Math 4200

Nested Intervals Theorem: Suppose $\{ [a_n , b_n] \}$ is a sequence of closed intervals such that for each n, $[a_{n+1} , b_{n+1}] \subseteq [a_n , b_n]$. Then the intersection of all of these intervals is either a closed interval or a single point.

Proof:

Consider the two sequences $\{a_n\}$ and $\{b_n\}$. It follows that $\{a_n\}$ is monotone increasing, and bounded above while $\{b_n\}$ is monotone decreasing and bounded below. Hence both sequences converge. Suppose $a_n \rightarrow a$ and $b_n \rightarrow b$.

Either $a \leq b$ or $b < a$. If $b < a$, let $\epsilon = \frac{a-b}{2}$. There exists $N_1 > 0$ such that if n is in J and $n > N_1$ then $a_n > a - \epsilon$. There exists $N_2 > 0$ such that if n is in J and $n > N_2$ then $b_n < b + \epsilon$. Let k be a natural number, $k \geq \max\{N_1 , N_2\}$. We now have $b_k < b + \epsilon = a - \epsilon < a_k$ and so $b_k < a_k$, a contradiction. Therefore, $a \leq b$.

Since $a_n \leq a \leq b \leq b_n$ for every n, it follows that $[a,b] \subseteq \cap [a_n , b_n]$. If $w < a$, then for some k, $w < a_k$ and so $w \notin \cap [a_k , b_k]$. If $b < y$, then for some j, $b_j < y$ and so $y \notin \cap [a_j , b_j]$. Thus, $[a,b] = \cap [a_n , b_n]$. Note that $[a,b]$ is a single point if and only if $b_n - a_n \rightarrow 0$.