1.5 ONE-VARIABLE SENTENCES: ALGEBRAIC AND GRAPHICAL TOOLS

> There is something very funny here. We can teach a computer to decide whether a mathematical formula is well formed or not. That's very easy. But we cannot teach a computer to talk, to form sentences. It is obviously a million times as hard. But take any kid who learns how to speak. If, as a kid, he hears two languages, he learns two languages. If he is mentally retarded, he still becomes bilingual. He will know fewer words, but he will know those words in both languages. He will form sentences. Now try to explain to him what is a well formed algebraic formula!
>
> *Lipman Bers*

In mathematics, language is the key to understanding while problem solving is the key to learning. The two are closely related since it is impossible to solve a problem without first understanding it. Learning to communicate is fundamental to all education. People express ideas in words, which they combine to form meaningful sentences. Sentences are the basic elements of communication.

> *I was definitely not the best student in my class. There were five of us who graduated, and there was one girl who was much smarter than I. There was another very bright kid. I was maybe third out of five. I went to the university thinking that I would make C's but I made A's without any trouble.*
>
> Mary Ellen Rudin

The language of mathematics is both precise and concise, often making use of symbols. However, mathematical symbols are combined together to form sentences having similar grammatical structures, including subjects and predicates, as sentences in our more familiar daily language.

The use of symbols allows us to write sentences in very compact form. For instance, in place of "The sum of 2 and 3 is 5," we write "$2 + 3 = 5$." Similarly, the symbolic sentence "$x \leq 4$" in everyday language means "x is less than or equal to 4," which is a complete sentence.

Statements and Open Sentences

One of our primary interests in mathematics is to determine the truth value of a statement, or to find all values of a particular variable that make a sentence true. Consider the following sentences.

(a) $2 + 5 = 7$ **(b)** $3 > \sqrt{16}$ **(c)** $x^2 - 2x - 3 = 0$

(d) Every even integer greater than 2 is the sum of two primes.

We can say that sentence **(a)** is true and sentence **(b)** is false (since $\sqrt{16} = 4$). Sentence **(d)** has a truth value; it is either true or false. But in more than 300 years no one has been able to prove either that it is true or to find a single counterexample to show that it is false. This famous unsolved problem is known as Goldbach's conjecture (see the Historical Note).

We cannot assign a truth value to sentence **(c)** above unless we replace x by a number. Replacing x by 2 gives $2^2 - 2 \cdot 2 - 3 = 0$, which is false. Replacing x by 3, however, gives $3^2 - 2 \cdot 3 - 3 = 0$, which is true. Sentence **(c)** is called an **open sentence;** there is no truth value until the variable x (a placeholder, an open spot) is filled. The remaining three sentences, because they are either true or false, are called **statements.**

Definition: statement

A **statement** is a sentence that has a truth value, either true or false.

HISTORICAL NOTE GOLDBACH, COUNTEREXAMPLES, AND UNSOLVED PROBLEMS

On page 38 we mentioned Goldbach's conjecture that "Every even integer greater than 2 is the sum of two primes." Goldbach made this assertion in 1742 in a letter to Euler. A quick look at the first cases shows how reasonable the conjecture seems:

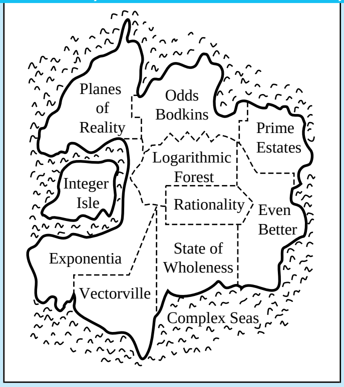
$$4 = 2 + 2, \qquad 6 = 3 + 3,$$
$$8 = 5 + 3, \qquad 10 = 7 + 3.$$

The question, of course, is how long this continues. Could we use computers to check?

Computer searches support Goldbach's assertion for *all even numbers up to a hundred million,* but forever? Who knows? A single *counterexample,* one even number that is not the sum of two primes, would prove Goldbach's conjecture false. Computers could conceivably show that Goldbach was wrong; no blind search process can ever prove him right.

An unsolved problem in mathematics does not necessarily mean there is no solution; it means that we cannot yet prove or disprove an assertion.

Each year, some long-standing questions are answered, and each answer raises more questions.



Four colors suffice to color even complicated maps.

Note: At a Cambridge University seminar in June, 1993, Professor Andrew Wiles of Princeton announced the proof of a conjecture about elliptic curves. His proof establishes that Fermat's Last Theorem is true.

Some recent milestones:

Four Color Theorem Four colors are enough to color any map; part of the proof required 1200 hours of computer time.

Classification of Simple Groups There are exactly 26 simple groups of a special type; the proof requires thousands of pages contributed by many mathematicians. The biggest group, called "The Monster," has more than 10^{53} elements.

