Intermediate Value Theorem. **If** $f : [a,b] \to \Re$ **is a continuous function and** M **is between** $f(a)$ **and** $f(b)$. **Then there exists** $w \in [a,b]$ **such that** $f(w) = M$.

$Proof:$

Case 1: $f(a) \le M \le f(b)$

Let $S = \{x \in [a,b] : f(x) \le M\}$. S is not empty since $a \in S$. Let $w = sup\,S$. Using the "back away principle" we can construct a sequence $\{x_n\}$ of points from S such that $w - \frac{1}{n} < x_n \le w$. Clearly $x_n \to w$. Since f is continuous, $f(x_n) \to f(w)$. Since $f(x_n) \le M$ for each n, it follows that $f(w) \le M$. We claim that $f(w) = M$.

If $f(w) < M$ then $w < b$ since $M \le f(b)$. Since f is continuous at $x = w$, for $\epsilon = M - f(w)$, there exists $\delta > 0$ with $\delta < b - w$, such that $f(w) - [M - f(w)] < f(x) < f(w) + [M - f(w)]$ for every x in $[w - \delta, w + \delta]$. Now we have $w < w + \frac{\delta}{2} < b$ and $f(w + \frac{\delta}{2}) < f(w) + [M - f(w)] = M$. So $w + \frac{\delta}{2}$ is an element of S. This is logical contradiction to the fact that $w = sup\,S$. Hence $f(w) = M$.

Case 2: $f(b) \le M \le f(a)$

The proof of this case is similar to the proof in case 1.