87. In Exercise 86, solve the problem if the ranger is 4 miles from P.

88. ***Looking Ahead to Calculus*** A solid has as its base the region in the xy-plane bounded by the circle $x^2 + y^2 = 4$.

(a) If every vertical cross section perpendicular to the x-axis is a semicircle, express the area K of the cross section at a distance u from the origin as a function of u.

(b) Repeat part **(a)** if each vertical cross section is an isosceles triangle with an altitude half as long as its base (not a semicircle).

(c) Repeat part **(a)** if each vertical cross section is an equilateral triangle.

(d) Repeat part **(a)** if each vertical cross section is a rectangle whose base is twice its vertical height.

2.6 COMBINING FUNCTIONS

> What is proved about numbers will be a fact in any universe.
>
> *Julia Robinson*

Just as we combine numbers to get other numbers, so we may combine functions to get other functions. The first four ways of combining functions give familiar sums, differences, products, or quotients, as we would expect. **Composition,** less familiar, is a key idea throughout much of what follows.

Definition: sum, difference, product, quotient functions

Suppose f and g are given functions. Functions denoted by $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$ are given by:

Sum: $(f + g)(x) = f(x) + g(x)$
Difference: $(f - g)(x) = f(x) - g(x)$
Product: $(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient: $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

The domain of each combined function is the set of all real numbers for which the right side of the equation is meaningful as a real number. Use parentheses as needed for clarity.

The definitions stated here are not mere formal manipulations of symbols. For instance, the plus sign in $f + g$ is part of the name of the function that assigns to each x the sum of two numbers, $f(x) + g(x)$.

Neyman . . . interviewed me [for a job at Berkeley and] said he would let me know. . . . I didn't really expect anything to happen. I had already written 104 letters of application to black colleges. Eventually I got a letter from [Neyman] saying something like "In view of the war situation and the draft possibilities, they have decided to appoint a woman to this position." [My eventual appointment here came 12 years later.]
David Blackwell

▶EXAMPLE 1 ***Combining functions*** If $f(x) = 4x - 6$ and $g(x) = 2x^2 - 3x$, write an equation for **(a)** $f - g$ and **(b)** $\frac{f}{g}$, and give the domain of each.

Solution

(a) $(f - g)(x) = f(x) - g(x) = (4x - 6) - (2x^2 - 3x) = -2x^2 + 7x - 6.$
The domain is the set of real numbers.

(b)
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{4x - 6}{2x^2 - 3x} = \frac{2(2x - 3)}{x(2x - 3)},$$

which simplifies to $\frac{2}{x}$ for $x \neq \frac{3}{2}$. Therefore $\left(\frac{f}{g}\right)(x) = \frac{2}{x}$, where the domain is $\{x \mid x \neq 0 \text{ and } x \neq \frac{3}{2}\}$. ◀

Composition of Functions

Another way to combine functions is used frequently and plays an important role in both precalculus and calculus.

Definition: composition of functions

Suppose f and g are functions. The **composition function, $f \circ g$,** read "**f of g**," is the function whose value at x is given by

$$(f \circ g)(x) = f(g(x)).$$

Thus to write a formula for $(f \circ g)(x)$, in the rule defining f,

replace each x in $f(x)$ by $g(x)$.

The **domain** of $f \circ g$ is the set of all real numbers x such that both $g(x)$ is defined, and $f(g(x))$ is defined.

The reason for calling the composition $f \circ g$ "f of g" is that the value of the composition function at a given number c is "f of $g(c)$."

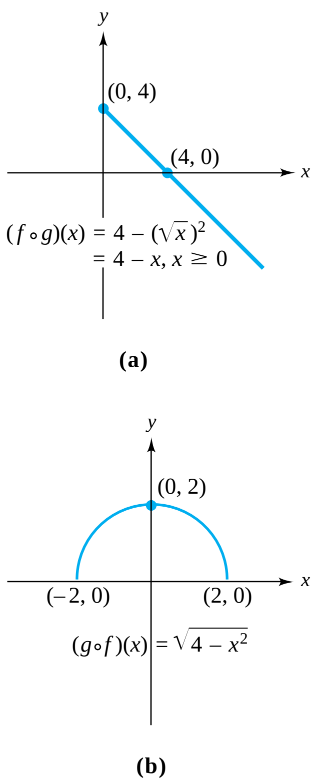


FIGURE 37

▶EXAMPLE 2 Two compositions If $f(x) = 4 - x^2$ and $g(x) = \sqrt{x}$, **(a)** write an equation and **(b)** draw a calculator graph of ***(i)*** $f \circ g$ ***(ii)*** $g \circ f$.

Solution

(a) For each composition, we follow the procedure given in the definition.

(i) $(f \circ g)(x) = f(g(x)) = 4 - (g(x))^2 = 4 - (\sqrt{x})^2$.

