Theorem.

Suppose f is defined in an open interval containing $x = a$. Then, f is continuous at $x = a$ if and only if given any sequence $\{a_n\}$, $f(a_n) \to f(a)$ whenever $a_n \to a$.

proof :

I. Suppose f is continuous at $x = a$. Let $\{a_n\}$ be a sequence and suppose $a_n \to a$. Let $\epsilon > 0$. Since f is continuous at $x = a$, there exists $\delta > 0$ such that

if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

Since $a_n \to a$, there exists $M > 0$ such that if $n > M$, then $|a_n - a| < \delta$. Hence,

if $n > M$, then $|f(a_n) - f(a)| < \epsilon$.

Therefore, $f(a_n) \to f(a)$.

II. Suppose that $f(a_n) \to f(a)$ whenever $a_n \to a$. By way of contradiction, assume f is <u>not</u> continuous at $x = a$. Then, there exists $\epsilon_1 > 0$ such that for every $\delta > 0$

there exists x in $(a - \delta, a + \delta)$ such that $|f(x) - f(a)| \geq \epsilon_1$.

For n in J , let $\delta = \frac{1}{n}$. Then,

there exists a_n in $(a - \frac{1}{n}, a + \frac{1}{n})$ such that $|f(a_n) - f(a)| \geq \epsilon_1$.

We now have a sequence $\{a_n\}$ such that $a_n \to a$ and $\{f(a_n)\}$ does not converge to $f(a)$, a logical contradiction. Therefore, f must be continuous at $x = a$.