

Sequences of Real Numbers

Definition: A sequence of real numbers is simply a function $f : J \rightarrow R$. Sequences are usually denoted by $f(n) = a_n$ for each n (the n -th term of the sequence), and we write $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ to represent the entire sequence.

Definition: The sequence $\{a_n\}$ is said to converge to a if

for every $\varepsilon > 0$, there exists $N > 0$ such that if $n > N$ then $|a_n - a| < \varepsilon$

The notation is $\lim_{n \rightarrow \infty} a_n = a$ or $a_n \rightarrow a$.

Theorem: Let $\{a_n\}$ be a sequence of real numbers. Then $a_n \rightarrow a$ if and only if every open interval containing a , contains all but finitely many terms of the sequence.

Theorem: Every bounded monotone sequence converges.

Definition of e :

For each n , let $a_n = \left(1 + \frac{1}{n}\right)^n$. Since $\{a_n\}$ is increasing and bounded above, it must converge and we define $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$.

Basic Theorems on Sequences:

- i) Every convergent sequence of real numbers is bounded.
- ii) Suppose $a_n \rightarrow a$ and $b_n \rightarrow b$. Then $a_n + b_n \rightarrow a + b$ and $a_n \cdot b_n \rightarrow a \cdot b$.
- iii) Suppose $a_n \rightarrow a$, $a \neq 0$, and $\forall n, a_n \neq 0$. Then $\frac{1}{a_n} \rightarrow \frac{1}{a}$.
- iv) Squeeze Theorem: If $a_n \rightarrow L$, $b_n \rightarrow L$, and $a_n \leq w_n \leq b_n$ for each n , then $w_n \rightarrow L$.

Nested Intervals Theorem: Suppose $\{[a_n, b_n]\}$ is a sequence of closed intervals such that for each n , $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$. Then the intersection of all of these intervals is either a closed interval or a single point.

Theorem (Bolzano-Weierstrass): Every bounded sequence has a convergent subsequence.

Definition: A sequence $\{a_n\}$ is said to be a Cauchy sequence if for each $\varepsilon > 0$, there exists $M > 0$ such that if $i > M$, $j > M$ then $|a_i - a_j| < \varepsilon$

Theorem: A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.