Fermat's Last Theorem The equation $x^n + y^n = z^n$ has no solutions in integers if $n > 2$. (For $n = 2$ there are lots; see Explore and Discover in Section 5.2.) A major step was taken by a German mathematician in 1983, marking the most progress in more than a hundred years. The problem is over three hundred years old.

What makes such progress exciting is more than just the solution of an unsolved problem. Work on one problem can help us understand others, and light shed in one corner of mathematics often lights up whole new vistas whose existence we may not even have suspected previously.

Solving Open Sentences

Open sentences include equations and inequalities. Because many of the methods of solution are the same, we treat equations and inequalities together. By solving an open sentence we mean finding all the admissible replacement values for the variable that make the sentence *true.* The **domain** or **replacement set** for a variable is the set of numbers that the problem allows as replacements for the variable. Any restrictions on the domain must be clearly stated; otherwise we adopt the following convention regarding domains.

Domain convention

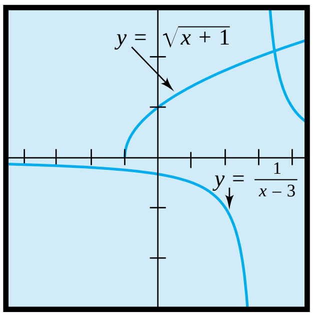
If no restrictions are stated, the domain of a variable is assumed to be the set of all real numbers that give meaningful real number statements. This excludes any division by zero or square roots of negative numbers.

▶EXAMPLE 1 ***Finding domain*** Find the domain of the variable x in the open sentence:

$$\sqrt{x + 1} \geq \frac{1}{x - 3}.$$

Solution

Algebraic Assuming the domain convention, we require $x - 3 \neq 0$ or $x \neq 3$, and we must have $x + 1 \geq 0$, or $x \geq -1$. Taking the conditions together, the domain D of the variable consists of all real numbers greater than or equal to -1, except for 3. In interval notation, D is $[-1, 3) \cup (3, \infty)$.



[−4.5, 4.5] by [−3, 3]

FIGURE 23

Graphical Graphing calculators can be used to confirm conclusions about the domains of equations. If we graph the two expressions that appear in the above inequality, $y = \sqrt{x + 1}$ and $y = 1/(x - 3)$, we get the two graphs in Figure 23. Tracing along each curve, the calculator shows that there is no y-value for $y = \sqrt{x + 1}$ when x is less than -1. Similarly, there is no y-value for the other curve when $x = 3$. We can thus literally *see* that the domain of the open sentence consists of all real numbers greater than -1, except for 3. ◀

The **solution set** for an open sentence is the set of all numbers in the domain that yield true statements. To solve an equation or inequality means to find the solution set, and the *roots* of an equation are the numbers in the solution set.

Solving equations and inequalities is not always easy, but to simplify this work we most generally perform operations that give us equivalent open sentences, hoping to reach a sentence whose solution set is obvious. For example, $2x - 3 = 5$ is equivalent to $2x = 8$, which is equivalent to $x = 4$. The solution to $2x - 3 = 5$ is 4. Equivalent open sentences have the same solution set. The following equivalence operations on open sentences yield equivalent open sentences.

Equivalence operations

1. *Replace* any expression in the sentence by another expression identically equal to it.
2. *Add* or *subtract* the same quantity on both sides.
3. For an equation, *multiply* or *divide* both sides by the same nonzero quantity.
4. For an inequality, *multiply* or *divide* both sides by the same positive quantity, or multiply or divide both sides by the same negative quantity and reverse the direction of the inequality.

The last equivalence operation for inequalities points up one of the major differences between equations and inequalities: multiplication by a negative number reverses the direction of an inequality. To avoid the necessity of treating separate cases, we suggest that you *never* multiply an inequality by an expression involving a variable.

Linear Equations and Inequalities

A **linear open sentence** is one that is equivalent to

$ax + b \;\square\; 0$, with $\square$ replaced by $=, <, >, \leq$, or $\geq$,

where a and b are constants and a is not zero.

The equivalence operations allow us to find the solution set for any linear open sentence.

Strategy: **(a)** First use Equivalence Operation 2 to get all x-terms on one side and the constants on the other (i.e., subtract x and 4 from both sides).
(b) Similarly, use Operation 2 to collect the x-terms on one side and constants on the other.

▶EXAMPLE 2 ***Solving linear open sentences*** Find the solution set.

(a) $3x + 4 = x - 1$
(b) $2 - 3x \leq 4$

Solution

Follow the strategy.

(a) $3x - x = -1 - 4, \quad$ or $\quad 2x = -5.$

Divide both sides by 2 (Equivalence Operation 3), giving $x = -\frac{5}{2}$. The solution set is $\{-\frac{5}{2}\}$.

