58. If $f(x) = 2x + 3$ and the domain D of f is given by $D = \{x \mid x^2 + x - 2 \leq 0\}$, then
(a) draw a graph of f, and
(b) find the range of f. (*Hint:* What values of y occur on the graph?)

59. The length L of a metal rod is a linear function of its temperature T, where L is measured in centimeters and T in degrees Celsius. The following measurements have been made: $L = 124.91$ when $T = 0$, and $L = 125.11$ when $T = 100$.
(a) Find a formula that gives L as a function of T.
(b) What is the length of the rod when its temperature is 20° Celsius?
(c) To what temperature should the rod be heated to make it 125.17 cm long?

60. ***Store Prices*** The owner of a grocery store finds that, on average, the store can sell 872 gallons of milk per week when the price per gallon is \$1.98. When the price per gallon is \$1.75, sales average 1125 gallons a week. Assume that the number N of gallons sold per week is a linear function of the price P per gallon.
(a) Find a formula that gives N as a function of P.
(b) If the price per gallon is \$1.64, how many gallons should the store owner expect to sell per week?
(c) To sell 1400 gallons per week, what price should the store set per gallon?

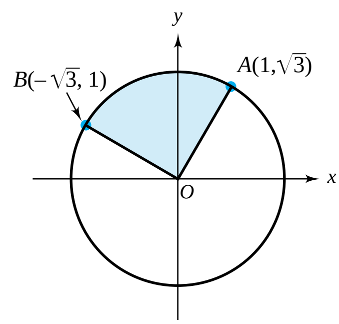
61. ***Equilateral Triangle*** For what value(s) of m will the triangle formed by the three lines $y = -2$, $y = mx + 4$, and $y = -mx + 4$ be equilateral?

62. ***Perimeter*** Find m such that the three lines $y = mx + 6$, $y = -mx + 6$, and $y = 2$ form a triangle with a perimeter of 16.

63. ***Area*** Given the three lines $y = m(x + 4)$, $y = -m(x + 4)$, and $x = 2$, for what value of m will the three lines form a triangle with an area of 24?

64. **(a)** Is the point $(-1, 3)$ on the circle $x^2 + y^2 - 4x + 2y - 20 = 0$?
(b) If the answer is yes, find an equation for the line that is tangent to the circle at $(-1, 3)$; if the answer is no, find the distance between the x-intercept points of the circle.

65. **(a)** Show that points $A(1, \sqrt{3})$ and $B(-\sqrt{3}, 1)$ are on the circle $x^2 + y^2 = 4$.
(b) Find the area of the shaded region in the diagram. (*Hint:* Show that segments $\overline{OA}$ and $\overline{OB}$ are perpendicular to each other.)



66. The horizontal line $y = 2$ divides $\triangle ABC$ into two regions. Find the area of the two regions for the triangle with these vertices:
(a) $A(0, 0)$, $B(4, 0)$, $C(0, 4)$
(b) $A(0, 0)$, $B(4, 0)$, $C(8, 4)$

67. The vertices of $\triangle ABC$ are $A(0, 0)$, $B(4, 0)$, $C(0, 4)$. If $0 < k < 4$, then the horizontal line $y = k$ will divide the triangle into two regions. Draw a diagram showing these regions for a typical k. Find the value of k for which the areas of the two regions are equal.

2.5 QUADRATIC FUNCTIONS, PARABOLAS, AND PROBLEM SOLVING

> It is only fairly recently that the importance of nonlinearities has intruded itself into the world of the working scientist. Nonlinearity is one of those strange concepts that is defined by what it is not. As one physicist put it, "It is like having a zoo of nonelephants."
>
> *B. J. West*

Much of this course, and calculus courses to follow, deals with other inhabitants of the "zoo" of nonlinear functions, including families of polynomial, exponential, logarithmic, and trigonometric functions. All are important, but quadratic functions are among the simplest nonlinear functions mathematicians use to model the world.

I had mathematical curiosity very early. My father had in his library a wonderful series of German paperback books . . . . One was Euler's Algebra. I discovered by myself how to solve equations. I remember that I did this by an incredible concentration and almost painful and not-quite conscious effort. What I did amounted to completing the square in my head without paper or pencil.
Stan Ulam

Definition: quadratic function

A **quadratic function** is a function with an equation equivalent to

$$f(x) = ax^2 + bx + c, \tag{1}$$

where a, b, and c are real numbers and a is not zero.

For example, $g(x) = 5 - 4x^2$ and $h(x) = (2x - 1)^2 - x^2$ are quadratic functions, but $F(x) = \sqrt{x^2 + x + 1}$ and $G(x) = (x - 3)^2 - x^2$ are not. (In fact, G is linear.)

Basic Transformations and Graphs of Quadratic Functions

The graphing techniques we introduced in Section 2.3 are collectively called **basic transformations.** The graph of any linear function is a line, and we will show that that graph of any quadratic function can be obtained from the core parabola, $f(x) = x^2$, by applying basic transformations. We apply terminology from the core parabola to parabolas in general. The point $(0, 0)$ is called the **vertex** of the core parabola, and the y-axis is the **axis of symmetry.** The axis of symmetry is a help in making a hand-sketch of a parabola. Whenever we locate a point of the parabola on one side of the axis of symmetry, we automatically have another point located symmetrically on the other side.

