Math 4200

Theorem. Suppose $a_n \to a$, $a \neq 0$, and $\forall n,\ a_n \neq 0$. Then $\dfrac{1}{a_n} \to \dfrac{1}{a}$.

Proof.

Suppose $a > 0$. Since $a_n \to a$ there exists $N_1 > 0$ such that if $n > N_1$ then $|a_n - a| < \dfrac{a}{3}$. So, if $n > N_1$ then $a - \dfrac{a}{3} < a_n < a + \dfrac{a}{3}$. That is, if $n > N_1$ then $\dfrac{2}{3} a < a_n$. Let $\varepsilon > 0$. Since $a_n \to a$, there exists $N_2 > 0$ such that if $n > N_2$ then $|a_n - a| < \varepsilon \dfrac{2a^2}{3}$. Let $N = \max\{N_1, N_2\}$. If $n > N$ we have $n > N_1,\ n > N_2$ and so

$$\left| \frac{1}{a_n} - \frac{1}{a} \right| = \left| \frac{a - a_n}{a_n a} \right| = \frac{|a - a_n|}{a_n a} < \frac{|a - a_n|}{\frac{2}{3} a^2} < \frac{\varepsilon \frac{2a^2}{3}}{\frac{2}{3} a^2} = \varepsilon .$$

Hence $\dfrac{1}{a_n} \to \dfrac{1}{a}$.

The proof for the case $a < 0$ is similar.