(b) $-3x \leq 4 - 2, \quad$ or $\quad -3x \leq 2.$

By Equivalence Operation 4, we can divide both sides by -3 if we reverse the direction of the inequality, getting $x \geq -\frac{2}{3}$. The solution set is $\{x \mid x \geq -\frac{2}{3}\}$, or in interval notation, $[-\frac{2}{3}, \infty)$. ◀

Factorable Equations and Inequalities

An equation or inequality that can be written in the form of a product (or quotient) of *linear* factors on one side and 0 on the other side, can be solved using a variety of techniques. All methods we discuss for finding the solution set of such open sentences rely on the Signed Product Principles.

Signed product principles

Zero-product Principle A product of factors *equals zero* if and only if *at least one factor equals zero.*
Positive-product Principle A product of two factors is *positive* if and only if they have the *same sign.*
Negative-product Principle A product of two factors is *negative* if and only if they have *opposite signs.*

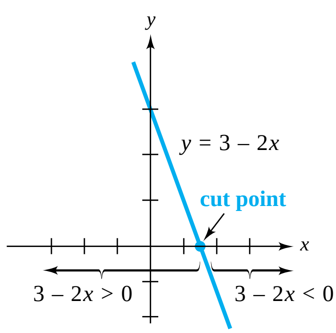


FIGURE 24

Associated with each linear factor is what we call a **cut point,** the point on the number line where *the factor equals 0.* A linear expression is always positive in one direction from its cut point and negative in the other direction. For example, $3 - 2x = 0$ when $x = \frac{3}{2}$, so $\frac{3}{2}$ is the cut point. When we replace x by any number greater than $\frac{3}{2}$ (to the right of $\frac{3}{2}$ on the number line), $3 - 2x$ is negative; for any x less than $\frac{3}{2}$, $3 - 2x$ is positive. The name *cut point* reminds us that $\frac{3}{2}$ *cuts* the number line into a piece where $3 - 2x$ is positive and a piece where $3 - 2x$ is negative. Looking at the graph of the line $y = 3 - 2x$ (Figure 24), we can see where the line cuts the x-axis, separating the portion to the left of $(\frac{3}{2}, 0)$, where the y-coordinates are positive $(3 - 2x > 0)$, from the portion to the right.

Quadratic Equations and Inequalities

A **quadratic open sentence** is one that is equivalent to

$$ax^2 + bx + c \;\square\; 0, \text{ with } \square \text{ replaced by } =, <, >, \leq, \text{ or } \geq,$$

where a, b, and c are constants and a is not zero.

▶EXAMPLE 3 ***Quadratic open sentences*** Find the solution set.

(a) $2x^2 - 3x - 2 = 0$ **(b)** $2x^2 - 3x - 2 < 0$

Solution

(a) $2x^2 - 3x - 2 = (2x + 1)(x - 2)$, and so by Equivalence Operation 1 the given equation is equivalent to

$$(2x + 1)(x - 2) = 0.$$

By the zero-product principle,

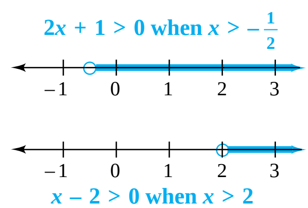
$$2x + 1 = 0 \quad \text{or} \quad x - 2 = 0.$$

Each of these equations determines a cut point for the linear factor and hence a *root of the original equation* (Check!), so the solution set is $\{-\frac{1}{2}, 2\}$.

(b) ***Algebraic*** Equivalence Operation 1 says that the inequality is equivalent to

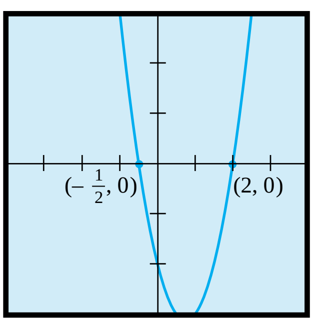
$$(2x + 1)(x - 2) < 0.$$

In the solution to part **(a)**, we found the cut points for the expression, $-\frac{1}{2}$ from $2x + 1 = 0$, and 2 from $x - 2 = 0$. We use the cut points to visualize the sign pattern for the product. The factor $2x + 1$ is positive when $x > -\frac{1}{2}$, that is, to the right of $-\frac{1}{2}$ on the number line, and the factor $x - 2$ is positive to the right of 2. We show this information on a pair of number lines.



By the negative-product principle, the desired inequality holds when the two factors have *opposite signs*. From the two number lines, we can see that the two factors have opposite signs, $2x + 1$ positive, and $x - 2$ negative, between $-\frac{1}{2}$ and 2. Thus the solution set is the open interval, $(-\frac{1}{2}, 2)$.

Graphical Having the given inequality in factored form (so we can identify the cut points), we can use a graphing calculator to see where the product is positive or negative. We enter $y = (2x + 1)(x - 2)$ and graph. See Figure 25. From the figure it is apparent that for any number x between the cut points, the y-value is negative, which is the condition we want. We can see that the solution set is the open interval, $(-\frac{1}{2}, 2)$. ◀



[−4, 4] by [−3, 3]

FIGURE 25

For quadratic expressions that cannot be factored readily, we can use the quadratic formula to find the zeros and hence to identify the cut points.

