

Squeeze Theorem for Sequences

If $\lim_{n \rightarrow \infty} a_n = L$, $\lim_{n \rightarrow \infty} b_n = L$, and for all n , $a_n \leq c_n \leq b_n$,
then $\lim_{n \rightarrow \infty} c_n = L$.

Proof:

Let $\epsilon > 0$. Since $\lim_{n \rightarrow \infty} a_n = L$, there exists $M_1 > 0$ such that if $n > M_1$,
then $|a_n - L| < \epsilon$. That is, if $n > M_1$, then $L - \epsilon < a_n < L + \epsilon$.
Since $\lim_{n \rightarrow \infty} b_n = L$, there exists $M_2 > 0$ such that if $n > M_2$,
then $|b_n - L| < \epsilon$. That is, if $n > M_2$, then $L - \epsilon < b_n < L + \epsilon$.

Let $M = \max\{M_1, M_2\}$. Now if $n > M$, we have
 $L - \epsilon < a_n \leq c_n \leq b_n < L + \epsilon$. So, if $n > M$, then $L - \epsilon < c_n < L + \epsilon$.
That is, if $n > M$, $|c_n - L| < \epsilon$. Hence $\lim_{n \rightarrow \infty} c_n = L$.