Math 4200

Theorem. A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.

Proof.

1. Suppose $\{a_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. There exists $N_1 > 0$ such that if $i > N_1,\ j > N_1$ then $\left| a_i - a_j \right| < \frac{\varepsilon}{2}$. Since $\{a_n\}$ is a Cauchy sequence it is bounded and hence must have a convergent subsequence. Suppose $\{a_{k_n}\}$ converges to a. There exists $N_2 > 0$ such that if $n > N_2$ then $\left| a_{k_n} - a \right| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. If $n > N$ we have $n > N_1,\ k_n > N_1,\ n > N_2$ and so
$\left| a_n - a \right| = \left| a_n - a_{k_n} + a_{k_n} - a \right| \le \left| a_n - a_{k_n} \right| + \left| a_{k_n} - a \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

2. Suppose that $\{a_n\}$ converges to a. Let $\varepsilon > 0$. There exists $N > 0$ such that if $n > N$ then $\left| a_n - a \right| < \frac{\varepsilon}{2}$. Suppose $i > N,\ j > N$. Then
$\left| a_i - a_j \right| = \left| a_i - a + a - a_j \right| \le \left| a_i - a \right| + \left| a - a_j \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.