For $\sqrt{x}$ to be a real number we must have $x \geq 0$, and when $x \geq 0$, we can simplify the equation for $f \circ g$:

$$(f \circ g)(x) = 4 - x, \text{ where } x \geq 0.$$

(ii) $(g \circ f)(x) = g(f(x)) = \sqrt{f(x)} = \sqrt{4 - x^2}$.

Again, the domain is limited: for $4 - x^2 \geq 0$, we have $-2 \leq x \leq 2$.

(b) With a graphing calculator we can always enter the compositions in the form we wrote above, Y1 = 4 − (√X)² and Y2 = √(4 − X²).

If your calculator has a Y = menu where you can enter several functions, there are other options. For example, having entered f and g as Y1 = 4 − X² and Y2 = √X, since $f(g(x)) = 4 - (g(x))^2$, we can enter $f \circ g$ as Y3 = 4 − Y2² and $g \circ f$ as Y4 = √Y1. Observe that for $f \circ g$ we follow the defining rule for composition functions: replace each x in $f(x)$ by $g(x)$.

The calculator graphs are shown in Figure 37. Note that the limitations on the domain are obvious from the graphs and that we can also read off the ranges. The range of $f \circ g$ is $(-\infty, 4]$, and the range of $g \circ f$ is the closed interval $[0, 2]$.

Alternate Solution Sometimes it is easier to verbalize the rules that define functions. The rules for f and g state that, for any given input, f squares the input and subtracts the result from 4, while g takes the square root of its input. Thus, suppose $\sqrt{x}$ is the input. The function f squares $\sqrt{x}$ and subtracts the result from 4: $4 - (\sqrt{x})^2$. Similarly, when g is applied to $f(x)$, g takes the square root of $f(x)$. The output is: $g(f(x)) = \sqrt{f(x)} = \sqrt{4 - x^2}$. ◀

Example 2 shows that $f \circ g$ and $g \circ f$ are not the same function. In general $f \circ g$ and $g \circ f$ are different, although there are important exceptions, as the next example demonstrates.

►EXAMPLE 3 ***Equal compositions*** If $f(x) = 3x - 8$ and $g(x) = \frac{x+8}{3}$, write an equation that gives the rule of correspondence for **(a)** $f \circ g$ **(b)** $g \circ f$.

Solution

Here the rule for f is "triple the input and then subtract 8;" for g it is "add 8 to the input and then divide the sum by 3."

(a) $$(f \circ g)(x) = f(g(x)) = f\left(\frac{x+8}{3}\right) = 3\left(\frac{x+8}{3}\right) - 8 = x.$$

(b) $$(g \circ f)(x) = g(f(x)) = g(3x - 8) = \frac{(3x-8)+8}{3} = x.$$

Thus $(f \circ g)(x) = (g \circ f)(x)$ for every number x. We say that the two functions $f \circ g$ and $g \circ f$ are equal, $f \circ g = g \circ f$. ◄

►EXAMPLE 4 ***Composition equations*** If $f(x) = x^2 - 2x$ and $g(x) = 3 - x$, solve the equations.

(a) $(f \circ g)(x) = 0$ **(b)** $(g \circ f)(x) + x^2 + 5 = 0$

Strategy: Write each equation in a more familiar form.

Solution

(a) $(f \circ g)(x) = f(g(x)) = f(3 - x) = (3 - x)^2 - 2(3 - x) = x^2 - 4x + 3$. Thus the given equation becomes

$$x^2 - 4x + 3 = 0 \quad \text{or} \quad (x-1)(x-3) = 0.$$

The solutions are 1 and 3.

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 - 2x) = 3 - (x^2 - 2x) = -x^2 + 2x + 3$. Replacing $(g \circ f)(x)$ by $-x^2 + 2x + 3$, the given equation becomes

$$(-x^2 + 2x + 3) + x^2 + 5 = 0 \quad \text{or} \quad 2x + 8 = 0.$$

The solution is -4. ◄

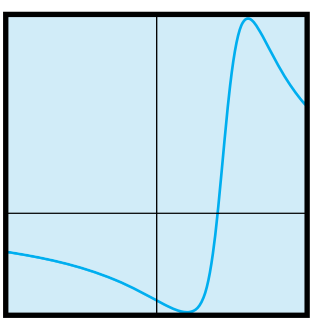
►EXAMPLE 5 ***Maxima and minima from calculator graphs*** Let F denote the composition $g \circ f$ on the limited domain $D = [-5, 5]$, where

$$f(x) = \frac{2x - 4}{x^2 - 4x + 5} \quad \text{and} \quad g(x) = x^2 + 3x.$$

(a) Draw a calculator graph of $F(x) = g(f(x))$.
(b) From your graph, find the maximum and minimum values of F.
(c) Find the solution set for $g(f(x)) > 0$.

Solution

(a) Writing a formula for the composition $g(f(x))$ requires us to replace each x in $x^2 + 3x$ by the entire $f(x)$. The process is messy, to say the least, but some calculators are designed to make composition much easier. See the Technology Tip following this example. Not knowing the range beforehand, we may set an x-range of $[-5, 5]$ to match the domain and adjust as necessary. A calculator graph is shown in Figure 38.



