Math 4200

Chain Rule:

Suppose f and g are differentiable functions and $w = g \circ f$. Then w is also differentiable and $w'(x) = g'[f(x)] \cdot f'(x)$.

Proof.

Let $h(y) = \begin{cases} \frac{g(y)-g(y_0)}{y-y_0} & \text{if } y \neq y_0, \\ g'(y_0) & \text{if } y = y_0 . \end{cases}$

It follows that h is continuous at y_0.

Let $t(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{if } x \neq x_0, \\ f'(x_0) & \text{if } x = x_0 . \end{cases}$

It follows that t is continuous at x_0.

We have $\frac{w(x)-w(x_0)}{x-x_0} = \frac{g[f(x)]-g[f(x_0)]}{x-x_0} = (h \circ f)(x) \cdot t(x)$, and

so $\lim_{x \to x_0} \frac{w(x)-w(x_0)}{x-x_0} = (h \circ f)(x_0) \cdot t(x_0) = g'[f(x_0)] \cdot f'(x_0)$.