We will derive a **transformation form** for a general quadratic function, an equation that identifies the vertex and axis of symmetry of the graph, but to graph any particular quadratic, you may not need all of the steps. Indeed, because a graphing calculator graphs any quadratic function, we could ask why we need the transformation form at all. Because a calculator graph is dependent on the window we choose and the pixel coordinates, we need an algebraic form from which we can read more exact information.

To begin, we factor out the coefficient a from the x-terms, and then add and subtract the square of half the resulting x-coefficient to complete the square on x.

$$\begin{aligned} f(x) = ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left[\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2}\right] + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right). \end{aligned}$$

The final equation has the form

$$f(x) = a(x - h)^2 + k \tag{2}$$

which we recognize as a core parabola shifted so that the vertex is at the point (h, k) and the axis of symmetry is the line $x = h$.

Parabola Features

Looking at the derivation of Equation (2), we can make some observations about the graphs of quadratic functions.

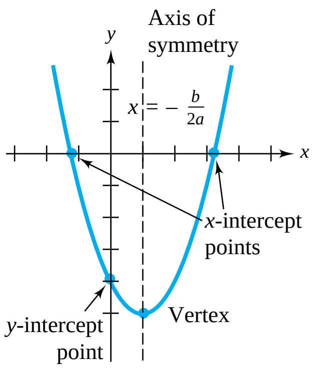


FIGURE 26

Graphs of quadratic functions

For the quadratic function $f(x) = ax^2 + bx + c$: The graph is a parabola with **axis of symmetry** $x = \dfrac{-b}{2a}$.

The parabola opens *upward if* $a > 0$, *downward if* $a < 0$.

To *find the coordinates of the vertex,* set $x = \dfrac{-b}{2a}$. Then the y-coordinate is given by $y = f\left(\dfrac{-b}{2a}\right)$.

The graph of every quadratic function intersects the y-axis (where $x = 0$), but it need not have any x-intercept points. To find any x-intercepts, we solve the equation $f(x) = 0$. By its nature, every quadratic function has a *maximum* or a *minimum* (depending on whether the parabola opens down or up) that occurs at the vertex of the parabola. See Figure 26.

▶EXAMPLE 1 ***Locating the vertex of a parabola*** The graph of the quadratic function is a parabola. Locate the vertex in two ways: **(i)** by writing the function in the form of Equation (2) and **(ii)** by setting $x = -\dfrac{b}{2a}$. Sketch the graph.

(a) $f(x) = x^2 - 2x - 3$ **(b)** $f(x) = 2x^2 + 4x - 1$

Solution

(a) **(i)** Complete the square on the x-terms.

$$\begin{aligned} f(x) &= x^2 - 2x - 3 = (x^2 - 2x + 1) - 1 - 3 \\ &= (x - 1)^2 - 4, \end{aligned}$$

From this form, the graph is the core parabola shifted 1 unit right and 4 units down; the vertex is at $(1, -4)$.

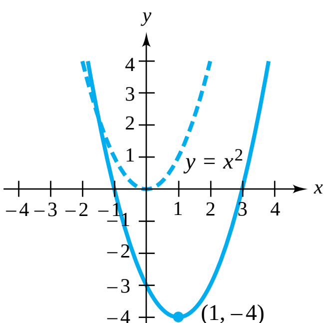
(ii) From $f(x) = x^2 - 2x - 3$, $-\frac{b}{2a} = -\frac{-2}{2 \cdot 1} = 1$. Substituting 1 for x, $f(1) = 1 - 2 \cdot 1 - 3 = -4$, so again the vertex is the point $(1, -4)$. The graph is shown in Figure 27(a).

(b) **(i)** Before completing the square, we factor out 2 from the x-terms:

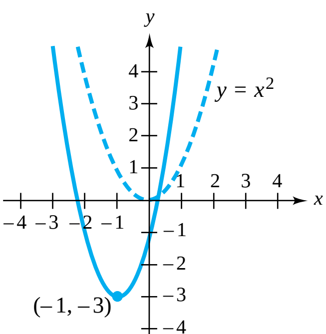
$$\begin{aligned} f(x) &= 2x^2 + 4x - 1 = 2(x^2 + 2x) - 1 \\ &= 2(x^2 + 2x + 1 - 1) - 1 = 2(x + 1)^2 - 3. \end{aligned}$$

The graph is obtained by shifting the core parabola 1 unit left, stretching by a factor of 2, and translating the stretched parabola down 3 units; the vertex is at $(-1, -3)$.

(ii) From $f(x) = 2x^2 + 4x - 1$, $-\frac{b}{2a} = -\frac{4}{4} = -1$, and $f(-1) = 2(-1)^2 + 4(-1) - 1 = -3$, and the vertex is at $(-1, -3)$. The graph appears in Figure 27(b). ◀



(a) $f(x) = x^2 - 2x - 3$
$= (x - 1)^2 - 4$



(b) $f(x) = 2x^2 + 4x - 1$
$= 2(x + 1)^2 - 3$

FIGURE 27

▶EXAMPLE 2 ***Graph, maximum or minimum, and intercept points*** Find the maximum or minimum value of f and the intercept points both from a calculator graph and algebraically.