Strategy: Use the quadratic formula and determine whether or not the solutions are real numbers.

▶EXAMPLE 4 ***Using the quadratic formula*** Apply the quadratic formula to solve $2x^2 + 4x + 3 = 0$, where the domain set is

(a) the set of real numbers **(b)** the set of complex numbers.

Solution

Substituting 2 for a, 4 for b, and 3 for c in the quadratic formula gives

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(3)}}{2 \cdot 2} = \frac{-4 \pm \sqrt{-8}}{4} = \frac{-2 \pm \sqrt{2}i}{2}.$$

Since the solutions are imaginary numbers, we conclude that:

(a) The given equation has no solutions in R.
(b) In C the solutions are:

$$\frac{-2 + \sqrt{2}i}{2} \quad \text{and} \quad \frac{-2 - \sqrt{2}i}{2}. \quad ◀$$

More on Quadratic Inequalities

The solution sets for the inequality and the equation in Example 3 illustrate a general relationship that applies to a broad class of quadratic inequalities. Given an inequality $ax^2 + bx + c \;\square\; 0$, we speak of the **related equation,** $ax^2 + bx + c = 0$. If the related equation has two distinct real roots, say $r_1 < r_2$, then the solution set for the inequality consists either of

all points *between* r_1 and r_2, or all points *outside* the interval (r_1, r_2).

The numbers r_1 and r_2 are included if the inequality sign is either $\leq$ or $\geq$.

Strategy: Find the roots of the related equation $-x^2 + 2\sqrt{2}x + 2 = 0$ by using the quadratic formula, with $a = -1$, $b = 2\sqrt{2}$, $c = 2$ to find r_1 and r_2. Pick a test number to see whether the solution set is inside or outside the interval (r_1, r_2).

▶EXAMPLE 5 ***Quadratic open sentence*** Find the solution set for $-x^2 + 2\sqrt{2}x + 2 \leq 0$.

Solution

Follow the strategy.

$$x = \frac{-2\sqrt{2} \pm \sqrt{(2\sqrt{2})^2 - 4(-1)(2)}}{-2}$$
$$= \frac{-2\sqrt{2} \pm 4}{-2}$$
$$= \sqrt{2} \pm 2.$$

This gives us cut points $r_1 = \sqrt{2} - 2$ and $r_2 = \sqrt{2} + 2$ (about -0.59 and 3.41), both of which are included in the solution set S. Using either an analysis of the sign pattern or from a graph of $y = -x^2 + 2\sqrt{2}x + 2$, we find that S consists of r_1, r_2, and everything outside the interval (r_1, r_2). That is,

$$S = (-\infty, \sqrt{2} - 2] \cup [\sqrt{2} + 2, \infty). \quad ◀$$

More Applications of the Product Principles

We can use the zero-product principle whenever we have a product equal to zero. The next example illustrates some typical uses.

Strategy: Use Equivalence Operation 2 to get a zero on one side of the open sentence. **(a)** Factor as far as possible and apply the zero-product principle. **(b)** Get a single fraction and identify cut points.

▶EXAMPLE 6 ***Solving other open sentences*** Find the solution set.

(a) $x^3 = x^2 + 4x$ **(b)** $1 < \dfrac{3}{x+1}$

Solution

(a) Subtract $x^2 + 4x$ from both sides to get zero on one side and factor.

$$x^3 - x^2 - 4x = 0 \qquad \text{or} \qquad x(x^2 - x - 4) = 0$$

By the zero-product principle, either $x = 0$ or $x^2 - x - 4 = 0$. We can use the quadratic formula to find the roots of the second equation:

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-4)}}{2} = \frac{1 \pm \sqrt{17}}{2}.$$

The solution set is

$$\left\{0, \frac{1 + \sqrt{17}}{2}, \frac{1 - \sqrt{17}}{2}\right\}.$$

(b) First, we get a zero on one side by subtraction.

$$1 - \frac{3}{x+1} < 0.$$

Combining fractions we get $1 - \dfrac{3}{x+1} = \dfrac{x-2}{x+1}$. Therefore, by Equivalence Operation 1, the given inequality is equivalent to

$$\frac{x-2}{x+1} < 0.$$

The sign properties for quotients are the same as for products; to be negative, the two factors, $x - 2$ and $x + 1$, must have opposite signs. Thus -1 and 2 are *cut points*. Choose a test number in each of the three intervals, $(-\infty, -1)$, $(-1, 2)$, or $(2, \infty)$, say -2, 1, and 5. Go back to the original inequality and replace x by each test number. The results are, respectively, $1 < -3$ (false), $1 < \frac{3}{2}$ (true), and $1 < \frac{3}{6}$ (false). The solution set is the interval $(-1, 2)$. ◀

WARNING: If we were to "clear fractions" by multiplying both sides of the inequality in Example 6(b) by $x + 1$, **we would not get an equivalent inequality.** In Equivalence Operation 4, the inequality may **remain or reverse,** depending on the sign of the multiplier, and if there is a variable, the sign may change.

