T} \le 2\pi$$

Parabola

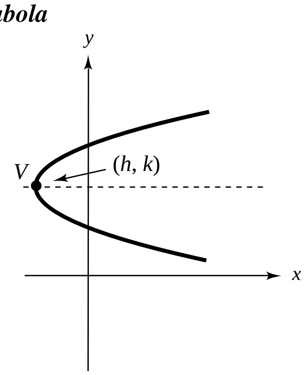


Vertex: (h, k)

Axis: $x = h$

$$(x - h)^2 = 4p(y - k)$$

$$\begin{cases} \mathrm{X = H + T,} \\ \mathrm{Y = K + (1/(4P))\,T^2} \end{cases}$$



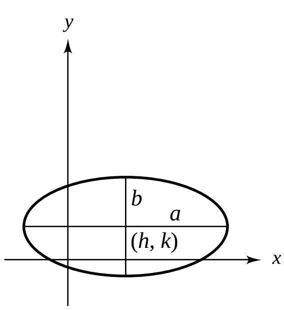
Vertex: (h, k)

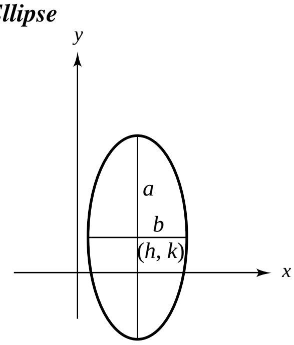
Axis: $y = k$

$$(y - k)^2 = 4p(x - h)$$

$$\begin{cases} \mathrm{X = H + (1/(4P))\,T^2} \\ \mathrm{Y = K + T} \end{cases}$$

Ellipse





$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \qquad \frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$

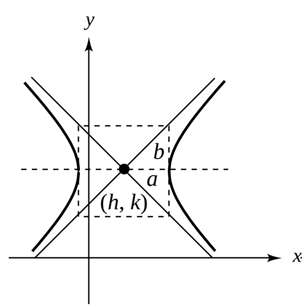
Center: (h, k), $a > b$,

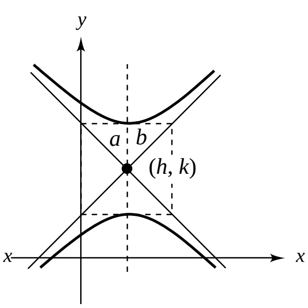
$c^2 = a^2 - b^2$

$$\begin{cases} \mathrm{X = H + A\ COS\ T} \\ \mathrm{Y = K + B\ SIN\ T} \end{cases} \qquad \begin{matrix} \mathrm{X = H + B\ COS\ T} \\ \mathrm{Y = K + A\ SIN\ T} \end{matrix}$$

$$0 \le \mathrm{T} \le 2\pi$$

Hyperbola





Focal axis: $y = k$

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

Focal axis: $x = h$

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

Center: (h, k), $c^2 = a^2 + b^2$

$$\begin{cases} \mathrm{X = H + A/COS\ T} \\ \mathrm{Y = K + B\ TAN\ T} \end{cases} \qquad \begin{matrix} \mathrm{X = H + B\ TAN\ T} \\ \mathrm{Y = K + A/COS\ T} \end{matrix}$$

$$0 \le \mathrm{T} \le 2\pi$$

Exponential Properties

If $b > 0$ and u and v are any real numbers, then

$$b^u b^v = b^{u+v} \qquad b^0 = 1$$
$$b^u/b^v = b^{u-v} \qquad b^1 = b$$
$$(b^u)^v = b^{uv}$$

Logarithm Properties

If b, u, and v are positive ($b \neq 1$) and p is any real number, then

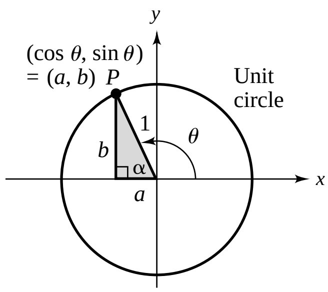
$$\log_b(uv) = \log_b u + \log_b v \qquad \log_b 1 = 0$$
$$\log_b(u/v) = \log_b u - \log_b v \qquad \log_b b = 1$$
$$\log_b(u^p) = p \log_b u$$

Natural Exponential and Logarithmic Functions

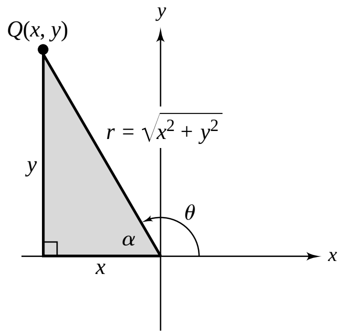
The base of the natural exponential function (e^x) and the natural logarithmic function ($\ln x$) is the number $e \approx 2.718281828$. These functions are related:

$$\ln x = y \text{ if and only if } e^y = x \qquad \begin{cases} e^{\ln x} = x & \text{for every positive number } x \\ \ln e^x = x & \text{for every real number } x \end{cases}$$