[−5, 5] by [−2.1, 4.1]

FIGURE 38
$F(x) = g(f(x))$

(b) Using the TRACE function on the graph of $y = F(x)$, we find the low point near $(1, -2)$ and the high point near $(3, 4)$. Shifting a decimal window, we confirm that the the maximum value of F is 4 and the minimum value is -2.

(c) The graph crosses the x-axis at $(2, 0)$, as is easily verified by evaluating $F(2)$, and is above the x-axis for all values of x (from the domain of F) greater than 2. Thus the solution set for $g(f(x)) > 0$ is the interval $(2, 5]$. ◀

TECHNOLOGY TIP ◆ *Graphing compositions and defining functions*

When composing functions, let the calculator do the hard work. In Example 5, to enter $g(f(x)) = (f(x))^2 + 3f(x)$, we need *lots* of parentheses:

Y = ((2X − 4)/(X² − 4X + 5))² + 3((2X − 4)/(X² − 4X + 5)).

If your calculator allows you to enter a list of functions, Y1, Y2, . . . , (TI and Casio) then you can enter the composition function much more simply. First enter f as Y1 and then use Y1 to enter $g(f(x))$ as Y2 = g(Y1):

Y1 = (2X − 4)/(X² − 4X + 5)   Y2 = Y1² + 3Y1.

HP–38 Having entered functions F1(X) = (2 * X − 4)/(X² − 4 * X + 5) and F2(X) = X² + 3 * X, write the composition as F3(X) = F2(F1(X)) and graph.

HP–48 On the home screen, enter each function as an equation, 'F(X) = (2 * X − 4)/(X² − 4 * X + 5)'. Then press the DEF key (above STO). Similarly for 'G(X) = X² + 3 * X'. Then on the PLOT screen enter the function as 'G(F(X))' and graph.

▶EXAMPLE 6 ***Composition inequality*** If $f(x) = x^2 - 9$ and $g(x) = 2x - 5$, find the solution for $f(g(x)) < 0$.

Strategy: Get simpler expressions for the composition function, substitute, and solve.

Solution

$f(g(x)) = f(2x - 5) = (2x - 5)^2 - 9$. Therefore the given inequality may be written as $(2x - 5)^2 - 9 < 0$. This is equivalent to

$$(2x - 5)^2 < 9 \quad \text{or} \quad -3 < 2x - 5 < 3 \quad \text{or} \quad 1 < x < 4.$$

The solution set is $\{x \mid 1 < x < 4\}$.

Alternate Solution *Graphical* We have seen often that calculator graphs allow us to read the solution set for an inequality such as $(2x - 5)^2 - 9 < 0$ or $4x^2 - 20x + 16 < 0$. We can graph Y = (2X − 5)² − 9 or we can simplify the inequality to an equivalent form, by dividing through by 4, getting $x^2 - 5x + 4 < 0$, and graph Y = X² − 5X + 4. In either case we have a parabola that crosses the x-axis at $(1, 0)$ and $(4, 0)$. See Figure 39. We read the solution set as $\{x \mid 1 < x < 4\}$. ◀

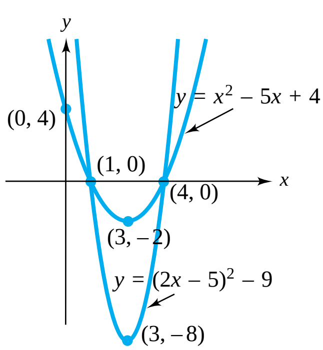


FIGURE 39

▶EXAMPLE 7 ***Applied composition*** An oil spill on a lake assumes a circular shape with an expanding radius r given by $r = \sqrt{t + 1}$, where t is the number of minutes after measurements are started and r is measured in meters.

(a) Find a formula that gives the area A of the circular region at any time t.

(b) What is the area at the beginning measurement $(t = 0)$? What is the area 3 minutes later?

Strategy: **(a)** Since $r = \sqrt{t + 1}$ is a function of t and $A = \pi r^2$ is a function of r, then by composing functions we can express A as a function of t.

Solution

(a) Follow the strategy.

$$A = \pi(\sqrt{t + 1})^2 = \pi(t + 1).$$

Thus A as a function of t is

$$A = \pi t + \pi.$$

When t *is* 0, $A = \pi \cdot 0 + \pi = \pi$ square meters. When t is 3, $A = 3\pi + \pi = 4\pi$ square meters. ◀

Calculator Evaluations

Many function evaluations by calculator actually involve composition of functions, especially with calculators that use "Reverse Polish" operations. With such a calculator, to evaluate $F(x) = \sqrt{x^2 + 1}$ when x is 3, we enter 3, square it, and add 1, after which we take the square root. This amounts to treating F as a composition $f \circ g$, where $g(x) = x^2 + 1$ and $f(x) = \sqrt{x}$. We accomplish the same thing if we have a graphing calculator using an Algebraic Operating System when we use the ANS key. Using the same example, if we evaluate $3^2 + 1$ and ENTER, the calculator displays 10. If we then evaluate √ANS, we are taking the composition of the square root function with the previously evaluated $x^2 + 1$ function.

