Theorem. (Bolzano-Weierstrass)

Every bounded sequence has a convergent subsequence.

proof:

Let $\{w_n\}$ be a bounded sequence. Then, there exists an interval $[\,a_1, b_1\,]$ such that $a_1 \le w_n \le b_n$ for all n.

Either $[\,a_1, \frac{a_1+b_1}{2}\,]$ or $[\,\frac{a_1+b_1}{2}, b_1\,]$ contains infinitely many <u>terms</u> of $\{w_n\}$. That is, there exists infinitely many n in J such that a_n is in $[\,a_1, \frac{a_1+b_1}{2}\,]$ or there exists infinitely many n in J such that a_n is in $[\,\frac{a_1+b_1}{2}, b_1\,]$. If $[\,a_1, \frac{a_1+b_1}{2}\,]$ contains infinitely many terms of $\{w_n\}$, let $[\,a_2, b_2\,] = [\,a_1, \frac{a_1+b_1}{2}\,]$. Otherwise, let $[\,a_2, b_2\,] = [\,\frac{a_1+b_1}{2}, b_1\,]$.

Either $[\,a_2, \frac{a_2+b_2}{2}\,]$ or $[\,\frac{a_2+b_2}{2}, b_2\,]$ contains infinitely many terms of $\{w_n\}$. If $[\,a_2, \frac{a_2+b_2}{2}\,]$ contains infinitely many terms of $\{w_n\}$, let $[\,a_3, b_3\,] = [\,a_2, \frac{a_2+b_2}{2}\,]$. Otherwise, let $[\,a_3, b_3\,] = [\,\frac{a_2+b_2}{2}, b_2\,]$. By mathematical induction, we can continue this construction and obtain a sequence of intervals $\{[\,\mathrm{a}_n, b_n\,]\}$ such that

$i)$ for each n, $[\,\mathrm{a}_n, b_n\,]$ contains infinitely many terms of $\{w_n\}$,
$ii)$ for each n, $[\,\mathrm{a}_{n+1}, b_{n+1}\,] \subseteq [\,\mathrm{a}_n, b_n\,]$, and
$iii)$ for each n, $\mathrm{b}_{n+1} - a_{n+1} = \frac{1}{2} \cdot (b_n - a_n)$.

The nested intervals theorem implies that the intersection of all of the intervals $[\,\mathrm{a}_n, b_n\,]$ is a single point w. We will now construct a subsequence of $\{w_n\}$ which will converge to w.

Since $[\,\mathrm{a}_1, b_1\,]$ contains infinitely many terms of $\{w_n\}$, there exists k_1 in J such that w_{k_1} is in $[\,\mathrm{a}_1, b_1\,]$. Since $[\,\mathrm{a}_2, b_2\,]$ contains infinitely many terms of $\{w_n\}$, there exists k_2 in J, $k_2 > k_1$, such that w_{k_2} is in $[\,\mathrm{a}_2, b_2\,]$. Since $[\,\mathrm{a}_3, b_3\,]$ contains infinitely many terms of $\{w_n\}$, there exists k_3 in J, $k_3 > k_2$, such that w_{k_3} is in $[\,\mathrm{a}_3, b_3\,]$. Continuing this process by induction, we obtain a sequence $\{w_{k_n}\}$ such that w_{k_n} is in $[\,\mathrm{a_n}, b_\mathrm{n}\,]$ for each n. The sequence $\{w_{k_n}\}$ is a subsequence of $\{w_n\}$ since $k_{\mathrm{n+1}} > k_\mathrm{n}$ for each n. Since $\mathrm{a}_n \to w$, $b_n \to w$, and $a_n \le w_n \le b_n$ for each n, the squeeze theorem implies that that $w_{k_n} \to w$.