Trigonometric Functions



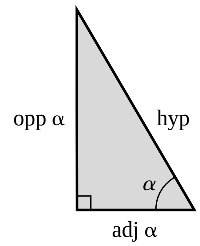
For any $P(\theta)$ on the unit circle,
$\begin{cases} \text{the } x\text{-coordinate is } \cos\theta \\ \text{the } y\text{-coordinate is } \sin\theta \end{cases}$



Angles in Standard Position

$\cos\theta = x/r \quad \sin\theta = y/r$

$\tan\theta = y/x$

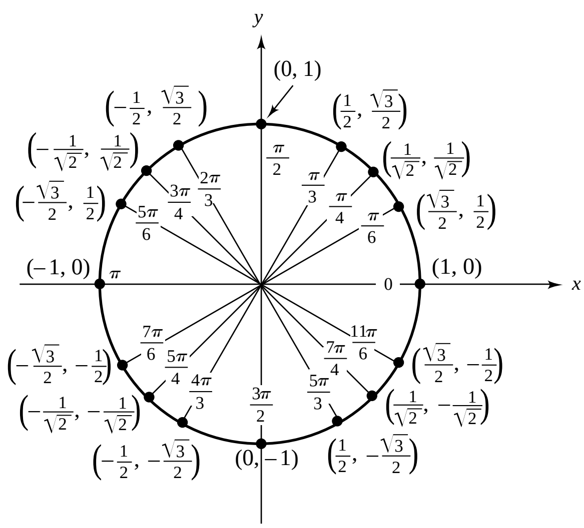


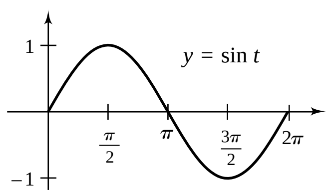
Angles in Right Triangle

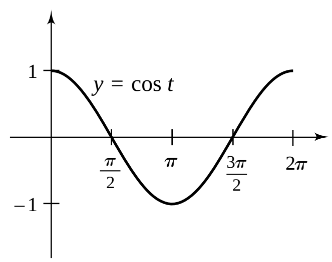
$\cos\alpha = \text{adj/hyp} \quad \sin\alpha = \text{opp/hyp}$

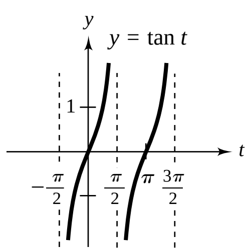
$\tan\alpha = \text{opp/adj}$

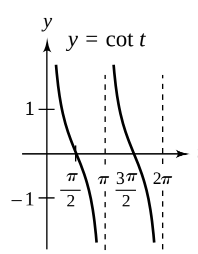
Graphs

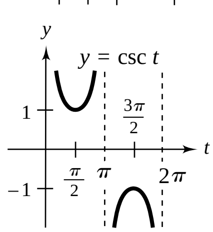


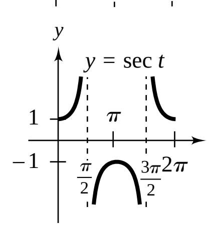






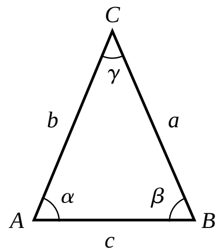






Law of Sines

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$



Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Basic Identities

$$\tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t}$$

$$\sin(-t) = -\sin t \qquad \cos(-t) = \cos t$$

$$\tan(-t) = -\tan t \qquad \sin^2 t + \cos^2 t = 1$$

$$\tan^2 t + 1 = \sec^2 t \qquad 1 + \cot^2 t = \csc^2 t$$

Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

Product-Sum Identities

$$2 \sin \alpha \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$$

$$2 \cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$$

$$2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

Double Angle Identities

$$\sin 2t = 2 \sin t \cos t$$

$$\cos 2t = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t = 2\cos^2 t - 1$$

$$\tan 2t = \frac{2 \tan t}{1 - \tan^2 t}$$

Half Angle Identities

$$\sin \frac{t}{2} = \pm\sqrt{\frac{1 - \cos t}{2}} \qquad \cos \frac{t}{2} = \pm\sqrt{\frac{1 + \cos t}{2}}$$

$$\tan \frac{t}{2} = \frac{\sin t}{1 + \cos t} = \frac{1 - \cos t}{\sin t}$$

$$\sin^2 t = \frac{1 - \cos 2t}{2} \qquad \cos^2 t = \frac{1 + \cos 2t}{2}$$

Sum-to-Product Identities

$$\sin x - \sin y = 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$$

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

Complex Numbers

Trigonometric Form

$a + bi = r(\cos \theta + i \sin \theta)$,

$r = \sqrt{a^2 + b^2}$, $\tan \theta = b/a$

Powers and Roots

$(a + bi)^n = [r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$

The n^{th} roots of $r(\cos \theta + i \sin \theta)$ are given by

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + k \cdot 360°}{n}\right) + i \sin\left(\frac{\theta + k \cdot 360°}{n}\right)\right],$$

where $k = 0, 1, 2, \ldots, n - 1$.

