

Division Algorithm:

If a and b are whole numbers, $b < a$, and $b \neq 0$, then there exists whole numbers q and r such that $a = bq + r$ and $0 \leq r < b$.

Proof:

If $b = 1$ then $a = 1 \cdot a + 0$ and we are done. Suppose $b > 1$.

Let $W = \{b, 2b, 3b, 4b, \dots\}$. Now let $S = \{k \text{ in } J : a < kb\}$. $S \neq \emptyset$ since $a \in S$.

Let m be the least element of S . Then $a < mb$ and $(m-1)b \leq a$. Let $q = m-1$ and $r = a - (m-1)b$. Now $a = bq + r$. We must show that $0 \leq r < b$.

Since $(m-1)b \leq a$, $r \geq 0$. Suppose $r \geq b$. Then

$a - (m-1)b \geq b$, $(m-1)b + b \leq a$, $[(m-1)+1]b \leq a$, $mb \leq a$, a contradiction since

$mb > a$. Therefore, $a = bq + r$ and $0 \leq r < b$.