Equations and Inequalities Involving Absolute Values

When working with an open sentence such as $|2x + 1| \le 3$ or $|x - 1| > \sqrt{2}$, it is often easier to replace the sentence with an equivalent one without absolute values. To understand the appropriate replacements, it is helpful to see a picture.

TECHNOLOGY TIP ◆ ***Entering absolute values***

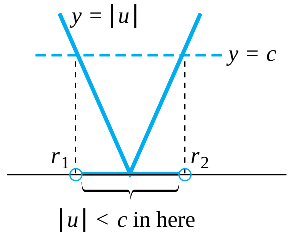
On the TI-82 and TI-81 the absolute value key is easy to spot in the left column, 2nd ABS. On the HP-38 ABS is above the $-x$ key. On other calculators the ABS key is hidden:

TI-85, 2nd MATH F1(NUM) F5 (Abs) .

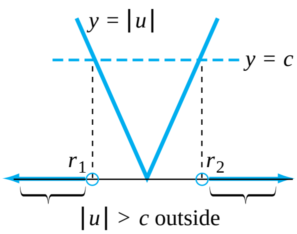
Casio, SHIFT MATH F3(NUM) F1 (Abs) .

HP-48, MTH REAL NXT ABS.

On all calculators except the HP–48, the ABS key enters the function on the home screen or on the function menu, and the HP–48 does the same thing when you write a function in tick marks.



(a)



(b)

FIGURE 27

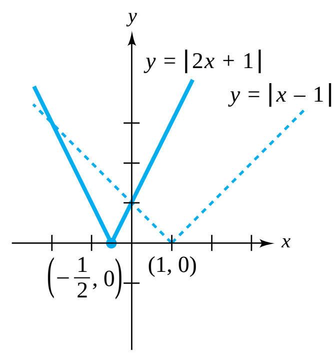


FIGURE 26

Figure 26 shows graphs of absolute value expressions $y = |2x + 1|$ and $y = |x - 1|$. The graph of any expression of the form $y = |ax + b|$ (entered on most graphing calculators as Y = ABS(AX + B)) is some sort of "vee" with its corner on the x-axis. The absolute value graph meets a horizontal line $y = c$ (for any positive number c) at two points, (r_1, c) and (r_2, c). The absolute value graph is *below the horizontal line between* r_1 and r_2, and the solution set for $|ax + b| < c$ is interval (r_1, r_2). *Outside the interval* from r_1 to r_2, the absolute value graph is *above the line* and the solution set for $|ax + b| > c$ is the union $(-\infty, r_1) \cup (r_2, \infty)$. See Figure 27. Finding the solution set for the type of open sentence inequalities we are considering is simply a matter of finding the numbers r_1 and r_2 and thinking about a graph, as illustrated in the next example.

Strategy: For **(a)**, recall that $|-3| = |3| = 3$, so if $|u| = 3$ then u must be 3 or -3.

▶EXAMPLE 7 *Absolute value inequalities*

(a) Solve the equations $|2x + 1| = 3$ and $|x - 1| = \sqrt{2}$.

(b) Use the solutions from part **(a)** to find the solution set for $|2x + 1| \le 3$ and $|x - 1| > \sqrt{2}$.

Solution

(a) Follow the strategy, $2x + 1$ must be either 3 or -3. That is, we must solve two equations,

$$2x + 1 = 3 \quad \text{or} \quad 2x + 1 = -3, \text{ from which}$$

$$x = 1 \quad \text{or} \quad x = -2.$$

Using the same reasoning, for the second equation, we have $x - 1 = \sqrt{2}$ or $x - 1 = -\sqrt{2}$. The solutions are $1 + \sqrt{2}$ and $1 - \sqrt{2}$.

(b) The solutions in part **(a)** give us the numbers r_1 and r_2 that we need for the graphs above.

From Figure 28a the absolute value graph is below the line $y = 3$, which means that $|2x + 1| \leq 3$ when x is between -2 and 1. The solution set for the inequality $|2x + 1| \leq 3$ is the closed interval $[-2, 1]$. *(Why are the endpoints -2 and 1 included in the solution set?)*

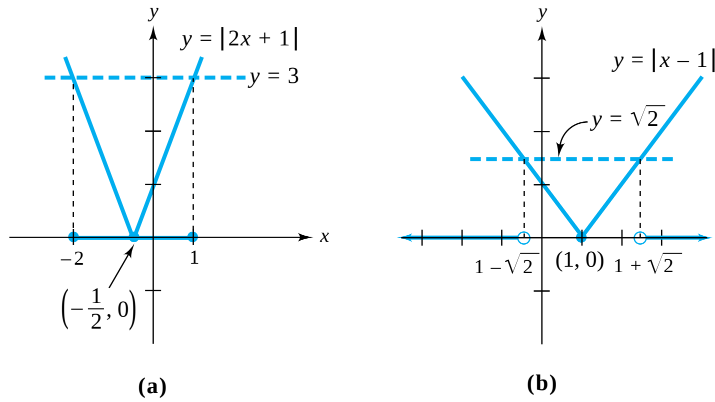


FIGURE 28

From Figure 28b, the absolute value graph is above the line $y = \sqrt{2}$ when x is any number *outside* the interval (r_1, r_2), so in interval notation, the solution set for the second inequality is $(-\infty, 1 - \sqrt{2}) \cup (1 + \sqrt{2}, \infty)$. *(Why are the endpoints $1 \pm \sqrt{2}$ not included?)* ◀

