Poisson Approximation to the Binomial Distribution

Suppose X is a binomial distribution with parameters *n* and *p* and consider the case when n is large, *p* is small, and $np = \lambda$.

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} =$$

$$\frac{n!}{k!(n-k)!}\left(\frac{\lambda^k}{n^k}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k} =$$

$$\frac{\lambda^k}{k!}\,\frac{n!}{(n-k)!n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k} =$$

$$\frac{\lambda^k}{k!}\,\frac{n(n-1)(n-2)...(n-k+1)}{n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k} =$$

$$\frac{\lambda^k}{k!}\,\frac{n}{n}\,\frac{n-1}{n}\,\frac{n-2}{n}\cdots\frac{n-k+1}{n}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k} \;\rightarrow\; \frac{\lambda^k}{k!}\,e^{-\lambda}$$