Theorem (Product Rule)

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then $\lim_{x\to a} f(x)\cdot g(x) = L\cdot M$.

$Proof$:

Let $\epsilon > 0$. There exists $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1,$ then $|f(x) - L| < 666.$ Then $L - 666 < f(x) < L + 666$ $whenever$ $0 < |x-a| < \delta_1.$ Let $B = max\{|L+666|, |L-666|\}$. We now have $|f(x)| < B$ $whenever$ $0 < |x-a| < \delta_1$.

There exists $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2,$ then $|g(x) - M| < \frac{\epsilon}{2B}$.

There exists $\delta_3 > 0$ such that if $0 < |x-a| < \delta_3,$ then $|f(x) - L| < \frac{\epsilon}{2(|M|+1|)}$.

Let $\delta = min\{\delta_1, \delta_2, \delta_3\}.$ If $0 < |x-a| < \delta$ then

$|f(x)\cdot g(x) - L\cdot M| = |f(x)\cdot g(x) - f(x)\cdot M + f(x)\cdot M - L\cdot M| \leq$

$|f(x)||g(x) - M| + |M||f(x) - L| \leq B\,|g(x) - M| + |M||f(x) - L| <$

$B\cdot\frac{\epsilon}{2B} + |M|\cdot\frac{\epsilon}{2(|M|+1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.