(a) $f(x) = x^2 - 2x - 3$ **(b)** $f(x) = 2x^2 + 4x - 1$

Solution

(a) From Example 1(a) and the graph in Figure 27(a), the minimum value of f is the y-coordinate of the vertex, -4. For the y-intercept, $f(0) = -3$, so the y-intercept point is $(0, -3)$. Solving the equation $x^2 - 2x - 3 = 0$, we have $(x - 3)(x + 1) = 0$, so the x-intercept points are $(3, 0)$ and $(-1, 0)$. While a calculator graph in any window shows that the x-intercept points are near $x = 3$ and $x = -1$, in trace mode, we do not see the x-intercepts exactly (where the y-coordinate equals 0) unless we have a decimal window. We can zoom in as needed, though, to get as close to $(3, 0)$ and $(-1, 0)$ as desired.

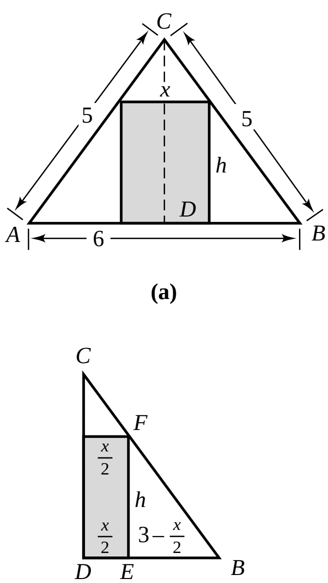
(b) From Example 1(b) and the graph in Figure 27(b), the function has a minimum value of -3. Since $f(0) = -1$, the y-intercept point is $(0, -1)$. Tracing along a calculator graph, we find that the x-intercepts are near 0.2 and -2.2. We can get closer approximations by zooming in, but the equation $2x^2 + 4x - 1 = 0$ does not factor with rational numbers, so we cannot read exact coordinates of the x-intercept points on any calculator graph. We use the quadratic formula to solve the equation and find the x-intercept points exactly:

$$\left(\frac{-2 + \sqrt{6}}{2}, 0\right) \text{ and } \left(\frac{-2 - \sqrt{6}}{2}, 0\right). \quad ◀$$

Quadratic Functions with Limited Domain

According to the domain convention the domain of any quadratic function is the set of all real numbers unless there is some restriction. Many applications place natural restrictions on domains, as illustrated in the next two examples.

Strategy: **(a)** Draw a separate diagram (Figure 28b) to show a right triangle formed by altitude CD. $\triangle CDB$ is similar to $\triangle FEB$. Use ratios of the sides to relate h and x.



(a)

(b)

FIGURE 28

►EXAMPLE 3 ***Limited domain*** A rectangle is inscribed in an isosceles triangle ABC, as shown in Figure 28a, where $|\overline{AB}| = 6$ and $|\overline{AC}| = |\overline{BC}| = 5$. Let x denote the width, h the height, and K the area of the rectangle. Find an equation for **(a)** h as a function of x, **(b)** K as a function of x. **(c)** Find the domain of each.

Solution

(a) Following the strategy, the ratios $\dfrac{|\overline{CD}|}{|\overline{DB}|}$ and $\dfrac{|\overline{FE}|}{|\overline{EB}|}$ are equal. $|\overline{DB}| = 3$, $|\overline{FE}| = h$, and $|\overline{EB}| = 3 - \dfrac{x}{2}$. For $|\overline{CD}|$ we can apply the Pythagorean theorem to $\triangle CDB$: $|\overline{CD}| = \sqrt{5^2 - 3^2} = 4$. Therefore

$$\frac{|\overline{FE}|}{|\overline{EB}|} = \frac{|\overline{CD}|}{|\overline{DB}|} \quad \text{or} \quad \frac{h}{3 - \frac{x}{2}} = \frac{4}{3} \quad \text{or} \quad h = 4 - \frac{2x}{3}.$$

(b) The area K is the product of x and h:

$$K = x \cdot h = x\left(4 - \frac{2x}{3}\right) = 4x - \frac{2}{3}x^2.$$

(c) From the nature of the problem, there is no rectangle unless x is a positive number less than 6. Hence the domain of both h and K is $\{x \mid 0 < x < 6\}$. ◀

▶EXAMPLE 4 ***Minimizing area*** Find the dimensions (x and h) for the rectangle with the maximum area that can be inscribed in the isosceles triangle ABC in Figure 28a.

Strategy: Graph K and find the highest point on the graph (the vertex).

Solution

Follow the strategy. In Example 3, we found the area K as a quadratic function of x:

$$K(x) = -\frac{2}{3}x^2 + 4x,\ 0 < x < 6.$$

Graphing K as a function of x, we get *part* of a parabola that opens down. The graph of K is shown in Figure 29. The maximum value of K occurs at the vertex of the parabola, where

$$x = -\frac{b}{2a} = \frac{-4}{2(-2/3)} = 3.$$

When $x = 3$, $h = 4 - \frac{2x}{3} = 4 - (\frac{2}{3})3 = 2$, and $K(3) = 6$. Therefore the inscribed rectangle with the largest area has sides of lengths 3 and 2, and area 6. ◀

Solving Quadratic Inequalities

In Section 1.5, we solved quadratic inequalities by factoring quadratic expressions. Here we look at the more general situation of solving quadratic inequalities with the use of graphs, as illustrated in the following example.