▶EXAMPLE 8 ***Function as a composition*** If $F(x) = \frac{1}{x^2 + 1}$, express F as a composition of two functions.

Solution

Let $f(x) = \frac{1}{x}$ and $g(x) = x^2 + 1$. Then

$$f(g(x)) = f(x^2 + 1) = \frac{1}{x^2 + 1}.$$

Thus, $F(x)$ is given by $F(x) = (f \circ g)(x)$. ◀

In problems of the type discussed in Example 8, be aware that there are many different solutions. For example, we could have taken

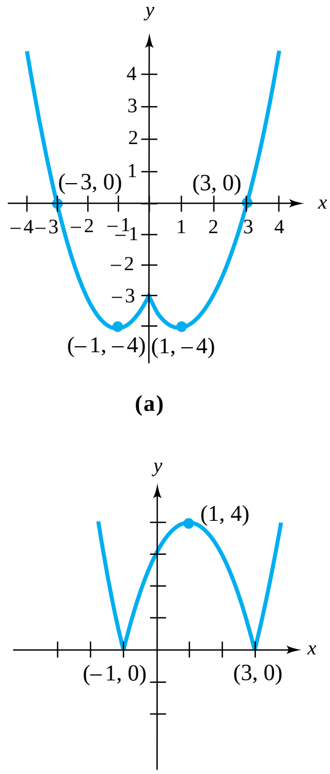
$$f(x) = \frac{1}{x + 1} \qquad \text{and} \qquad g(x) = x^2.$$

Then

$$f(g(x)) = f(x^2) = \frac{1}{x^2 + 1}.$$

Composition with Absolute Value

Composition of functions with the absolute value function affects graphs in a consistent fashion giving us two more useful basic transformations. It is easiest to look at a specific example.



(a)
(b)
FIGURE 40

►EXAMPLE 9 *Composing absolute value with a quadratic function*

Let $f(x) = x^2 - 2x - 3$ and $g(x) = |x|$. Write an equation and draw a calculator graph of **(a)** $f \circ g$ **(b)** $g \circ f$.

Solution

(a) $(f \circ g)(x) = f(|x|) = (|x|)^2 - 2|x| - 3 = x^2 - 2|x| - 3$, since $|x|^2 = x^2$. For a calculator graph, we enter Y = X² − 2 abs (X) − 3, and get a graph like that in Figure 40a. We observe that $x^2 - 2|x| - 3$ is an *even function.* The graph in Figure 40a consists of the portion of the parabola $y = x^2 - 2x - 3$ to the right of the x-axis, together with its reflection through the y-axis.

(b) $(g \circ f)(x) = g(x^2 - 2x - 3) = |x^2 - 2x - 3|$. For the graph we enter Y = abs(X² − 2X − 3). The graph is shown in Figure 40b. Since the absolute value of a number cannot be negative (recall that $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$), any portion of the graph of the parabola $y = x^2 - 2x - 3$ *below the y-axis* is reflected upward through the x-axis. ◄

The effects we observe in the graphs in Figure 40 are applicable in general. For any function $f(x)$, the function $f(|x|)$ is an even function whose graph is symmetric about the y-axis. Thus the graph of $y = f(|x|)$ consists of the graph of $y = f(x)$ for $x \geq 0$, together with the horizontal reflection of this portion about the y-axis.

Similarly, since $|f(x)| \geq 0$, for the graph of $y = |f(x)|$, any part of the graph of $y = f(x)$ that lies above the x-axis is unchanged; whatever part of the graph lies below the x-axis is reflected upward through the x-axis.

These transformations are consistent with the basic transformations of Section 2.3. A transformation operation on the "outside," $|f(x)|$, affects the vertical aspects of the graph; an operation on the argument, "inside," $f(|x|)$, affects the graph horizontally.

Composition of a function with the absolute value function

From the graph of $y = f(x)$, the graph of

$y = \lvert f(x) \rvert$	$y = f(\lvert x \rvert)$
is a **vertical reflection:** the part above the x-axis is unchanged; any portion below the x-axis is reflected up, through the x-axis.	is a **horizontal reflection:** function is even; the portion to the right of the y-axis is unchanged and is also reflected to the left, through the y-axis.

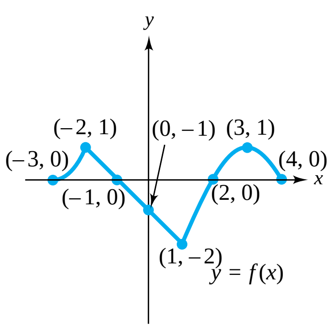


FIGURE 41

►EXAMPLE 10 *Composition with absolute value*

The graph of a function f is shown in Figure 41. If $g(x) = |x|$, draw a graph of **(a)** $f \circ g$ **(b)** $g \circ f$, identifying the points corresponding to the labeled points in Figure 41. Explain the thinking used to get each graph.

