Math 4200

Linear Approximation Theorem:

Suppose f is defined in an open interval containing x_0. Then f is differentiable at x_0 if and only if there exists a linear (affine) function

$A(x) = a(x - x_0) + f(x_0)$ that approximates f near x_0 in the sense that

$$\lim_{x \to x_0} \frac{f(x) - A(x)}{|x - x_0|} = 0 .$$

Proof.

I. Suppose f is differentiable at $x = x_0$. Let $A(x) = f'(x_0)(x - x_0) + f(x_0)$.

$$\lim_{x \to x_0} \frac{|f(x) - A(x)|}{|x - x_0|} = \lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| = 0 .$$ So,

$$\lim_{x \to x_0} \frac{f(x) - A(x)}{|x - x_0|} = 0 .$$

II. Suppose there exists $A(x) = a(x - x_0) + f(x_0)$ such that

$\lim_{x \to x_0} \frac{f(x) - A(x)}{|x - x_0|} = 0$. Then, $\lim_{x \to x_0} \left| \frac{f(x) - A(x)}{x - x_0} \right| = 0$ and

$$\lim_{x \to x_0} \left| \frac{f(x) - f(x_0) - a(x - x_0)}{x - x_0} \right| = 0 ,$$

$\lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} - a \right| = 0$. This implies that $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = a$.