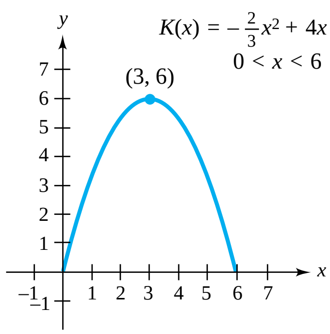


FIGURE 29

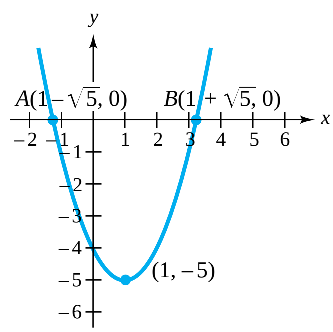


FIGURE 30

▶EXAMPLE 5 ***Using a graph*** Find the domain of the function $f(x) = \sqrt{x^2 - 2x - 4}$.

Strategy: Use a graph of $y = x^2 - 2x - 4$ to see where the y-values are non-negative.

Solution

Follow the strategy. Let $y = x^2 - 2x - 4$ and draw a graph. This gives a parabola that opens upward, with vertex at $(1, -5)$. To find the x-intercept points, solve the equation

$$x^2 - 2x - 4 = 0$$

by the quadratic formula to get $x = 1 \pm \sqrt{5}$. Thus the intercept points are $A(1 - \sqrt{5}, 0)$ and $B(1 + \sqrt{5}, 0)$, as shown in Figure 30. Use the graph to read off the solution set to the inequality that defines the domain of f:

$$\begin{aligned} D &= \{x \mid x \le 1 - \sqrt{5} \quad \text{or} \quad x \ge 1 + \sqrt{5}\} \\ &= (-\infty, 1 - \sqrt{5}] \cup [1 + \sqrt{5}, \infty). \end{aligned}$$ ◀

Maximum and Minimum Values of a Function

When we use mathematical models to answer questions about an applied problem, we frequently need to determine the maximum or minimum values of a function. The general problem can be very difficult, even using the tools of calculus, but when quadratic functions are involved one can simply read off a maximum or minimum value from a graph.

Definition: maximum or minimum value of a function

Suppose f is a function with domain D.

If there is a number k in D such that $f(k) \geq f(x)$ for every x in D, then $f(k)$ is the **maximum value of *f***.
If there is a number k in D such that $f(k) \leq f(x)$ for every x in D, then $f(k)$ is the **minimum value of *f***.

Strategy: First draw a graph, being aware of the given domain. From the graph, read off the minimum or maximum values of y.

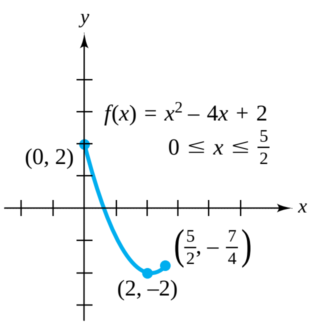


FIGURE 31

►EXAMPLE 6 ***A function with limited domain*** For the function $f(x) = x^2 - 4x + 2$, with domain $D = \{x \mid 0 \leq x \leq \frac{5}{2}\}$ find **(a)** the maximum and minimum values of f, and **(b)** the set of values where $f(x) > 0$.

Solution

(a) The graph of f is the part of the parabola $y = x^2 - 4x + 2$ on the interval $[0, \frac{5}{2}]$. A calculator graph may not allow us to read all needed points in exact form, so we first locate the vertex and then evaluate the function at the ends of the domain.

Using $x = \frac{-b}{2a} = \frac{4}{2} = 2$, we have $f(2) = -2$, so the point $(2, -2)$ is the vertex. At the endpoints of the domain, $f(0) = 2$, and $f(\frac{5}{2}) = -\frac{7}{4}$, so the graph has endpoints $(0, 2)$ and $(\frac{5}{2}, -\frac{7}{4})$.

We get the graph shown in Figure 31. The maximum value of f is 2, which occurs at the left end of the graph, and the minimum value is -2, at the vertex of the parabola.

(b) To solve the inequality $f(x) > 0$, we need the x-intercept point between 0 and 1. From the quadratic formula, $f(x) = 0$ when $x = 2 \pm \sqrt{2}$. Since the function is only defined on the interval $[0, \frac{5}{2}]$, $f(x) > 0$ on the interval $[0, 2 - \sqrt{2}]$. ◄

►EXAMPLE 7 ***Ranges of transformed functions*** For the function $f(x) = x^2 - 4x + 2$, with domain $D = \{x \mid 0 \leq x \leq \frac{5}{2}\}$, find the range of

(a) $y = f(x) + 2$ **(b)** $y = -f(x)$ **(c)** $y = \dfrac{3}{2}f(x)$.

Solution

(a) The graph of $y = f(x) + 2$ is shifted 2 units up from the graph in Figure 31 and is shown in Figure 32a. The range of f is the interval $[-2, 2]$, so the range of $y = f(x) + 2$ is also shifted 2 units up, to $[0, 4]$.

(b) To get the graph of $y = -f(x)$, we reflect the graph of f in the x-axis, as in Figure 32b. The range is still $[-2, 2]$.