Solution

(a) $(f \circ g)(x) = f(g(x)) = f(|x|)$. From the box above, $f(|x|)$ is an even function. Knowing the graph of the function for positive x-values, the rest of the graph is obtained by taking the horizontal reflection in the y-axis. Each labeled point (a, b) is reflected to the point $(-a, b)$. The resulting graph is shown in Figure 42a.

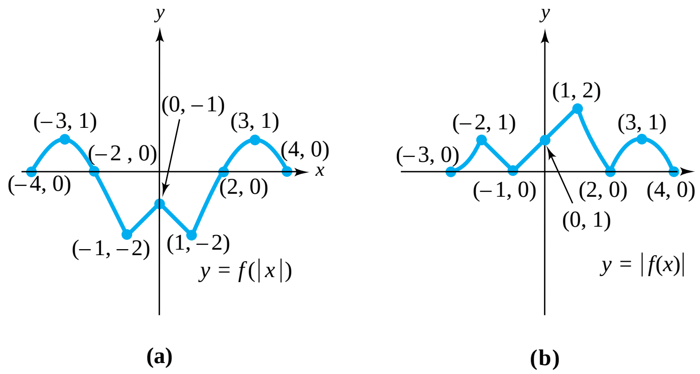


FIGURE 42

(b) $(g \circ f)(x) = g(f(x)) = |f(x)|$. Since $|f(x)|$ can never be negative, the graph can contain no points below the x-axis. Whenever the graph of f dips below the x-axis, the graph of $|f(x)|$ is reflected back up, above the axis. Any point $(a, -b)$ on the graph of f below the x-axis is reflected to the point (a, b). The graph is shown in Figure 42b. ◀

EXERCISES 2.6

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. If $f(x) = x^2$ and $g(x) = x^2 - 1$, then $g \circ f$ is a quadratic function in x.

2. If $f(x) = x^2$ and g is any function for which the domain of $g \circ f$ is not the empty set, then the function $g \circ f$ must be an even function.

3. If $f(x) = 2x - 1$, then $f(a + b) = f(a) + f(b)$ for all real numbers a and b.

4. If $g(x) = 3x$, then $g(c + d) = g(c) + g(d)$ for all real numbers c and d.

5. If $f(x) = x^2 + 1$ and $g(x) = x + 3$, then the graph of $y = (f \circ g)(x)$ contains no points below the x-axis.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. If $f(x) = 2x + 5$ and $g(x) = Int(x)$, then $(g \circ f)(\sqrt{5}) =$ ______.

7. If $f(x) = 2x + 5$ and $g(x) = Int(x)$, then $(f \circ g)(\sqrt{5}) =$ ______.

8. If $f(x) = 2x + 1$ and $g(x) = x^2$, then $(f \circ g)(-1) =$ ______.

9. If $f(x) = 2x - 1$ and $g(x) = x^2 - 3x - 4$, then the zeros of $g \circ f$ are ______.

10. If $f(x) = x^2 - 5x + 4$ and $g(x) = x^2$, then the sum of the roots of the equation $(f \circ g)(x) = 0$ is equal to ______.

Develop Mastery

Exercises 1–4 Evaluate **(a)** $(f - g)(-1)$ **(b)** $(f \cdot g)(0.5)$.

1. $f(x) = 2x$, $g(x) = 1 - 2x$

2. $f(x) = x^2 - 3$, $g(x) = \sqrt{x + 4}$

3. $f(x) = |x - 2|$, $g(x) = x + 1$

4. $f(x) = x^2 - x$, $g(x) = 3|1 - x|$

Exercises 5–8 ***Sum and Quotient Functions*** *Find an equation to describe the rule for* **(a)** $(f + g)(x)$ *and* **(b)** $(\frac{f}{g})(x)$. *In each case state the domain.*

5. $f(x) = x - \frac{1}{x}$, $g(x) = x$

6. $f(x) = x^2 - 1$, $g(x) = 1 - x$

7. $f(x) = \sqrt{x} - 2$, $g(x) = 1 - \sqrt{x}$

8. $f(x) = x - 4$, $g(x) = \frac{1}{x}$

Exercises 9–10 ***Composition Functions*** *Use $f(x) = x + 2$ and $g(x) = x^2 - 2x$.*

9. Evaluate **(a)** $(f \circ g)(-1)$ **(b)** $(g \circ f)(3)$ **(c)** $(f \circ f)(4)$

10. Find an equation to describe **(a)** $(f \circ g)$ **(b)** $(g \circ f)$.

11. ***Composition from Tables*** The domain of function f is $\{-3, -1, 0, 1, 3\}$ and the domain of g is $\{-1, 0, 1, 3, 5\}$. The rules for f and g are given in tabular form:

x	-3	-1	0	1	3
$f(x)$	-1	0	2	3	5

x	-1	0	1	3	5
$g(x)$	-2	-1	2	3	4

(a) Complete the following table for $g \circ f$. If an entry is undefined write U.

x	-3	-1	0	1	3
$(g \circ f)(x)$					

What is the domain of
(b) $g \circ f$? **(c)** $f \circ g$?