Reduction Formulas

$$\sin\left(\frac{\pi}{2} \pm t\right) = \cos t \qquad \sin(\pi \pm t) = \mp\sin t$$

$$\cos\left(\frac{\pi}{2} \pm t\right) = \mp\sin t \qquad \cos(\pi \pm t) = -\cos t$$

Geometric Sequence

First term $a_1 = a$, common ratio r, $a_n = ar^{n-1}$

Sum $a_1 + a_2 + \ldots + a_n = \dfrac{a}{1 - r}(1 - r^n)$

Geometric series $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}$, if $|r| < 1$

Arithmetic Sequence

First term $a_1 = a$, common difference d, $a_n = a + (n - 1)d$

Sum $a_1 + a_2 + \ldots + a_n = n\dfrac{a_1 + a_n}{2} = n\dfrac{2a + (n - 1)d}{2}$

Page 25, Ex. 47: should be $(1+\sqrt{3}i)^2$

Page 88, Ex. 38: should be . . . $R=[4,\infty)$

Page 105, Ex. 27: $f(x)=2x^2-4x+1$

Page 129, Exs. 35-36: should be (a)f^{-1} (b)f

Page 166, Line 14: should be 2nd Alpha y1)

Page 168, One line above #25: should be $f(x)$

Page 169, $Exercises\,62-65$, end of line 3: insert $or\,equal\,to$

Page 172, Line 3: should be $\frac{1\pm\sqrt{7}i}{2}$

Page 209, Exs. 65, 68: replace 0.5 by 1.4

Page 235, Ex. 31: $f(x)=4\cdot 10^{1-x}$

Page 248, #26, 36, 37: replace $g(x)$ by y

Page 277, Line 2: should position point $P(\theta)$

Page 284, Figure 40: label $S(2,0)$

Page 364, Figure 40: the fundamental cycle starts on axis

Page 366, Historical Note, paragraph 4, line 2: should be saxaphone

Page 375, Ex. 12: replace $cos\,\frac{x}{2}$ by $sin\frac{x}{2}$

Page 417, Technology Tip, number 4, line 2: should be . . . $w_n=w_0$. Trace . . .

Page 431, Ex. 53: should be $8\sqrt{2}(1+i)$

Page 449, Exs. 11-22: replace $>$ by $\geq$
Ex. 34: $a_n=\frac{(-1)^n(2n)}{n+1}$

Page 450, Exercises 61-62, line 4: $converges,\ and$

Page 460, Ex. 64: replace 0.96 by 0.72

Page 483, Ex. 2: . . . terms in the ex<u>pan</u>sion of . . .

Page 538, Technology Tip: delete initial letter of lines 2^- and 5^-

Page 596: Final graph is $r^2 = 4\,cos 2\theta$

A-2, Exercises 1.5, 63: (a) 7 (b) 7
Exercises 1.6, 36: $6ft$; Area is 3
Exercises 1, Review 45: (c) $(0, -3)$

A-5, Exercises 2.6, 11: (a) first entry is -3
63: replace f by g

A-6, Exercises 2.8, 45: we have denoted area by K, not k

A-7, Exercises 3.1, 11, 13: replace (c) by (iii)

A-8 Exercises 3.4 85: (a) 8.2 by 24.6 by 12.3

A-9, Exercises 4.2, 49: (a) $f^{-1}(x) = e^{x-1} - 2$

A-12, Exercises 5.2, 51: add an answer (a) $y = \pm\frac{\sqrt{3}}{2}$ (b) $cos\,t = \frac{1}{2}$, $sin\,t = \pm\frac{\sqrt{3}}{2}$

A-15, Exercises 6.5, 13: change "left" to "right"
15: (b) $g(x) = 3cos(2x + 3 - 2\pi)$
51: Add $f(x) = sin(2x + 2 - 2\pi)$

A-17, Exercises 7.4, 65: replace i by 1
Exercises 7.5, CYU 7: delete $<$

A-18, Exercises 8.3, 11: (b) $\frac{1}{2187}$

77: $\sqrt{\frac{1+\sqrt{5}}{2}}$

A-21, Exercises 9.6,
49: (a) $\begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$,
$B^{-1}A^{-1} = (AB)^{-1}$

A-22, Exercises 9.6, 63: (a) lower left entry should be -2

A-24, Exercises 10.5, 55: delete minus sign

Index I-1: Circular motion, 8, 259

Index I-2: Fibonacci sequence, 436, 442, 449