

Math 4200

Theorem: If F is an ordered field then F must contain infinitely many elements.

Proof:

Let $f_1 = 1$, and for each $n \geq 2$, let $f_{n+1} = f_n + 1$. We will show that $\forall i, j \in J, f_i < f_j$ whenever $i < j$. Suppose $\exists i$ and $\exists j$ such that $i < j$ and $f_j \leq f_i$. Let $S = \{n \in J : i < n \text{ and } f_n \leq f_i\}$. $S \neq \emptyset$ because $j \in S$. By mathematical induction, S contains a least element, call it k . Consider f_{k-1} . Since $k-1 < k$, we have $k-1 \notin S$ and so $f_i < f_{k-1}$. Since $0 < 1$, we have $f_i + 0 < f_{k-1} + 1$. That is, $f_i < f_k$, a contradiction. So $S = \emptyset$ and $\forall i, j \in J, f_i < f_j$ whenever $i < j$. Therefore, F contains infinitely many elements.