Exercises 12–13 ***Domain of Composition*** *Use $f(x) = \sqrt{x}$ and $g(x) = x^2 - 4$.*

12. Give an equation to describe $f \circ g$. State the domain.

13. Give an equation to describe $g \circ f$. State the domain.

Exercises 14–19 ***Solving Equations*** *For functions f and g,*

$$f(x) = x^2 - 2x - 3 \quad \text{and} \quad g(x) = 2x - 3,$$

solve the equation.

14. $(f + g)(x) = 10$ **15.** $\left(\frac{f}{g}\right)(x) = x + 1$

16. $(f \cdot g)(x) = 0$ **17.** $(g \circ f)(x) = 3x$

18. $(f \circ g)(x) = 5$ **19.** $(g \circ f)(x) + x^2 = 0$

Exercises 20–25 ***Solving Inequalities*** *For functions f and g,*

$$f(x) = -x^2 - x + 1 \quad \text{and} \quad g(x) = 3 - x,$$

find the solution set.

20. $(f + g)(x) < 1$ **21.** $(f - g)(x) \geq -4$

22. $(f \circ g)(x) + x \leq 1$ **23.** $(f \circ g)(x) + x^2 \geq 0$

24. $(g \circ f)(x) + 1 > 0$ **25.** $(g \circ g)(x) \geq x - 1$

Exercises 26–29 ***Graph of Compositions*** ***(a)*** *Draw a graph of the function $y = (f \circ g)(x)$.* ***(b)*** *Give the x and y intercept points. (Solve graphically and then verify algebraically.)*

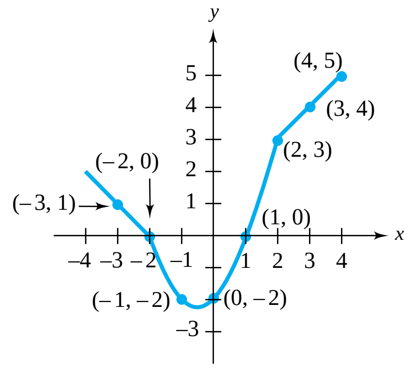
26. $f(x) = x^2 - 3x$, $g(x) = \sqrt{x}$

27. $f(x) = x^2 + x$, $g(x) = x^2 - 2x$

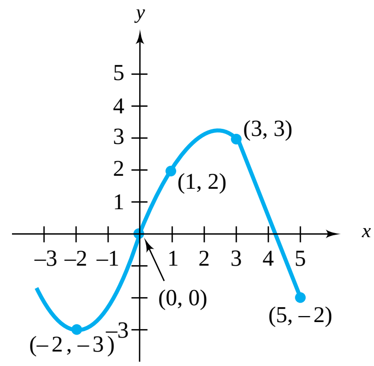
28. $f(x) = x^2 - 2x$, $g(x) = x^2 + x$

29. $f(x) = 9 - x^2$, $g(x) = x^2 - 4$

The following graphs apply to Exercises 30–36.



$y = f(x)$



$y = g(x)$

30. ***Reading Graphs*** Graphs of the functions f and g are shown. Complete the following tables.

x	-3	0	1	2	4
$(g \circ f)(x)$					

x	-2	0	1	3	5
$(f \circ g)(x)$					

Exercises 31–32 *Is the statement true or is it false?*

31. **(a)** $(f \circ g)(0.9) > 0$ **(b)** $-2 < (f \circ g)(4) < 2$

32. **(a)** $(g \circ f)(0.75) > 0$ **(b)** $2 < (f \circ g)(3.1) < 3$

Exercises 33–36 ***Related Graphs*** *Use the graphs in Exercise 30 to draw a graph of h. First give a verbal description of the strategy you plan to use. Label the coordinates of five points that must be on the graph of h. Can a graph of h be used as a check? Explain.*

33. **(a)** $h(x) = f(-x)$ **(b)** $h(x) = -g(x)$

34. **(a)** $h(x) = f(|x|)$ **(b)** $h(x) = |g(x)|$

35. **(a)** $h(x) = f(x - 2)$ **(b)** $h(x) = g(x) - 2$

36. **(a)** $h(x) = g(|x|)$ **(b)** $h(x) = |f(x)|$

*Exercises 37–40 Given functions f and g, **(a)** find equations that describe the composition functions $f \circ g$ and $g \circ f$. **(b)** Are the functions $f \circ g$ and $g \circ f$ equal? That is, do they have the same domain D, and is $(f \circ g)(x) = (g \circ f)(x)$ for every x in D?*

37. $f(x) = 3x - 1$, $g(x) = \dfrac{x + 1}{3}$

38. $f(x) = 4 - 3x$, $g(x) = \dfrac{4 - x}{3}$

39. $f(x) = x^2 + 1$, $g(x) = \sqrt{x - 1}$

40. $f(x) = x^2 + 1$, $g(x) = \sqrt{x}$

Exercises 41–44 For the given function f, find a function g such that $f(g(x)) = x$ for every value of x. (Hint: In Exercise 41, $f(g(x)) = 2g(x) - 5$; solve the equation $2g(x) - 5 = x$ for $g(x)$.)

