Joint Distributions

Let X and Y be random variables. Their *joint cumulative distribution function* is given by $F_{X,Y}(x,y) = P(X \le x, Y \le y)$.
Note that

$$P(x_1 \le X \le x_2 \; , \; y_1 \le Y \le y_2) = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) \; .$$

If X and Y are discrete random variables, then their *joint probability mass function* is given by

$$p_{X,Y}(i,j) = P(X = i, Y = j) \; .$$

Note that $p_X(i) \; = \sum_j p_{X,Y}(i,j) \;\; and \;\; p_Y(j) \; = \sum_i p_{X,Y}(i,j)$

The continuous random variables X and Y are said to be *jointly continuous* if there exists a function $f_{X,Y} : R^2 \to [0,\infty)$ such that for any set A,

$$P\{(X,Y) \in A\} = \iint_A f_{X,Y}(x,y)\,dxdy \quad .$$

$f_{X,Y}$ is called the *joint density function* for the random variables X and Y.

Note that

$$\frac{\partial^2 F}{\partial x \partial y}(x,y) = f_{X,Y}(x,y) \;\; and \;\; f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \;\; and \;\; f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \; .$$

The random variables X and Y are said to be *independent* if

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \;\; for \; all \;\; x, y$$

Remark: The above definitions are easily generalized to the situation where we are considering many random variables: $X_1, X_2, X_3, \ldots, X_n$.