Summing up our observations, we have some guidelines for equivalent absolute value expressions and finding solution sets.

Absolute value equivalents

Suppose c is any positive number and u is an expression involving the variable x. Then

$|u| = c$ may be replaced by the two equations $u = c$ or $u = -c$,

$|u| < c$ may be replaced by the two inequalities $-c < u < c$ (the solution set consists of the numbers between r_1 and r_2),

$|u| > c$ may be replaced by the two inequalities $u < -c$ or $u > c$ (the solution set consists of the numbers *outside* $[r_1, r_2]$).

Strategy: Consider two cases: **(i)** for $x \geq 0$, replace $|x|$ by x; **(ii)** for $x < 0$, replace $|x|$ by $-x$. In each case solve the related equation to get cut points.

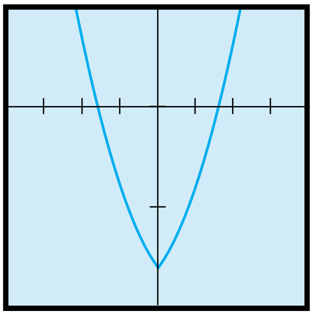
►*EXAMPLE 8* ***Integer solutions*** What integers satisfy the inequality $x^2 + 2|x| - 8 < 0$?

Solution

Follow the strategy.

(i) $x \geq 0$: $\quad x^2 + 2x - 8 = 0, \qquad$ or $\qquad (x + 4)(x - 2) = 0.$

We have two roots, -4 and 2, but only 2 satisfies $x \geq 0$.



$[-5, 5]$ by $[-10, 5]$
$y = x^2 + 2|x| - 8$

FIGURE 29

(ii) $x < 0$: $x^2 - 2x - 8 = 0$ or $(x - 4)(x + 2) = 0.$

Again, there are two roots, 4 and -2, but only -2 satisfies $x < 0$.

The cut points for the original inequality are 2 and -2. Checking test points in the original, we find that the solution set is the interval $(-2, 2)$. The integers in the interval $(-2, 2)$ are -1, 0, and 1.

Graphical If we look at a calculator graph of $y = x^2 + 2|x| - 8$ in the window $[-5, 5] \times [-10, 5]$ (see Figure 29), we see that the graph is below the x-axis (meaning that the y-coordinates are less than zero, or that $x^2 + 2|x| - 8 < 0$) on an interval from about -2 to 2. Without a decimal window, when we trace, we may not be able to tell exactly where the graph crosses the axis. We still need to do some analysis as above to identify the endpoints precisely. From the graph we can easily see that y is negative at the integer values 0, 1, and -1, and we can evaluate $x^2 + 2|x| - 8$ to verify that $y = 0$ at 2 and -2. Thus the integer values that satisfy the given inequality are 0, 1, and -1. ◀

EXERCISES 1.5

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. The two equations $x^3 - 2x^2 - 5x = 0$ and $x^2 - 2x - 5 = 0$ have the same solution set.

2. The sum of all the integers in the set $\{x \mid -7 < 3x - 1 < 14\}$ is 14.

3. The number -2 is in the solution set for $x^2 - 3x - 5 > |x|$.

4. The solution set for $(x + 3)^2 = 1$ is the same as the solution set for $x + 3 = 1$.

5. The solution set for $x < \frac{4}{x}$ is the same as the solution set for $x^2 < 4$.

6. The solution set for $x < |x|$ is the set of negative numbers.

Exercise 7–10 Fill in the blank so that the resulting statement is true.

7. The largest prime number in the set $\{x \mid |x - 3| \leq 21\}$ is ______.

8. The smallest positive integer that is not in the set $\{x \mid x^2 - 4x - 5 < 0\}$ is ______.

9. If $S = \{x \mid (x - 3)(x + 2) \leq 0\}$, then the sum of all integers in S is ______.

10. If k is any positive number, then the number of real roots for $x^2 + 2x - k = 0$ is ______.

Develop Mastery

If not specified, the domain of the variable is assumed to be R.

