Fundamental Theorem of Calculus I: Suppose f is continuous on the interval I. Let $a \in I$, and $w(x) = \int_a^x f(t)dt$, for x in I. Then w is differentiable and $w'(x) = f(x)$.

Proof.

$$w'(x) \; = \; \lim_{h\to 0}\frac{w(x+h)-w(x)}{h} \; = \; \lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt - \int_a^{x} f(t)dt}{h} \; = \; \lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h} \; =$$

$\lim_{h\to 0}\frac{f(x_h)\cdot h}{h}$ where x_h is between x and $x+h$ (Intermediate Value Theorem). Since f is continuous,

$\lim_{h\to 0} f(x_h) \; = \; f(x)$. Then, $w'(x) \; = \; f(x)$.

Fundamental Theorem of Calculus II: Suppose f is continuous on the interval [a , b]. Suppose F is an antiderivative of f; that is, $F'(x) = f(x)$ for each x in [a , b]. Then $\int_a^b f(x)dx \; = \; F(b) - F(a)$.

Proof.

Let $w(x) = \int_a^x f(t)dt + F(a)$. Then w is differentiable, $w(a) = F(a)$, and $w'(x) \; = \; f(x)$. Since w and F have the same derivative and agree at a point, they must be identical as functions on [a , b]. Therefore, $F(b) = w(b) = \int_a^b f(t)dt \; + \; f(a)$ and it follows that $\int_a^b f(x)dx \; = \; F(b) - F(a)$.