41. $f(x) = 2x - 5$

42. $f(x) = 3 - 4x$

43. $f(x) = \dfrac{2x}{x - 2}$

44. $f(x) = \dfrac{-3x}{2x + 3}$

*Exercises 45–48 **Evaluating Combined Functions** If $f(x) = \sqrt{x}$ and $g(x) = \frac{x}{x - 1}$, evaluate the expression and round off the result to two decimal places.*

45. $(f + g)(\sqrt{3})$

46. $(f \circ g)(1.63)$

47. $(g \circ f)(5)$

48. $\left(\dfrac{f}{g}\right)(0.37)$

*Exercises 49–52 **Function as a Composition** Function F is given. Find two functions f and g so that $F(x) = (f \circ g)(x)$. Solutions to these problems are not unique.*

49. $F(x) = \dfrac{1}{x^2 + 5}$

50. $F(x) = \sqrt{x^2 - 3x + 5}$

51. $F(x) = |5x + 3|$

52. $F(x) = \dfrac{4}{x^2} + 1$

*Exercises 53–56 **Functions of Your Choice (a)** Give formulas for functions f (quadratic) and g (linear) of your choice that satisfy the specified conditions. **(b)** Determine a formula for $f \circ g$ and draw its graph. **(c)** What are the coordinates of the lowest or highest point on the graph of $f \circ g$?*

53. Function f has a positive and a negative zero; g has a zero between 1 and 3.

54. The graph of f opens downward and has no x-intercept points; the graph of g has a positive slope.

55. The graph of f contains points (0, 2) and (3, 0); the graph of g passes through (0, −2) and has negative slope.

56. Function f has no real zeros and its graph contains the point (1, −2); function g has a zero at −3.

57. For functions $f(x) = 2x + 5$ and $g(x) = Int(x)$, find the solution set for **(a)** $(f \circ g)(x) = 0$ and **(b)** $(g \circ f)(x) = 0$

58. For functions $f(x) = Int(x)$ and $g(x) = x^2 - x - 6$, find the minimum value of $f \circ g$.

*Exercises 59–60 **Graphing Composition Functions** Graph $g \circ f$ where $g(x) = x^2 - 4$ and f is the given function. For Y1 enter the formula for $f(x)$, for Y2 enter $x^2 - 4$, and for Y3 enter Y1² − 4 (that is, $(g \circ f)(x)$). Before drawing the graphs, make a prediction about how Y2 and Y3 are related. Review material in Section 2.3 (Operating on the "inside") if necessary. Draw the graphs of Y2 and Y3 simultaneously and see if your prediction is correct.*

59. **(a)** $f(x) = x - 2$ **(b)** $f(x) = 2x$ **(c)** $f(x) = 0.5x$

60. **(a)** $f(x) = x + 4$ **(b)** $f(x) = 1.5x$ **(c)** $f(x) = 0.3x$

Exercises 61–62 Replace $g(x)$ in Exercises 59–60 with $g(x) = x^2 - 4x - 2$.

61. **(a)** $f(x) = x - 2$ **(b)** $f(x) = x + 2$ **(c)** $f(x) = -x$

62. **(a)** $f(x) = x - 3$ **(b)** $f(x) = x + 1$ **(c)** $f(x) = -x - 1$

*Exercises 63–64 **Graphs** Draw graphs of $g(x) = x^2 - 4x - 2$ and $f \circ g$ for the given function f. Before you draw the graphs, predict how they are related. See Section 2.3.*

63. **(a)** $f(x) = x + 2$ **(b)** $f(x) = |x|$

64. **(a)** $f(x) = x - 2$ **(b)** $f(x) = -x$

*Exercises 65–66 **Highest and Lowest Points** Draw a calculator graph of $g \circ f$. Find the coordinates of the **(a)** highest point and **(b)** the lowest point. See Example 5. (Hint: Let Y1 = $f(x)$ and Y2 = g(Y1) and draw the graph of Y2.)*

