PROBABILITY: The Mathematical Analysis of Chance Processes.

Definitions and Notation:

S = sample space or outcome space.

An event is a subset of S.

A random variable is a function $X : S \to \Re$.

Chance processes are described and analyzed mathematically using random variables.

Discrete Random Variables:

$X : S \to \Re$ is a discrete random variable if X assumes at most countable many values. (The values of X can be listed in a sequence.)

$p_X(x) = P(X = x)$ is called the probability frequency function or probability mass function for X.

$F_X(x) = P(X \le x)$ is called the distribution function or cumulative distribution for X.

Note: $\lim_{x \to -\infty} F(x) = 0$, and $\lim_{x \to \infty} F(x) = 1$.

F is non decreasing and right continuous.

Expected value of $X = E(X) = \sum_k x_k P(X = x_k) = \sum_k x_k p_X(x) = \mu_X$

Note: Expected value of $g(X) = \sum_k g(x_k)\ p_X(x)$

Variance of $X = E(X - \mu_X)^2 = E(X^2) - \mu_X^2 = \sigma_X^2$.

Standard Deviation of $X = \sigma_X$.

Some Important Discrete Random Variables:

1. Geometric:

X = number of independent trials until the first success where
p = probability of success on any given trial.

$p(k) = P(X=k) = (1-p)^{k-1}p = q^{k-1}p$, $k = 1, 2, 3, \ldots$

$E(X) = \frac{1}{p}$, $\text{var}(X) = \frac{q}{p^2}$

2. Binomial:

X = number of successes in n independent trials where
p = probability of success on any given trial.

$p(k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, 2, \ldots, n$

$E(X) = np$, $\text{var}(X) = npq$

3. Negative Binomial:

X = number of independent trials until r successes where
p = probability of success on any given trial.

$p(k) = \binom{k-1}{r-1} p^r q^{k-r}$, $k = r, r+1, r+2, \ldots$

$E(X) = \frac{r}{p}$, $\text{var}(X) = \frac{r(1-p)}{p^2}$

4. Hypergeometric:

Draw n balls, without replacement, from a box containing N balls, of which m are white and N-m are black. Let X = number of white balls selected.

$$p(k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}} \quad , \quad k = 0, 1, 2, \ldots , n$$

$$E(X) = \frac{nm}{N} \quad , \quad var(X) = \frac{N-n}{N-1} np(1-p)$$

Note: In statistics, the hypergeometric distribution is called "sampling without replacement".

5. Poisson:

X = the number of rare events occurring in any fixed interval or region..

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad , \quad k = 0, 1, 2, \ldots.$$

$$E(X) = \lambda \quad , \quad Var(X) = \lambda$$

Note: For n large and p small , the binomial distribution is approximately a Poisson distribution with $\lambda = np$.