Math 4200
Are there really uncountably many real numbers?

Let S denote the collection of all sequences of zeros and ones. Then S is an uncountable set.

Proof:

Suppose S is a countable set. Then the elements of S can listed as follows.

$$
\begin{array}{ll}
S: & a_{11}, a_{12}, a_{13}, a_{14}, \ldots, a_{1n}, \ldots \\
 & a_{21}, a_{22}, a_{23}, a_{24}, \ldots, a_{2n}, \ldots \\
 & a_{31}, a_{32}, a_{33}, a_{34}, \ldots, a_{3n}, \ldots \qquad \text{where } \forall i, j \quad a_{ij} = 0 \text{ or } 1 \\
 & \quad \ldots \\
 & \quad \ldots \\
 & a_{n1}, a_{n2}, a_{n3}, a_{n4}, \ldots, a_{nn}, \ldots \\
 & \quad \ldots \\
 & \quad \ldots
\end{array}
$$

Let $b_1, b_2, b_3, b_4, \ldots, b_n, \ldots$ be a sequence defined by $b_k = \begin{cases} 0 & \text{if } a_{kk} = 1 \\ 1 & \text{if } a_{kk} = 0 \end{cases}$. This sequence belongs to S and must therefore be listed. However, $b_1, b_2, b_3, b_4, \ldots, b_n, \ldots$ is not the first sequence listed since $b_1 \neq a_{11}$. It is not the second sequence listed since $b_2 \neq a_{22}$. In fact, for each n, $b_1, b_2, b_3, b_4, \ldots, b_n, \ldots$ is not the *n-th* sequence listed since $b_n \neq a_{nn}$. Hence, $b_1, b_2, b_3, b_4, \ldots, b_n, \ldots$ is not listed.

This is a logical contradiction since it was assumed that all the elements of S could be listed. The assumption that S is a countable set led to this contradiction; therefore S is an uncountable set.

Yes, there really are that many real numbers:

Let $W = \{ .w_1 w_2 w_3 \ldots w_i \ldots$ where for each i, $w_i = 0$ or $w_i = 1\}$. Now, W is equivalent to the set S above. The 1-1 correspondence is obtained by associating each sequence of zeros and ones $b_1, b_2, b_3, b_4, \ldots, b_n, \ldots$ with the decimal expansion $.b_1\ b_2\ b_3\ b_4\ \ldots\ b_n, \ldots$. The real line now contains an uncountable subset so it must also be uncountable.