(c) Stretching the graph of f vertically by a factor of $\frac{3}{2}$, we get part of another parabola, $y = \frac{3}{2}x^2 - 6x + 3$, as shown in Figure 32c. The range of $y = \frac{3}{2}f(x)$ is $[-3, 3]$. ◄

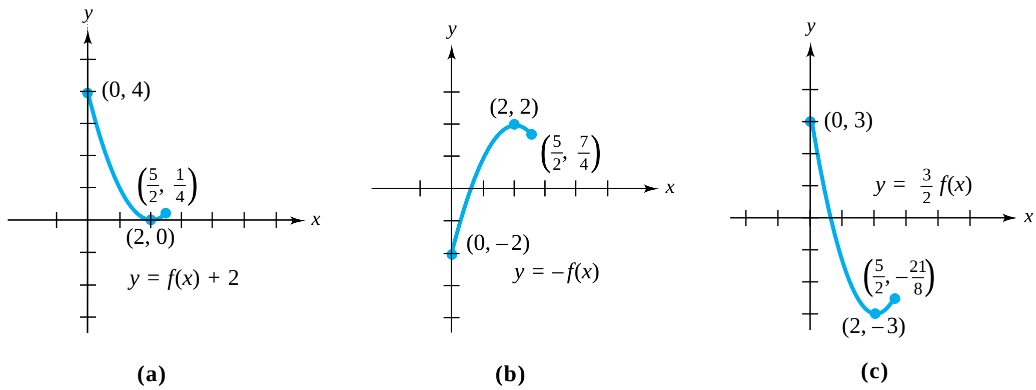


FIGURE 32

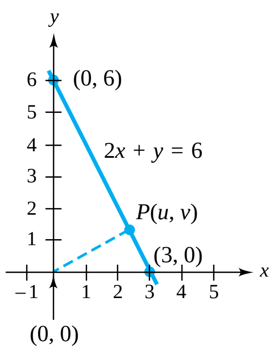


FIGURE 33

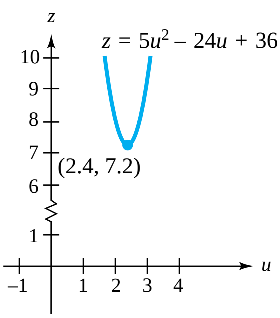


FIGURE 34

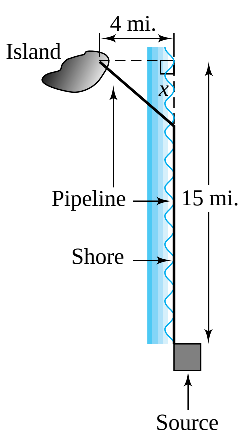


FIGURE 35

▶EXAMPLE 8 *Distance from a point to a line*

(a) Find the minimum distance from the origin to the line L given by $2x + y = 6$.
(b) What are the coordinates of the point Q on L that is closest to the origin?

Solution

First draw a diagram that will help formulate the problem (Figure 33). Since $P(u, v)$ is on L, then $2u + v = 6$, or $v = 6 - 2u$.

$$d = \sqrt{(u-0)^2 + (v-0)^2} = \sqrt{u^2 + v^2} = \sqrt{u^2 + (6 - 2u)^2}$$
$$= \sqrt{5u^2 - 24u + 36}.$$

(a) The minimum value of d will occur when the expression under the radical is a minimum. Determine the minimum of the function

$$z = 5u^2 - 24u + 36,$$

whose graph is shown in Figure 34. The lowest point on the parabola occurs where

$$u = -\frac{b}{2a} = -\frac{-24}{10} = 2.4$$

When $u = 2.4$, $z = 7.2$. Therefore, the minimum value of z is 7.2, so the minimum distance from the origin to the line L is $\sqrt{7.2}$ (≈ 2.68).

(b) The point Q on L that is closest to the origin is given by $u = 2.4$ and $v = 6 - 2(2.4) = 1.2$. Thus, Q is point $(2.4, 1.2)$. ◀

Looking Ahead to Calculus

Not all problems lead to quadratic functions. The applied problem in the next example requires calculus techniques to find an exact solution. With a graphing calculator, however, we can find an excellent approximation from a graph of a cost function.

▶EXAMPLE 9 *Reading a solution from a graph*

A freshwater pipeline is to be built from a source on shore to an island 4 miles offshore as located in the diagram (Figure 35). The cost of running the pipeline along the shore is \$7500 per mile, but construction offshore costs \$13,500 per mile.

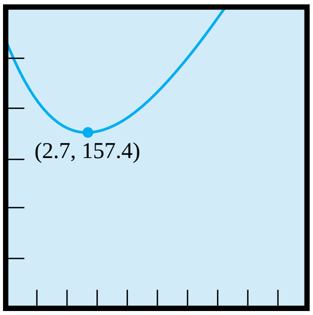
(a) Express the construction cost C (in thousands of dollars) as a function of x.

(b) Select an appropriate window and graph $y = C(x)$ to find the approximate distance x that minimizes the cost of construction.

Solution

(a) If we begin the underwater construction x miles from the point nearest the island, as in Figure 35, then we have $(15 - x)$ miles along the coast, at a cost of $(7500)(15 - x)$ dollars. The distance offshore is then $\sqrt{x^2 + 4^2}$, so the offshore construction cost is $(13{,}500)\sqrt{x^2 + 16}$ dollars.
The total cost, in thousands of dollars, is given by

$$C(x) = 7.5(15 - x) + 13.5\sqrt{x^2 + 16}.$$



[0, 10] by [140, 170]