Exercises 1–8 ***Solving Equations*** *Solve. Simplify the result.*

1. $5 - 3x = 7 + x$
2. $5x - 1 = \sqrt{3}$
3. $(x - 2)^2 = x^2 - 2$
4. $(1 - 2x)^2 = 4x^2 - x$
5. $3x^2 + 2x - 1 = 0$
6. $2x^2 + x = 10$
7. $6x + 5 = 9x^2 - 3$
8. $\sqrt{3}x - 4 = x$

Exercises 9–10 Assume the replacement set is the set of complex numbers. Solve. Simplify the result.

9. $4x^2 + 4x - 15 = 0$
10. $2x^2 + 4x + 5 = 0$

Exercises 11–26 ***Solving Inequalities*** *Solve. Use a graph as a check.*

11. $3x - 1 > 5$
12. $\frac{1 - 2x}{-3} > \frac{1}{2}$
13. $-0.1 \leq 2x + 1 \leq 0.1$
14. $-1 \leq \frac{x + 3}{-2} \leq 1$
15. $(2 - x)(1 + x) \geq 0$
16. $2x^2 - x - 3 > 0$
17. $\frac{4 - x^2}{x + 3} \geq 0$
18. $x + 1 > \frac{2}{x}$
19. $2 \leq 3x - 1 \leq 8$
20. $0 \leq x^2 - 1 \leq 8$
21. $|x| > x$
22. $x^4 + 4x^3 \geq 12x^2$
23. $|2x - 3| > 5$
24. $|x - 4| + x \leq 6$
25. $|x - 2| + 2x \leq 4$
26. $x^2 - 71x - 10{,}296 < 0$

Exercises 27–30 ***Solution Set*** *Find the solution set and show it on a number line. Use a graph as a check.*

27. $5x - 1 > 3 + 7x$
28. $x^2 - x > 12$
29. $\frac{x + 2}{x^2 - 9} > 0$
30. $2x + 1 > \frac{2}{x}$

Exercises 31–34 ***Absolute Value Inequalities*** *Find the solution set and express it in interval notation.*

31. $|x - 1| < 2$ **32.** $|2x + 1| > 3$

33. $|x| + 1 < \sqrt{2}$ **34.** $|1 - x| \leq 0.1$

Exercises 35–36 ***Discriminant*** *Use the discriminant to determine the number of real roots.*

35. $x^2 - 15x + 8 = 0$

36. $4x^2 + 4\sqrt{3}x + 3 = 0$

Exercises 37–46 ***Solving Equations*** *Solve. Use a graph to support your answer.*

37. $x^4 + 3x^2 - 10 = 0$

38. $2|x + 3| - 1 = 5$

39. $|5 - x| - 5 = 3$

40. $(\sqrt{x})^2 - 2\sqrt{x} - 3 = 0$

41. $|x|^2 - 2|x| = 3$

42. $\sqrt{2x + 3} = 1$

43. $\sqrt{x^2 + 4x} = x + 2$

44. $\frac{1}{x} - \frac{3}{x} = \frac{1}{2} + \frac{1}{4}$

45. $\sqrt{x^4 - 5x^2 - 35} = 1$

46. $x - 2\sqrt{x} - 8 = 0$

Exercises 47–50 *Find the solution set. Assume that the replacement set is the set of integers.*

47. $-4 \leq 3x - 2 \leq 4$ **48.** $|2 - 3x| < 4$

49. $2x^2 + x < 15$ **50.** $\sqrt{(x - 2)^2} \leq 3$

Exercises 51–52 *Determine the values of x for which the expression yields a real number.*

51. $\sqrt{-x^2 - 4x - 3}$ **52.** $\sqrt{x - \frac{4}{x}}$

Exercises 53–54 *Determine the values of x for which the expression yields complex nonreal numbers.*

53. $\sqrt{-x^2 - 5x - 6}$ **54.** $\sqrt{4 - 2|x|}$

Exercises 55–56 *Find the solution set.* (*Hint:* *Recall* $\sqrt{u^2} = |u|$.)

55. $\sqrt{(2x - 1)^2} = 5$ **56.** $\sqrt{x^2} = -x$

Exercises 57–58 *Find the solution set.*

57. $x + 2 < 3$ and $x + 2 > -3$

58. $2x - 3 \leq -1$ and $2x - 3 > -4$

Exercises 59–62 ***Zero-product Principle*** *Use the zero-product principle to find a quadratic equation with the pair of roots.*

59. $-2, -4$ **60.** $-2, \frac{1}{2}$

61. $1 + \sqrt{2}, 1 - \sqrt{2}$ **62.** $1 + i, 1 - i$

Exercises 63–64 ***(a)*** *Determine the number of integers in the set.* ***(b)*** *Find the sum of all the integers in the set.*

63. $\{x \mid 2x + 5 > 0 \text{ and } -3x + 16 > 3\}$

64. $\{x \mid |x - 3| < \sqrt{5}\}$

Exercises 65–66 *Use the zero-product principle to find the solution set.*

65. $(x^2 - 9)(x^2 + x - 6) = 0$

66. $(|x| - 1)(3 - |x + 1|) = 0$

67. How many prime numbers are contained in the set $\{x \mid x^2 - 15x \leq 0\}$? What is the largest one?

68. Find the largest integer k for which the equation $kx^2 + 10x + 3 = 0$ will have real roots.

69. Find the smallest integer c for which the equation $x^2 + 5x - c = 0$ will have real roots.

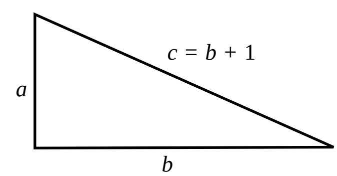
70. What is the sum of all the positive integers x for which $x^2 - 2x - 17$ is negative?

