Definition. A sequence $\{a_n\}$ is *increasing* if and only if $a_n \leq a_{n+1}$ for all positive integers *n*. A sequence $\{a_n\}$ is *decreasing* if and only if $a_n \geq a_{n+1}$ for all positive integers *n*. A sequence is *monotone* if and only if it is either increasing or decreasing.

Theorem. Every bounded monotone sequence converges.

Proof:

Suppose $\{a_n\}$ is a bounded increasing sequence. Then there exists $M > 0$ such that $|a_n| \leq M$ for all n . If $S = \{a_n : n \in J\}$, then S is bounded above by M. By the completeness axiom, S has a least upper bound, call it a . We will now show that the sequence $\{a_n\}$ converges to a .

$Let\ \epsilon > 0$. Using the backaway principle, there exists a_k such that $a - \epsilon < a_k \leq a$. Consider a_n where $n > k$. Since $\{a_n\}$ is increasing, $a_k \leq a_n$. Since a is an upper bound for S , $a_n \leq a$. For $n > k$, $a - \epsilon < a_k \leq a_n \leq a < a + \epsilon$, and $a - \epsilon < a_k < a + \epsilon$. Thus, for $n \geq k$ we have $|a_n - a| < \epsilon$.

Suppose $\{a_n\}$ is a bounded decreasing sequence. The proof is similar to the one above except you use the greatest lower bound and the backaway principle for infima. You could also consider the sequence $\{-a_n\}$ and apply the above result .