65. $f(x) = \dfrac{2x}{x^2 + 1}$ $g(x) = x^2 + 3x$

66. $f(x) = \dfrac{2x}{x^2 + 1}$ $g(x) = x^2 + 3x + 1$

Exercises 67–70 Express the given function as a composition of two of these four functions

$$f(x) = x - 4 \qquad g(x) = x^2 + 1$$
$$h(x) = \frac{1}{x} \qquad k(x) = |x|.$$

67. $F(x) = |x| - 4$

68. $G(x) = \dfrac{1}{|x|}$

69. $H(x) = \dfrac{1}{x^2} + 1$

70. $K(x) = x^2 - 3$

71. If $g(x) = 2x - 3$ and $f(g(x)) = 4x^2 - x$, find $f(-5)$.

72. If $g(x) = 4 - x^2$ and $f(g(x)) = \dfrac{3 - x^2}{x^2}$, find $f(3)$.

73. If $g(x) = x - 5$ and $f(g(x)) = \sqrt{x + 1}$, find $f(3)$.

74. If $g(x) = 2x + 5$ and $f(g(x)) = x^2 + 4$, find $f(-1)$.

Exercises 75–76 ***Iteration Evaluations*** *A function f is given. New functions are denoted* $f^{(1)}, f^{(2)}, f^{(3)}, \ldots$, *where* $f^{(1)}(x) = f(x), f^{(2)}(x) = f(f(x)), f^{(3)}(x) = f(f(f(x))), \ldots$. *Observe that the notation* $f^{(n)}$ *indicates repeated composition of f, not multiplication; that is,* $f^{(n)}(x)$ *is not the same as* $(f(x))^n$.

75. $f(x) = \dfrac{-3x}{2x + 3}$

(a) Evaluate $f^{(1)}(-1), f^{(2)}(-1), f^{(3)}(-1), f^{(4)}(-1)$.

(b) Based on your observations, what is $f^{(16)}(-1)$? $f^{(23)}(-1)$?

76. $f(x) = \dfrac{2x}{x - 2}$

(a) Evaluate $f^{(1)}(3), f^{(2)}(3), f^{(3)}(3), f^{(4)}(3)$.

(b) Based on your observations, what is $f^{(24)}(3)$? $f^{(47)}(3)$?

77. ***Maximum Cost*** A manufacturer determines that the cost C (in dollars) to build x graphing calculators is described by the equation

$$C = 80 + 48x - x^2 \qquad \text{for } 0 \le x \le 40.$$

Also, it is known that in t hours, the number x of calculators that can be produced is

$$x = 4t, \text{ where } 0 \le t \le 10.$$

(a) Express C as a function of t.

(b) What is the cost when the factory operates four hours?

(c) For what time t is the cost the greatest?

78. A rock is thrown into a lake causing a ripple in the shape of an expanding circle whose radius r is given by $r = \sqrt{t}$, where t is the number of seconds after the rock hits the water and r is measured in feet.

(a) What are the radius, circumference, and area of the circle when $t = 4$?

(b) Express the circumference C and area A as functions of t.

(c) At what time t is the circumference 8 feet?

(d) At what time t is the area 36 square feet?

79. A spherical balloon is being inflated in such a way that the diameter d is given by $d = \frac{t}{2}$, where t is measured in seconds and d in centimeters.

(a) Express the volume V of the balloon as a function of t.

(b) At what time t will the volume be 20 cubic centimeters?

80. ***Cost, Revenue, Profit*** A manufacturing company sells toasters to a retail store for \$25 each plus a fixed handling charge of \$15 on each order. The retailer applies a 30 percent markup to the total price paid to the manufacturer.

(a) Suppose the order consists of 20 toasters. How much does the retailer pay for the order? What is the retailer's total revenue from the sale of the 20 toasters? How much profit per toaster does the retailer make?

(b) Suppose C is the cost to the retailer for an order of x toasters, R is the total revenue from the sale of x toasters, and P is the profit per toaster. Find formulas that give C, R, and P as functions of x.

81. A circle is shrinking in size in such a way that the radius r (in feet) is a function of time t (in minutes), given by the equation $r = f(t) = \dfrac{1}{t + 1}$. The area of the circle is given by $A(r) = \pi r^2$, so the area is also a function of time, given by $(A \circ f)(t)$.

(a) Write a formula for $(A \circ f)(t)$.

(b) What is the area at the end of one minute? Two minutes?

(c) For what value of t is the area $\dfrac{\pi}{25}$?

82. ***Number of Bacteria*** The number of bacteria in a certain food is a function of the food's temperature. When refrigerated, the number is $N(T)$ at a temperature T degrees Celsius, described by the equation

$$N(T) = 10\,T^2 - 60\,T + 800, \text{ for } 3 \le T \le 13.$$

When the food is removed from the refrigerator the temperature increases and t minutes later the temperature is $T = 2t + 3$, for $0 \le t \le 5$.

(a) Determine an equation that describes the number of bacteria t minutes after the food is removed from the refrigerator.

(b) How many bacteria are in the food three minutes after it is removed from the refrigerator?

(c) How many minutes after the food is taken out of the refrigerator will it contain 2150 bacteria? Check graphically.

83. ***Volume of Balloon*** A spherical weather balloon is being inflated in such a way that the radius is $r = f(t) = 0.25t + 3$, where t is in seconds and r is in feet. The volume V of the balloon is the function $V(r) = 4\pi r^3/3$.

(a) What is the radius when the inflation process begins?

(b) Write an equation to describe the composition $V \circ f$ that gives the volume at t seconds after inflation begins.

(c) What is the volume of the balloon 10 seconds after inflation begins?

(d) In how many seconds will the volume be 400 cubic feet? Check graphically.