Cost: $C(x) = 7.5(15 - x) + 13.5\sqrt{x^2 + 16}$

FIGURE 36

(b) Because of the greater cost of offshore construction, it appears that x should be less than 10, so we might try an x-range of $[0, 10]$. We can try a value for x to evaluate C, say $x = 5 : C(5) \approx 161.4$, just over \$161,000. To safely bracket that value, let's try a y-range of $[140, 170]$. We get a graph as shown in Figure 36. Tracing to find the low point on the graph, we get a minimum of about 157.4 near $x = 2.67$. Furthermore, we observe that the graph is quite flat near the low point, that the cost differs only by a few dollars for x near 2.7. We conclude that we should build along the coast for about 12.3 miles and then head directly toward the island. The cost will be about \$157,400. ◀

EXERCISES 2.5

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. If we translate the graph of $y = x^2$ two units to the right and one unit down, the result will be the graph of $y = x^2 - 4x + 3$.
2. The y-intercept point for the graph of $y = x^2 + x - 3$ is above the x-axis.
3. The maximum value of $f(x) = 15 - 2x - x^2$ is 12.
4. The graphs of $y = x^2 - 5x - 4$ and $y = 8 + 3x - x^2$ intersect at points in Quadrants II and IV.
5. If we translate the graph of $y = x^2$ three units down it will be the graph of $y = 2x^2 - 6$.

Exercises 6–8 Fill in the blank so that the resulting statement is true. The number of points at which the two graphs intersect is ______.

6. $f(x) = x^2 - 5x - 5$, $g(x) = 8 + 3x - x^2$
7. $f(x) = 3 + 3x - x^2$, $g(x) = 8 - x$
8. $f(x) = 3 + 3x - x^2$, $g(x) = |x - 2| - 2x$

Exercises 9–10 Draw a graph of function f using a $[-10, 10]$ by $[-10, 10]$ window. The number of x-intercept points visible in this window is ______.

9. $f(x) = 0.3x^2 - 4x - 1$
10. $f(x) = 3 - 3x - 0.3x^2$

Develop Mastery

Exercises 1–12 ***Intercept, Vertex*** *Find the coordinates of the intercept points and vertex algebraically, and then draw a graph as a check.*

1. $f(x) = x^2 - 3$
2. $f(x) = -x^2 + 3$
3. $g(x) = 2(x - 1)^2$
4. $g(x) = 2(x + 1)^2$
5. $f(x) = (x + 1)^2 - 3$
6. $f(x) = (x - 3)^2 + 1$
7. $f(x) = -x^2 - 2x + 2$
8. $f(x) = x^2 + 4x + 1$
9. $f(x) = 2x^2 - 4x + 2$
10. $f(x) = -2x^2 + 8x - 5$
11. $f(x) = \frac{1}{2}x^2 + 2x$
12. $f(x) = -\frac{1}{2}x^2 - 2x - 1$
13. ***Explore*** For each real number b, the graph of $f(x) = x^2 - bx - 1$ is a parabola. Choose several values of b greater than or equal to 1 and in each case draw the corresponding graph. Describe the role that b plays. Where are the x and y-intercept points located? What about the vertex?
14. Repeat Exercise 13 for $f(x) = x^2 + bx - 1$.

15. ***Explore*** Try several values of b, positive and negative, and in each case draw a graph of $f(x) = x^2 - b|x| + 4$. What do you observe about the zeros of f?

16. ***Explore*** For $f(x) = x^2 - 4x + c$, choose several values of c and in each case draw the corresponding graph. Describe the role that c plays. How many x-intercept points are there?

Exercises 17–20 Find the equation (in slope-intercept form) for the line containing the vertex and the y-intercept point of the graph of f. Draw a graph of f and the line as a check.*

17. $f(x) = x^2 - 4x + 1$ **18.** $f(x) = x^2 - 3x$

19. $f(x) = -2x^2 - 8x + 3$

20. $f(x) = -3x^2 - 12x - 8$

Exercises 21–24 ***Graphs and Quadrants*** *Draw a graph and determine the quadrants through which the graph of the function passes.*

21. $y = x^2 - 4x + 3$ **22.** $y = 2x^2 + 7x + 3$

23. $y = x^2 + 4x + 5$ **24.** $y = -x^2 - 2x - 1$

Exercises 25–28 ***Distance Between Intercepts*** *Find the distance between the x-intercept points for the graph of the function. Solve algebraically and then check with a graph.*

25. $f(x) = x^2 - 4x - 3$ **26.** $f(x) = x^2 + 2x - 8$

27. $f(x) = x^2 - 4x + 1$

28. $f(x) = -2x^2 + 4x + 3$

Exercises 29–32 ***Inequalities*** *Find the solution set for the given inequality.* ***(a)*** *Draw a graph of the left side and read off the answer.* ***(b)*** *Use algebra to justify your answer.*

29. $x^2 - 4x + 3 > 0$

30. $x^2 + 5x + 4 < 0$

31. $2x^2 - x - 3 < 0$

32. $-x^2 + 2x + 4 \leq 0$

Exercises 33–34 ***Intercepts to Vertex*** *The intercept points for the graph of a quadratic function f are specified. Find the coordinates of the vertex.*

33. $(-1, 0), (3, 0)\ (0, -3)$ **34.** $(-3, 0), (2, 0), (0, 6)$

Exercises 35–38 ***Range*** *Determine the range of the function. State your answer using* ***(a)*** *set notation and* ***(b)*** *interval notation. A graph will help.*