71. What is the smallest integer k such that $3x(kx - 4) - x^2 + 4 = 0$ has no real roots?

72. What is the largest integer x such that the reciprocal of $x + 4$ is greater than $x - 4$?

73. What is the sum of all prime numbers in the set $\{x \mid 3x + 4 < 5x + 7 < 4x + 15\}$?

74. In the right triangle in the diagram, $c = b + 1$, and the perimeter is 12. Find a, b, and c.



75. In a triangle having sides a, b, and c, if $a = 10$, $b = 12$, and $b^2 = a^2 + c^2 - ac$, find all possible values for c.

76. **(a)** If $x + y = 8$ and $x^2 + 4xy + 3y^2 = 48$, find $2x + 6y$. (*Hint:* Factor the left-hand side.)

(b) If $a^2b + ab^2 + a + b = 72$ and $a \cdot b = 8$, then find the sum $a + b$ and the sum $a^2 + b^2$. (*Hint:* First factor the left side, and then use $(a + b)^2 = a^2 + 2ab + b^2$.)

77. A certain chemical reaction takes place when the temperature is between 5° and 20° Celsius. What is the corresponding temperature on the Fahrenheit scale? (*Hint:* $F = \frac{9}{5}C + 32$.)

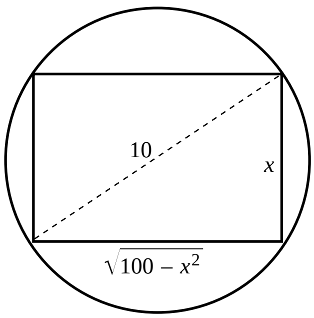
78. In 1990 the population of Newbury increased by 1600 people. During 1991 the population decreased by 12 percent and the town ended up with 56 fewer people than had lived there before the 1600 increase. What was the original population?

79. A boy walking to school averages 90 steps per minute, each step 3 feet in length. It takes him 15 minutes to reach school. His friend walks to school along the same route averaging 100 steps per minute, each step covering 2.5 feet. How long does it take for the friend to walk to school?

80. A beaker contains 100 cc of water. Suppose x cc of water are removed and replaced by x cc of pure acid. From the resulting mixture, another x cc are removed and replaced by x cc of acid. In the final mixture the ratio of water to acid is 16 to 9. Find x, and the final volume of acid.

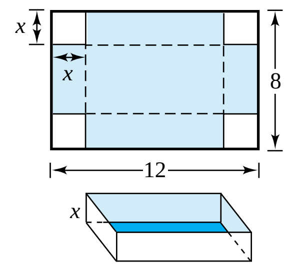
81. A farmer has 200 feet of fencing to enclose a rectangular garden. If the width of the garden is x feet, find an equation that gives the area A of the garden in terms of x. For what values of x is the equation meaningful?

82. ***Largest Area*** A rectangle is inscribed in a circle of diameter 10 inches, as shown.



(a) Using the information shown in the diagram, find an equation that gives the area A of the rectangle in terms of x.
(b) For what values of x is the equation meaningful?
(c) Use a graph to read the value of x (one decimal place) that will give the largest value of A.

83. ***Maximum Volume*** A box with an open top is to be made from a rectangular piece of tin 8 inches by 12 inches, by cutting a square from each corner and bending up the sides as shown in the diagram. Let x be the length of the sides of each square.
(a) Show that the volume V of the box is given by $V = 4x^3 - 40x^2 + 96x$.
(b) For what values of x is $V > 0$?
(c) What is the value of x (one decimal place) that gives the largest value of V? What is the maximum volume?



1.6 MODELS AND PROBLEM SOLVING

> One cadet, who had a private airplane pilot's license, was failing mathematics. When he was asked how much gas he would need to carry if he were going to fly two hundred miles at so many miles per gallon, he didn't know whether to multiply or divide. How, the officers asked, was he able to get the right answer? He replied that he did it both ways and took the reasonable answer. They felt that anybody who knew what was a reasonable answer had promise, so they gave him a second chance.
>
> *Ralph P. Boas, Jr.*

Problem solving is the key to learning mathematics. In this section we consider problems that are somewhat different from some you may have met earlier. Here we try to draw on what you already know, and to develop reasoning and strategy to attack a given problem. Always try your own approach; do not just follow an example in the book or mimic a solution from someone else. Genuine learning takes place when you think for yourself.