

5 Probability: The Study of Randomness

5.1 Randomness

Randomness: a phenomenon is random if individual outcomes are uncertain but there is a regular distribution of outcomes in a large number of repetitions

Probability: the probability of an outcome of a random phenomenon is the proportion of times this outcome would occur in a very large number of repetitions

Independence: the outcome of one trial (repetition) does not depend on the outcome of any of the other trials (repetitions)

Example:

Which of these phenomena are independent?

- Flipping a coin twice – yes/no
- Drawing two numbers in a lottery (but **not** placing back the first number into the urn after it has been drawn) – yes/no
- Drawing two numbers in a lottery (but placing back the first number into the urn after it has been drawn) – yes/no

5.2 Probability Models

Sample Space: we use the symbol S (or Ω) to denote the set of all possible outcomes of a random phenomenon

Event: events are subsets that consist of one or multiple possible outcomes of a random phenomenon

Example:

- (i) Toss a regular “fair” dice.
Possible outcomes: $S = \dots$

- (ii) Toss a “fair” coin.
Possible outcomes: $S = \dots$

Probability Rules:

- (i) The probability of any event A is a number between 0 and 1, i.e., $0 \leq P(A) \leq 1$.
- (ii) All possible outcomes together must have probability 1, i.e., $P(S) = 1$.
- (iii) Complement Rule:
The probability that an event A does not occur is 1 minus the probability that the event A occurs, i.e., $P(A^C) = 1 - P(A)$.
- (iv) Addition Rule for Disjoint Events:
If two events A and B have no outcomes in common, i.e., if they are disjoint, the probability that A or B occurs is the sum of their individual probabilities, i.e., $P(A \text{ or } B) = P(A) + P(B)$.

Example: “Fair” Dice

$$P(\text{not } 5) =$$

$$P(3 \text{ or } 4) =$$

Venn Diagram:

a graphical way to represent the relationship of events

Assigning Probabilities: Equally Likely Outcomes

If a random phenomenon has k possible outcomes that are all equally likely, then the probability for each individual outcome is $\frac{1}{k}$. The probability of any event A is

$$P(A) = \frac{\# \text{ outcomes in } A}{\# \text{ outcomes in } S} = \frac{\# \text{ outcomes in } A}{k}.$$

Probability Rules (continued):

(v) Multiplikation Rule for Independent Events:

The probability that two independent events A and B occur in two trials is the product of their individual probabilities, i.e., $P(A \text{ and } B) = P(A) \cdot P(B)$.

Example: "Fair" Dice

$$P(\text{first a 5, then a 6}) =$$

$$P(\text{first an odd number, then a number } \leq 2) =$$