35. $f(x) = x^2 + 3x - 4$

36. $g(x) = -2x^2 + 4x + 1$

37. $f(x) = (4 - x)(2 + x)$

38. $f(x) = 2x^2 + 4\sqrt{3}\,x$

Exercises 39–42 ***Verbal to Formula*** *Give a formula for a quadratic function f that satisfies the specified conditions. The answer is not unique.*

39. Both zeros of f are positive and $f(0) = -2$.

40. The graph of f does not cross the x-axis and $f(0) = -3$.

41. A zero of f is between -2 and -1, and the other zero is 3.

42. A zero of f is greater than 1, the other zero is less than -1, and the graph contains the point $(0, 2)$.

Exercises 43–46 ***Verbal to Formula*** *Determine the quadratic function whose graph satisfies the given conditions.*

43. The axis of symmetry is $x = 2$, the point $(-1, 0)$ is on the graph, and $(0, 5)$ is the y-intercept point. (*Hint:* Use symmetry to find the other x-intercept point, and then express $f(x)$ in factored form.)

44. The vertex is $(3, -4)$ and one of the x-intercept points is $(1, 0)$. (See the hint in Exercise 43.)

45. The graph is obtained by translating the core parabola 3 units left and 2 units down.

46. The graph is obtained by reflecting the core parabola about the x-axis, then translating to the right 2 units.

Exercises 47–50 ***Area*** *Let A be the y-intercept point and B, C be the x-intercept points for the graph of the function. Draw a diagram and then find the area of $\triangle ABC$.*

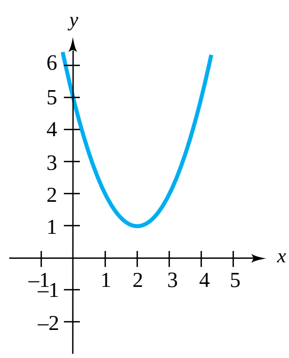
47. $f(x) = -x^2 - x + 6$ **48.** $f(x) = x^2 - 6x + 8$

49. $f(x) = -x^2 - 2x + 8$

50. $f(x) = -x^2 - 6x + 8$

Exercises 51–54 ***Graph to Verbal and Formula*** *Each of the graphs shown began with the core parabola ($y = x^2$) followed by one or more basic transformations.* ***(a)*** *Give a verbal description of the transformation used.* ***(b)*** *Give an equation for the function. Check by using a graph.*

51.



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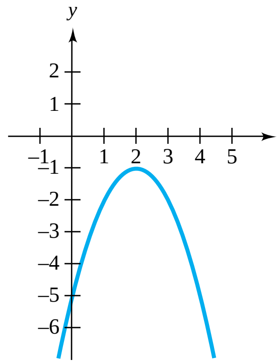

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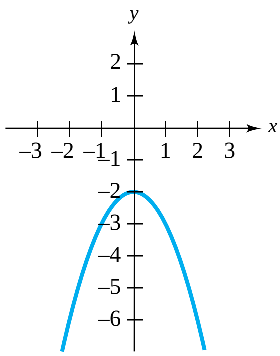

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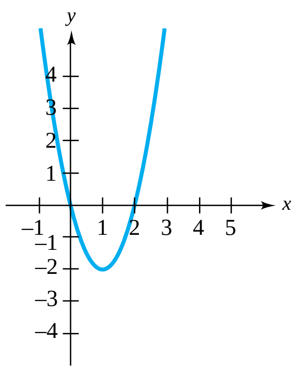


Exercises 55–58 ***Range, Limited Domain*** *A formula for a function is given along with its domain, D. Find the range of the function. Draw a graph.*

55. $f(x) = x^2 + 2x + 5;$ $D = \{x \mid -3 \le x \le 0\}$

56. $f(x) = x^2 + 2x + 5;$ $D = \{x \mid 0 \le x \le 2\}$

57. $g(x) = -x^2 - 4x + 4;$ $D = \{x \mid -3 < x < 1\}$

58. $g(x) = -x^2 + 2x + 4;$ $D = \{x \mid 0 \le x \le 3\}$

Exercises 59–64 ***Maximum, Minimum*** *Find the maximum and/or minimum value(s) of the function. A graph will be helpful.*

59. $f(x) = x^2 - 3x - 4$

60. $g(x) = -x^2 + 4x + 3$

61. $g(x) = x^2 - x, -1 \le x \le 4$

62. $g(x) = 3x - x^2, 0 < x < 4$

63. $f(x) = 6x - x^2, x \ge 0$

64. $f(x) = 12x - 3x^2, x \ge 0$

Exercises 65–66 ***Translations*** *Describe horizontal and vertical translations of the graph of f so that the result will be a graph of $y = x^2$. (Hint: Complete the square.)*

65. $f(x) = x^2 - 4x + 1$

66. $f(x) = x^2 + 4x + 5$

Exercises 67–70 ***Maximum, Minimum*** *Find the maximum and/or minimum value(s) of the function. Solve algebraically by considering the expression under the radical as a quadratic function with restricted domain. Use a graph of f as a check.*

67. $f(x) = \sqrt{3 + 2x - x^2}$

68. $f(x) = \sqrt{5 + 4x - x^2}$

69. $g(x) = \sqrt{3 + 2x + x^2}$

70. $g(x) = \sqrt{4 - 2x + x^2}$

71. ***Explore*** At the beginning of this section we noted that $F(x) = \sqrt{x^2 + x + 1}$ is not a quadratic function.

