Math 4200

Definition of e

For each n, let $a_n = \left(1+\frac{1}{n}\right)^n$. We will show that $\{a_n\}$ is increasing and bounded above.

Then, $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$.

1. $\{a_n\}$ is bounded above:

By the binomial theorem,

$$a_n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \ldots + \frac{n(n-1)\ldots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k + \ldots + \frac{n!}{n!}\left(\frac{1}{n}\right)^n$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\ldots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)$$

$$< 1+1+\frac{1}{2!}+\ldots+\frac{1}{k!}+\ldots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}$$

$$< 1+\sum_{i=0}^{\infty}\frac{1}{2^i} = 3$$

2. 1. $\{a_n\}$ is increasing:

As was done in part 1, write out the binomial expansion for both a_n and a_{n+1}. Consider the k-th term for each of these sums. We have

$$\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right) < \frac{1}{k!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\ldots\left(1-\frac{k-1}{n+1}\right).$$

Since a_{n+1} has one more term, we obtain $a_n < a_{n+1}$.