Theorem. If $f : [a,b] \longrightarrow \Re$ is a continuous function, then there exists w in $[a,b]$ such that $f(x) \leq f(w)$ for all x in $[a,b]$.

$Proof$:

We will first show that the function f is bounded above over $[a,b]$. Then we will consider the set $S = \{f(x) : x$ is in $[a,b]$ $\}$. With $M = sup\, S$, we will then show that there exists w in $[a,b]$ with $f(w) = M$. Since $M = sup\, S$, it will follow that $f(x) \leq f(w)$ for all x in $[a,b]$.

Suppose that f is not bounded above over $[a,b]$. Then for each positive integer n , there exists x_n in $[a,b]$ such that $f(x_n) > n$. Now the sequence $\{x_n\}$ is a bounded sequence and so it has a convergent subsequence $\{x_{k_n}\}$. Suppose $\{x_{k_n}\}$ converges to the point x in $[a,b]$. Since f is continuous, it must preserve convergent sequences and so $f(x_{k_n}) \to f(x)$. But, $f(x_{k_n}) > k_n$ for each n and so $\{f(x_{k_n})\}$ does not converge, a logical contradiction. Hence, f must be bounded above.

Let $S = \{f(x) : x$ is in $[a,b]$ $\}$. We have shown that S is bounded above, so it has a least upper bound $M = sup\, S$. We must show there exists w in $[a,b]$ with $f(w) = M$. Using the "back away principle for suprema", for each positive integer n , there exists w_n in $[a,b]$ such that $M - 1/n < f(w_n) \leq M$. We now have a sequence $\{w_n\}$ such that $f(w_n) \to M$. Since $\{w_n\}$ is a bounded sequence, it has a convergent subsequence $\{w_{k_n}\}$. Suppose $\{w_{k_n}\}$ converges to the point w in $[a,b]$. Since f is continuous, it must preserve convergent sequences and so $f(w_{k_n}) \to f(w)$. Now $\{f(w_{k_n})\}$ is a subsequence of $\{f(w_n)\}$ which converges to M. So, $f(w_{k_n}) \to M$. Because a sequence cannot converge to two different points, we must have $f(w) = M$.

Since $f(w) = M$, and $M = sup\, S$ we have $f(x) \leq f(w)$ for all x in $[a,b]$.