Math 4200

Theorem. Every convergent sequence of real numbers is bounded.

proof:

Let $\{a_n\}$ be a sequence and suppose $a_n \to L$. For $\epsilon = 13$, there exists a positive integer $M > 0$ such that if $n > M$ then $| a_n - L| < 13$. That is, if $n > M$ then $L - 13 < a_n < L + 13$.

Now let $B = max\{| a_1|, | a_2|, \dots , | a_M|, | L - 13 |, | L + 13 | \}$. If $1 \leq n \leq M$, then clearly $| a_n| \leq B$. If $n > M$ then $L - 13 < a_n < L + 13$, and so $| a_n| \leq max\{ | L - 13 |, | L + 13 | \}$. Hence, for all n, $| a_n| \leq B$. Therefore, $\{a_n\}$ is bounded.

Theorem. If $\{a_n\}$ and $\{b_n\}$ are sequences having limits A and B respectively, then the sequence $\{a_n \cdot \mathrm{b}_n\}$ converges to $A \cdot B$.

proof:

Since $\{a_n\}$ is bounded, there exists $K > 0$ such that $| a_n| \leq K$ for each n. Let $\epsilon > 0$. Since $\{\mathrm{b}_n\}$ converges to B, there exists $M_1 > 0$ such that if n $> M_1$ then $| b_n - B | < \frac{\epsilon}{2\mathrm{K}}$. Since $\{a_n\}$ converges to A, there exists $M_2 > 0$ such that if n $> M_2$ then $| \mathrm{a}_n - \mathrm{A} | < \frac{\epsilon}{2(|\mathrm{B}|+1)}$. Choose $M = max\{M_1, M_2\}$. For $n > M$,

$| a_n b_n - AB | = | a_n b_n - a_n B + a_n B - AB | \leq | a_n b_n - a_n B | + | a_n B - AB | =$
$| a_n| \, | b_n - B | + | B | \, | a_n - A | < K \cdot \frac{\epsilon}{2\mathrm{K}} + | B | \cdot \frac{\epsilon}{2(|\mathrm{B}|+1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence, the sequence $\{a_n \cdot \mathrm{b}_n\}$ converges to $A \cdot B$.