(a) In a decimal window, graph, in turn, each of the following:

$$f(x) = \sqrt{x^2 + x - 1}$$
$$F(x) = \sqrt{x^2 + x + 1}$$
$$G(x) = \sqrt{x^2 + 2x + 1}$$

(b) Write a paragraph to describe some of the differences between the graphs of f, F, and G. Note any symmetries and comment on domains and on ways each graph differs from a parabola.

(c) The graph of G should look familiar. Explain.

72. Solve the problem in Exercise 71 where the functions are

$$f(x) = \sqrt{x^2 - x - 1}$$
$$F(x) = \sqrt{x^2 - x + 1}$$
$$G(x) = \sqrt{x^2 - 2x + 1}$$

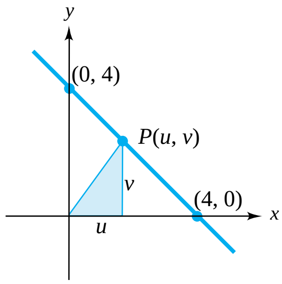
Exercises 73–74 ***Minimum Distance*** *Find the minimum distance from point P to the graph of $y = x^2 + 4x - 8$. (Hint: Let $Q(u, v)$ be any point on the parabola. Determine a formula that gives the distance d from point P as a function of u (do not simplify). Draw a graph and use* TRACE.*)*

73. $P(0, 1)$

74. $P(0, 2)$

75. ***Maximum Area*** Point $P(u, v)$ is in the first quadrant on the graph of the line $x + y = 4$. A triangular region is shown in the diagram.

(a) Express the area A of the shaded region as a function of u.



(b) For what point P will the area of the region be a maximum?

(c) What is the maximum area?

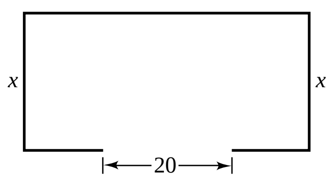
76. Repeat Exercise 75 for point $P(u, v)$ on the line segment joining (0, 3) and (4, 0).

77. **(a)** A wire 36 inches long is bent to form a rectangle. If x is the length of one side, find an equation that gives the area A of the rectangle as a function of x.

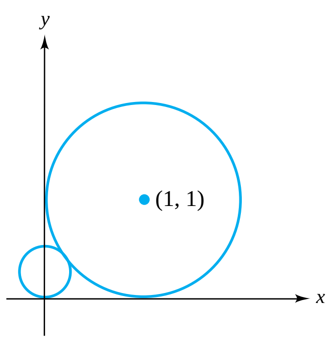
(b) For what values of x is the equation valid?

78. ***Maximum Area*** Of all the rectangles of perimeter 15 centimeters, find the dimensions (length and width) of the one with greatest area.

79. ***Maximum Area*** A farmer has 800 feet of fencing left over from an earlier job. He wants to use it to fence in a rectangular plot of land except for a 20-foot strip that will be used for a driveway (see the diagram, where x is the width of the plot). Express the area A of the plot as a function f of x. What is the domain of f? For what x is A a maximum?



80. In the diagram the smaller circle is tangent to the x-axis at the origin and it is tangent to the larger circle, which has a radius of 1 and center at (1, 1). What is the radius of the smaller circle?



81. ***Maximum Revenue*** A travel agent is proposing a tour in which a group will travel in a plane of capacity 150. The fare will be \$1400 per person if 120 or fewer people go on the tour; the fare per person for the entire group will be decreased by \$10 for each person in excess of 120. For instance, if 125 go, the fare for each will be $\$1400 - \$10(5) = \$1350$. Let x represent the number of people who go on the tour and T the total revenue (in dollars) collected by the agency. Express T as a function of x. What value of x will give a maximum total revenue? It will be helpful to draw a graph of the function.

82. ***Minimum Area*** A piece of wire 100 centimeters long is to be cut into two pieces; one of length x centimeters, to be formed into a circle of circumference x, and the other to be formed into a square of perimeter $100 - x$ centimeters. Let A represent the sum of the areas of the circle and the square.

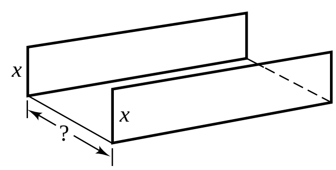
(a) Find an equation that gives A as a function of x.

(b) For what value of x will A be the smallest? What is the smallest area?

83. Solve the problem in Exercise 82 if the two pieces are to be formed into a square and an equilateral triangle.

84. Solve the problem in Exercise 82 if the two pieces are to be formed into a circle and an equilateral triangle.

85. ***Maximum Capacity*** A long, rectangular sheet of galvanized tin, 10 inches wide, is to be made into a rain gutter. The two long edges will be bent at right angles to form a rectangular trough (see diagram, which shows a cross section of the gutter with height x inches).



(a) Find a formula that gives the area A of the cross section as a function of x. What is the domain of this function?

(b) What value of x will give a gutter with maximum cross sectional area? Solve algebraically and use a graph as a check.

86. ***Minimum Time*** A forest ranger is in the forest 3 miles from the nearest point P on a straight road. His car is parked down the road 5 miles from P. He can walk in the forest at a rate 2 mi/hr and along the road at 5 mi/hr.

(a) Toward what point Q on the road between P and his car should he walk so that the total time T it takes to reach the car is the least?

(b) How long will it take to reach the car? (*Hint:* Let $|PQ| = x$, then use a graph to help you solve the problem.)